UMA204: Introduction to Basic Analysis

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Contents

Lecture 01

Mon 01 Jan '24

We assume the following.

- Basics of set theory
- Existence of $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the usual operations $+$ and $\cdot$

For a recap, refer lectures 1 to 3 of UMA101.

1 Number Systems

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

1.1 Recap of the Naturals

$\mathbb{N}$ is the unique minimal inductive set granted by the ZFC axioms. Addition and multiplication are defined by the recursion principle and showed that they

- are associative and commutative,
- admit identity elements 0 and 1 respectively,
- satisfy the distributive law,

- satisfy cancellation laws,
- but do not admit inverses.

1.2 Relations

(Recall) A relation on a set A is a subset $R \subseteq A \times A$. We write $a \, R \, b$ to denote $(a, b) \in R$.

Definition 1.1 (Partial order)**.** A relation R on A is called a *partial order* if it is

- reflexive, *i.e.* $a \, R \, a$ for all $a \in A$;
- antisymmetric, *i.e.* if $a \, R \, b$ and $b \, R \, a$ then $a = b$ for all $a, b \in A$;
- transitive, *i.e.* if $a \, R \, b$ and $b \, R \, c$ then $a \, R \, c$ for all $a, b, c \in A$.

Additionally, if for all $x, y \in A$, $x \, R \, y$ or $y \, R$, then R is called a *total order*.

A set A equipped with a partial order $\leq$ is called a *partially ordered set* (or *poset*).
A set A equipped with a total order $\leq$ is called a *totally ordered set* or simply an *ordered set*.

Examples.

- $(\mathbb{N}, \leq)$ where we say that $a \leq b$ if $\exists \, c \in \mathbb{N}$ such that $a + c = b$.
- $(\mathbb{N}, |)$ where we say that $a \mid b$ if $\exists \, c \in \mathbb{N}$ such that $a \cdot c = b$.

In UMA101, we defined order slightly differently, where we said that either $a \leq b$ or $b \leq a$ but never both. This is a "strict order". We will denote a weak partial order by $\leq$ and a strict partial order by $<$. (the notation is suggestive of how to every order there is a corresponding strict order and vice versa).

Definition 1.2 (Equivalence)**.** An *equivalence relation* on a set A is a relation R satisfying

- reflexivity;
- symmetry, *i.e.* if $a \, R \, b$ then $b \, R \, a$ for all $a, b \in A$;
- transitivity.

Notation. We write $[x]_R$ to denote the set $\{y \in A \mid x \, R \, y\}$.

Proposition 1.3. The collection $\mathscr{A} = \{[x]_R \mid x \in A\}$ partitions A.

Proof. For every $x \in A$, $x \in [x]_R$ and so $\bigcup \mathscr{A} = A$.

Let $[x]_R \cap [y]_R \neq \varnothing$, where $x, y \in A$. Then there exists $z \in A$ such that $x \; R \; z$ and $y \; R \; z$, from which it follows that $x \; R \; y$ and $[x]_R = [y]_R$. $\square$

1.3 Integers

We cannot solve $3 + x = 2$ in $\mathbb{N}$. We introduce $\mathbb{Z}$ to solve this problem.

Consider the relation R on $\mathbb{N} \times \mathbb{N}$ given by

$$(a, b) \; R \; (c, d) \iff a + d = b + c.$$

(check that this is an equivalence relation trivial).

Definition 1.4. We define $\mathbb{Z}$ to be the set of equivalence classes of R, notated $\mathbb{N} \times \mathbb{N}/R$.

Further, define

- $[(a, b)] +_{\mathbb{Z}} [(c, d)] \coloneqq [(a + c, b + d)]$;
- $[(a, b)] \cdot_{\mathbb{Z}} [(c, d)] \coloneqq [(ac + bd, ad + bc)]$.
- $z_1 \leq_{\mathbb{Z}} z_2$ iff there exists $n \in \mathbb{N}$ such that $z_1 +_{\mathbb{Z}} [(n, 0)] = z_2$ (alternatively, $[(a, b)] \leq_{\mathbb{Z}} [(c, d)]$ iff $a + d \leq b + c$).

We need to check that these are well-defined. What does this mean? Consider

$$[(1, 2)] +_{\mathbb{Z}} [(3, 4)] = [(4, 6)]$$
$$[(3, 4)] +_{\mathbb{Z}} [(3, 4)] = [(6, 8)]$$

Our definition must ensure that $[(4, 6)] = [(6, 8)]$.
In general, the definitions are well-defined if they are independent of the choice of representatives.

Lecture 02

Tue 02 Jan '24

Proposition 1.5. The operations $+_{\mathbb{Z}}$, $\cdot_{\mathbb{Z}}$ and the order $\leq_{\mathbb{Z}}$ are well-defined.

Proof. Suppose $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. Then

$$\begin{aligned}
a + b' &= a' + b \\
c + d' &= c' + d \\
(a + c) + (b' + d') &= (a' + c') + (b + d) \\
[(a + c, b + d)] &= [(a' + c', b' + d')]
\end{aligned}$$

Since $\leq_{\mathbb{Z}}$ is defined in terms of $+_{\mathbb{Z}}$, it is also well-defined. □

Definition 1.6 (Ring)**.** A *ring* is a set S with two binary operations $+$ and $\cdot$ such that for all $a, b, c \in S$,

(i) addition is associative,

(ii) addition is commutative,

(iii) there exists an additive identity 0,

(iv) there exists an additive inverse $-a$,

(v) multiplication is associative,

(vi) there exists a multiplicative identity 1,

(vii) multiplication is distributive over addition (on both sides).

A ring in which multiplication is commutative is called a *commutative ring*.

Note that inverses are unique, since if $a + b = 0$ and $a + b' = 0$, then $b = (b' + a) + b = b' + (a + b) = b'$.

Definition 1.7 (Ordered Ring)**.** An *ordered ring* is a ring S with a total order $\leq$ such that for all $a, b, c \in S$,

(i) $a \leq b$ implies $a + c \leq b + c$,

(ii) $0 \leq a$ and $0 \leq b$ implies $0 \leq ab$.

Theorem 1.8.

- $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, \leq_{\mathbb{Z}})$ is an ordered (commutative) ring.
- The map $f = n \mapsto [(n,0)]$ from $\mathbb{N}$ to $\mathbb{Z}$ is an injective map that respects $+$, $\cdot$ and $\leq$. That is, for all $n, m \in \mathbb{N}$,

  (i) $f(n+m) = f(n) +_{\mathbb{Z}} f(m)$,

  (ii) $f(nm) = f(n) \cdot_{\mathbb{Z}} f(m)$,

  (iii) $n \leq m \iff f(n) \leq_{\mathbb{Z}} f(m)$.

  In other words, f is an ordered commutative ring isomorphism onto a subset of $\mathbb{Z}$.

Thus, we may view $(\mathbb{N}, +, \cdot, \leq)$ as a subset of $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, \leq_{\mathbb{Z}})$, denote $[(n,0)]$ as n and drop $\mathbb{Z}$ in the subscript. We further define $-[(a,b)] \coloneqq [(b,a)]$ and $z_1 - z_2 \coloneqq z_1 + (-z_2)$.

Moreover, we have the following properties.

Proposition 1.9.

- There are no zero divisors in $\mathbb{Z}$. That is, for all $a, b \in \mathbb{Z}$, $ab = 0$ implies $a = 0$ or $b = 0$.
- The cancellation laws hold: for all $a, b, c \in \mathbb{Z}$, $a + b = a + c$ implies $b = c$, and $ab = ac$ implies $a = 0$ or $b = c$.
- (trichotomy) For all $z \in \mathbb{Z}$, $z = n$ or $z = -n$ for some $n \in \mathbb{N}$.

2 Rationals

We cannot solve $3x = 2$ in $\mathbb{Z}$.

Proof. Suppose $3x = 2$ for some $x = [(a,b)] \in \mathbb{Z}$. Then

$$\begin{aligned} 3x &= 2 \\ [(3,0)] \cdot [(a,b)] &= [(2,0)] \\ [(3a,3b)] &= [(2,0)] \\ 3a &= 3b + 2 \end{aligned}$$

What now? $\square$

We define $\mathbb{Z}^*$ to be $\mathbb{Z} \backslash \{0\}$ and define the relation R on $\mathbb{Z} \times \mathbb{Z}^*$ by $(a,b)R(c,d)$ if $ad = bc$. Then R is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$.

Definition 2.1. We define $\mathbb{Q}$ to be the set of equivalence classes of R, notated $\mathbb{Z} \times \mathbb{Z}^*/R$.

We define operations $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ on $\mathbb{Q}$ by

$$[(a,b)] +_{\mathbb{Q}} [(c,d)] \coloneqq [(ad+bc, bd)]$$
$$[(a,b)] \cdot_{\mathbb{Q}} [(c,d)] \coloneqq [(ac, bd)]$$

Since there are no zero divisors in $\mathbb{Z}$, $bd \neq 0$.

We define an order $\leq_{\mathbb{Q}}$ on $\mathbb{Q}$ by

$$[(a,b)] \leq_{\mathbb{Q}} [(c,d)] \iff (ad-bc)bd \leq 0.$$