UMA204: Introduction to Basic Analysis

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Contents

Lecture 03: Wed 03 Jan '24

Definition 0.1 (Exponentiation)**.** The recursion principle guarantees the existence of $\mathrm{pow} : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ such that for all $n, m \in \mathbb{N}$,

$$\mathrm{pow}(m, 0) = 1$$
$$\mathrm{pow}(m, n+1) = m \cdot \mathrm{pow}(m, n)$$

We extend this to $\mathrm{pow} : \mathbb{Q}^* \times \mathbb{Z} \to \mathbb{Q}$ as follows.

$$\mathrm{pow}\left(\frac{a}{b}, m\right) \coloneqq \begin{cases} a^m/b^m & \text{if } m \in \mathbb{N} \\ b^m/a^m & \text{if } -m \in \mathbb{N} \end{cases}$$

We write z^n to denote $\mathrm{pow}(z, n)$.

Remarks. Note that we have defined 0^0 to be 1, but we don't really care.

Proposition 0.2. Expoentiation is well-defined.

Proof. Let $a/b = \tilde{a}/\tilde{b} \in \mathbb{Q}$. That is, $a\tilde{b} = b\tilde{a} \in \mathbb{Z}$. For $m \in \mathbb{N}$, thus $a^m\tilde{b}^m = b^m\tilde{a}^m$ (easily proved by induction).

Similarly if $-m \in \mathbb{N}$. $\square$

Theorem 0.3. There exists no $x \in \mathbb{Q}$ such that $x^2 = 2$.

We first make note of the following lemma.

Lemma 0.4. Let $x \in \mathbb{Q}$. Then there exists $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$ such that $x = p/q$.

In particular, if $x > 0$, then $x = p/q$ for some $p \in \mathbb{N}$, $q \in \mathbb{N}^*$.

Proof. Let $x = a/b$. If $b \in \mathbb{N}$, we are done. Otherwise, $x = -a/-b$ and $-b \in \mathbb{N}$. $\square$

We will make use of the well-ordered property of $(\mathbb{N}, \leq)$ proved below in theorem 0.5.

Proof of theorem 0.3. Suppose there exists such an x. By the field properties, $(-x)^2 = x^2$. Thus we may assume $x \geq 0$. Let $x = p/q$ for some $q \in \mathbb{N}^*$. Since $x \geq 0$, we have $p \geq 0 \iff p \in \mathbb{N}$.

Let $A = \{q \in \mathbb{N}^* \mid x = p/q \text{ for some } p \in \mathbb{N}\}$. A is non-empty.

By the well-ordering principle, A has a least element q_0. Let $p_0 \in \mathbb{N}$ such that $x = p_0/q_0$.

We know that $1 < x < 2$ [why? because $(\cdot)^2$ is an increasing function on positive reals (why? difference of squares)] and so $0 < p_0 - q_0 < q_0$. Now

$$\begin{aligned} \frac{2q_0 - p_0}{p_0 - q_0} &= \frac{2 - x}{x - 1} \\ &= \frac{(2-x)(x+1)}{x^2 - 1} \\ &= 2x + 2 - x^2 - x \\ &= x, \end{aligned}$$

in contradiction to the minimality of q_0. $\square$

Theorem 0.5 (Well-ordering principle)**.** Every non-empty subset of $\mathbb{N}$ has a least element.

Proof. Let $S \subseteq \mathbb{N}$ be non-empty. We define $P(n)$ to be "if $n \in S$, then S has a least element". Clearly $P(0)$ holds.

Suppose $P(k)$ holds for all $k \leq n \in \mathbb{N}$.
If $n + 1 \notin S$, $P(n + 1)$ holds vacuously.
If $\exists\, m \in S(m < n + 1)$, then $P(n + 1)$ holds by virtue of $P(m)$.
Otherwise $n + 1 \in S$ and $\forall\, m \in S(n + 1 \leq m)$, so that $n + 1$ is the least element of S.

In any case, $P(n + 1)$ holds. $\square$

Theorem 0.6. Let

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\}$$
$$B = \{x \in \mathbb{Q} \mid x^2 > 2, x > 0\}$$

Then A has no largest element and B has no smallest element.

Proof. Let $a \in A$. $a > -2$ since otherwise $a^2 \geq 4$. Let $c = a + \frac{2-a^2}{2+a}$. Clearly $c > a$. Now

$$c = \frac{2a+2}{2+a}$$
$$c^2 = \frac{4a^2+8a+4}{4+4a+a^2}$$
$$c^2 - 2 = \frac{2a^2-4}{(2+a)^2} < 0$$

Thus $c \in A$.

For B, let $c = b + \frac{2-b^2}{2+b} = \frac{2b+2}{2+b}$. Clearly $0 < c < b$ and $c^2 - 2 = \frac{2b^2-4}{(2+b)^2} > 0$. Thus $c \in B$. $\square$

Corollary 0.7. $(\mathbb{Q}, \leq)$ does not have the least upper bound property.

Proof. Let b be an upper bound of A. Clearly $b > 0$. b cannot be in A since A has no largest element. b cannot have square 2 by theorem 0.3. Thus $b \in B$. But since B has no smallest element, there is a $b' \in B$ which is less than b.

For any $a \in A$, if $a < 0$ then $a < b'$. Otherwise, $0 < (b')^2 - a^2 = (b'-a)(b'+a)$ and so $a < b'$. Thus b' is an upper bound of A which is less than b.

Since b was arbitrary, A cannot have a least upper bound. $\square$

0.1 Ordered Fields with LUB

(Recall from UMA101 Lecture 6) Given an ordered set $(X, \leq)$, a subset $S \subseteq X$ is said to be *bounded above* (resp. below) if there exists $x \in X$ such that for all $s \in S$, $s \leq x$ (resp. $x \leq s$), and any such x is called an *upper* (resp. lower) *bound* of S.

A (The) *supremum* or least upper bound of S is an element $x \in X$ such that x is an upper bound of S and for all upper bounds y of S, $x \leq y$. Similarly, infimum or greatest lower bound.

$(X, \leq)$ is said to have the least upper bound property if every non-empty bounded above subset of X admits a supremum.

Proposition 0.8. $(\mathbb{Q}, \leq)$ does not have the least upper bound property.

Proof. From theorem 0.6, we know that A has no largest element and B has no smallest element.

Let s be a supremum of A. Since there is no largest element in A, $s \notin A$. From theorem 0.3, we know that $s^2 \neq 2$. Thus by trichotomy, $s^2 > 2$ and so $s \in B$. But then there is an $s' \in B$ which is less than s but also an upper bound of A. This is a contradiction. $\square$

Theorem 0.9. Every ordered field F "contains" $\mathbb{Q}$, *i.e.*, there exists an injective map $f : \mathbb{Q} \to F$ that respects $+$, $\cdot$ and $\leq$.

We will notate this statement as $\mathbb{Q} \subseteq F$.

Proof. Let $f : \mathbb{Z} \to F$ be defined as

$$f(n) = \begin{cases} 0_F & \text{if } n = 0 \\ \underbrace{1_F + \cdots + 1_F}_{n \text{ times}} & \text{if } n > 0 \\ \underbrace{(-1_F) + \cdots + (-1_F)}_{m \text{ times}} & \text{if } n = -m, m > 0 \end{cases}$$

Note that $f(-n) = -f(n)$ for all $n \in \mathbb{N}$. Let us show that $f(n+m) = f(n) + f(m)$ for all $n, m \in \mathbb{Z}$.

Case 1: $n = 0$ or $m = 0$. Immediate.

Case 2: $n > 0$ and $m > 0$. Then

$$\begin{aligned} f(n+m) &= \underbrace{1_F + \cdots + 1_F}_{n+m \text{ times}} \\ &= \underbrace{1_F + \cdots + 1_F}_{n \text{ times}} + \underbrace{1_F + \cdots + 1_F}_{m \text{ times}} \\ &= f(n) + f(m) \end{aligned}$$

Case 3: $n < 0$ and $m < 0$. Then $f(n+m) = -f((-n) + (-m)) = -(f(-n) + f(-m)) = f(n) + f(m)$.

Case 4: $nm < 0$. WLOG, let $m < 0 < n$. Suppose $0 < n + m$. Then $f(n+m) + f(-m) = f(n+m-m) = f(n)$ from case 2. Now suppose

$n+m<0$. Then $f(n)+f(-n-m)=f(n-n-m)=-f(m)$ from case 3. In either case, $f(n+m)=f(n)+f(m)$.

Now consider $f(nm)$. If $nm=0$, then $f(nm)=0_F=f(n)f(m)$. If $0<n,m$, then

$$\begin{aligned}
f(nm) &= \overbrace{1_F+\cdots+1_F}^{nm \text{ times}} \\
&= \underbrace{\overbrace{(1_F+\cdots+1_F)}^{n \text{ times}}+\cdots+\overbrace{(1_F+\cdots+1_F)}^{n \text{ times}}}_{m \text{ times}} \\
&= \underbrace{(1_F+\cdots+1_F)}_{n \text{ times}}\cdot\underbrace{(1_F+\cdots+1_F)}_{m \text{ times}} \\
&= f(n)f(m)
\end{aligned}$$

If either of n, m is negative, then we take the negative sign out and use the above case.

Thus f respects $+$ and $\cdot$.

Suppose that $m<n$. Then $f(n)-f(m)=f(n)+f(-m)=f(n-m)=(n-m)1_F$ (where $z1_F$ is notation for 1_F added z times). $n-m$ is positive, but 1_F added to itself a positive number of times must be positive. This is because $0_F<1_F$ (UMA101) and so $k1_F<(k+1)1_F$ for all $k\in\mathbb{N}^+$. Induction gives $0_F<k1_F$ for all $k\in\mathbb{N}^+$. Thus $f(m)<f(n)$ and so f respects $<$ (and hence $\leq$).

Finally, injectivity of f follows from order preservation.

We extend f to $\mathbb{Q}$ by defining $f(a/b)=f(a)f(b)^{-1}$. This continues to be an isomorphism. $\square$

Lecture 04: Wed 10 Jan '24

Definition 0.10 (Archimedean property)**.** An ordered field F is said to have the *Archimedean property* if for every $x,y>0$, there exists an $n\in\mathbb{N}\subseteq F$ such that $nx>y$.

Theorem 0.11. $\mathbb{Q}$ has the Archimedean property.

Proof. Let $x,y>0$ be rationals. If $x>y$, $n=1$ works. Suppose $x\leq y$. It suffices to show that $\exists\, n\in\mathbb{N}(nr>1)$, where $r=x/y$. Since r is positive, we have $p,q\in\mathbb{N}^*$ such that $r=p/q$. Let $n=2q$. This gives $nr>1$. $\square$

Remarks. Not all ordered fields have the Archimedean property.

Theorem 0.12. Let F be an ordered field with the LUB property. Then F has the Archimedean property.

Proof. Let $x, y > 0$. Suppose $\forall n \in \mathbb{N}(nx \leq y)$. Let $A = \{nx \mid n \in \mathbb{N}\}$. Clearly A is non-empty and bounded above. Then $\sup A$ exists and so there exists an $m \in \mathbb{N}$ such that $\sup A - x < mx$. Thus $\sup A < (m+1)x \in A$, a contradiction. $\square$

Theorem 0.13. Let F be an ordered field with the LUB property. Then $\mathbb{Q}$ is dense in F, *i.e.*, given $x < y \in F$, there exists a rational $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. Follows from theorem 0.11 and problem 4 on assignment 1. $\square$

0.2 The Reals

Theorem 0.14 (Dedekind/Cauchy)**.** There exists a unique (up to isomorphism) ordered field with the LUB property.

Proof of uniqueness. Let F and G be OFWLUB. Let h be identity on $\mathbb{Q} \subseteq F, G$. Let $z \in F$ and

$$A_z = \{w \in \mathbb{Q} \mid w \leq_F z\}.$$

Claim: A_z is non-empty and bounded above when viewed as a subset of G. First, A_z is non-empty by density applied to $z - 1_F, z$ or Archimedean applied to $-z$. Secondly, by Archimedean (or density) there exists a *rational* upper bound q of A_z in F. Thus A_z has a supremum in G.

We define $h(z) = \sup_G A_z$.

We check that h preserves order. Let $z < w \in F$. By density of $\mathbb{Q}$ in F, there exist rationals r, s, t such that $z < r < s < t < w$. Then $A_z \subsetneq A_w$ as subsets of F and hence of G. Thus

$$h(z) = \sup_G A_z \leq_G r < s < t \leq_G \sup_G A_w = h(w).$$

$\square$

0.2.1 Dedekind's Construction

Definition 0.15 (Dedekind cut)**.** A *Dedekind cut* is a non-empty proper subset $A \subsetneq \mathbb{Q}$ such that

(i) if $a \in A$, then $b \in A$ for all $b \in \mathbb{Q}$ with $b < a$.

(ii) if $a \in A$, then there exists a $c \in A$ such that $a < c$.

Definition 0.16 ($\mathbb{R}$)**.** We define

$$\mathbb{R} := \left\{A \in 2^{\mathbb{Q}} \mid A \text{ is a Dedekind cut}\right\}.$$

Further,

(i) $A \leq B \iff A \subseteq B$;

(ii) $A + B = \{a + b \mid a \in A, b \in B\}$. The additive identity $0 = \{x \in \mathbb{Q} \mid x < 0\}$;

(iii) for $A, B > 0$,

$$A \cdot B = \{q \in \mathbb{Q} \mid q \leq rs \text{ for some } r \in A, s \in B\}.$$

If $A < 0$ but $B > 0$, then $A \cdot B = -((-A) \cdot B)$. If $B < 0$ but $A > 0$, then $A \cdot B = -(A \cdot (-B))$. If $A < 0$ and $B < 0$, then $A \cdot B = (-A) \cdot (-B)$.

Lecture 05: Thu 11 Jan '24

Theorem 0.17. $\mathbb{R}$ has the least upper bound property.

Proof. Let α be a non-empty subset of $\mathbb{R}$ that is bounded above. We claim that $S = \bigcup_{A \in \alpha} A$ is the supremum of α.

s is a cut: Since S is a union of a non-empty set of non-empty sets, it is non-empty. Since S is bounded above, say by some cut C, we have $S \subseteq C \subsetneq \mathbb{Q}$ and so $S \neq \mathbb{Q}$. If $a \in S$, then $a \in A$ for some $A \in \alpha$. Since A is a cut, every rational smaller than a is contained in A and thereby in S. Moreover, there exists an $a' \in A$ which is larger than a. Thus $a' \in S$ is larger than a.

upper bound: $A \subseteq S$ for all $A \in \alpha$.

least upper bound: For any $D \subsetneq S$, let $b \in S \setminus D$. But since $b \in A$ for some $A \in \alpha$, D is not an upper bound of α. □

Dedekind's construction is an "order completion". Thus all the order properties (LUB, density) are nice, but arithmetic is ugly.

0.2.2 Cauchy's Construction

There seem to be sequences in $\mathbb{Q}$ that "should" have a limit (*e.g.*, a monotone and bounded sequence) but do not (within $\mathbb{Q}$). We construct equivalence classes of sequences which "converge" to the same number, and define reals by those classes.

Definition 0.18 (Sequence)**.** A sequence of rational numbers is a $f\colon \mathbb{N} \to \mathbb{Q}$. We usually denote $f(k)$ by a_k and call it the k-th term of the sequence. The function f is usually written as $(a_k)_{k\in\mathbb{N}}$.

Definition 0.19. A sequence $(a_k)_{k\in\mathbb{N}} \subseteq \mathbb{Q}$ is said to be

(i) $\mathbb{Q}$-bounded if there exists an $M \in \mathbb{Q}$ such that $|a_k| \leq M$ for all $k \in \mathbb{N}$.

(ii) $\mathbb{Q}$-Cauchy if for every rational $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon$ for all $m, n \geq N$.

(iii) convergent in $\mathbb{Q}$ if there exists an $L \in \mathbb{Q}$ such that for all (rational) $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

Exercise 0.20. Show that if a sequence is convergent in $\mathbb{Q}$, then it is $\mathbb{Q}$-Cauchy, and if it is $\mathbb{Q}$-Cauchy, then it is $\mathbb{Q}$-bounded.

Remarks. From UMA101, we know that if a sequence is convergent in $\mathbb{Q}$, the limit is unique. We also know arithmetic laws of limits (which we proved over $\mathbb{R}$, but they hold over $\mathbb{Q}$as well).

Definition 0.21. Two sequences $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ are said to be *equivalent* if their difference converges to 0.

Proposition 0.22. Let $\mathcal{C}$ denote the space of $\mathbb{Q}$-cauchy sequences. Then $\sim$ given by $a \sim b$ if a and b are equivalent (as per the previous definition) is an equivalence relation.

Proof. Reflixivity and symmetry are immediate. Transitivity follows from the triangle inequality. $\square$

Definition 0.23 ($\mathbb{R}$)**.** We define

$$\mathbb{R} \coloneqq \mathcal{C}/\sim.$$

Further,

(i) $[a] +_{\mathbb{R}} [b] \coloneqq [a + b]$.

(ii) The additive identity $0 = [(0)_{n\in\mathbb{N}}]$.

(iii) $[a] \cdot_{\mathbb{R}} [b] \coloneqq [a \cdot b]$.

(iv) $[a] >_{\mathbb{R}} 0$ if there exists a rational $c > 0$ and an $N \in \mathbb{N}$ such that $a_n > c$ for all $n \geq N$. From positivity, we can define order as $[a] >_{\mathbb{R}} [b]$ iff there is some $[d] > 0$ such that $[a] + [d] = [b]$.