MA262: Introduction to Stochastic Processes

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Contents

Lecture 1. Thursday January 04

Chapter

The Course

Texts:

- *Markov Chains*, J. R. Norris
- *Introduction to Stochastic Processes*, Hoel, Port, Stone
- Karlin and Taylor

Grading:

(20%) 2 quizzes

(30%) 1 midterm

(50%) Final

Chapter I

Discrete time Markov Chains

Definition I.1 (Stochastic matrix)**.** Let $\mathcal{S}$ be a state set (at most countable). A matrix $P = (p_{xy})_{x,y\in\mathcal{S}}$ is called a *stochastic matrix* if $p_{xy} \geq 0$ for all $x, y \in \mathcal{S}$ and $\sum_{y\in\mathcal{S}} p_{xy} = 1$ for all $x \in \mathcal{S}$.

Definition I.2 (Markov chain)**.** Let $\mathcal{S}$ be a state set, $P = (p_{xy})$ a stochastic matrix, and μ_0 a probability distribution on $\mathcal{S}$, *i.e.*, $\mu_0(x) \geq 0$ for all $x \in \mathcal{S}$ and $\sum_{x\in\mathcal{S}} \mu_0(x) = 1$.

Suppose $X_0, X_1, \ldots$ are random variables defined on the same probability space taking values in $\mathcal{S}$. Then $(X_n)_{n\in\mathbb{N}}$ is called a *Markov chain* with initial distribution μ_0 and transition matrix P, denoted $MC(\mu_0, P)$, if X_0 has distribution μ_0 and for all $n \geq 0$,

$$\Pr(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0) = p_{x_n x_{n+1}}$$

whenever $\Pr(X_n = x_n, \ldots, X_0 = x_0) > 0$.

Notation. Whenever writing $\Pr(X_n \in A \mid (X_0, \ldots, X_{n-1}) \in B)$, it will be understood that only $\Pr((X_0, \ldots, X_{n-1}) \in B) > 0$ is considered.

Theorem I.3. *$(X_n)_{n=0}^N$ is $MC(\mu_0, P)$ iff*

$$\Pr(X_0 = x_0, \ldots, X_N = x_N) = \mu_0(x_0) p_{x_0 x_1} \ldots p_{x_{N-1} x_N}$$

for all $x_0, \ldots, x_N \in \mathcal{S}$.

Proof. Both directions are proven by induction.

Suppose $(X_n)_{n=0}^N$ is $MC(\mu_0, P)$. Then $\Pr(X_0 = x_0) = \mu_0(x_0)$.
If $\Pr(X_0 = x_0) > 0$, then $\Pr(X_0 = x_0, X_1 = x_1) = \mu_0(x_0) p_{x_0 x_1}$.
If $\Pr(X_0 = x_0) = 0$, then $\Pr(X_0 = x_0, X_1 = x_1) \leq \Pr(X_0 = x_0) = 0$, and so $\Pr(X_0 = x_0, X_1 = x_1) = 0 = \mu_0(x_0) p_{x_0 x_1}$.

Induction: Suppose

$$P_j \coloneqq \Pr(X_0 = x_0, \dots, X_j = x_j) = \mu_0(x_0) p_{x_0 x_1} \dots p_{x_{j-1} x_j}.$$

If this is zero, so is P_{j+1}, and so it is equal to $\mu_0(x_0) p_{x_0 x_1} \dots p_{x_{j-1} x_j} p_{x_j x_{j+1}}$. If not, then

$$\begin{aligned} P_{j+1} &= P_j \Pr(X_{j+1} = x_{j+1} \mid X_0 = x_0, \dots, X_j = x_j) \\ &= P_j p_{x_j x_{j+1}} \\ &= \mu_0(x_0) p_{x_0 x_1} \dots p_{x_{j-1} x_j} p_{x_j x_{j+1}}, \end{aligned}$$

closing the induction. In particular,

$$\Pr(X_0 = x_0, \dots, X_N = x_N) = \mu_0(x_0) p_{x_0 x_1} \dots p_{x_{N-1} x_N}.$$

Now for the converse, suppose

$$\Pr(X_0 = x_0, \dots, X_N = x_N) = \mu_0(x_0) p_{x_0 x_1} \dots p_{x_{N-1} x_N}$$

for all $x_0, \dots, x_N \in \mathcal{S}$. Then for any $x_0, \dots, x_{N-1} \in \mathcal{S}$,

$$\begin{aligned} \Pr(X_0 = x_0, \dots, X_{N-1} = x_{N-1}) &= \sum_{x_N \in \mathcal{S}} \Pr(X_0 = x_0, \dots, X_N = x_N) \\ &= \sum_{x_N \in \mathcal{S}} \mu_0(x_0) p_{x_0 x_1} \dots p_{x_{N-2} x_{N-1}} p_{x_{N-1} x_N} \\ &= \mu_0(x_0) p_{x_0 x_1} \dots p_{x_{N-2} x_{N-1}}. \end{aligned}$$

We have by backwards induction that for all $1 \le i \le N$,

$$\Pr(X_0 = x_0, \dots, X_i = x_i) = \mu_0(x_0) p_{x_0 x_1} \dots p_{x_{i-1} x_i}$$

and $\Pr(X_0 = x_0) = \mu_0(x_0)$. This allows us to deduce that

$$\Pr(X_{i+1} = x_{i+1} \mid X_0 = x_0, \dots, X_i = x_i) = p_{x_i x_{i+1}}$$

by definition of conditional probability. □

Fact I.4 (Strong Law of Large Numbers)**.** *Suppose $Z_1, Z_2, \dots$ are iid $\mathbb{R}$-valued random variables and $\mathrm{E}[Z_1]$ exists. Then*

$$\frac{Z_1 + \dots + Z_n}{n} \xrightarrow{a.s.} \mathrm{E}[Z_1]$$

as $n \to \infty$, that is,

$$\Pr\left\{\omega \in \Omega : \lim_{n \to \infty} \frac{Z_1(\omega) + \dots + Z_n(\omega)}{n} = \mathrm{E}[Z_1]\right\} = 1.$$

Fact I.5 (Weak Law of Large Numbers)**.** *Suppose $Z_1, Z_2, \dots$ are iid $\mathbb{R}$-valued random variables and $\mathrm{E}[Z_1]$ exists. Then*

$$\frac{Z_1 + \dots + Z_n}{n} \rightsquigarrow \mathrm{E}[Z_1]$$

as $n \to \infty$, that is, for any $\delta > 0$,

$$\lim_{n\to\infty} \Pr\left\{\left|\frac{Z_1 + \cdots + Z_n}{n} - \mathrm{E}[Z_1]\right| \le \delta\right\} = 1.$$

Fact I.6 (Central Limit Theorem)**.** *Suppose $Z_1, Z_2, \ldots$ are iid $\mathbb{R}$-valued random variables and $\mathrm{E}[Z_1^2]$ exists. Then*

$$Y_n := \frac{\sqrt{n}}{\sqrt{\mathrm{Var}(Z_1)}}\left(\frac{Z_1 + \cdots + Z_n}{n} - \mathrm{E}[Z_1]\right) \xrightarrow{d} N(0,1)$$

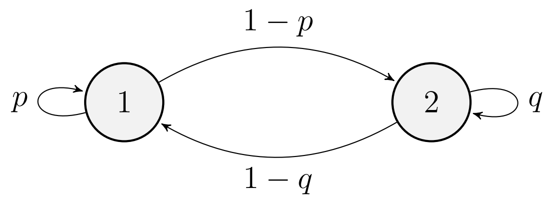
as $n \to \infty$, that is,

$$\lim_{n\to\infty} \Pr\{Y_n \le y\} = \Phi(y)$$

for all $y \in \mathbb{R}$, where Φ is the pdf of the standard normal distribution.

Examples.

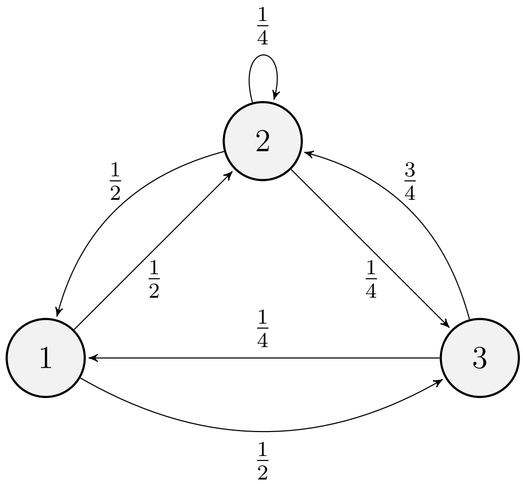
- A two-state Markov chain.



  This corresponds to the matrix

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}.$$

- A three-state Markov chain.



This has transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \end{pmatrix}.$$

- Simple random walk on $\mathbb{Z}$. Staring from some randomly chosen point, at each step, move right with probability p and left with probability $q := 1 - p$. That is,

$$\Pr(X_{n+1} = y \mid X_n = x) = \begin{cases} p & \text{if } y = x+1, \\ q & \text{if } y = x-1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Such a simple random walk is called symmetric if $p = q = \frac{1}{2}$. A special case is where $\mu_0 = \delta_x$ for some $x \in \mathbb{Z}$ (where δ_x is the Krönecker delta).

Suppose that $Y_1, Y_2, \dots$ are iid with distribution $\begin{pmatrix} 1 & -1 \\ p & 1-p \end{pmatrix}$. Each Y_i has expectation $2p-1$, and variance

$$\mathrm{E}[Y_1^2] - (\mathrm{E}[Y_1])^2 = 1 - (2p-1)^2 = 4pq.$$

We have that $(X_n)_{n\in\mathbb{N}} \stackrel{d}{=} (\sum_{j=1}^n Y_j)_{n\in\mathbb{N}}$.

Definition I.7. Suppose $Z_1, \dots, Z_k$ are random variables taking values in a state set $\mathcal{S}$ defined on a probability space $(\Omega, \mathcal{F}, \Pr)$. and $\tilde{Z}_1, \dots, \tilde{Z}_k$ are rvs taking values in a state set $\mathcal{S}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Then $(Z_1, \dots, Z_k)$ and $(\tilde{Z}_1, \dots, \tilde{Z}_k)$ are said to be identically distributed if

$$\Pr(Z_1 = x_1, \dots, Z_k = x_k) = \Pr\Big(\tilde{Z}_1 = x_1, \dots, \tilde{Z}_k = x_k\Big).$$

This is denoted as

$$(Z_1, \dots, Z_k) \stackrel{d}{=} (\tilde{Z}_1, \dots, \tilde{Z}_k).$$

Then from the weak law of large numbers,

$$\frac{X_n}{n} \to \mathrm{E}[Y_1] = 2p - 1.$$

From the central limit theorem,

$$\frac{X_n - n(p-q)}{\sqrt{n}\sqrt{4pq}} \xrightarrow{\mathrm{d}} N(0,1).$$

On a graph, a simple symmetric random walk is a random walk on a graph where each

$$p_{xy} = \begin{cases} \frac{1}{\deg(x)} & \text{if } y \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

On $\mathbb{Z}^2$, a simple random walk is given by p_N, p_E, p_S, p_W, where $p_N + p_E + p_S + p_W = 1$. At each step, move up with probability p_N, right with probability p_E, down with probability p_S, and left with probability p_W.

- Consider a shooting game with 4 modes: N (normal), D (distance), W (windy) and DW (distance and windy). The game changes mode randomly to a mode different from the current mode with direcred graph K_4 with some edge weights.

Theorem I.8. *If $(X_n)_{n\in\mathbb{N}}$ is a DTMC with transition matrix P, then*

$$\Pr_{\mu_0}(X_n = y) = (\mu_0 P^n)_y.$$

In particular, $\Pr_x(X_n = y) = (P^n)_{x,y} =: p_{xy}^{(n)}$.

Here, μ_0 is viewed as a row vector, and $\Pr_{\mu_0}$ is the distribution under the assumption that $X_0 \sim \mu_0$. Also, $\Pr_x$ is under the assumption that $\mu_0 = \delta_x$.

Proof.

$$\begin{aligned}\Pr_{\mu_0}(X_n = y) &= \sum_{x_0,\dots,x_{n-1}\in\mathcal{S}} \Pr(X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_n = y) \\ &= \sum_{x_0,\dots,x_{n-1}\in\mathcal{S}} (\mu_0)_{x_0} p_{x_0x_1} \cdots p_{x_{n-1}y} \\ &= (\mu_0 P^n)_y\end{aligned}$$

$\square$

Theorem I.9 (Markov property)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be $MC(\mu_0, P)$. Then for any $n \geq 0$, $l \geq 1$, $x_n, \dots, x_{n+l} \in \mathcal{S}$ and $A \subseteq \mathcal{S}^n$,*

$$\begin{aligned}&\Pr_{\mu_0}(X_i = x_i, n < i \leq n+l \mid X_n = x_n, (X_0, \dots, X_{n-1}) \in A) \\ &\qquad = \Pr_{x_n}(X_1 = x_{n+1}, \dots, X_l = x_{n+l})\end{aligned}$$

In other words, conditioning on $X_n = x_n$ and $(X_0, \dots, X_{n-1}) \in A$, the process $(X_n, X_{n+1}, \dots)$ is $MC(\delta_{x_n}, P)$.

Proof.

$$
\begin{aligned}
&\Pr_{\mu_0}(X_{n+l} = x_{n+l}, \dots, X_n = x_n, (X_0, \dots, X_{n-1}) \in A) \\
&\qquad = \sum_{(x_0,\dots,x_{n-1})\in A} p_{x_n x_{n+1}} \cdots p_{x_{n+l-1} x_{n+l}} \mu_0(x_0) p_{x_0 x_1} \cdots p_{x_{n-1} x_n} \\
&\qquad = \left(\sum_{(x_0,\dots,x_{n-1})\in A} \mu_0(x_0) p_{x_0 x_1} \cdots p_{x_{n-1} x_n} \right) p_{x_n x_{n+1}} \cdots p_{x_{n+l-1} x_{n+l}} \\
&\qquad = \Pr_{\mu_0}(X_n = x_n, (X_0, \dots, X_{n-1}) \in A) \cdot p_{x_n x_{n+1}} \cdots p_{x_{n+l-1} x_{n+l}}.
\end{aligned}
$$

By the definition of conditional probability,

$$
\begin{aligned}
&\Pr_{\mu_0}(X_{n+l} = x_{n+l}, \dots, X_n = x_n \mid X_n = x_n, (X_0, \dots, X_{n-1}) \in A) \\
&\qquad = p_{x_n x_{n+1}} \cdots p_{x_{n+l-1} x_{n+l}} \\
&\qquad = \delta_{x_n}(x_n) p_{x_n x_{n+1}} \cdots p_{x_{n+l-1} x_{n+l}} \\
&\qquad = \Pr_{x_n}(X_1 = x_{n+1}, \dots, X_l = x_{n+l}).
\end{aligned}
$$

□

Lecture 2. Tuesday January 09

Definition I.10 (Sigma algebra)**.** A *σ-algebra* over a set Ω is a collection $\mathcal{F}$ of subsets of Ω such that

(i) $\varnothing \in \mathcal{F}$,

(ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,

(iii) if $\mathcal{A} \subseteq \mathcal{F}$ is countable, then $\bigcup \mathcal{A} \in \mathcal{F}$.

Proposition I.11. *Let $\mathcal{F}$ be a σ-algebra over Ω. Then if $\mathcal{A} \subseteq \mathcal{F}$ is countable, then $\bigcap \mathcal{A} \in \mathcal{F}$.*

Proof. Let $B = \bigcap \mathcal{A}$. Then $B^c = \bigcup\{A^c : A \in \mathcal{A}\}$. By closure under complements, each $A^c \in \mathcal{F}$. By closure under countable unions, $B^c \in \mathcal{F}$. Thus $B = (B^c)^c \in \mathcal{F}$. □

Definition I.12 (Probability space)**.** A *probability space* is a triple $(\Omega, \mathcal{F}, \Pr)$, where Ω is a set, $\mathcal{F}$ is a σ-algebra over Ω, and $\Pr$ is a probability measure over $(\Omega, \mathcal{F})$.

Definition I.13 (Random variable)**.** Given a probability space $(\Omega, \mathcal{F}, \Pr)$ and a measurable space $(E, \mathcal{E})$, a *random variable* is a measurable function $X : \Omega \to E$, which means that for all $B \in \mathcal{E}$, $X^{-1}(B) \in \mathcal{F}$.

For our purposes, $E = \mathcal{S}$ and $\mathcal{E} = 2^{\mathcal{S}}$. Notice that if $(X_i)_{i=1}^n$ are random variables, then for any $B \in \mathcal{E}^n$,

$$\begin{aligned}(X_1, \ldots, X_n)^{-1}(B) = \{\omega \in \Omega : X_1(\omega) \in B_1, \ldots, X_n(\omega) \in B_n\} \\ = X_1^{-1}(B_1) \cap \cdots \cap X_n^{-1}(B_n) \in \mathcal{F},\end{aligned}$$

by closure under intersections. Thus $(X_1, \ldots, X_n)$ is a random variable onto the product space $(E^n, \mathcal{E}^n)$.

In fact this holds for any countable collection of random variables.

Definition I.14. Let X_1, X_2, $\ldots$ and X be random variables over a probability space $(\Omega, \mathcal{F}, \Pr)$. We define

Almost sure convergence. $X_n \xrightarrow{\text{a.s.}} X$ if

$$\Pr\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = 1.$$

Convergence in probability. $X_n \rightsquigarrow X$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \Pr\{|X_n - X| \leq \varepsilon\} = 1.$$

Convergence in distribution. $X_n \xrightarrow{\text{d}} X$ if for every x,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

where F's are cumulative distribution functions and F_X is continuous.

Exercise I.15 (Chapman-Kolmogorov equation)**.** *Recall that* $p_{ij}^{(n)} := \Pr_i(X_n = j)$. *Derive the Chapman-Kolmogorov equation*

$$p_{xy}^{(m+n)} = \sum_{z\in\mathcal{S}} p_{xz}^{(m)} p_{zy}^{(n)}$$

for all $x, y \in \mathcal{S}$ *and* $m, n \in \mathbb{N}$.

Solution. We have

$$\begin{aligned} p_{xy}^{(m+n)} &= P_{xy}^{m+n} \\ &= (P^m P^n)_{xy} \\ &= \sum_{z\in\mathcal{S}} P_{xz}^m P_{zy}^n \\ &= \sum_{z\in\mathcal{S}} p_{xz}^{(m)} p_{zy}^{(n)}. \end{aligned}$$ ■

Exercise I.16. *Let* $(X_n)_{n\in\mathbb{N}}$ *be* $MC(\mathcal{S}, \mu_0, P)$. *Show that for any* $k \geq 1$, $(X_{kn})_{n\in\mathbb{N}}$ *is* $MC(\mathcal{S}, \mu_0, P^k)$.

Solution. Let $Y_n = X_{kn}$. Let $N \in \mathbb{N}$. Then

$$\begin{aligned}\Pr_{\mu_0}(Y_0 = y_0, \ldots, Y_N = y_N) &= \Pr_{\mu_0}(X_0 = y_0, \ldots, X_{kN} = y_N) \\ &= \mu_0(y_0) p^{(k)}_{y_0 y_1} \cdots p^{(k)}_{y_{kN} y_N} \\ &= \mu_0(y_0)(P^k)_{y_0 y_1} \cdots (P^k)_{y_{kN} y_N}.\end{aligned}$$

Thus by theorem I.3, $(Y_n)_{n\in\mathbb{N}}$ is $MC(\mathcal{S}, \mu_0, P^k)$. ■

Definition I.17 (Communication)**.** Let $(X_n)_{n\in\mathbb{N}}$ be $MC(\mathcal{S}, \mu_0, P)$. For $x, y \in \mathcal{S}$, we say that x *leads to* y if $\Pr_x(X_n = y \text{ for some } n \geq 0) > 0$.

We say that x *communicates with* y if $x \to y$ and $y \to x$, and write $x \leftrightarrow y$.

Theorem I.18. *Suppose $x, y \in \mathcal{S}$ and $x \neq y$. Then the following are equivalent:*

(i) $x \to y$,

(ii) there exists an $n \geq 1$ such that $(P^n)_{xy} = p^{(n)}_{xy} > 0$,

(iii) there exists an $n \geq 1$ and $x_1, \ldots, x_{n-1} \in \mathcal{S}$ such that

$$p_{xx_1} p_{x_1x_2} \cdots p_{x_{n-1}y} > 0.$$

Proof.

$$\Pr_x(X_n = y \text{ for some } n \geq 0) \leq \sum_{n=0}^{\infty} \Pr_x(X_n = y) = \sum_{n=0}^{\infty} p^{(n)}_{xy}.$$

The sum is zero iff all terms are zero. Thus (i) and (ii) are equivalent.

Now since $p^{(n)}_{xy} = (P^n)_{xy}$, we have

$$p^{(n)}_{xy} = \sum_{x_1,\ldots,x_{n-1}\in\mathcal{S}} p_{xx_1} p_{x_1x_2} \cdots p_{x_{n-1}y}.$$

Thus $p^{(n)}_{xy}$ is zero iff all paths from x to y of length n have zero probability. This proves that (ii) $\iff$ (iii). □

Corollary I.19. *If $x \to y$ and $y \to z$, then $x \to z$.*

Proof. By theorem I.18, there exist $n, m \geq 1$ such that $p^{(n)}_{xy} > 0$ and $p^{(m)}_{yz} > 0$. Then by exercise I.15, $p^{(n+m)}_{xz} > 0$ so $x \to z$. □

Corollary I.20. *Communication is an equivalence relation.*

Proof. Reflexivity and symmetry are immediate. Transitivity follows from the previous corollary. □

Definition I.21 (Communicating class)**.** The equivalence classes of $\leftrightarrow$ are called *communicating classes.*

A communicating class $\mathcal{C}$ is *closed* if for all $x \in \mathcal{C}$ and $y \in \mathcal{S}$, $x \to y$ implies $y \in \mathcal{C}$.

A state x is *absorbing* if $\{x\}$ is a closed communicating class.

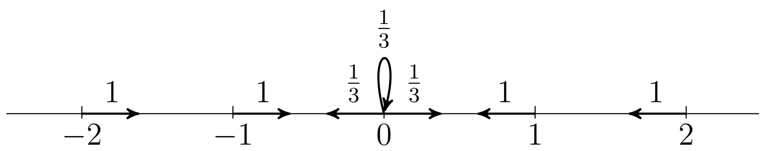
A markov chain is *irreducible* if its state space is itself a communicating class.

Definition I.22 (Period)**.** Let $(X_n)_{n\in\mathbb{N}}$ be $MC(\mathcal{S}, \mu_0, P)$. For each $x \in \mathcal{S}$, let $F_x = \left\{n \in \mathbb{N}^* : p_{xx}^{(n)} > 0\right\}$. The *period* of x is defined as $d_x = \gcd F_x$, where $\gcd \varnothing$ is considered to be 0.

A state x is *aperiodic* if $d_x = 1$. A Markov chain is *aperiodic* if all its states are aperiodic.

Examples.

- The simple random walk on $\mathbb{Z}$ is periodic with period 2.
- Consider the walk on $\mathbb{Z}$ given by



0 is aperiodic. 0's aperiodicity induces aperiodicity on all other states. Thus the chain is aperiodic.

Theorem I.23. *If $x \leftrightarrow y$, then $d_x = d_y$.*

Proof. Trivial when $x = y$. Suppose $x \neq y$ and let $n, m \in \mathbb{N}$ be lengths of paths from x to y and from y to x, respectively. Note that $d_x, d_y \neq 0$. By the Chapman-Kolmogorov equation, $p_{xx}^{(n+m)} \geq p_{xy}^{(n)} p_{yx}^{(m)} > 0$, so $d_x \mid n + m$.

Now let p be a path length from y to itself. Then $p_{xx}^{(n+m+p)} \geq p_{xy}^{(n)} p_{yy}^{(p)} p_{yx}^{(m)} > 0$, so $d_x \mid n + m + p$. This implies $d_x \mid p$. Since p was arbitrary, $d_x \mid d_y$.

By symmetry, $d_y \mid d_x$, so $d_x = d_y$. $\square$

Theorem I.24. *If $d_x \geq 1$, then there exists an $N \in \mathbb{N}^*$ such that for all $n \geq N$, $p_{xx}^{(nd_x)} > 0$.*

As a special case, if p is aperiodic, then $p_{xx}^{(n)} > 0$ for all large enough n.

We first prove a general number-theoretic result.

Theorem I.25 (Schur's Lemma)**.** *Suppose $S \subseteq \mathbb{N}^*$ and denote* $\gcd(S)$ *by g_S. Then there exists an $m_s \in \mathbb{N}^*$ such that for all $m \geq m_s$, there exist $k \in \mathbb{N}^*$, $e_1, \ldots, e_k \in \mathbb{N}^*$ and $s_1, \ldots, s_k \in S$ such that $mg_S = \sum_{i=1}^k e_i s_i$.*

We prove the following lemma to restrict S to a finite set.

Lemma I.26. *Let $S \subseteq \mathbb{N}^*$. Then there exists a finite set $S' \subseteq S$ such that* $\gcd(S) = \gcd(S')$*.*

Proof. Let $g_S = \gcd(S)$. For any finite set $S' \subseteq S$, we either have $\gcd(S') = g_S$ in which case we are done, or $\exists s \in S \setminus S'$ such that $\gcd(S') \nmid s$. In the latter case, we can add s to S' and continue, producing a sequence of finite sets with *strictly decreasing* gcds. Since the gcd can decrease only a finite number of times, this process must terminate with a finite set whose gcd is g_S. □

We will also use the following characterization of the gcd.

Lemma I.27. *Let $X \subseteq \mathbb{N}^*$ and let $Y = X \cup \{n\}$. Then* $\gcd(Y) = \gcd\{\gcd(X), n\}$*.*

Proof. Let $g = \gcd(Y)$ and $\tilde{g} = \gcd\{\gcd(X), n\}$.

- Since $\tilde{g} \mid \gcd(X)$ and $\tilde{g} \mid n$, we have $\tilde{g} \mid y$ for all $y \in Y$. Thus $g \mid \tilde{g}$.
- Since $g \mid y$ for all $y \in Y$, we have $g \mid \gcd(X)$ and $g \mid n$. Thus $\gcd\{\gcd(X), n\} = \tilde{g} \mid g$. □

We are now ready to prove Schur's Lemma.

Proof of Schur's Lemma. Let $S = \{s_1, s_2, \ldots, s_k\}$. Define $\tilde{g}_S$ to be the minimum positive linear combination of S over $\mathbb{Z}$. That is,

$$\tilde{g}_S = \min\left([1, \infty) \cap \left\{\sum_{i=1}^{j} a_i x_i \mid 1 \leq j \leq k, a_i \in \mathbb{Z}, x_i \in S\right\}\right).$$

We claim that $\tilde{g}_S = g_S$.

- $g_S \mid \tilde{g}_S$ by definition.
- Let $s \in S$ be decomposed as $s = q\tilde{g}_S + r$ with $0 \leq r < \tilde{g}_S$. Then $r = s - q\tilde{g}_S$. However, this is a linear combination of S over $\mathbb{Z}$, so $r = 0$. Thus $\tilde{g}_S \mid g_S$.

Thus we can write $g_S = \sum_{s \in S} a_s s$ where $a_s \in \mathbb{Z}$. . . .

Lecture 3.
Thursday
January 11

Lecture 3

.... First consider the case $|S| = 2$. Let $S = \{s_1, s_2\}$. We know that $g_S = as_1 + bs_2$ for some $a, b \in \mathbb{Z}$. Now for any $m \in \mathbb{N}^*$,

$$\begin{aligned} mg_S &= mas_1 + mbs_2 + ks_1s_2 - ks_1s_2 \\ &= (ma - ks_2)s_1 + (mb + ks_1)s_2 \end{aligned}$$

Choose $k \in \mathbb{N}$ such that $0 \leq ma - ks_2 < s_2$. We can write $mg_S = a_m s_1 + b_m s_2$ where $0 \leq a_m < s_2$.

Let m_0 be such that $m_0 g_S > s_1 s_2$. Then for all $m \geq m_0$, $mg_S - a_m s_1 > (s_2 - a)s_1 > 0$, so that $b_m > 0$. Thus $m_S = m_0$ works.

Suppose the lemma holds for all sets of size $l - 1$.
Let $S = \{s_1, s_2, \ldots, s_l\}$ and $F = S \setminus \{s_l\}$. Then by the previous lemma, $g_S = \gcd(g_F, s_l)$.

Let m_0 be such that $mg_S - m_F g_F \geq m_{\{g_F, s_l\}}$. Then

$$\begin{aligned} mg_S - m_F g_F &= ag_F + bs_l \text{ for some } a \in \mathbb{N}, b \in \mathbb{N}^* \\ mg_S &= (a + m_F)g_F + bs_l \end{aligned}$$

but $a + m_F \geq m_F$, so we can write

$$mg_S = \sum_{i=1}^{l-1} a_i s_i + bs_l$$

where all a_is are non-negative integers. This closes the induction. $\square$

We can now prove theorem I.24.

Proof. Applying Schur's lemma to F_x, we have the existence of an N such that nd_x can be written as a non-negative integer combination of elements of F_x, for all $n \geq N$. Let $nd_x = \sum_{i=1}^k a_i f_i$. Then since $p_{ij}^{(m)} = (P^m)_{ij}$,

$$\begin{aligned} p_{xx}^{(nd_x)} &\geq \underbrace{p_{xx}^{(f_1)} \ldots p_{xx}^{(f_1)}}_{a_1 \text{ times}} \cdots \underbrace{p_{xx}^{(f_k)} \ldots p_{xx}^{(f_k)}}_{a_k \text{ times}} \\ &\geq \prod_{i=1}^k (p_{xx}^{(f_i)})^{a_i} \\ &> 0 \end{aligned}$$

$\square$

I.1 Stopping Times

Definition I.28 (The Extended Reals)**.** The *extended real line* is the set of real numbers along with 2 formal sumbols $+\infty$ and $-\infty$, denoted by

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

$\overline{\mathbb{R}}$ will be endowed with the order

$$-\infty < x < \infty \text{ for all } x \in \mathbb{R},$$

along with the usual order on $\mathbb{R}$. We extend the algebraic operations on $\mathbb{R}$ to $\overline{\mathbb{R}}$.

- $x + \infty = +\infty$, $x - \infty = -\infty$ for all $x \in \mathbb{R}$.
- $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$ for all $x \in \mathbb{R}$, $x > 0$.
- $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$ for all $x \in \mathbb{R}$, $x < 0$.
- $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$, for all $x \in \mathbb{R}$.

If $E \subseteq \mathbb{R}$ is not bounded above in $\mathbb{R}$, we say $\sup E = +\infty$. If $E = \varnothing$, we say $\inf E = +\infty$.

Definition I.29 (Filtration)**.** Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space. A collection $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of σ-algebras over Ω is called a *filtration* if $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}$ for all $n \in \mathbb{N}$.

Definition I.30 (Natural filtration)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{S}$-valued random variables defined on $(\Omega, \mathcal{F}, \Pr)$. For $n \in \mathbb{N}$, define

$$\mathcal{F}_n = \left\{(X_0, X_1, \dots, X_n)^{-1}(A) \mid A \subseteq \mathcal{S}^{n+1}\right\} \quad = \sigma(X_0, X_1, \dots, X_n)$$

Here, $(X_0, \dots, X_n)^{-1}(A) = \{\omega \in \Omega \mid (X_0(\omega), \dots, X_n(\omega)) \in A\}$. This sequence of σ-algebras is called the *natural filtration* of $(X_n)_{n\in\mathbb{N}}$.

Remark. Note that elements of $\mathcal{F}_n$ are subsets of Ω, not elements of Ω.

(GUESS) $\mathcal{F}_n$ is the set of all subsets of Ω subject to one condition: if $(X_0, \dots, X_n)(\omega_1) = (X_0, \dots, X_n)(\omega_2)$, then any set in $\mathcal{F}_n$ containing ω_1 must also contain ω_2.

$$\begin{aligned}\mathcal{F}_n = \{A \subseteq \Omega \mid \forall \omega_1, \omega_2 \in \Omega :&\\ &(\omega_1 \in A) \oplus (\omega_2 \in A) \to (X_0, \dots, X_n)(\omega_1) \neq (X_0, \dots, X_n)(\omega_2)\}\end{aligned}$$

Why is $\mathcal{F}_n$ a σ-algebra? The empty set is in $\mathcal{F}_n$ because $\varnothing \in \mathcal{S}^{n+1}$, and the set of pre-images of the null set is the null set. The complement of any set in $\mathcal{F}_n$ is in $\mathcal{F}_n$ because $(X_0, \dots, X_n)^{-1}(A)^c = (X_0, \dots, X_n)^{-1}(A^c)$.

Why is $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$? For any $A \subseteq \mathcal{S}^{n+1}$, we have

$$(X_0, \dots, X_n)^{-1}(A) = (X_0, \dots, X_n, X_{n+1})^{-1}(A \times \mathcal{S}).$$

Definition I.31 (Stopping time)**.** Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{S}$-valued random variables on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \Pr)$ where $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the natural filtration of $(X_n)_{n\in\mathbb{N}}$.

Then $\tau \colon \Omega \to \mathbb{N} \cup \{\infty\}$ is called a *stopping time* with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$,

$$\{\omega \in \Omega \mid \tau(\omega) \leq n\} \in \mathcal{F}_n.$$

This is equivalent to saying that for all $n \in \mathbb{N}$,

$$\mathbf{1}_{\{\tau \le n\}} = \mathbf{1}_{\{(X_0, \dots, X_n) \in A\}} \quad \text{for some } A \in \mathcal{S}^{n+1}.$$

(GUESS) In keeping with the previous guess, it is equivalent to the following: if $\omega, \omega' \in \Omega$ with $\tau(\omega) = n < \tau(\omega')$, then

$$(X_0, \dots, X_n)(\omega) \ne (X_0, \dots, X_n)(\omega').$$

Intuitively, a stopping time is a time at which we can decide whether or not to stop the process based on the information available up to that time (in measurable terms).

Consider the simple random walk $(X_n)_{n \in \mathbb{N}}$ on $\mathbb{Z}$. Then the event that the hitting time of 10 is at most n is

$$\{T_{10} \le n\} = \bigcup_{i=1}^{n} \{X_i = 10\}.$$

Examples.

- Let $(X_n)_{n \in \mathbb{N}}$ be an $\mathcal{S}$-valued stochastic process and let $A \subseteq \mathcal{S}$. Let $T_A := \inf\{n \ge 1 \mid X_n \in A\}$ (where we take $\inf \varnothing$ to be $+\infty$). Then T_A is a stopping time with respect to the natural filtration associated with $(X_n)_{n \in \mathbb{N}}$. That is, for all $n \in \mathbb{N}$,

  $$\{T_A \le n\} = \bigcup_{i=1}^{n} \{X_i \in A\} \in \mathcal{F}_n.$$

  Intuitively, say we stop as soon as we hit a desired state. Then we can decide whether or not to stop at time n based on the information available up to time n.

- SRW(p) started at the origin. Then $L = \sup\{n \ge 1 \mid X_n < 7\}$ is NOT a stopping time. Intuitively, L is the *last* time we are below 7, which cannot be determined based on the information available from the past.

Proposition I.32. *τ is a stopping time iff for all $n \in \mathbb{N}$,*

$$\{\tau = n\} \in \mathcal{F}_n.$$

Proof. Suppose τ is a stopping time. Then $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. This is because

$$\{\tau = n\} = \{\tau \le n\} \cap \{\tau \le n-1\}^c$$

where both sets are in $\mathcal{F}_n \supseteq \mathcal{F}_{n-1}$, and therefore so is their intersection (de Morgan's law).

Conversely, suppose $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n$, we have $\{\tau = i\} \in \mathcal{F}_n$ for all $i \leq n$. Hence so is

$$\{\tau \leq n\} = \bigcup_{i=0}^{n} \{\tau = i\}. \qquad \square$$

Proposition I.33. *If τ_1 and τ_2 are stopping times, then so are $\tau_1 \wedge \tau_2$, $\tau_1 \vee \tau_2$ and $\tau_1 + \tau_2$.*

Proof. We have

$$\begin{aligned}
\{\tau_1 \wedge \tau_2 \leq n\} &= \{\tau_1 \leq n\} \cup \{\tau_2 \leq n\} \\
\{\tau_1 \vee \tau_2 \leq n\} &= \{\tau_1 \leq n\} \cap \{\tau_2 \leq n\} \\
\{\tau_1 + \tau_2 \leq n\} &= \bigcup_{i=0}^{n} \{\tau_1 \leq i\} \cap \{\tau_2 \leq n - i\}
\end{aligned} \qquad \square$$

We can interpret the above operations as follows.

- $\tau_1 \wedge \tau_2$ is the stopping time if we stop when either of our conditions are met.
- $\tau_1 \vee \tau_2$ is the stopping time if we stop when both of our conditions are met.
- $\tau_1 + \tau_2$ is the stopping time if we stop when we have waited for τ_1 before we started looking for τ_2.

Exercise I.34. *Give an example of two stopping times τ_1 and τ_2 such that $\Pr(\tau_1 \leq \tau_2) = 1$ but $\tau_2 - \tau_1$ is not a stopping time.*

Solution. Consider the SRW(p) started at the origin, with

$$\begin{aligned}
\tau_1 &= \inf\{n \geq 1 \mid X_n = 1\}, \\
\tau_2 &= \inf\{n \geq 1 \mid X_n = 2\}.
\end{aligned} \qquad \blacksquare$$

Theorem I.35 (Strong Markov property)**.**
Let $(X_n)_{n\in\mathbb{N}}$ be $MC(\mu_0, P)$, and let τ be a stopping time. Let $A \subseteq \mathbb{N}$. Then

$$\begin{aligned}
&\Pr_{\mu_0}(X_{\tau+1} = x_1, \ldots, X_{\tau+n} = x_n \mid \tau \in A, X_\tau = x) \\
&\qquad\qquad = \Pr_x(X_1 = x_1, \ldots, X_n = x_n)
\end{aligned}$$

Remark. The SMP is equivalent to

$$\boxed{\mathrm{E}_{\mu_0}\Big[f\Big((X_{\tau+j})_{j\in\mathbb{N}}\Big) \mid \tau \in A, X_\tau = x\Big] = \mathrm{E}_x\Big[f\Big((X_j)_{j\in\mathbb{N}}\Big)\Big]}$$

for any bounded function $f \colon \mathcal{S}^\infty \to \mathbb{R}$.

Proof.

$$\begin{aligned}
&\Pr_{\mu_0}(X_{\tau+1} = x_1, \dots, X_{\tau+n} = x_n, \tau \in A, X_\tau = x) \\
&\quad = \sum_{m\in A} \Pr_{\mu_0}(X_{m+1} = x_1, \dots, X_{m+n} = x_n, \tau = m, X_m = x) \\
&\quad = \sum_{m\in A} \Pr_{\mu_0}(\tau = m, X_m = x) \\
&\qquad\qquad\qquad \times \Pr_{\mu_0}(X_{m+1} = x_1, \dots, X_{m+n} = x_n \mid \tau = m, X_m = x) \\
&\quad = \sum_{m\in A} \Pr_{\mu_0}(\tau = m, X_m = x) \Pr_x(X_1 = x_1, \dots, X_n = x_n)
\end{aligned}$$

by the Markov property, since $\tau = m$ is equivalent to $(X_0, \dots, X_m)$ belonging to some set in $\mathcal{F}_m$. Summing over A gives

$$= \Pr_{\mu_0}(\tau \in A, X_\tau = x) \Pr_x(X_1 = x_1, \dots, X_n = x_n)$$

and dividing by $\Pr_{\mu_0}(\tau \in A, X_\tau = x)$ yields the result. □

Definition I.36. Suppose X takes values in $\mathbb{N} \cup \{\infty\}$, and let $p_k := \Pr(X = k)$, $k \in \mathbb{N} \cup \{\infty\}$. Then the *probability generating function* of X is defined as

$$\begin{aligned}
G_X(s) &= p_0 + p_1 s + p_2 s^2 + \dots, \qquad s \in (-1, 1) \\
&= \mathrm{E}[s^X]
\end{aligned}$$

where we take $s^\infty = 0$ for $|s| < 1$.

Remark. The left limit $G_X(1^-) = \lim_{s\uparrow 1} G_X(s) = 1 - p_\infty$.
If $p_\infty > 0$, then $\mathrm{E}[X] = \infty$. Otherwise,

$$\mathrm{E}[X] = \sum_{k=0}^{\infty} k p_k = \sum_{k=0}^{\infty} p_k k (1)^{k-1} = G'_X(1^-)$$

Theorem I.37. *Let X, Y be two random variables taking values in $\mathbb{N} \cup \{\infty\}$ and $G_X(s) = G_Y(s)$ $\forall s \in (-1, 1)$. Then $X \overset{d}{=} Y$.*

Proof. Since $G_X, G_Y \in C^\infty(-1, 1)$, we have

$$G_X^{(n)} = G_Y^{(n)} \text{ for all } n \in \mathbb{N}.$$

But $G_X^{(n)}(0) = n! p_n^X$. Thus $p_n^X = p_n^Y$ for all $n \in \mathbb{N}$. □

Exercise I.38. *Suppose $(X_n)_{n\in\mathbb{N}}$ is an SRW(p) on $\mathbb{Z}$ started at the origin. Find $G_{T_{-1}}$, where T_{-1} is the first hitting time of -1.*

Solution. Denote $G_{T_{-1}}$ by G for simplicity.

$$\begin{aligned} G(s) &= \mathrm{E}_0[s^{T_{-1}}] \\ &= p\,\mathrm{E}_0[s^{T_{-1}} \mid X_1 = 1] + q\,\mathrm{E}_0[s^{T_{-1}} \mid X_1 = -1] \\ &= p\,\mathrm{E}_1[s^{1+T_{-1}}] + qs \\ &= ps\,\mathrm{E}_1[s^{T_{-1}}] + qs \\ &= ps\,\mathrm{E}_0[s^{T_{-2}}] + qs \end{aligned}$$

Since $s^\infty = 0$ by our convention, we have

$$\begin{aligned} \mathrm{E}_0[s^{T_{-2}}] &= \mathrm{E}_0[s^{T_{-2}}\mathbf{1}_{T_{-1}<\infty}] \\ &= \sum_m \mathrm{E}_0[s^{T_{-2}}\mathbf{1}_{T_{-1}=m}] \\ &= \sum_m \Pr_0(T_{-1} = m)\,\mathrm{E}_{-1}[s^{m+T_{-2}}] \\ &= \sum_m \Pr_0(T_{-1} = m)s^m\,\mathrm{E}_{-1}[s^{T_{-2}}] \\ &= \mathrm{E}_0[s^{T_{-1}}]\sum_m \Pr_0(T_{-1} = m)s^m \\ &= G(s)^2 \end{aligned}$$

Thus

$$\begin{aligned} G(s) &= psG(s)^2 + qs \\ G(s) &= \frac{1 \pm \sqrt{1-4pqs^2}}{2ps} \end{aligned}$$

Claim: $G(s) = \frac{1-\sqrt{1-4pqs^2}}{2ps}$ for all $s \in (-1,1)\setminus\{0\}$. ■

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We get several results from this exercise. The probability of ever hitting -1 is

$$\begin{aligned} 1 - \Pr_0(T_{-1} = \infty) &= \lim_{s\uparrow 1} G(s) \\ &= \frac{1-\sqrt{1-4pq}}{2p} \\ &= \frac{1-|2p-1|}{2p} \\ &= \begin{cases} 1 & \text{if the walk is left-biased,} \\ \frac{q}{p} & \text{otherwise.} \end{cases} \\ &= \frac{q}{p} \wedge 1 \end{aligned}$$

Another way to see the left-biased case is to note that X_n's are sums of iid

$\mathrm{Ber}_{\pm}(p)$ random variables, and so by the strong law of large numbers,

$$\frac{X_n}{n} \xrightarrow{\text{a.s.}} p - q.$$

Thus if $p < q$, then $X_n \xrightarrow{\text{a.s.}} -\infty$.

Exercise I.39. *Consider the SRW(p) on $\mathbb{Z}$ started at the origin. Show that $T_{-2} \overset{d}{=} \tau_1 + \tau_2$, where τ_1 and τ_2 are iid copies of T_{-1}.*

Solution.

$$\begin{aligned}
\mathrm{E}_0[s^{T_{-2}}] &= \mathrm{E}_0[s^{T_{-2}} \mathbf{1}_{T_{-1}<\infty}] \\
&= \sum_m \mathrm{E}_0[s^{T_{-2}} \mathbf{1}_{T_{-1}=m}] \\
&= \sum_m \Pr_0(T_{-1} = m)\, \mathrm{E}_{-1}[s^{m+T_{-2}}] \\
&= \sum_m \Pr_0(T_{-1} = m) s^m\, \mathrm{E}_{-1}[s^{T_{-2}}] \\
&= \mathrm{E}_0[s^{T_{-1}}] \sum_m \Pr_0(T_{-1} = m) s^m \\
&= G(s)^2
\end{aligned}$$

On the other hand, $\mathrm{E}_0[s^{\tau_1+\tau_2}] = \mathrm{E}_0[s^{\tau_1}]^2 = G(s)^2$ by independence.

Thus, $T_{-2} \overset{d}{=} \tau_1 + \tau_2$ by theorem I.37. ∎

Exercise I.40. *Consider the SRW(p) on $\mathbb{Z}$ started at the origin. Find* $\Pr_0(T_{-1} = n)$, $n \in \mathbb{N}$.

Solution. We have $G(s) = \frac{1-\sqrt{1-4pqs^2}}{2ps}$. We first need a nice expression for $\binom{1/2}{k}$ in order to use the binomial theorem.

$$\begin{aligned}
\binom{1/2}{k} &= \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-k+1)}{k!} \\
&= \frac{1}{2^k}\frac{1(1-2)\dots(1-2k+2)}{k!} \\
&= \frac{(-1)^{k-1}}{2^k}\frac{(2k-3)!!}{k!}
\end{aligned}$$

We can rewrite $(2k-3)!!$ as

$$\begin{aligned}
(2k-3)!! &= \frac{(2k-3)!}{(2k-4)!!} \\
&= \frac{(2k-3)!}{2^{k-2}(k-2)!}
\end{aligned}$$

so that

$$\binom{1/2}{k} = \frac{(-1)^{k-1}}{2^{2k-2}} \frac{(2k-3)!}{k!(k-2)!}$$
$$= \frac{(-1)^{k-1}}{2^{2k-2}k} \binom{2k-3}{k-2}$$

but multiplying by $\frac{2k-2}{2(k-1)}$ yields an even nicer

$$= \frac{(-1)^{k-1}}{2^{2k-1}} \frac{1}{k} \binom{2k-2}{k-1}$$

The expression doesn't make sense for $k = 0$, for which the coefficient is 1, and the derivation doesn't make sense for $k = 1$, but for which the expression happens to match the coefficient.

But if we look closely, we see that this is just

$$\frac{(-1)^{k-1}}{2^{2k-1}} C_{k-1}$$

where C_{k-1} is the $(k-1)$th Catalan number!

From the binomial theorem,

$$(1-x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-x)^k$$
$$= 1 - \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} C_{k-1} x^k$$

but more interestingly,

$$(1-4x)^{\frac{1}{2}} = 1 - 2\sum_{k=1}^{\infty} C_{k-1} x^k$$

In fact, this gives that

$$\frac{1-\sqrt{1-4x}}{2x} \text{ is the generating function for } (C_k)_{k\in\mathbb{N}}.$$

Getting back to the problem at hand, we have

$$G(s) = \frac{1-\sqrt{1-4pqs^2}}{2ps}$$
$$= qs \sum_{k=0}^{n} C_k (pqs^2)^k$$

So we have

$$\Pr_0(T_{-1} = n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ p^k q^{k+1} C_k & \text{if } n = 2k+1 \end{cases}$$

In the case of $p = q = \frac{1}{2}$, this in fact proves that the number of SRW paths of length $2n$ from the origin to the origin, that never go below the origin, is the nth Catalan number. Why? Because each such path can be extended bijectively to a path of length $2n+1$ that hits -1 for the first time at time $2n+1$. ■

I.2 Transience & Recurrence

Definition I.41. Let $(X_n)_{n\in\mathbb{N}}$ be $MC_{\mathcal{S}}(\mu_0, P)$. Define $T_y = ??? = \inf\{n \in \mathbb{N}^* \mid X_n = y\}$, where we take $\inf \varnothing$ to be $+\infty$.

For $x, y \in \mathcal{S}$, define $f_{xy} = \mathrm{Pr}_x(T_y < \infty)$. A state $x \in \mathcal{S}$ is said to be *recurrent* if $f_{xx} = 1$, and *transient* otherwise.

A state x is said to be *absorbing* if $f_{xy} > 0$ only when $x = y$. (This is equivalent to $\{x\}$ being a communicating class.)

We further define

$$N_y = \#\{n \in \mathbb{N}^* \mid X_n = y\},$$
$$G(x, y) = \mathrm{E}_x[N_y].$$

$G\colon \mathcal{S}^2 \to \mathbb{R}$ is called the *Green's function.*

Lemma I.42. *For all $x, y \in \mathcal{S}$, $G(x, y) = \sum_{n\in\mathbb{N}} p_{xy}^{(n)}$.*

Proof. We write N_y as $\sum_{n=1}^{\infty} \mathbf{1}_{X_n=y}$. Then,

$$\begin{aligned}
G(x, y) &= \mathrm{E}_x\left[\sum_{n=1}^{\infty} \mathbf{1}_{X_n=y}\right] \\
&= \sum_{n=1}^{\infty} \mathrm{E}_x[\mathbf{1}_{X_n=y}] \qquad \text{(MCT)} \\
&= \sum_{n=1}^{\infty} \Pr_x(X_n = y) \\
&= \sum_{n=1}^{\infty} p_{xy}^{(n)}
\end{aligned}$$

The interchange of the sum and the expectation is justified by the monotone convergence theorem stated below. □

Fact I.43 (Monotone convergence theorem)**.** *Let $(\Omega, \mathcal{F}, \mathrm{Pr})$ be a probability space. Let $X_n\colon \Omega \to [0, \infty]$ be a sequence of random variables and $X\colon \Omega \to [0, \infty]$ be another random variable. Suppose that $X_n(\omega) \le X_{n+1}(\omega)$ for each n and ω, and that $X_n(\omega) \to X(\omega)$ for each ω. Then, $\mathrm{E}[X_n] \to \mathrm{E}[X]$.*

Remark. The statement holds even if $X_n \xrightarrow{\text{a.s.}} X$.

Proof. We consider the case where $X_n\colon \Omega \to [0,\infty)$ and $X\colon \Omega \to [0,\infty)$.

Since $X_n \le X_{n+1} \le X$, we have

$$\mathrm{E}[X_n] \le \mathrm{E}[X_{n+1}] \le \mathrm{E}[X].$$

Thus the sequence $(\mathrm{E}[X_n])_{n\in\mathbb{N}}$ is increasing and bounded and so converges to some limit $L \le \mathrm{E}[X]$. . . .

What does this have to do with the Green's function?

$$\mathrm{E}_x\left[\sum_{n=1}^{\infty} \mathbf{1}_{X_n=y}\right] = \mathrm{E}_x\left[\lim_{n\to\infty} S_n\right]$$

where S_n's are the partial sums. Note that $S_n \le S_{n+1}$ and $S_n \to \sum_{n=1}^{\infty} p_{xy}^{(n)}$. Thus applying the monotone convergence theorem, we get

$$\begin{aligned}\mathrm{E}_x\left[\sum_{n=1}^{\infty} \mathbf{1}_{X_n=y}\right] &= \lim_{n\to\infty} \mathrm{E}_x[S_n] \\ &= \lim_{n\to\infty} \sum_{m=1}^{n} \mathrm{E}_x[\mathbf{1}_{X_m=y}] \\ &= \sum_{n=1}^{\infty} \mathrm{E}_x[\mathbf{1}_{X_n=y}].\end{aligned}$$

Theorem I.44. *For all $x, y \in \mathcal{S}$,*

$$\Pr_x(N_y = m) = \begin{cases} 1 - f_{xy} & \textit{if } m = 0 \\ f_{xy} f_{yy}^{m-1}(1 - f_{yy}) & \textit{if } m \in \mathbb{N}^* \\ f_{xy}[f_{yy} = 1] & \textit{if } m = +\infty \end{cases}$$

Proof. $N_y = 0$ if and only if $T_y = +\infty$. This occurs with probability $1 - f_{xy}$.

We define $T_y^{(1)} = T_y$ and for $m \ge 1$,

$$T_y^{(m+1)} = \inf\left\{n > T_y^{(m)} \mid X_n = y\right\}.$$

Note that $T_y^{(m)} = +\infty$ implies $T_y^{(m+1)} = +\infty$. Now

$$\begin{aligned}\Pr_x(T_y^{(m+1)} < \infty) &= \Pr_x(T_y^{(m)} < \infty \text{ and } T_y^{(m+1)} < \infty) \\ &= \Pr_x(T_y^{(m)} < \infty) \Pr_y(T_y < \infty) \quad \text{(Strong Markov property)}\end{aligned}$$

and by induction,

$$\begin{aligned}\Pr_x(N_y \ge m) &= \Pr_x(T_y^{(m)} < \infty) \\ &= f_{xy} f_{yy}^{m-1}. \qquad (*)\end{aligned}$$

The result follows by taking the difference. Or more directly,

$$\Pr_x(N_y = m) = \Pr_x(T_y^{(m)} < \infty) \Pr_y(T_y = +\infty) = f_{xy} f_{yy}^{m-1}(1 - f_{yy}).$$

Finally,

$$\begin{aligned}
\Pr_x(N_y = +\infty) &= 1 - \sum_{m=0}^{\infty} \Pr_x(N_y = m) \\
&= 1 - (1 - f_{xy}) - f_{xy}(1 - f_{yy}) \sum_{m=0}^{\infty} f_{yy}^m \\
&= \begin{cases} f_{xy} & \text{if } f_{yy} = 1 \\ 0 & \text{if } f_{yy} < 1 \end{cases} \\
&= f_{xy}[f_{yy} = 1].
\end{aligned}$$

□

Theorem I.45.

(1) Suppose y is transient. Then for all $x \in \mathcal{S}$, $\Pr_x(N_y < \infty) = 1$ and $G(x, y) = \frac{f_{xy}}{1-f_{yy}} < \infty$.

(2) Suppose y is recurrent. Then $\Pr_y(N_y = \infty) = 1$ and $G(y, y) = +\infty$. Further, for all $x \in \mathcal{S} \setminus \{y\}$, $\Pr_x(N_y = \infty) = f_{xy}$ and

$$G(x, y) = \begin{cases} 0 & \textit{if } f_{xy} = 0, \\ \infty & \textit{if } f_{xy} > 0. \end{cases}$$

Proof.

(1) Since y is transient, $f_{yy} < 1$. Thus by the previous theorem, $\Pr_x(N_y = \infty) = 0$. Then,

$$\begin{aligned}
G(x, y) &= \sum_{m=1}^{\infty} m \Pr_x(N_y = m) \\
&= f_{xy}(1 - f_{yy}) \sum_{m=1}^{\infty} m f_{yy}^{m-1} \\
&= f_{xy}(1 - f_{yy}) \frac{1}{(1 - f_{yy})^2} \\
&= \frac{f_{xy}}{1 - f_{yy}}.
\end{aligned}$$

Alternatively, we use equation (∗) to write

$$\begin{aligned}
G(x, y) &= \sum_{m=1}^{\infty} \Pr_x(N_y \geq m) \\
&= \sum_{m=1}^{\infty} f_{xy} f_{yy}^{m-1} \\
&= \frac{f_{xy}}{1 - f_{yy}}.
\end{aligned}$$

(2) Since y is recurrent, $f_{yy} = 1$. By the previous theorem, for any $x \in \mathcal{S}$,

$$\Pr_x(N_y = m) = \begin{cases} 1 - f_{xy} & \text{if } m = 0, \\ 0 & \text{if } m \in \mathbb{N}^*, \\ f_{xy} & \text{if } m = +\infty. \end{cases}$$

Thus $G(x, y) = +\infty$ if $f_{xy} > 0$ and 0 otherwise. □

Corollary I.46. *A state x is recurrent iff $G(x, x) = \sum_{m=1}^{\infty} p_{xx}^{(m)} = +\infty$.*

Definition I.47. A DTMC is said to be *recurrent* (resp. *transient*) if all its states are recurrent (resp. transient).

Theorem I.48. *If $|\mathcal{S}| < \infty$, then there exists a recurrent state.*

Proof. Suppose not. Then for all $x, y \in \mathcal{S}$, $G(x, y) = \sum_{m=0}^{\infty} p_{xy}^{(m)} < \infty$. Then the individual terms of the series must tend to 0. Thus

$$\begin{aligned} &\sum_{y \in \mathcal{S}} \lim_{m \to \infty} p_{xy}^{(m)} = 0 \\ \implies &\lim_{m \to \infty} \sum_{y \in \mathcal{S}} p_{xy}^{(m)} = 0. \end{aligned}$$

The interchange of the limit and the sum is justified since the sum is finite. But $\sum_{y \in \mathcal{S}} p_{xy}^{(m)} = 1$ for all m. Thus we have a contradiction. □

Theorem I.49. *Suppose $x \neq y \in \mathcal{S}$, x is recurrent and $x \to y$. Then y is recurrent, $y \to x$ and $f_{xy} = f_{yx} = 1$.*

Proof. Since $x \to y$, there exists $n \in \mathbb{N}^*$ and $x_1, \ldots, x_{n-1}$ distinct from x such that $p_{xx_1} p_{x_1 x_2} \ldots p_{x_{n-1} y} > 0$. Since x is recurrent,

$$0 = \Pr_x(T_x = +\infty) \geq p_{xx_1} p_{x_1 x_2} \ldots p_{x_{n-1} y} \Pr_y(T_x = +\infty)$$

so $\Pr_y(T_x = +\infty)$ must be 0. Thus $y \to x$ with $f_{yx} = 1$. If y is recurrent, then f_{xy} would be 1 by the same argument. Thus we need only show that y is recurrent. We can show this by showing that $G(y, y) = +\infty$. Let

$p_{yx}^{(n_1)} > 0$ and $p_{xy}^{(n_2)} > 0$.

$$\begin{aligned} G(y,y) &\geq \sum_{m=n_1+n_2+1}^{\infty} p_{yy}^{(m)} \\ &\geq \sum_{r=1}^{\infty} p_{yx}^{(n_1)} p_{xx}^{(r)} p_{xy}^{(n_2)} \\ &= p_{yx}^{(n_1)} p_{xy}^{(n_2)} G(x,x) \\ &= +\infty. \end{aligned}$$

□

Lecture 5. Sunday February 04

Definition I.50. A recurrent state x is said to be *null recurrent* if $\mathrm{E}_x[T_x] = \infty$.

It is said to be *positive recurrent* if $\mathrm{E}_x[T_x] < \infty$.

Theorem I.51. *Suppose $x, y \in \mathcal{S}$. Then let*

$$N_y^{(n)} = \sum_{j=1}^{n} \mathbf{1}_{X_j = y},$$

$$G^{(n)}(x,y) = \mathrm{E}_x[N_y^{(n)}] = \sum_{j=1}^{n} p_{xy}^{(j)}.$$

Then under Pr_x,

$$\frac{N_y^{(n)}}{n} \xrightarrow{a.s.} \frac{\mathbf{1}_{T_y<\infty}}{\mathrm{E}_y[T_y]}.$$

Further,

$$\frac{G^{(n)}(x,y)}{n} \to \frac{f_{xy}}{\mathrm{E}_y[T_y]}.$$

We are using the convention that $\frac{1}{\infty} = 0$, which makes both the limits zero for transient and null recurrent states.

Proof. Suppose y is transient. Then

$$G(x,y) = \mathrm{E}_x[N_y] = \frac{f_{xy}}{1 - f_{yy}} < \infty$$

so

$$\frac{N_y^{(n)}}{n} \leq \frac{N_y}{n} \xrightarrow{\text{a.s.}} 0.$$

Similarly

$$\frac{G^{(n)}(x,y)}{n} \leq \frac{G(x,y)}{n} \to 0.$$

Now suppose y is recurrent. Define $T_y^{(1)} = T_y$, and for $m \geq 2$, define

$$T_y^{(m)} = \inf\{n > T_y^{(m-1)} \mid X_n = y\}.$$

Then
$$N_y^{(n)} = m \iff T_y^{(m)} \le n < T_y^{(m+1)}$$
If $T_y = \infty$, then obviously $N_y^{(n)} = 0$ for all n, so
$$\frac{N_y^{(n)}}{n} \xrightarrow{\text{a.s.}} 0.$$
Suppose $T_y < \infty$.
$$\frac{T_y^{(m)}}{m} \le \frac{n}{m} < \frac{T_y^{(m+1)}}{m} < \frac{T_y^{(m+1)}}{m+1}$$
Then as $n \to \infty$, $m \to \infty$, and
$$\frac{T_y^{(m)}}{m} \xrightarrow{\text{a.s.}} \mathrm{E}_y[T_y].$$
So
$$\frac{n}{m} \xrightarrow{\text{a.s.}} \mathrm{E}_y[T_y].$$
Thus
$$\frac{N_y^{(n)}}{n} \xrightarrow{\text{a.s.}} \frac{1}{\mathrm{E}_y[T_y]}.$$
Similarly
$$\begin{aligned}\mathrm{E}_x[N_y^{(n)}] &= \mathrm{E}_x[\mathbf{1}_{T_y<\infty} N_y^{(n)}] \\ &= \mathrm{E}_x[\mathbf{1}_{T_y<\infty}]\,\mathrm{E}_x[N_y^{(n)} \mid T_y < \infty] \\ \frac{G^{(n)}(x,y)}{n} &\to \frac{f_{xy}}{\mathrm{E}_y[T_y]}.\end{aligned}$$
□

Fact I.52 (Dominated convergence theorem)**.** *Suppose Z, X and $(X_n)_{n\in\mathbb{N}}$ are $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \Pr)$. Further assume that*
$$X_n \xrightarrow{a.s.} X$$
$$\forall n\Big(|X_n| \le Z \textit{ almost surely}\Big)$$
$$\mathrm{E}[Z] < \infty.$$
Then
$$\mathrm{E}[X_n] \to \mathrm{E}[X] \in \mathbb{R}.$$

Theorem I.53. *Suppose $x \to y$ and x is positive recurrent. Then y is positive recurrent.*

Proof. We will use the previous theorem. We know from theorem I.49 that y is recurrent and $y \to x$. Let $p_{xy}^{(n_1)} > 0$ and $p_{yx}^{(n_2)} > 0$.

Then $p_{yy}^{(n_1+m+n_2)} \geq p_{xy}^{(n_1)} p_{xx}^{(m)} p_{yx}^{(n_2)}$.

$$\begin{aligned}\frac{G^{(n)}(y,y)}{n} &\geq \frac{1}{n} \sum_{m=0}^{n-n_1-n_2} p_{xy}^{(n_1)} p_{xx}^{(m)} p_{yx}^{(n_2)} \\ &= \frac{1}{n} p_{xy}^{(n_1)} p_{yx}^{(n_2)} G^{(n-n_1-n_2)}(x,x)\end{aligned}$$

Taking limits,

$$\frac{1}{\mathrm{E}_y[T_y]} \geq \frac{p_{xy}^{(n_1)} p_{yx}^{(n_2)}}{\mathrm{E}_x[T_x]} > 0.$$

Thus $\mathrm{E}_y[T_y] < \infty$. □

Corollary I.54. *Let $\mathcal{C}$ be a communicating class. Then either all states in $\mathcal{C}$ are transient, or all are null recurrent, or all are positive recurrent.*

Theorem I.55. *Let $\mathcal{C}$ be a finite communicating class. Then all states in $\mathcal{C}$ are positive recurrent.*

Proof. Let $x \in \mathcal{C}$.

$$\begin{aligned}\sum_{y \in \mathcal{C}} \frac{G^{(n)}(x,y)}{n} &= \frac{1}{n} \sum_{y \in \mathcal{C}} \sum_{j=1}^{n} p_{xy}^{(j)} \\ &= \frac{1}{n} \sum_{j=1}^{n} \sum_{y \in \mathcal{C}} p_{xy}^{(j)} \\ &= 1.\end{aligned}$$

Taking limits,

$$\sum_{y \in \mathcal{C}} \frac{f_{xy}}{\mathrm{E}_y[T_y]} = 1.$$

The interchange of the sum and limit is justified by the fact that the sum is finite. If all $\mathrm{E}_y[T_y]$ were infinite, then the sum would be zero. Thus there exists at least one positive recurrent state in $\mathcal{C}$, which forces all states to be positive recurrent. □

I.2.1 Transience and Recurrence of the SSRW on $\mathbb{Z}^d$

Theorem I.56. *The SSRW on $\mathbb{Z}^d$ is* null *recurrent if $d = 1, 2$ and transient if $d \geq 3$.*

I.3 Stationary distributions

Let $\mathcal{S}$ be a countable state space and P a transition matrix on $\mathcal{S}$. Then any function $\lambda\colon \mathcal{S} \to [0, \infty]$ corresponds to a measure on $\mathcal{S}$, by setting

$$\lambda(A) = \sum_{x \in A} \lambda(x).$$

Conversely, any measure on $\mathcal{S}$ is of this form. (Why?)

A measure λ can be thought of as a row vector $(\lambda(x))_{x \in \mathcal{S}}$. Then

$$(\lambda P)_y = \sum_{x \in \mathcal{S}} \lambda(x) P(x, y)$$

also gives a measure on $\mathcal{S}$.

A *probability distribution* π on $\mathcal{S}$ is a measure such that $\pi(\mathcal{S}) = 1$.

Definition I.57. A measure λ on $\mathcal{S}$ is *invariant* for a DTMC with transition matrix P is $\lambda P = \lambda$. That is,

$$\sum_{x \in \mathcal{S}} \lambda(x) P(x, y) = \lambda(y) \quad \forall y \in \mathcal{S}.$$

If λ is a probability distribution, it is called a *stationary distribution.*

Proposition I.58. *If $(X_n)_{n \in \mathbb{N}}$ is $MC(\pi, P)$ where π is a stationary distribution, then*

$$X_0 \overset{d}{=} X_1 \overset{d}{=} \cdots \sim \pi.$$

Chapter II

Branching Processes

Lecture 6. Thursday February 08

Definition II.1 (Branching process)**.** Let $\underline{p} = (p_i)_{i\in\mathbb{N}}$ be a probability distribution on $\mathbb{N}$ and let $X_{n,i} \sim \underline{p}$ be iid random variables for each $n, i \in \mathbb{N}$. We define the *branching process* $(Z_n)_{n\in\mathbb{N}}$ by

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}.$$

$\underline{p}$ is called the *offspring/progeny distribution*. The associated random tree (the ith node on the nth level, if it exists, having $X_{n,i}$ children) is called the *Galton-Watson tree* or the *Bienaymé tree*.

Clearly Z_{n+1} depends only on Z_n, so the process is a Markov chain.

Definition II.2 (Extinction)**.** The *extinction probability* of a branching process $(Z_n)_{n\in\mathbb{N}}$ is

$$\eta = \Pr(Z_n = 0 \text{ for some } n \in \mathbb{N}).$$

Proposition II.3 (Expectation)**.** *Let* $\mu = \mathrm{E}_{X\sim\underline{p}}[X] = \mathrm{E}[Z_1]$. *Then* $\mathrm{E}[Z_n] = \mu^n$.

Proof. By induction, $\mathrm{E}[Z_0] = 1 = \mu^0$. Then

$$\mathrm{E}[Z_{n+1} \mid Z_n] = \mathrm{E}\left[\sum_{i=1}^{Z_n} X_{n,i} \mid Z_n\right] = Z_n \mathrm{E}[X_{n,1}] = Z_n\mu.$$

So

$$\mathrm{E}[Z_{n+1}] = \mathrm{E}[\mathrm{E}[Z_{n+1} \mid Z_n]] = \mathrm{E}[Z_n\mu] = \mu\,\mathrm{E}[Z_n]$$

and by induction follows the result. □

Proposition II.4. *If* $\mathrm{E}[Z_1] < 1$*, then the process becomes extinct with probability 1.*

Proof. Markov's inequality gives

$$\Pr(Z_n \geq 1) \leq \mathrm{E}[Z_n] = \mu^n.$$

So

$$\lim_{n\to\infty} \Pr(Z_n \neq 0) = 0. \qquad \square$$

Theorem II.5. *Consider a branching process with $p_1 < 1$. Let G be the pgf of Z_1. Then the extinction probability η is the smallest solution to $G(s) = s$ in $[0,1]$.*

Further,

$$\eta = 1 \text{ if } \mathrm{E}[Z_1] \leq 1,$$
$$\eta < 1 \text{ if } \mathrm{E}[Z_1] > 1.$$

Proof. Let A be the event that the branching process goes extinct. Then

$$\eta = \Pr(A) = \mathrm{E}[\Pr(A \mid Z_1)] = \mathrm{E}[\eta^{Z_1}] = G(\eta).$$

Thus the extinction probability is a fixed point of G. The second last equality is because

$$\Pr(A \mid Z_1 = k) = \Pr\{\text{each of the } k \text{ children goes extinct}\},$$

which are independent events.

Let $\eta_n := \Pr(Z_n = 0)$. Since $\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\} \subseteq A$, $\eta_n \leq \eta_{n+1} \leq \eta$ and $\eta_n \uparrow \eta$. $\square$

Definition II.6 (Criticality)**.** An offspring distribution is called *critical* if $\mathrm{E}[Z_1] = 1$. It is called *subcritical* (resp. *supercritical*) if $\mathrm{E}[Z_1] < 1$ (resp. $\mathrm{E}[Z_1] > 1$).

Lecture 7.
Tuesday
February 13

II.1 The Structure of GWB Trees

Definition II.7 (Conjugate distribution)**.** Let $\underline{p} = (p_i)_{i\in\mathbb{N}}$ be a pmf on $\mathbb{N}$ with $p_0 > 0$ (hence $\eta > 0$). Then $\underline{\tilde{p}} = (\eta^{i-1} p_i)_{i\in\mathbb{N}}$ is called the *conjugate distribution* of $\underline{p}$.

Exercise II.8. *Show that $\underline{\tilde{p}}$ is a pmf.*

Solution.

$$\begin{aligned}
\sum_{i=1}^{\infty} \eta^{i-1} p_i &= \frac{1}{\eta} \sum_{i=1}^{\infty} \eta^i p_i \\
&= \frac{1}{\eta} \mathrm{E}_{X\sim\underline{p}}[\eta^X] \\
&= \frac{1}{\eta} G(\eta)
\end{aligned}$$

but $G(\eta) = \eta$

$$= 1.$$

For a more intuitive proof, recall that $G(\eta) = \eta$ *because* $\mathrm{E}[\eta^{Z_1}] = \eta$. ■

Exercise II.9. *Show that $\underline{\tilde{p}}$ is a critical or subcritical offspring distribution.*

Proof. If $\eta = 1$, then $\underline{\tilde{p}} = \underline{p}$. But $\eta = 1 \iff \mathrm{E}[Z_1] \leq 1$. Thus $\underline{\tilde{p}}$ is critical or subcritical in this case.

Suppose $\eta < 1$, so that $p_0 + p_1 < 1$ (otherwise the process would die off, since each node produces either one child, or, with positive probability, no children. Alternatively, $\mathrm{E}[Z_1] < 1$). Then

$$\begin{aligned}\mathrm{E}[Z_1] &= \sum k\eta^{k-1}p_k \\ &= \frac{\mathrm{d}}{\mathrm{d}\eta}\mathrm{E}[\eta^X] \\ &= G'(\eta)\end{aligned}$$

If $G'(\eta) > 1$, then $G'(s) > 1\ \forall s \in (\eta, 1)$. Why? Because $p_0 + p_1 < 1$, so G is strictly convex. But $G(\eta) = \eta$, so there exists a $\xi \in (\eta, 1)$ such that $G'(\xi) = 1$, a contradiction. Thus $G'(\eta) \leq 1$ so $\mathrm{E}[Z_1] \leq 1$. □

Theorem II.10. *Let $\underline{\tilde{p}}$ be the conjugate distribution of $\underline{p}$. Let $\mathcal{T}_{\underline{p}}$ and $\mathcal{T}_{\underline{\tilde{p}}}$ be the GWB trees with offspring distributions $\underline{p}$ and $\underline{\tilde{p}}$ respectively. Then*

$$(\mathcal{T}_{\underline{p}} \mid \textit{it is finite}) \stackrel{d}{=} \mathcal{T}_{\underline{\tilde{p}}}.$$

Exercise II.11. *Find the conjugate distribution of $Bin(r, p)$ where $p \in (\frac{1}{r}, 1)$.*

Definition II.12 (Breadth-first walk)**.** Let $\underline{t}$ be a plane (rooted) tree. Label its vertices $1, 2, \ldots, n$ in breadth-first order. Let $C_j(\underline{t})$ be the number of children of vertex j in $\underline{t}$. Then the *breadth-first walk* of $\underline{t}$ is the sequence

$$\begin{aligned}S_j(\underline{t}) &= \begin{cases} 1 & \text{if } j = 0, \\ S_{j-1} + C_j(\underline{t}) - 1 & \text{otherwise.}\end{cases} \\ &= 1 + \sum_{i=1}^{j}(C_i(\underline{t}) - 1).\end{aligned}$$

It is obvious that $S_n(\underline{t}) = 0$.

Theorem II.13. *There exists a bijection between the set of finite plane trees and the set* $\mathcal{S}$ *of sequences* $(s_n)_{n\in\mathbb{N}}$ *such that*

- $s_0 = 1$,
- $s_{n+1} \geq s_n - 1$,
- *there is an* n_0 *such that* $s_n = 0 \iff n \geq n_0$.

This bijection is such that each tree corresponds to its breadth-first walk.