UMA204: Introduction to Basic Analysis

Naman Mishra

January 2024

Contents

Lecture 10: Wed 24 Jan '24

0.1 Compactness

Definition 0.1. A subset $E \subseteq (X, d)$ is said to be bounded if there exists a $p \in X$ and $M > 0$ such that $E \subseteq B(p; M)$.

Consider $E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}$. Then E is both closed and bounded in $(\mathbb{Q}, |\cdot|)$. However, continuous functions on E are neither uniformly continuous nor bounded.

Definition 0.2. Let $E \subseteq (X, d)$. An open cover $\{\mathcal{U}_\alpha\}_{\alpha\in\Lambda}$ of E in X is a collection of open sets $\mathcal{U}_\alpha$ such that $E \subseteq \bigcup_{\alpha\in\Lambda} \mathcal{U}_\alpha$.

Definition 0.3. A subset $E \subseteq (X, d)$ is said to be compact if any open cover $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in\Lambda}$ of E in X admits a finite subcover of E, *i.e.*, there exist $\alpha_1, \ldots, \alpha_k \in \Lambda$ such that $E \subseteq \bigcup_{i=1}^k \mathcal{U}_{\alpha_i}$.

Examples.

- $E \subseteq (X, d)$ is finite. Let $\mathcal{U}$ be an open cover of $E = \{p_1, \ldots, p_n\}$. Then for each $p_j \in E$, there exists $\alpha_j \in \Lambda$ such that $p_j \in \mathcal{U}_{\alpha_j}$. Then $E \subseteq \bigcup_{j=1}^n \mathcal{U}_{\alpha_j}$.
- $E = (0, 1)$ is not compact in $(\mathbb{R}, |\cdot|)$.

  Proof. Let $\mathcal{U}_n = (\frac{1}{n+2}, \frac{1}{n})$ for $n \in \mathbb{N}^*$. Then $\mathcal{U} = \{\mathcal{U}_n\}_{n\in\mathbb{N}^*}$ is an open cover of E. However, $\mathcal{U}$ does not admit a finite subcover of E.

For any finite $\{\mathcal{U}_{n_1}, \dots, \mathcal{U}_{n_k}\}$, let $n_0 = \max\{n_j : 1 \le j \le k\}$. Then $\bigcup \mathcal{U}_{n_j} \subseteq (\frac{1}{n_0+2}, 1)$ and thus is not a cover of E. □

- $E = [0, 1]$ is compact in $(\mathbb{R}, |\cdot|)$. In fact, all rectangles (sets of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$) are compact in $(\mathbb{R}^n, \|\cdot\|)$.

Theorem 0.4. Let $E \subseteq (\mathbb{R}^n, \|\cdot\|)$. Then the following are equivalent:

(1) E is compact.

(2) E is closed and bounded.

(3) Every infinite subset of E admits a limit point in E.

Proof. We show (1) $\implies$ (2) in a general metric space (X, d). Let $E \subseteq X$ be compact. Let $z \in E^c$. For any $y \in E$, let $\delta_y = d(y, z)/2$. Note that $B(z, \delta_y) \cap B(y, \delta_y) = \varnothing$.

Then $\mathcal{U} = \{B(y; \delta_y) : y \in E\}$ is an open cover of E. Since E is compact, $\mathcal{U}$ admits a finite subcover of E. That is, there exist $y_1, \dots, y_k \in E$ such that $E \subseteq \bigcup_{i=1}^k B(y_i; \delta_{y_i})$. Let $\delta = \min\{\delta_{y_i}\}$. Then $B(z; \delta) \cap \bigcup_{i=1}^k B(y_i; \delta_{y_i}) = \varnothing$, so $B(z; \delta) \subseteq E^c$.

For boundedness, take the largest ball in the finite subcover of $\bigcup_{R>0} B(p; R)$ for some $p \in E$.

We show (2) $\implies$ (1) in $(\mathbb{R}^n, \|\cdot\|)$. We first show that for any $R \in \mathbb{R}$, the set $[-R, R]^n$ is compact. WLOG let $R = 1$. □

Lecture 11: Thu 25 Jan '24

Theorem 0.5. Let $\{K_\alpha\}_{\alpha \in \Lambda}$ be a collection of compact sets in (X, d) such that any non-empty finite subcollection has non-empty intersection. Then $\bigcap_{\alpha \in \Lambda} K_\alpha \neq \varnothing$.

Proof. Suppose $\bigcap_{\alpha \in \Lambda} K_\alpha = \varnothing$. No element in K_1 is in every other K_α. Let $\mathcal{U}_\alpha = K_\alpha^c$ for each α. Any point in K_1 is in at least one $\mathcal{U}_\alpha$. Then $\mathcal{U}_\alpha$ is an open cover of K_1. But since K_1 is compact, there is a finite subcover $\mathcal{U}_{\alpha_1}, \dots, \mathcal{U}_{\alpha_n}$. But then $K_1 \subseteq (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n})^c$, so $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \varnothing$. Contradiction. □

Theorem 0.6. Every closed subset of a compact set is compact.

Proof. Let $E \subseteq Y \subseteq (X, d)$ where Y is compact and E is closed. Let $\mathcal{U}$ be an open cover of E in X. Then $\mathcal{U} + E^c$ is an open cover of Y. Let $\mathcal{V}$ be a finite subcover of $\mathcal{U} + E^c$. Then $\mathcal{V} - E^c$ is a finite subcover of $\mathcal{U}$. This is because for any $x \in E$, $x \in \mathcal{V}$ (because $x \in Y$) but $x \notin E^c$, so $x \in \mathcal{V} - E^c$. $\square$

Theorem 0.7. Every infinite subset of a compact set has a limit point in the compact set.

Proof. Suppose $E \subseteq (X, d)$ is compact and $F \subseteq E$ is infinite. Suppose F has no limit point in E. Then for every $z \in E$, let $B(z, \varepsilon_z)$ be a neighbourhood of z that contains no point of F (except possibly z). Then $\{B(z, \varepsilon_z)\}_{z \in E}$ is an open cover of E. However, since E is compact, there is a finite subcover. Since each $B(z, \varepsilon_z)$ contains at most one point of F, there are only finitely many points of F. Contradiction. $\square$

Proof that (3) $\implies$ (2). Suppose (3) holds on some $E \subseteq (\mathbb{R}^n, \|\cdot\|)$ but E is not bounded. Let $x_0 \in E$. We can produce a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$ such that

$$\|x_{n+1}\| > \|x_n\| + 1 \text{ for all } n \in \mathbb{N}.$$

Now suppose (3) holds on E but E is not closed. Then there exists a $z \in E^c$ such that z is a limit point of E. Then there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$ such that $\|x_j - z\| < \frac{1}{j}$ for all $j \in \mathbb{N}$. The set $F = \{x_n\}_{n \in \mathbb{N}}$ is infinite (otherwise, the minimum distance is the infimum, which is zero, but $z \notin E$). Then F must have a limit point in E.

For any $y \in \mathbb{R}^n$,

$$\begin{aligned} \|x_j - y\| &\geq \|z - y\| - \|x_j - z\| \\ &\geq \|z - y\| - \frac{1}{j}. \end{aligned}$$

If $\|z - y\|$ is positive, then there are only finitely many x_j within a distance $\|z - y\|$ of y. Hence y can be a limit point of F only if $y = z$. $\square$

Theorem 0.8. Let $E \subseteq Y \subseteq (X, d)$ where Y is compact in X. Then E is compact in Y if and only if it is compact in X.

Lecture 12: Mon 29 Jan '24

0.2 Connected Sets

Definition 0.9.

(a) Let (X, d) be a metric space. A pair of sets $A, B \subseteq X$ are said to be *separated* in X if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.

(b) A set $E \subseteq X$ is said to be *disconnected* if it is the union of two separated sets in X.

(c) E is *connected* if it is not disconnected.

Examples.

- Sets $A = (-1, 0)$ and $B = (0, 1)$ are separated in $\mathbb{R}$. Note that sgn is continuous on $A \cup B$ but does not satisfy the intermediate value property.

  However, if $A = (-1, 0]$ instead, then all continuous functions on $A \cup B$ satisfy the intermediate value property.

- The empty set is connected.

- $\mathbb{Q}$ is disconnected in $\mathbb{R}$. The partition $\{\mathbb{Q} \cap (-\infty, \sqrt{2}), \mathbb{Q} \cap (\sqrt{2}, \infty)\}$ separates $\mathbb{Q}$.

- $\mathbb{Q}$ is disconnected even in $\mathbb{Q}$.

Exercise 0.10. Let $E \subseteq Y \subseteq (X, d)$. Then E is connected relative to Y iff E is cpnnected in X.

Theorem 0.11. Let $E \subseteq \mathbb{R}$. Then E is connected iff E is convex, *i.e.*, for all $x < y \in E$, $[x, y] \subseteq E$.

Proof. Suppose E is connected, but not convex, *i.e.*, there exist $x < y \in E$ and some $r \in (x, y)$ that is not in E. Then $A = (-\infty, r] \cap E$ and $B = [r, \infty) \cap E$ separate E.

Conversely, suppose E is convex but not connected. Then there exist $A, B \subseteq E$ that separate E. Let $x \in A$ and $y \in B$ and suppose WLOG that $x < y$. Note that $A \cap [x, y]$ is non-empty and bounded. Let $r = \sup(A \cap [x, y])$.

By the lemma below, $r \in \overline{A \cap [x, y]} \subseteq \overline{A} \cap [x, y]$ so $r \in \overline{A}$. Disconnectedness forces that $r \notin B \iff r \in A$ so $x \leq r < y$.

But since r is the supremum of $A \cap [x, y]$, $(r, y) \subseteq B$. This gives $r \in \overline{B}$, violating the separation of A and B. $\square$

0.3 The Cantor Set

Definition 0.12 (Perfect set)**.** A set $E \subseteq (X, d)$ is said to be *perfect* if every point of E is a limit point of E.

Note that $E = [0, 1]$ is perfect in $\mathbb{R}$. Can we produce a "sparse" perfect set? Throwing away isolated points makes the set open. Throwing away a finite number of open sets preserves perfectness, but there are still *intervals of positive length*.

Can we produce a perfect set such that

(i) it contains no intervals of positive length?

(ii) E is *nowhere dense*, *i.e.*, the interior of the closure of E is empty?

Note that the second condition implies the first.