Assignment 5

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Problem 5.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$, with $x_n \geq 0$ for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}^*$. Show that*

$$\lim_{n\to\infty} (x_n)^{1/k} = \left(\lim_{n\to\infty} x_n\right)^{1/k}.$$

Solution. Call the limit of $(x_n)_n$ L. Let $\varepsilon > 0$ and let $\varepsilon' = \frac{\varepsilon}{L^{1/k}}$. Then for sufficiently small ε',

$$(L^{1/k} - \varepsilon)^k \leq L(1 - k\varepsilon' + 2^k(\varepsilon')^2) < L$$
$$(L^{1/k} + \varepsilon)^k \geq L(1 + k\varepsilon') > L$$

But $x_n \to L$, so eventually $x_n \in (L(1 - k\varepsilon' + 2^k(\varepsilon')^2), L(1 + k\varepsilon'))$.
Then $x_n^{1/k} \in (L^{1/k} - \varepsilon, L^{1/k} + \varepsilon)$. ■

Solution. [Alternative] Let $L = \lim_{n\to\infty} x_n$. Then $\frac{x_n}{L} \to 1$. But notice that for any real $a > 0$,

$$\left|1 - a^{1/k}\right| \leq |1 - a|$$

because $x^k - 1 = (x - 1)(x^{k-1} + x^{k-2} + \cdots + 1)$ where the second term is obviously larger than 1.

But then

$$\left(\frac{x_n}{x}\right)^{1/k} \to 1$$

which proves the result. ■

Problem 5.2. *Let (X, d) be a complete metric space, and $Y \subseteq X$. Show that $(Y, d|_Y)$ is a complete metric space if and only if Y is closed in (X, d).*

Solution. Y is a complete metric space iff every Cauchy sequence in Y converges in Y. But X is complete, so every Cauchy sequence in Y converges in X. Thus, Y is complete iff every convergent sequence in Y (viewed as a sequence in X) converges in Y. This is true iff Y is closed in X. ∎

Problem 5.3. *Let (X, d) be a metric space and $A \subseteq X$ be a dense subset, i.e., $\overline{A} = X$. Show that if every Cauchy sequence in A converges to a limit in X, then X is a complete metric space.*

Solution. Let $(x_n)_n$ be a Cauchy sequence in X. For each $n \in \mathbb{N}$, there exists $a_n \in A$ such that $d(x_n, a_n) < \frac{1}{n}$. Then $(a_n)_n$ is a Cauchy sequence in A, so it converges to some $a \in X$. But $d(x_n, a_n) \to 0$, so $x_n \to a$. ∎

Problem 5.4. *For any real sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ show that*

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$
$$\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

Solution. Let $X = \limsup_{n\to\infty} x_n$ and $Y = \limsup_{n\to\infty} y_n$. Then for any $z > X + Y$, rewrite z as $(X + \delta) + (Y + \delta) + \delta$. Then there is an N such that for all $n \geq N$,

$$x_n < X + \delta \quad \text{and} \quad y_n < Y + \delta$$

so that

$$x_n + y_n < z - \delta.$$

But then $z - \delta$ cannot be a subsequential limit of $(x_n + y_n)_n$. Thus

$$\limsup_{n\to\infty} x_n + y_n \leq X + Y.$$ ∎

Problem 5.5. *Compute $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$, where the sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{R}$ is given by*

$$x_1 = 0,$$
$$x_{2m} = \frac{x_{2m-1}}{2}, \quad m \geq 1,$$
$$x_{2m+1} = \frac{1}{2} + x_{2m}, \quad m \geq 1.$$

Solution. **Claim:** $x_{2m+1} = 1 - \frac{1}{2^m}$.

Proof. Induction. $\square$

Corollary: $x_{2m} = \frac{1}{2} - \frac{1}{2^m}$.
Thus $\inf_{n \geq 2m} x_n = x_{2m}$. Then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{2^n} \right) = \frac{1}{2}.$$

For limsup, note that each term is less than 1, but 1 is a subsequential limit via the odd terms. Thus

$$\limsup_{n\to\infty} x_n = 1. \qquad \blacksquare$$