UMA205: Introduction to Algebraic Structures

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Contents

Lecture 09: Mon 23 Jan '24

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Theorem 0.1 (Multinomial theorem)**.** Let $n, k \in \mathbb{N}$ and $x_1, \ldots, x_k$ be indeterminates. Then

$$\sum_{\substack{0 \le a_1, \ldots, a_k \le n \\ a_1 + \cdots + a_k = n}} \binom{n}{a_1, \ldots, a_k} x_1^{a_1} \ldots x_k^{a_k} = (x_1 + \cdots + x_k)^n$$

Proof. □

Example.

$$\begin{aligned} \binom{1/2}{n} &= \frac{(1/2)(-1/2)\ldots(1/2-n+1)}{n} \\ &= \frac{(-1)^{n-1}(2n-3)!!}{2^n n!} \end{aligned}$$

Definition 0.2. A *weak composition* of $n \in \mathbb{N}$ is a sequence $(a_i)_{i=1}^k$ where $a_i \in \mathbb{N}$ and $a_1 + \cdots + a_k = n$. If each $a_i > 0$, then it is called a *(strict) composition.*

Example. For $n = 3$, its strict compositions are $(1,1,1)$, $(1,2)$, $(2,1)$ and (3).

Proposition 0.3. The number of weak compositions of n into k parts is $\binom{n+k-1}{k-1}$.

Proof. □

Corollary 0.4. The number of compositions of n into k parts is $\binom{n-1}{k-1}$.

Proof. Each box must get at least one ball, so use proposition 0.3 with $n \mapsto n-k$. $\square$

Corollary 0.5. The total number of compositions is 2^{n-1}.

Proof. $\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$. $\square$

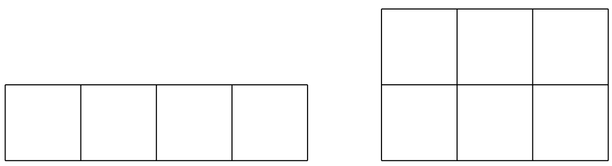
Definition 0.6 (Partitions)**.** An *(integer) partition* of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of weakly decreasing positive integers which sum to n. We write $\lambda \vdash n$. Each λ_i is called a *part* and the number of parts is called the *length*, denoted $\ell(\lambda)$. We write $p(n)$ for the number of partitions of n.

Example. The partitions of 5 are (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$ and $(1,1,1,1,1)$. Thus $p(5) = 7$.

Proposition 0.7. The number of partitions of n into exactly (resp. at most) k parts is the same as the number of partitions of n with largest part exactly (resp. at most) k.

Definition 0.8. The *Young/Ferrers diagram* of a partition is a left-justified array of boxes with λ_i boxes in the ith row.

Example. The Young diagrams of $(4,1)$ and $(3,2)$ are



Definition 0.9 (Conjugate)**.** The *conjugate* of a partition λ, denoted λ', is the partition whose Young diagram is the transpose of that of λ. That is,

$$\lambda'_i = \#\{j \in \mathbb{N} : \lambda_j \geq i\}$$

Proof of proposition 0.7. If λ has length k, then λ' has largest part k. $\square$

Theorem 0.10. The number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

Proof. $\square$

Theorem 0.11 (Euler)**.** The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Fact 0.12 (Hardy-Ramanujan Formula)**.**

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Definition 0.13. A *set partition* of $[n]$ is a collection of pairwise disjoint non-empty subsets/blocks whose union is $[n]$. The number of set partitions of $[n]$ into k (non-empty) blocks is called the *Stirling number of the second kind* and denoted $\left\{ {n \atop k} \right\}$, read "$n$ set k".

Example. The set partitions of [3] are 123, 12|3, 13|2, 1|23 and 1|2|3.

Lecture 11: Mon 29 Jan '24

We define, by convention, $\left\{ {n \atop 0} \right\} = \delta_{n,0}$ and $\left\{ {n \atop k} \right\} = 0$ for $k > n$.

We immediately have that $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$ for $n \neq 0$. We enumerate some more values below.

n \ k	1	2	3	4
1	1			
2	1	1		
3	1	3	1	
4	1	6	7	1

Table 1: Stirling numbers of the second kind

Proposition 0.14. For $1 \leq k \leq n$, $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$

Proof. We split the partitions into two cases:

(i) The partition contains $\{n\}$ as a singleton. There are $\left\{ {n-1 \atop k-1} \right\}$ such partitions.

(ii) n belongs to some other block. There are $\left\{ {n-1 \atop k} \right\}$ ways to partition the remaining elements, and k ways to choose which block n belongs to.

$\square$

Proposition 0.15. The number of surjections from $[n]$ to $[k]$ is $k!\left\{ {n \atop k} \right\}$.

Proof. Any surjection is determined by a sequence $(p_1, p_2, \ldots, p_k)$ of preimages of $1, 2, \ldots, k$ respectively. These are simply permutations of k blocks of $[n]$. $\square$

Corollary 0.16. For all $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} \left\{ {n \atop j} \right\} x^{\underline{j}} = x^n$$

Proof. For $x \in \mathbb{N}$, the RHS counts functions from $[n]$ to $[x]$. We split these functions by the size of the image.

For functions of image size j, there are $\binom{n}{j}$ ways to choose the image, and $j!\left\{ {n \atop j} \right\}$ ways to choose the preimage. But $\binom{n}{j} j!$ is precisely $n^{\underline{j}}$.

Thus both sides agree at infinitely many points, and so they are equal. $\square$

Definition 0.17 (Bell numbers)**.** The number of set partitions of $[n]$ is called the nth *Bell number*, denoted $B_n \coloneqq \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$.

Exercise 0.18. Prove that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

Solution. Let b_k be the number of partitions of $[n+1]$ with $n+1$ in a block of size $k+1$. Then $b_k = \binom{n}{k} B_{n-k}$. This gives the desired result by the re-indexing $k \mapsto n-k$.

Though this seems like a recurrence, it is not (for this course). By “recurrence” we will mean a recurrence relation dependent upon at most M previous terms, for some fixed M.

0.1 Permutations as Cycles

Let S_n be the set of all permutations of $[n]$. Recall that any permutation $\pi \in S_n$ can be written as a product of cycles. A useful convention is to skip cycles of length 1. Thus we write $\sigma = 6754132$ as $(1635)(27)$. This allows us to consider π as just a product (under composition) of permutations which are cyclic on some subset of $[n]$. *E.g.* $\pi = (1635) \circ (27)$, where (27) for example is the permutation which swaps 2 and 7 and fixes everything else.

Lemma 0.19. Let $\sigma \in S_n$ and $j \in [n]$. Then there exists an $i \in \mathbb{N}^*$ such that $\sigma^i(j) = j$.

Proof. Consider the sequence $(\sigma(j), \dots, \sigma^n(j))$. If any of these are equal to j, we are done. Otherwise, by the pigeonhole principle, there exist $k < l$ such that $\sigma^k(j) = \sigma^l(j)$. Then $\sigma^{l-k}(j) = j$ (since σ is a bijection). □

Corollary 0.20. Let $\sigma \in S_n$. Then $\sigma^{n!} = \mathrm{id}$.

Proof. By the lemma, for each $j \in [n]$, there exists an $i_j \in [n]$ such that $\sigma^{i_j}(j) = j$. Since $i_j \mid n!$ for all j, we have $\sigma^{n!}(j) = j$ for all j. □

Notation. We will write cyclic decompositions of permutations as follows:

- Each cycle has its smallest element first.
- Cycles are written in increasing order of their smallest elements.

Definition 0.21. The *cycle type* of a permutation σ, denoted $\mathrm{type}(\sigma)$, is the partition formed by arranging its cycle lengths in weakly decreasing order.

Definition 0.22. The number of permutations in S_n with k cycles is called the (unsigned) *Stirling number of the first kind*, denoted $\left[{n \atop k}\right]$.