UM 204 HOMEWORK ASSIGNMENT 5

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(NOT FOR SUBMISSION)

- These problems are for self-study. Try these **on your own** before seeking hints.
- Some of these problems will be (partially) discussed at the next tutorial.
- A 15-min. quiz based on this assignment will be conducted at the end of the tutorial section.

Problem 1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$, with $x_n \geq 0$, for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}_{>0}$. Show that

$$\lim_{n\to\infty} (x_n)^{\frac{1}{k}} = \left(\lim_{n\to\infty} x_n\right)^{\frac{1}{k}}.$$

Problem 2. Let (X, d) be a metric space, and $Y \subseteq X$. Show that $(Y, d|_Y)$ is a complete metric space if and only if Y is closed in (X, d).

Problem 3. Let (X, d) be a metric space and $A \subseteq X$ be a dense subset, i.e., $\overline{A} = X$. Show that if every Cauchy sequence in A converges to a limit in X, then X is a complete metric space.

Problem 4. For any real sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ show that

$$\begin{aligned}
\limsup_{n\to\infty}(x_n + y_n) &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n, \\
\liminf_{n\to\infty}(x_n + y_n) &\geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.
\end{aligned}$$

Problem 5. Compute $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$, where $\{x_n\}_{n\in\mathbb{N}_+} \subset \mathbb{R}$ is given by

$$\begin{aligned}
x_1 &= 0 \\
x_{2m} &= \frac{x_{2m-1}}{2}, \quad m \geq 1, \\
x_{2m+1} &= \frac{1}{2} + x_{2m}, \quad m \geq 1.
\end{aligned}$$