UMA204: Introduction to Basic Analysis

Naman Mishra

January 2024

Contents

Lecture 04: Wed 10 Jan '24

Definition 0.1 (Archimedean property)**.** An ordered field F is said to have the *Archimedean property* if for every $x, y > 0$, there exists an $n \in \mathbb{N} \subseteq F$ such that $nx > y$.

Theorem 0.2. $\mathbb{Q}$ has the Archimedean property.

Proof. Let $x, y > 0$ be rationals. If $x > y$, $n = 1$ works. Suppose $x \leq y$. It suffices to show that $\exists\, n \in \mathbb{N}(nr > 1)$, where $r = x/y$. Since r is positive, we have $p, q \in \mathbb{N}^*$ such that $r = p/q$. Let $n = 2q$. This gives $nr > 1$. □

Remarks. Not all ordered fields have the Archimedean property.

Theorem 0.3. Let F be an ordered field with the LUB property. Then F has the Archimedean property.

Proof. Let $x, y > 0$. Suppose $\forall\, n \in \mathbb{N}(nx \leq y)$. Let $A = \{nx \mid n \in \mathbb{N}\}$. Clearly A is non-empty and bounded above. Then $\sup A$ exists and so there exists an $m \in \mathbb{N}$ such that $\sup A - x < mx$. Thus $\sup A < (m+1)x \in A$, a contradiction. □

Theorem 0.4. Let F be an ordered field with the LUB property. Then $\mathbb{Q}$ is dense in F, *i.e.*, given $x < y \in F$, there exists a rational $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. Follows from theorem 0.2 and problem 4 on assignment 1. $\square$

0.1 The Reals

Theorem 0.5 (Dedekind/Cauchy)**.** There exists a unique (up to isomorphism) ordered field with the LUB property.

Proof of uniqueness. Let F and G be OFWLUB. Let h be identity on $\mathbb{Q} \subseteq F, G$. Let $z \in F$ and

$$A_z = \{w \in \mathbb{Q} \mid w <_F z\}.$$

Claim: A_z is non-empty and bounded above when viewed as a subset of G, and therefore has a supremum in G.
First, A_z is non-empty by density applied to $(z - 1_F, z)$ o Archimedean applied to $-z$. Secondly, by Archimedean (or density) there exists a *rational* upper bound q of A_z in F. This q is also an upper bound of A_z in G.
By LUB, A_z has a supremum in G.

We define $h(z) := \sup_G A_z$. For this we need to show that $h(r) = r$ for all $r \in \mathbb{Q}$, so that the definitions coincide. Let $r \in \mathbb{Q}$ so that $A_r = \{w \in \mathbb{Q} \mid w <_F r\}$. Clearly r is an upper bound of A_r in G. For any $g \in G$, there is some $q \in \mathbb{Q}$ such that $g <_G q <_G r$ (by density of $\mathbb{Q}$ in G). Thus g cannot be an upper bound of $A_r \subseteq G$. Thus $r = \sup_G A_r = h(r)$.

Claim: h preserves order.
Let $z < w \in F$. By density of $\mathbb{Q}$ in F, there exist rationals r, s, t such that $z < r < s < t < w$. Then $A_z \subsetneq A_w$ as subsets of F and hence of G. Thus

$$h(z) = \sup_G A_z \leq_G r < s < t \leq_G \sup_G A_w = h(w).$$

Claim: h preserves addition.
It is sufficient to show that $A_{x+y} = A_x + A_y$, where set addition is defined pairwise. If a rational $q \in A_x + A_y$, then clearly $q <_F x + y$ and so $q \in A_{x+y}$. Let $q \in A_{x+y} \iff q <_F x + y$. Then $q - x \in A_y$. Since A_y has no largest element (by density), there exists an $r \in A_y$ with $q - x < r < y$. Then $q - r < x$ and so $q - r \in A_x$. Thus $q = (q - r) + r \in A_x + A_y$ which gives equality of the sets.

Since $\sup A_x + \sup A_y = \sup(A_x + A_y) = \sup A_{x+y}$, h preserves addition.

Claim: h preserves multiplication.

$\square$

0.1.1 Dedekind's Construction

Definition 0.6 (Dedekind cut)**.** A *Dedekind cut* is a non-empty proper subset $A \subsetneq \mathbb{Q}$ such that

(i) if $a \in A$, then $b \in A$ for all $b \in \mathbb{Q}$ with $b < a$.

(ii) if $a \in A$, then there exists a $c \in A$ such that $a < c$.

Definition 0.7 ($\mathbb{R}$)**.** We define

$$\mathbb{R} := \left\{A \in 2^{\mathbb{Q}} \mid A \text{ is a Dedekind cut}\right\}.$$

Further,

(i) $A \le B \iff A \subseteq B$;

(ii) $A + B = \{a + b \mid a \in A, b \in B\}$. The additive identity $0 = \{x \in \mathbb{Q} \mid x < 0\}$;

(iii) for $A, B > 0$,

$$A \cdot B = \{q \in \mathbb{Q} \mid q \le rs \text{ for some } r \in A, s \in B\}.$$

If $A < 0$ but $B > 0$, then $A \cdot B = -((-A) \cdot B)$. If $B < 0$ but $A > 0$, then $A \cdot B = -(A \cdot (-B))$. If $A < 0$ and $B < 0$, then $A \cdot B = (-A) \cdot (-B)$.

Lecture 05: Thu 11 Jan '24

Theorem 0.8. $\mathbb{R}$ has the least upper bound property.

Proof. Let α be a non-empty subset of $\mathbb{R}$ that is bounded above. We claim that $S = \bigcup_{A \in \alpha} A$ is the supremum of α.

s **is a cut:** Since S is a union of a non-empty set of non-empty sets, it is non-empty. Since S is bounded above, say by some cut C, we have $S \subseteq C \subsetneq \mathbb{Q}$ and so $S \neq \mathbb{Q}$. If $a \in S$, then $a \in A$ for some $A \in \alpha$. Since A is a cut, every rational smaller than a is contained in A and thereby in S. Moreover, there exists an $a' \in A$ which is larger than a. Thus $a' \in S$ is larger than a.

upper bound: $A \subseteq S$ for all $A \in \alpha$.

least upper bound: For any $D \subsetneq S$, let $b \in S \setminus D$. But since $b \in A$ for some $A \in \alpha$, D is not an upper bound of α. □

Dedekind's construction is an "order completion". Thus all the order properties (LUB, density) are nice, but arithmetic is ugly.

0.1.2 Cauchy's Construction

There seem to be sequences in $\mathbb{Q}$ that "should" have a limit (*e.g.*, a monotone and bounded sequence) but do not (within $\mathbb{Q}$). We construct equivalence classes of sequences which "converge" to the same number, and define reals by those classes.

Definition 0.9 (Sequence)**.** A sequence of rational numbers is a $f\colon \mathbb{N} \to \mathbb{Q}$. We usually denote $f(k)$ by a_k and call it the k-th term of the sequence. The function f is usually written as $(a_k)_{k\in\mathbb{N}}$.

Definition 0.10. A sequence $(a_k)_{k\in\mathbb{N}} \subseteq \mathbb{Q}$ is said to be

(i) $\mathbb{Q}$-bounded if there exists an $M \in \mathbb{Q}$ such that $|a_k| \leq M$ for all $k \in \mathbb{N}$.

(ii) $\mathbb{Q}$-Cauchy if for every rational $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon$ for all $m, n \geq N$.

(iii) convergent in $\mathbb{Q}$ if there exists an $L \in \mathbb{Q}$ such that for all (rational) $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

Exercise 0.11. Show that if a sequence is convergent in $\mathbb{Q}$, then it is $\mathbb{Q}$-Cauchy, and if it is $\mathbb{Q}$-Cauchy, then it is $\mathbb{Q}$-bounded.

Remarks. From UMA101, we know that if a sequence is convergent in $\mathbb{Q}$, the limit is unique. We also know arithmetic laws of limits (which we proved over $\mathbb{R}$, but they hold over $\mathbb{Q}$as well).

Definition 0.12. Two sequences $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ are said to be *equivalent* if their difference converges to 0.

Proposition 0.13. Let $\mathcal{C}$ denote the space of $\mathbb{Q}$-cauchy sequences. Then $\sim$ given by $a \sim b$ if a and b are equivalent (as per the previous definition) is an equivalence relation.

Proof. Reflixivity and symmetry are immediate. Transitivity follows from the triangle inequality. $\square$

Definition 0.14 ($\mathbb{R}$)**.** We define

$$\mathbb{R} := \mathcal{C}/\sim.$$

Further,

(i) $[a] +_{\mathbb{R}} [b] := [a + b]$.

(ii) The additive identity $0 = [(0)_{n \in \mathbb{N}}]$.

(iii) $[a] \cdot_{\mathbb{R}} [b] := [a \cdot b]$.

(iv) $[a] >_{\mathbb{R}} 0$ if there exists a rational $c > 0$ and an $N \in \mathbb{N}$ such that $a_n > c$ for all $n \geq N$. From positivity, we can define order as $[a] >_{\mathbb{R}} [b]$ iff there is some $[d] > 0$ such that $[a] + [d] = [b]$.

Proposition 0.15. The operations $+\mathbb{R}$ and $\cdot_{\mathbb{R}}$ and the relationn $>_{\mathbb{R}}$ are well-defined.

Proof. Let $a \sim a'$ and $b \sim b'$. Then $a+b-(a'+b') = (a-a')+(b-b') \to 0$. ☐

Lecture 06: Mon 15 Jan '24

We define an isomorphism from $\mathbb{Q}$ into $\mathbb{R}$ as

$$r \in \mathbb{Q} \mapsto [(r, r, \dots)] \in \mathbb{R}.$$

Theorem 0.16. $(\mathbb{R}, +, \cdot, \leq)$ satisfies the Archimedean property.

Proof. Let $[a], [b] > 0$ be in $\mathbb{R}$. Since $[b]$ is $\mathbb{Q}$-Cauchy, there exists a positive $M \in \mathbb{Q}$ such that $b_n < M$ for all $n \in \mathbb{N}$.

Since $[a] > 0$, let $c \in \mathbb{Q}^+$ and $N \in \mathbb{N}$ be such that $a_n > c$ for all $n \geq N$. By the Archimedean property of $\mathbb{Q}$, there exists an $m \in \mathbb{N}$ such that $mc > M$. Thus $b_n < M < mc < ma_n$ for all $n \geq N$. Thus $(m+1)a_n - b_n > ma_n - b_n + c > c$ for all $n \geq N$ and so $[m+1][a] > [b]$. ☐

Theorem 0.17. $(\mathbb{R}, +, \cdot, \leq)$ satisfies the LUB property.

Proof. Let $A \subseteq \mathbb{R}$ be a non-empty bounded above set.

For $n \in \mathbb{N}^*$, let $U_n = \left\{m \in \mathbb{Z} : \frac{m}{n} \text{ is an upper bound of } A\right\}$. From the Archimedean property of $\mathbb{R}$, U_n is non-empty and bounded below. By well-ordering, U_n has a minimum $m(n)$. Let $a_n = \frac{m(n)}{n}$ for each $n \in \mathbb{N}^*$.

Claim: $(a_n)_{n\in\mathbb{N}^*}$ is $\mathbb{Q}$-Cauchy.
Let ε be a positive rational number. By Archimedean, there $\frac{1}{n} < \varepsilon$ for all n above some N in $\mathbb{N}$. Note that for any $n \in \mathbb{N}^*$, a_n is an upper bound of A, and $a_n - \frac{1}{n}$ is not an upper bound of A.

Thus for any $n, n' \geq N^*$, we have

$$\begin{aligned} \frac{m(n)}{n} &> \frac{m(n')}{n'} - \frac{1}{n'} & \qquad \frac{m(n')}{n'} &> \frac{m(n)}{n} - \frac{1}{n} \\ a_n - a_{n'} &> -\frac{1}{n'} & a_n - a_{n'} &< \frac{1}{n} \end{aligned}$$

and so $|a_n - a_{n'}| < \max\{\frac{1}{n}, \frac{1}{n'}\} < \varepsilon$.

Claim: $[(a_n)]$ is an upper bound of A.
Suppose there exists some $[x] > [a]$. That is, there is some positive rational c such that $c < x_n - a_n$ for all n larger than some $N_1 \in \mathbb{N}^*$. Since (x_n) is $\mathbb{Q}$-Cauchy, $-c/2 < x_n - x_m < c/2$ for all n, m larger than some $N_2 \in \mathbb{N}^*$. $\square$