UMA205: Introduction to Algebraic Structures

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Contents

Lecture 17: Fri 09 Feb '24

Proposition 0.1. Let T be a tree.

(1) Deleting any edge in T disconnects it.

(2) Adding a new edge to T creates a cycle.

(3) For any $v, w \in V(T)$, there is a *unique* path from u to v.

Proof. (3) holds because if there is more than one path from u to v, then we must create a cycle.

For (2), suppose $\{v, w\} \notin E(T)$ and we add it. By (3), there is a unique path from v to w in $E(T)$, and so adding this edge creates a cycle.

For (1), suppose that removing an edge $\{v, w\}$ from T still left it connected. Then we would have two paths from v to w, contradicting (3). □

Definition 0.2. A vertex with degree 1 is called a *leaf* or *pendant vertex.*

Lemma 0.3. Every tree on $n \geq 2$ vertices has at least 2 leaves.

Proof. Let the longest path in the tree be $(v_1, \ldots, v_k)$. Then v_1 and v_k must be leaves, for otherwise the path could be made longer. □

Theorem 0.4. All trees on n vertices have $n - 1$ edges.

Proof. This is clearly true for the singleton tree. Let T be a tree with $n+1$ vertices, and let l be a leaf (by the previous lemma). Removing l and its incident edge gives a tree of n vertices with $n-1$ edges. Thus, T has n edges. Winduction. $\square$

Theorem 0.5. Any connected graph on n vertices with $n-1$ edges is a tree.

Proof. True for $n = 1$. $\square$

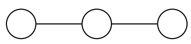
Definition 0.6. A *forest* is a graph with no cycles.

A tree is a connected forest.

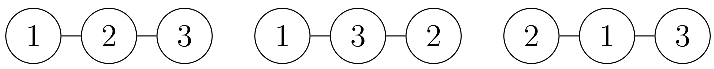
We wish to count the number of trees on vertices labelled $[n]$.

Examples.

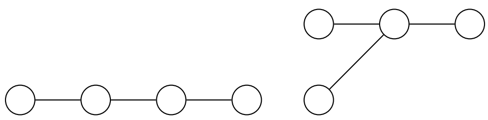
- For $n = 3$, first note that there is exactly one unlabelled tree,

  This gives rise to 3 labelled trees

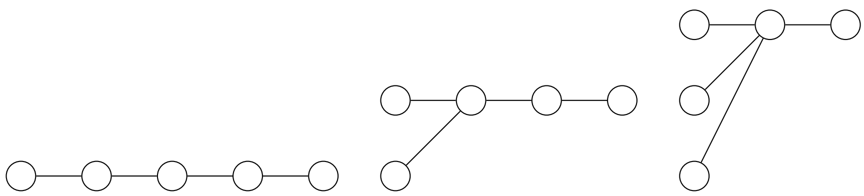
  

- For $n = 4$, there are 2 unlabelled trees.

  The first tree gives rise to $4!/2 = 12$ labelled trees, and the second gives rise to 4 labelled trees. Total: 16 labelled trees.

- For $n = 5$, there are 3 unlabelled trees.

  The first tree gives rise to $5!/2 = 60$ labelled trees, the second to $5!/2! = 60$, and the last to 5. Total: 125 labelled trees.

We observe the pattern that the number of labelled trees on n vertices is n^{n-2}.

Theorem 0.7 (Cayley's formula)**.** The number of trees labelled $[n]$ is n^{n-2}.

Remark. The book presents a bijective proof.

Definition 0.8. A *rooted tree* $T(v)$ is a tree with a marked vertex v, called the *root*.

A *branching* of a rooted tree $T(v)$ is an orientation of T, *i.e.*, an assignment of directions to the edges of T, in which every edge is directed away from v.

A *rooted forest* is one where every component has a root, and we can constuct branchings in the same way.

We will show that the number of branchings which is equal to the number of rooted trees is n^{n-1}.

Proof of Cayley's formula. We start with the empty graph over n vertices and add edges one at a time to form a branching. Initially, there are n components. At the k^{th} stage, we will have $n-k$ components. Consider the following algorithm:

> For $1 \leq k \leq n-1$, at the k^{th} stage, add an oriented edge (u, v) from any vertex to the root of one of the components to which it does not belong.

At the first stage, we have n choices for u and $n-1$ choices for v. At the second stage, we have n choices for u and $n-2$ choices for v, and so on. Thus at the k^{th} stage, we have $n(n-k)$ ways of forming a rooted forest. Continuing this way, we get that the number of branchings is $n^{n-1}(n-1)!$.

But note that every branching occurs $(n-1)!$ times, because of different orderings of the edges. Thus the total number of rooted trees is n^{n-1}. $\square$

Cayley's formula follows as a corollary. (A factor of n comes from the choice of root.)

Exercise 0.9. The number of rooted forests on n vertices is $(n+1)^{n-1}$.

Proof. Introduce a special vertex v_{-1}, and consider all rooted trees on $n+1$ vertices with root v_{-1}. Removing v_{-1} gives a rooted forest on n vertices, and every rooted forest on n vertices arises in this way. Thus by Cayley's formula, the number of rooted forests on n vertices is $(n+1)^{n-1}$. $\square$

Definition 0.10. Let $G = (V, E)$ and $|V| = n$. The *adjacency matrix* A is the $n \times n$ matrix indexed by V whose entries are

$$A_{v,w} = \mathbf{1}_{\{v,w\} \in E}.$$

Proposition 0.11. Let G be a graph and A be its adjacency matrix. Then $(A^k)_{v,w}$ counts the number of walks from v to w of length k.

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We aim to generalise Cayley's formula.

Definition 0.12 (Subgraph)**.** Let $G = (V, E)$. A *subgraph* of G is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E \cap 2^{V'}$.

Definition 0.13 (Spanning tree)**.** A *spanning tree* T of a graph $G = (V, E)$ is a subgraph with vertex set V such that T is a tree.

Example. A spanning tree of the complete graph K_5 with vertex set $[5]$ has the edges $\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}$.

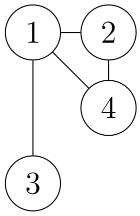
Definition 0.14 (Complete graph)**.** The *complete graph* K_n is the graph on n vertices with an edge between every pair of vertices.

In the language of spanning trees, Cayley's formula states that the number of spanning trees of K_n is n^{n-2}.

Definition 0.15 (Laplacian)**.** The *Laplacian* of a graph $G = (V, E)$ is the matrix given by $L = D - A$, where A is the adjacency matrix of G and $D = \operatorname{diag}(\deg(v_1), \dots, \deg(v_n))$.

The *reduced Laplacian* L_0 is obtained by deleting the last row and column of L.

Example. Let G be given by



Theorem 0.16 (Kirchoff's matrix tree theorem)**.** Let G be a graph and L_0 its reduced Laplacian. Then the number of spanning trees of G is given by $\det(L_0)$.

Definition 0.17. Let $G = (V, E)$ be a graph, $V = [n]$, and $m = |E|$, with the edges labelled by $[m]$. Suppose the edges of G are oriented in some way. Then the *incidence matrix* $\mathcal{I}(G) = \mathcal{I}$ is the $n \times m$ matrix given by

$$\mathcal{I}_{v,e} = \begin{cases} 1 & \text{if } v \text{ is the head of } e, \\ -1 & \text{if } v \text{ is the tail of } e, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 0.18. Let $G = (V, E)$ be a graph with Laplacian L. Let $\mathcal{I}_0$ be the incidence matrix with the last row removed, for any orientation of the edges of G. Then, independent of the choice of orientation,

- $L = \mathcal{I}\mathcal{I}^\top$,
- $L_0 = \mathcal{I}_0\mathcal{I}_0^\top$.

Theorem 0.19 (Cauchy-Binet formula)**.** Let A be an $n \times m$ matrix and B an $m \times n$ matrix with $n < m$. For an n-sized subset S of $[m]$, let $A_{[n],S}$ (resp. $B_{S,[n]}$) be the $n \times n$ submatrix of A (resp. B) formed by choosing the columns of A (resp. rows of B) with indices in S. Then

$$\det AB = \sum_{S \in \binom{[m]}{n}} \det A_{[n],S} \det B_{S,[n]}.$$

Proof of theorem 0.16. We will use the face that $L_0 = \mathcal{I}_0\mathcal{I}_0^\top$ and Cauchy-Binet. Fix a subset S of $[m]$, *i.e.*, edges in G, of size $n - 1$. Let $X = (\mathcal{I}_0)_{[n-1],S}$ Then the summand on the right-hand side of Cauchy-Binet for the determinant of $L_0 = \mathcal{I}_0\mathcal{I}_0^\top$ is $\det X \det X^\top = (\det X)^2$.

We claim that $(\det X)^2 = [(V, S) \text{ is a tree}]$.

Suppose there exists a vertex i of degree 1 in $G' = (V, S)$. Then the i^{th} row in X has only one non-zero entry, either 1 or -1. Expand $\det X$ using that row and use the induction hypothesis. The remaining graph is a tree iff G' is a tree.

If there are no vertices of degree 1 in G', then G' cannot be a tree. Since G' has $n-1$ edges, it is disconnected and must contain a cycle. The columns of X corresponding to the cycle must be linearly dependent, so $\det X = 0$.

Thus the claim is proved, and therefore $\det L_0 = \det \mathcal{I}_0 \mathcal{I}_0^\top$ gets a contribution of 1 from each spanning tree of G. □

Corollary 0.20 (Cayley's formula)**.** The number of spanning trees of K_n is n^{n-2}.

Proof. Let $G = K_n$. Then

$$\begin{aligned}
\det L_0 &= \det \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}_{(n-1)\times(n-1)} \\
&= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix} \\
&= n^{n-2}.
\end{aligned}$$

□

Lecture 19: Wed 14 Feb '24

We will now prove the Cauchy-Binet formula.

Lemma 0.21 (Sylvester's determinant identity)**.** Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$. Then

$$\lambda^m \det(\lambda I_n + AB) = \lambda^n \det(\lambda I_m + BA).$$

Proof. Use 2×2 block matrices. Note that

$$\begin{aligned}\begin{pmatrix}\lambda I_n & A\\ B & \lambda I_m\end{pmatrix} &= \begin{pmatrix}I_n & 0\\ B & I_m\end{pmatrix}\begin{pmatrix}\lambda I_n & 0\\ 0 & \lambda I_m - BA\end{pmatrix}\begin{pmatrix}I_n & A\\ 0 & I_m\end{pmatrix}\\ &= \begin{pmatrix}I_n & A\\ 0 & I_m\end{pmatrix}\begin{pmatrix}\lambda I_n - AB & 0\\ B & \lambda I_m\end{pmatrix}\begin{pmatrix}I_n & 0\\ B & I_m\end{pmatrix}.\end{aligned}$$

Fact 0.22. For such block matrices, the determinant of an upper or lower triangular block matrix is the product of the determinants of the diagonal blocks.

Then we have

$$\det\begin{pmatrix}\lambda I_n & A\\ B & \lambda I_m\end{pmatrix} = \lambda^n$$

□

Proof of Cauchy-Binet. Compare the coefficient of λ^{m-n} in the two sides of

$$\lambda^{m-n}\det(\lambda I_n + AB) = \det(\lambda I_m + BA).$$

□

0.1 Planar Graphs

Definition 0.23 (Planar graph)**.** A graph which can be drawn in the plane without edges intersecting in non-vertices is called a *planar graph*. We can allow loops and parallel edges.

A planar graph together with its planar embedding is called a *plane graph*.

Examples.

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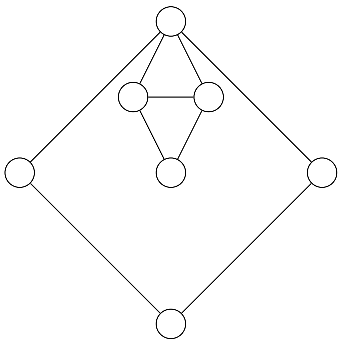
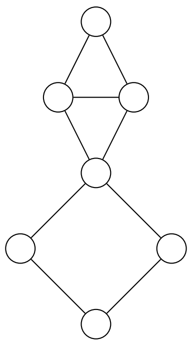
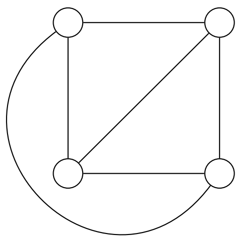
 and 

are isomorphic as planar graphs, but not as plane graphs.

- Edges need not be straight lines. Thus K_4 is planar.



Theorem 0.24 (Jordan curve theorem)**.** A plane graph partitions the plane into disjoint regions, which we call *faces*. We also include the unbounded face.

Theorem 0.25 (Euler's theorem)**.** Let G be a connected planar graph with V vertices, E edges, and F faces. Then

$$V - E + F = 2.$$

Proof. We induct on E. If $E = 0$, then $V = 1$ and $F = 1$, so the result holds. Suppose the result holds for all connected planar graphs with $E - 1$ edges.

We find an edge e such that removing e from G gives a connected graph G'. Removing e will merge the two faces on either side of e. Then G' has V vertices, $E-1$ edges and $F-1$ faces. Then $V-E+F = V-(E-1)+(F-1) = 2$.

If such an edge does not exist, *i.e.*, removing any edge disconnects the graph, then G is a tree. So $V - E + F = V - (V - 1) + 1 = 2$. □

Remark. Planar graphs can also be embedded on a sphere.

Definition 0.26 (Bipartite graph)**.** A *bipartite graph* $G = (V, E)$ is one where $V = V_1 \sqcup V_2$ such that no edge connects two vertices in the same set. The *complete bipartite graph* $K_{m,n}$ is the bipartite graph where $|V_1| = m$, $|V_2| = n$, and $\{v_1, v_2\} \in E$ for all $v_1 \in V_1$ and $v_2 \in V_2$.

Corollary 0.27. $K_{3,3}$ is not planar.

Proof. Suppose it were. $V = 6$, $E = 9$, so by Euler's theorem, $F = 5$. But each face must have at least 4 edges since $K_{3,3}$ is bipartite. Summing over all faces, we get $2E \geq 4F$, a contradiction. $\square$

Definition 0.28 (Minor)**.** A *minor* of a graph G is one obtained by deleting vertices or edges, or *contracting* edges. An edge is contracted by removing it and merging its two endpoints.

Fact 0.29 (Kuratowski's theorem)**.** A graph is planar iff it has no minor isomorphic to K_5 or $K_{3,3}$.

Definition 0.30 (Colouring)**.** A *(vertex) colouring* of a graph G is an assignment of colours to the vertices of G so that adjacent vertices have different colours.

Fact 0.31 (Four colour theorm)**.** Any planar graph can be coloured using at most 4 colours.