$$\min_{x \in \mathbb{R}^d} f(x) \qquad a_i \in \mathbb{R}^d, b_i \in \mathbb{R}$$

$$a_i^T x \geq b_i \qquad i = 1, \dots, m$$

$\{x \in \mathbb{R}^d : a_i^T x \geq b_i \quad i = 1, \dots, m\}$
is the constraint set

$f(x)$ is objective function
x is decision variable

minimizing a convex function
over a closed convex set
is called a convex optimization
problem.

$$f^* = \min_{x \in C \subseteq \mathbb{R}^d} f(x) \quad \Big| \quad f(x) \geq f^* \text{ for all } x \in C$$

$$x^* = \operatorname*{argmin}_{x \in C \subseteq \mathbb{R}^d} f(x) \quad \Big| \quad f(x^*) \leq f(x) \text{ for all } x \in C$$

$C \rightarrow$ Closed Convex Set

$f \rightarrow$ convex function

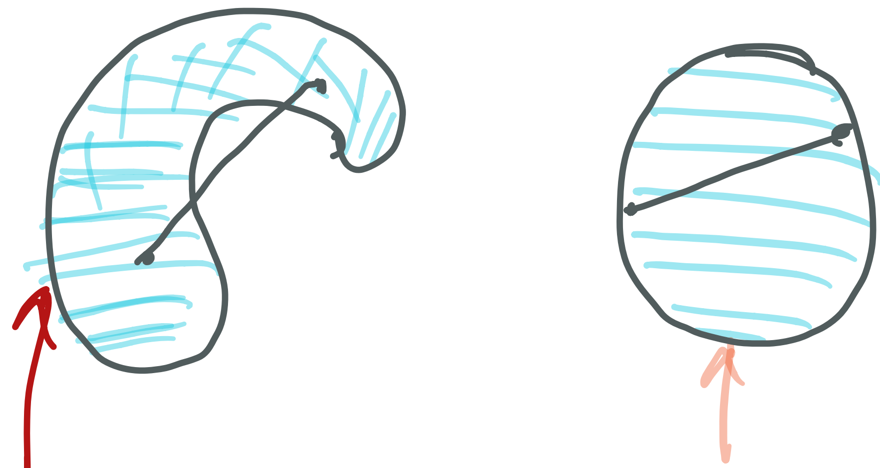
A set $C \subseteq \mathbb{R}^d$ is called a convex set

If for all $x^{(1)} \in C$ and $x^{(2)} \in C$

then $(1-\alpha)x^{(1)} + \alpha x^{(2)} \in C$

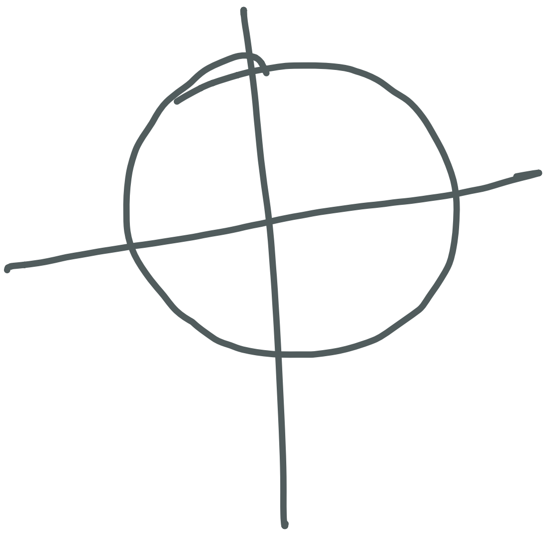
for all $0 < \alpha < 1$

$$x(\alpha) = x^{(1)} + \alpha\left(x^{(2)} - x^{(1)}\right)$$

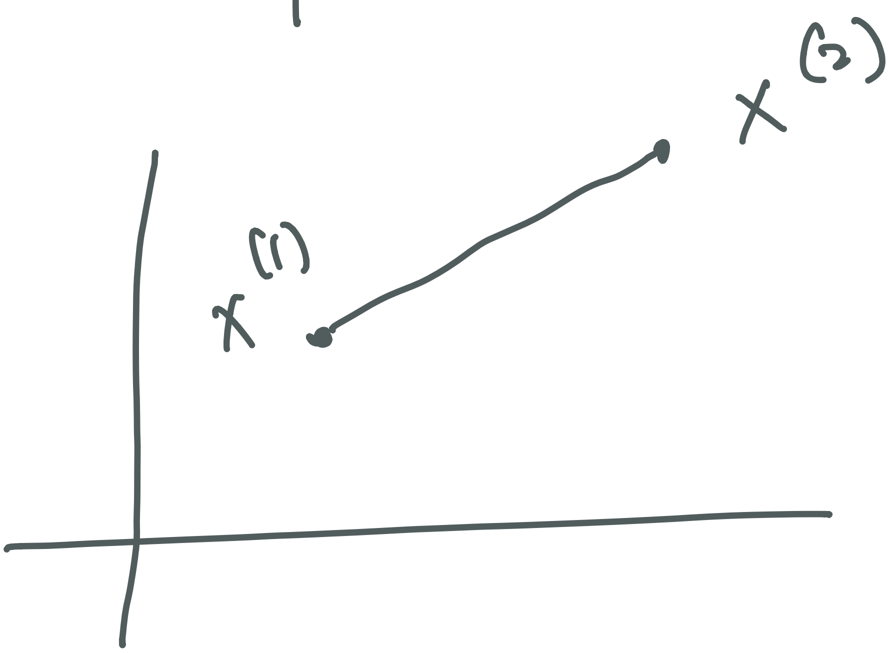


Not a Convex set

Convex set



$$x_1^2 + x_2^2 = 1$$



$[a, b] = \{x \mid a \leq x \leq b\}$

is a convex set

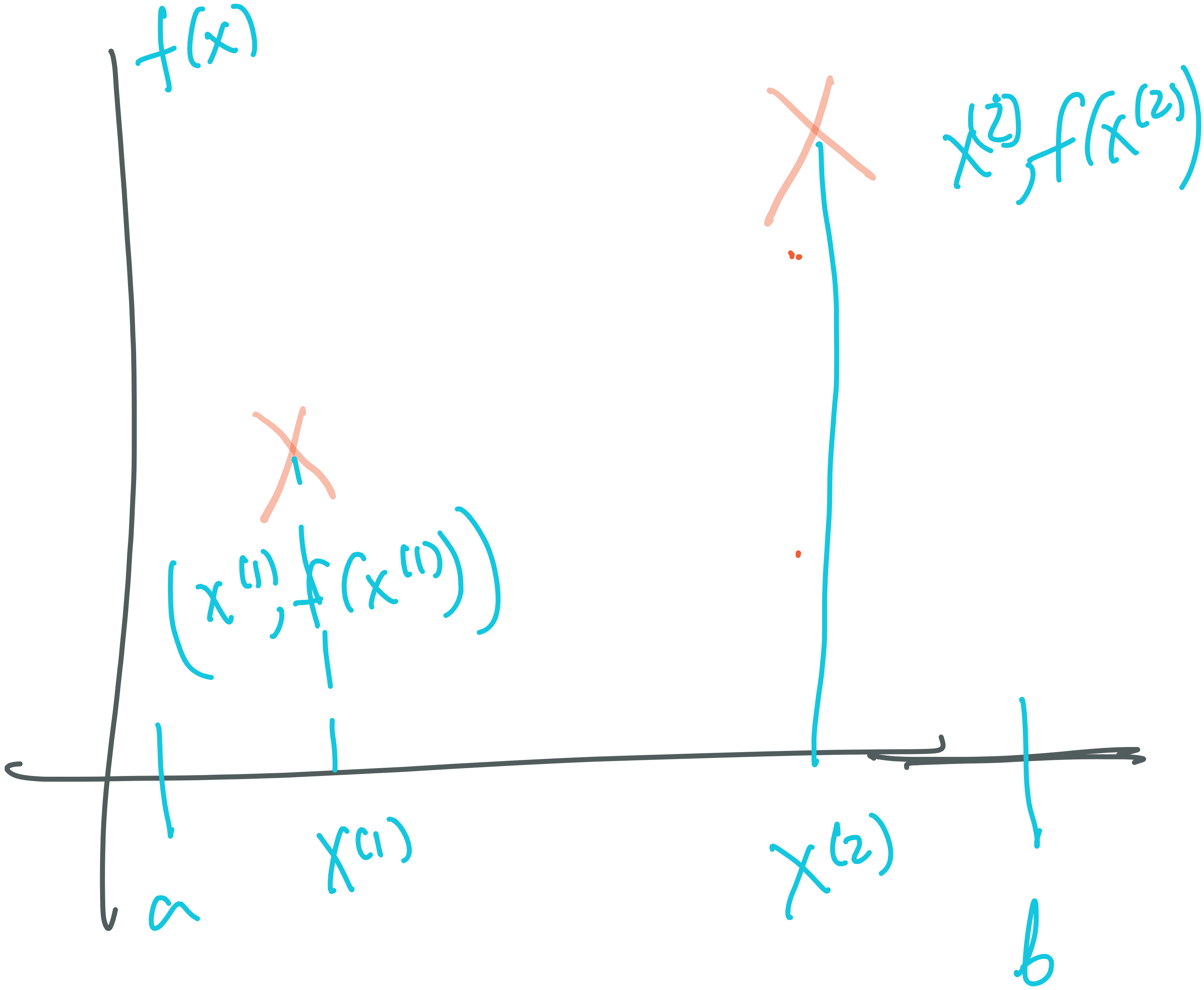
$f : [a, b] \to \mathbb{R}$ is said

to be convex if

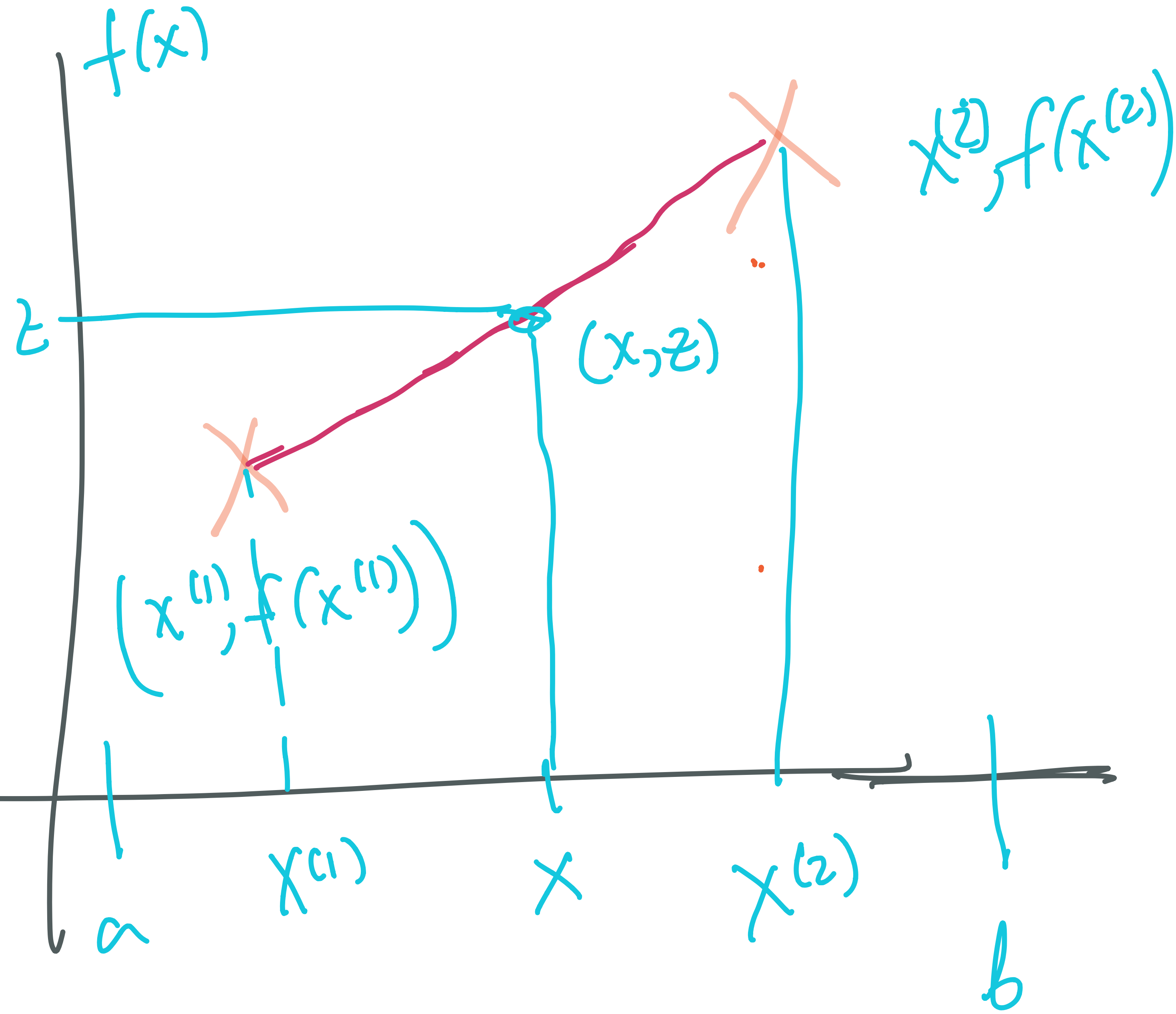
for all $x^{(1)} \neq x^{(2)} \in [a, b]$

$$f\left((1-\alpha)x^{(1)} + \alpha x^{(2)}\right)$$

$$\leq (1-\alpha) f(x^{(1)}) + \alpha f(x^{(2)})$$



$f(x)$
$x^{(2)}, f(x^{(2)})$
$(x^{(1)}, f(x^{(1)}))$
a
$x^{(1)}$
$x^{(2)}$
b



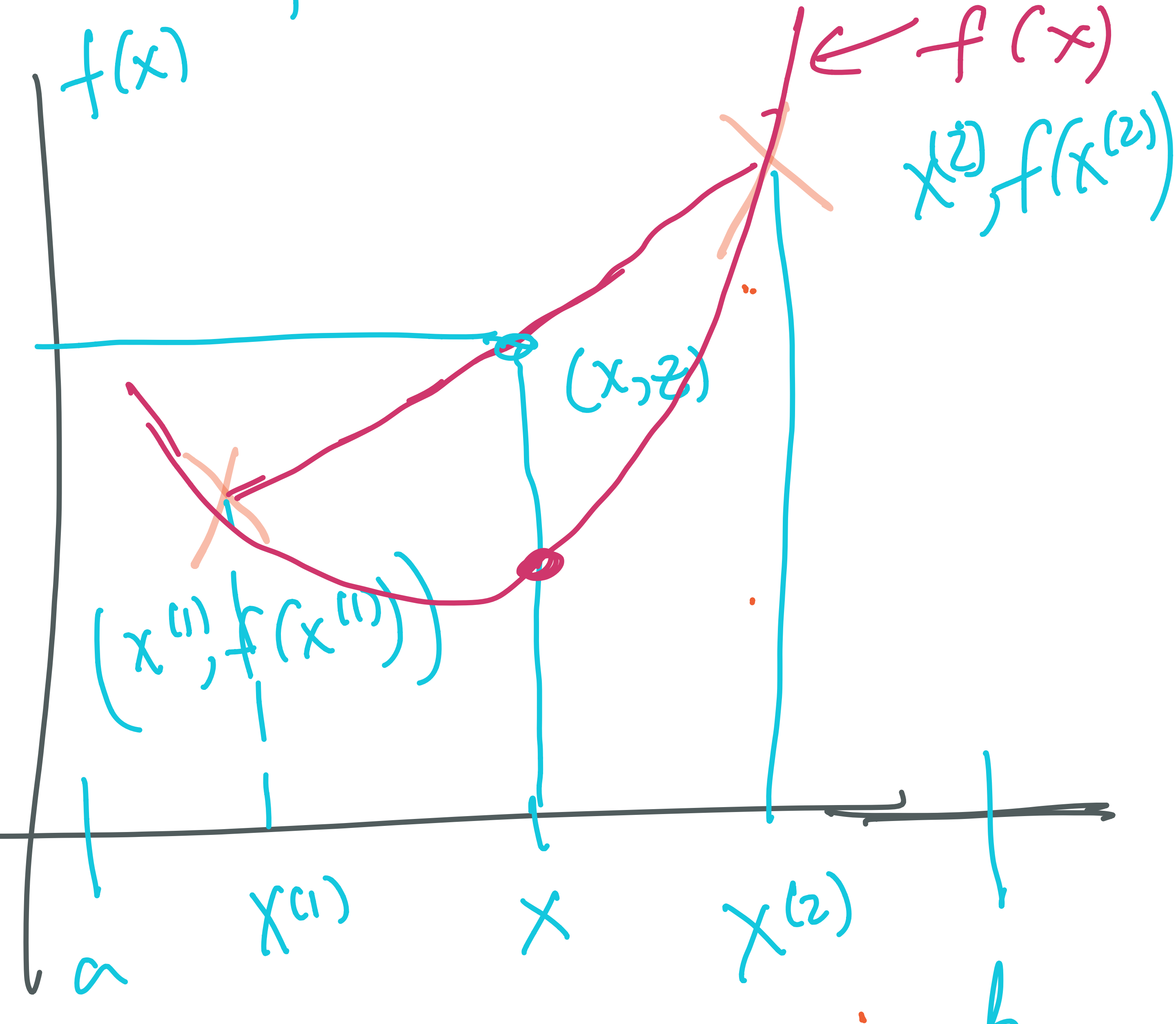
$$x = (1-\alpha)\,x^{(1)} + \alpha\,x^{(2)}$$

$$\alpha = \frac{x - x^{(1)}}{x^{(2)} - x^{(1)}}$$

$$z - f(x^{(1)}) = \frac{f(x^{(2)}) - f(x^{(1)})}{x^{(2)} - x^{(1)}} (x - x^{(1)})$$

$$= \alpha \left(f(x^{(2)}) - f(x^{(1)}) \right)$$

$$z = f(x^{(1)}) + \alpha \left(f(x^{(2)}) - f(x^{(1)}) \right)$$



$$f(x) \le z$$

$$f\left(\overbrace{(1-\alpha)x^{(1)} + \alpha x^{(2)}}^{x}\right)$$

$$\le \underbrace{(1-\alpha) f(x^{(1)}) + \alpha f(x^{(2)})}_{z}$$

Let $C \subseteq \mathbb{R}^d$ be a convex set. $f: C \to \mathbb{R}$ is a convex function if for all $x^{(1)} \neq x^{(2)} \in C$

$$f((1-\alpha)x^{(1)} + \alpha x^{(2)}) \leq (1-\alpha) f(x^{(1)}) + \alpha f(x^{(2)})$$

$f: \mathbb{R}^d \to \mathbb{R}$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{bmatrix}$$ is called

the gradient.

$\frac{\partial f}{\partial x_i}$ are continuous.

Let $f \in C^1$ defined over a convex set $C \subseteq \mathbb{R}^d$. f is convex over C iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

for all $x, y \in C$

If $f \in C^2$ defined over a convex set $C \subseteq \mathbb{R}^d$. f is convex iff

$$H(x) \succeq 0 \quad \text{for all } x \in C$$

$$\left(H(x)\right)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$H(x)$ is symmetric and p.s.d.

$$H(x) \geq 0$$

$$\Rightarrow \quad u^T H(x) u \geq 0.$$

$g : \mathbb{R}^d \to \mathbb{R}$ be a convex function

$\{x : g(x) \leq t\}$ is a convex set for all $t \in \mathbb{R}$

$$\begin{bmatrix} g((1-\alpha)x^{(1)} + \alpha x^{(2)}) \\ \leq (1-\alpha)\, g(x^{(1)}) + \alpha\, g(x^{(2)}) \\ \leq t \end{bmatrix}$$

f, f_i are convex functions

$$\left\{ \begin{array}{lll} \min\limits_{x \in \mathbb{R}^d} & f(x) & \\ & f_i(x) \leq 0 & i = 1, \dots, m \\ (P) & a_i^T x = b_i & j = 1, \dots, n \end{array} \right.$$

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{n} \mu_j (a_j^T x - b_j)$$

(1) $\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{j=1}^{n} \mu_j a_j = 0$

(2) $\lambda_i f_i(x) = 0$ $\qquad \lambda_i \geq 0$

(3) $f_i(x) \leq 0$ $\qquad a_j^T x = b_j$

$i = 1, \dots, m$ $\qquad j = 1, \dots, m$

If for any x^* there exists λ^*, μ^* such that (x^*, λ^*, μ^*) satisfy ① - ③ then x^* is a K.K.T. point of (P).

If x^* is a K.K.T. point then it is global minimum of (P)

$$\left.\begin{aligned} \min_{x \in \mathbb{R}^d} \ & \frac{1}{2}\|x - z\|^2 \\ & \omega^T x + b = 0 \end{aligned}\right\} \ (P^*)$$

$$\mathcal{L}(x, \mu) = \frac{1}{2}\|x - z\|^2 + \mu(\omega^T x + b)$$

$$x - z + \mu\omega = 0 \qquad x = z - \mu\omega$$

$$\omega^T x + b = 0$$

$$\omega^T z - \mu\|\omega\|^2 + b = 0$$

$$\mu^* = \frac{\omega^T z + b}{\|\omega\|^2} \qquad x^* = z - \mu^* \omega$$

$$\therefore \|x^* - z\| = \|\mu^* \omega\| = |\mu^*| \, \|\omega\|$$

$$= \frac{|\omega^T z + b|}{\|\omega\|}.$$

x^* is a K.K.T point of (P)

x^* is global minimum of (P).

$$\|x^* - z\| = \|\mu^* \omega\| = |\mu^*| \, \|\omega\|$$

$$|\mu^*| = \frac{|w^T z + b|}{\|w\|^2}$$

$$\Rightarrow \|x^* - z\| = \frac{|w^T z + b|}{\|w\|}.$$

Distance of z from

$w^T x + b = 0$

$$\min_i \frac{y_i (w^T x^{(i)} + b)}{\|w\|} = \gamma(w)$$

$$\max_{w, b} \gamma(w)$$

$$\frac{y_i (w^T x^{(i)} + b)}{\|w\|} \geq \gamma(w)$$