UMA204: Introduction to Basic Analysis

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January 2024

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Proof of (3) $\Rightarrow$ (2). Let $S \subseteq E$ be an infinite set. Thus there exists a sequence $(x_n)_n \subseteq S$ of distinct elements.

By (3), there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $(x_{n_k})_k$ is convergent to some $x \in E$. By the sequential characterization of closures, $x \in \overline{S}$.

Thus for all $\varepsilon > 0$, there exists a $k_\varepsilon \in \mathbb{N}$ such that for all $k \geq k_\varepsilon$, we have that $d(x_{n_k}, x) < \varepsilon$. Thus x is a limit point of S in E. □

0.1 Cauchy Sequences & Completeness

{sec:cauchy}

Recall the HW2 problem to show that the sequence $(x_n)_n$ given by

$$x_n = \begin{cases} 2 & \text{if } n = 0 \\ x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} & \text{if } n \geq 1 \end{cases}$$

is $\mathbb{Q}$-Cauchy but not convergent in $\mathbb{Q}$. This is an application of the Newton-Raphson method.

0.1.1 Newton-Raphson Method (Informal)

{sec:newton-raphson_method}

Given a function $f \colon \mathbb{R} \to \mathbb{R}$, we want to find a root of f. We pick some initial guess $x_0 \in \mathbb{R}$, and iterate via

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Under *some* assumptions on f and x_0, $(x_n)_n$ is Cauchy. Then

$$f(x_{n-1}) = f'(x_{n-1})(x_{n-1} - x_n) \to 0$$

If $\lim_{n\to\infty} x_n = l$, and f is continuous, then

$$f(l) = \lim_{n\to\infty} f(x_n) = 0.$$

{def:cauchy:sequence}

Definition 0.1 (Cauchy sequence)**.** Let $(x_n)_{n\in\mathbb{N}} \subseteq (X, d)$. We say that (x_n) is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n, m \geq N$, $d(x_n, x_m) < \varepsilon$.

{def:cauchy:completeness}

Definition 0.2 (Completeness)**.** (X, d) is said to be a *complete* metric space if every Cauchy sequence in (X, d) is convergent.

Theorem 0.3.

(a) Every convergent sequence is Cauchy.

(b) Every Cauchy sequence is bounded.

Proof. Trivial. □

Theorem 0.4. Every compact metric space is complete.

Proof. Let (X, d) be compact and let $(x_n)_n$ be a Cauchy sequence in X. Since X is compact, $(x_n)_n$ has a convergent subsequence $(x_{n_k})_k$ converging to some $x \in X$ (by **??**).

Then $(x_n)_n$ also converges to x by the triangle inequality. For large enough n, $d(x_n, x) \leq d(x_n, x_{n_n}) + d(x_{n_n}, x) < 2\varepsilon$. □

Theorem 0.5. $(\mathbb{R}^d, \|\cdot\|)$ is complete.

Proof. Let $(x_n)_n$ be a Cauchy sequence in $\mathbb{R}^d$. Then it must be bounded. Take a closed ball B centered at x_0 containing all elements of $(x_n)_n$. This is compact, and so the above theorem applies to give that $(x_n)_n$ has a limit in $B \subseteq \mathbb{R}^d$. □

Exercise 0.6. Every increasing and bounded above sequence in $\mathbb{Q}$ or $\mathbb{R}$ is Cauchy.

Proof. Suppose not. Then there exists an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n(N) > m(N) \geq N$ such that $\left|x_{n(N)} - x_{m(N)}\right| \geq \varepsilon$.

Let $m_0 = m(0)$ and $n_0 = n(0)$. For $k \geq 1$, let $m_k = m(n_{k-1})$ and $n_k = n(n_{k-1})$. Then

$$\begin{aligned} x_{n_k} &\geq x_{m_k} + \varepsilon \\ &\geq x_{n_{k-1}} + \varepsilon \end{aligned}$$

and so $(x_{n_k})_k$ is a subsequence with each term at least ε more than the last. Thus $x_{n_k} \geq x_0 + k\varepsilon$ for all $k \in \mathbb{N}$, which contradicts boundedness. □

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0.2 Sequences in $\mathbb{R}$

{def:sequences:R:extended}{sec:sequences:R}

Definition 0.7 (The Extended Reals)**.** The *extended real line* is the set of real numbers along with 2 formal sumbols $+\infty$ and $-\infty$, denoted by

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

$\overline{\mathbb{R}}$ will be endowed with the order

$$-\infty < x < \infty \text{ for all } x \in \mathbb{R},$$

along with the usual order on $\mathbb{R}$. We extend the algebraic operations on $\mathbb{R}$ to $\overline{\mathbb{R}}$.

- $x + \infty = +\infty$, $x - \infty = -\infty$ for all $x \in \mathbb{R}$.
- $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$ for all $x \in \mathbb{R}$, $x > 0$.
- $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$ for all $x \in \mathbb{R}$, $x < 0$.
- $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$, for all $x \in \mathbb{R}$.

If $E \subseteq \mathbb{R}$ is not bounded above in $\mathbb{R}$, we say $\sup E = +\infty$.

When constructing $\mathbb{R}$ through Dedekind cuts, $\overline{\mathbb{R}}$ can be constructed by relaxing the condition that a cut must be neither empty nor the whole of $\mathbb{Q}$. Then $\varnothing$ is a Dedekind cut represented as $-\infty$, and $\mathbb{Q}$ is a Dedekind cut represented as $+\infty$.

Definition 0.8. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Suppose that for all $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \geq M$. Then we say that $x_n \to +\infty$. If $-x_n \to +\infty$, we say that $x_n \to -\infty$.

Definition 0.9. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Let $E \subseteq \overline{\mathbb{R}}$ denote the set of subsequential limits of $(x_n)_n$ in the extended real line. The supremum of E is called the *upper limit* or *limit superior* of $(x_n)_n$, and is denoted by $\limsup_{n\to\infty} x_n$.

The infimum of E is called the *lower limit* or *limit inferior* of $(x_n)_n$, denoted $\liminf_{n\to\infty} x_n$.

Example. Let $(x_n = (-1)^n)_{n\in\mathbb{N}}$. Then $E = \{-1, +1\}$ so $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$.

{thm:sequences:R:limsup}

Theorem 0.10. Let $(x_n)_n \subseteq \mathbb{R}$ be a sequence and E be the set of subsequential limits of E in $\overline{\mathbb{R}}$.

(1) E is non-empty.

(2) $\sup E$ and $\inf E$ are contained in E.

(3) If $x > \sup E$ (resp. $x < \inf E$), then there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < x$ (resp. $x_n > x$).

(4) $\sup E$ (resp. $\inf E$) is the only element of $\overline{\mathbb{R}}$ statisfying both (2) and (3).

Proof. **(1)** If $(x_n)_n$ is bounded, then E is non-empty by **??**.

Let $(x_n)_{n\in\mathbb{N}}$ be unbounded above. Let $n_0 = 0$, and for $k \geq 0$, let

$$n_{k+1} = \min\{n > n_k \mid x_n > x_{n_k}\}$$

This exists since $(x_n)_n$ is unbounded above.

Suppose $m \notin (n_k)_{k\in\mathbb{N}}$. Let k be such that $n_k < m < n_{k+1}$. $x_m > x_{n_k}$ would imply $n_{k+1} \leq m$, so $x_m \leq x_{n_k}$. This shows that each x_m not in the subsequence is bounded above by some element of the subsequence.

Thus $(x_{n_k})_k$ is unbounded above, for if it weren't, all of $(x_n)_n$ would be bounded above. So for every $M \in \mathbb{R}$, there is a K such that $x_{n_K} > M$, but since the subsequence is increasing, $x_{n_k} > M$ for all $k \geq K$. Thus $\lim x_{n_k} = +\infty$.

(2) If $\sup E = +\infty$, then for all $M \in \mathbb{R}$, there is an $e_M \in E$ larger than $M+1$, so there is some x_n larger than M. Thus $(x_n)_n$ is unbounded above, so by the previous argument, $+\infty \in E$.

Now suppose $\sup E = x \in \mathbb{R}$. Let $\varepsilon_n = \frac{1}{2n}$. Let $n_0 = 0$. For $k > 0$, let e_k be an element of E larger than $x - \varepsilon_k$. Let $n_k > n_{k-1}$ be such that $x_{n_k} \in (e_k - \varepsilon_k, e_k + \varepsilon_k)$. Then $|x_{n_k} - x| < 2\varepsilon_k = \frac{1}{k}$. Thus $x_{n_k} \to x$, so $x \in E$. □

Example. Let $(x_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Then $E = \overline{\mathbb{R}}$.

Proof. Let $x \in \mathbb{R}$. Then for any $\varepsilon > 0$, there are infinitely many rationals that are ε-close to x. Thus $x \in E$.

For $x = \pm\infty$, replace "ε-close" with "larger/smaller than M". □

Theorem 0.11.

(1) Suppose $x_n \le y_n$ for all $n \in \mathbb{N}$. Then

$$\liminf x_n \le \liminf y_n \quad \text{and} \quad \limsup x_n \le \limsup y_n.$$

(2) $\lim x_n = x$ iff $\limsup x_n = \liminf x_n = x$.

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Proof of theorem 0.10 (continued). Let $\alpha^* = \sup E$.

(3) Suppose not. Let $x > \sup E$ such that for every $k \in \mathbb{N}$, there exists an $m(k) \ge k$ such that $x_{m(k)} \ge x$. Let $n_0 = m(0)$, and for $l \ge 1$, let $n_k = m(n_{k-1} + 1)$. Then $n_0 < n_1 < n_2 < \cdots$ and $x_{n_k} \ge x$ for all k. Thus $\gamma = (x_{n_k})_k$ is a subsequence of $(x_n)_n$, but all subsequential limits of γ are at least $x > \sup E$. But a subsequential limit of γ is a subsequential limit of $(x_n)_n$, so $\sup E \ge x$, a contradiction.

(4) Suppose $y < z$ in $\overline{\mathbb{R}}$ satisfy both (2) and (3). That is, both y and z are sequential limits of $(x_n)_n$, and if $x > y$ (or $x > z$), then there exists an $N \in \mathbb{N}$ such that $x_n < x$ for all $n \ge N$.

Choose

$$x = \begin{cases} 0 & \text{if } y = -\infty, z = +\infty \\ z - 1 & \text{if } y = -\infty, z \in \mathbb{R} \\ y + 1 & \text{if } y \in \mathbb{R}, z = +\infty \\ \frac{y+z}{2} & \text{if } y, z \in \mathbb{R} \end{cases}$$

In each case, $y < x < z$. By (3) applied to x, all but finitely many x_n are less than x. By (2) applied to z, infinitely many x_n are greater than x. Contradiction.

$\square$

Theorem 0.12.

(1) The following sequences admit limits in $\overline{\mathbb{R}}$.

$$y_n = \sup\{x_k : k \geq n\}$$
$$z_n = \inf\{x_k : k \geq n\}$$

(2) Moreover,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} z_n$$

where limits are taken in $\overline{\mathbb{R}}$.

Remark. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X. Define

$$A^* = \limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$
$$A_* = \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$$

Then $x \in A^*$ iff x is in infinitely many A_n, and $x \in A_*$ iff x is in all but finitely many A_n.

We say that $(A_n)_{n\in\mathbb{N}}$ *converges* if $A^* = A_*$.

We can characterize this using indicator functions.

$$\mathbf{1}_{A^*} = \limsup_{n\to\infty} \mathbf{1}_{A_n}$$
$$\mathbf{1}_{A_*} = \liminf_{n\to\infty} \mathbf{1}_{A_n}$$

which is to say that for each $x \in X$,

$$\mathbf{1}_{A^*}(x) = \limsup_{n\to\infty} \mathbf{1}_{A_n}(x)$$
$$\mathbf{1}_{A_*}(x) = \liminf_{n\to\infty} \mathbf{1}_{A_n}(x)$$

Proof. $(y_n)_n$ is a decreasing sequence, so it has a limit in $\overline{\mathbb{R}}$, since if it is not bounded, it converges to $-\infty$.

Let $y = \lim_{n\to\infty} y_n$. Since (y_n) is decreasing, given $k \in \mathbb{N}$, there exists an $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$,

$$y \leq y_n < y + \frac{1}{k}.$$

But $y_n = \sup\{x_i : i \geq n\}$, so for all $n \geq N(k)$, there exists an $m(k, n)$ such that $y_n - \frac{1}{k} < x_{m(k,n)} \leq y_n$.

Let

$$\begin{aligned}
n_1 &= m(1, N(1)) \\
n_2 &= m(2, n_1 \vee N(2) + 1) > n_1 \vee N(2) \\
&\vdots \\
n_k &= m(k, n_{k-1} \vee N(k) + 1) > n_{k-1} \vee N(k)
\end{aligned}$$

$\square$