Assignment 3

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Problem 3.1.

Solution (Case 1). Let $b > 1$ and $n > 0$. Let $B = \{t \in \mathbb{R} : t > 0, t^n < b\}$. B is non-empty since $1 \in B$.
Since $b > 1$, $b^n > b$. Moreover, since $0 < x < y \implies x^n < y^n$, we have that B is bounded above by b. Thus b has a supremum.

We now show that $(\sup B)^n = b$. For this we show that B has no largest element. For any $u \in B$, let $\delta = \min\left\{\frac{b-u^n}{2^n u^n}, 1\right\}$. Then

$$\begin{aligned}
(u(1+\delta))^n &= u^n(1 + n\delta + \cdots + \delta^n) \\
&\leq u^n\left(1 + \delta\sum_{j=1}^{n}\binom{n}{j}\right) \\
&< u^n + 2^n u^n \delta \\
&\leq u^n + b - u^n \\
&= b
\end{aligned}$$

Thus $u(1+\delta)$ is an element of B greater than u.

This implies that $\sup B \notin B$. Now let $u = \sup B$ and suppose $u^n > b$. Let $\delta = \min\left\{\frac{u^n-b}{2^n u^n}, 1\right\}$. Then

$$\begin{aligned}
(u(1-\delta))^n &\geq u^n(1 - n\delta) \\
&\geq u^n - 2^n u^n \delta \\
&= b.
\end{aligned}$$

Thus $u(1-\delta)$ is an upper bound of B less than u. This contradicts that u is the supremum.

Thus, $(\sup B)^n = b$. Using $0 < x < y \implies x^n < y^n$, we have that $\sup B$ is the only positive real number whose n-th power is b, so $t^n = b \implies t = \sup B$. ■

Solution (Case 2). Let $mq = np$. For b and t positive,

$$\begin{aligned} t^n < b^m &\implies t^{np} < b^{mp} \\ &\implies t^{mq} < b^{mp} \\ &\implies (t^q)^m < (b^p)^m \\ &\implies t^q < b^p. \end{aligned}$$

The last implication may seem hairy, but it also follows directly from $0 < x < y \implies x^m < y^m$. Similarly,

$$t^q < b^p \implies t^n < b^m.$$

Thus

$$\sup\{t \in \mathbb{R} : t > 0, t^n < b^m\} = \sup\{t \in \mathbb{R} : t > 0, t^q < b^p\}.$$

For the rest of the proof, we can give up. ■

Problem 3.2. Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p > 0$, define

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

(a) Show that if $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \|y\|_q, \quad \forall x, y \in \mathbb{R}^n.$$

You may directly use Young's inequality: if $a, b \ge 0$, then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

(b) Let $d_p(x, y) = \|x - y\|_p$, $x, y \in \mathbb{R}^n$. Show that $(\mathbb{R}^n, d_p)$ is a metric space if $p \ge 1$.

(c) Show that $(\mathbb{R}^n, d_p)$ is not a metric space if $p \in (0, 1)$.

Lemma 3.1 (Young's inequality)**.** *Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then for $a, b \ge 0$, $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.*

Proof. □

Solution.

(a) Apply Young's inequality on $\frac{|x_i|}{\|x\|_p}$ and $\frac{|y_i|}{\|y\|_q}$. Then

$$\begin{aligned}\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} &\leq \sum_{i=1}^{n} \left(\frac{1}{p} \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \frac{|y_i|^q}{\|y\|_q^q} \right) \\ &= \frac{1}{p} \sum_{i=1}^{n} \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \sum_{i=1}^{n} \frac{|y_i|^q}{\|y\|_q^q} \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1.\end{aligned}$$

■