Assignment 7

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Problem 7.1. *Consider the function* $f\colon \mathbb{R} \to [0,1]$ *given by*

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q},\ \textit{with } p \in \mathbb{Z} \textit{ and } q \in \mathbb{N}^*, \gcd(p,q) = 1, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Show that f is discontinuous at every rational number, and continuous elsewhere. (Here, we have used the fact that every rational number $x \in \mathbb{Q}$ admits a unique representation of the form p/q, with p and q as described above.)

Solution. Discontinuity at rational numbers is easy. Let x be a rational number, so $f(x) > 0$. Then, for any $\delta > 0$, there exists a $y = x + \frac{\delta}{\sqrt{2}} \in B(x;\delta)$ such that $f(y) = 0$.

For continuity at an irrational number x, let $\epsilon > 0$. Let $q \in \mathbb{N}^*$ be such that $\frac{1}{q} < \epsilon$. Choose

$$\delta = \inf\left\{ \left| x - \frac{p}{q!} \right| : p \in \mathbb{Z} \right\}.$$

This is positive since $B(x; \frac{1}{2q!})$ can contain at most one such $p/q!$, which cannot be x.

Note that for $f(y) = \frac{1}{n}$, $n \leq q$, y must be of the form m/n, and hence of the form $p/q!$ for some $p \in \mathbb{N}$.

Then $B(x;\delta)$ contains no such y. This is obvious by some light casework.

Thus for all $y \in B(x;\delta)$, $f(y) = 0$ if $y \notin \mathbb{Q}$, and $f(y) = \frac{1}{n} < \frac{1}{q}$ for some $n > q$ if $y \in \mathbb{Q}$. In any case, $|f(y) - f(x)| < \varepsilon$. ■

Problem 7.2. *Let (X,d) be a metric space and $A \subseteq X$ be a nonempty subset. Define*

$$f_A(x) = \inf\{d(x,y) \mid y \in A\}, \quad x \in X.$$

(a) Show that f_A is uniformly continuous on X.

(b) Prove that $x \in \overline{A}$ if and only if $f_A(x) = 0$.

Solution. Notation: $D_x = \{d(x,a) \mid a \in A\}$.

For uniform continuity, choose $\delta < \varepsilon$ for any given $\varepsilon > 0$. Let $x, y \in X$ be such that $d(x,y) < \delta$. By the triangle inequality, $d(x,a) \le d(x,y) + d(y,a) < d(y,a) + \varepsilon$ for every $a \in A$.

Then $f_A(x) \le d(x,a) < d(y,a) + \delta$ for every $a \in A$. Thus $f_A(x) - \delta \le d(y,a)$ for every $a \in A$. Hence $f_A(x) - \delta$ is a lower bound of D_y. So $f_A(y) \ge f_A(x) - \delta$.

Similarly $f_A(x) \ge f_A(x) - \delta$. Thus $|f_A(x) - f_A(y)| \le \delta < \varepsilon$.

For the second part,

$$\begin{aligned} x \in \overline{A} &\iff \forall \varepsilon > 0 \,\exists\, a \in A(d(x,a) < \varepsilon) \\ &\iff \forall \varepsilon > 0 \,\exists\, d \in D_x(d < \varepsilon) \\ &\iff f_A(x) = 0. \end{aligned}$$

■

Problem 7.3. *Show that uniformly continuous functions map Cauchy sequences to Cauchy sequences. Is the converse true?*

Proof. Let $f\colon X \to Y$ be uniformly continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X.

Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $d_X(x,y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. Since $(x_n)_n$ is Cauchy, there exists $N \in \mathbb{N}$ such that $d_X(x_n, x_m) < \delta$ for all $n, m \ge N$.

Then $d_Y(f(x_n), f(x_m)) < \varepsilon$ for all $n, m \ge N$.

Is the converse true? Take $f\colon \mathbb{R} \to \mathbb{R}$ to be $x \mapsto x^2$. f is continuous, so it maps Cauchy (hence convergent) sequences to convergent (hence Cauchy) sequences (sequential characterization of continuity). But f is not uniformly continuous, since for any $\delta > 0$, $f(\frac{1}{\delta} + \delta) - f(\frac{1}{\delta}) = 2 + \delta^2 > 1$.

What went wrong? Both the domain and codomain are unbounded. Which one is the problem? □

Proposition 7.1. *Let X and Y be metric spaces. Suppose that X is compact and Y is complete. Let $f: X \to Y$ be such that it maps every Cauchy sequence to a Cauchy sequence. Then f is uniformly continuous.*

Lemma 7.2. *Let f be a function from some metric space X to a complete metric space Y. Suppose that f maps Cauchy sequences to Cauchy sequences. Then f is continuous.*

Proof. Let $x \in X$. Let $(x_n)_n$ be a sequence in X converging to x. Then $(x_n)_n$ is Cauchy. Since f maps Cauchy sequences to Cauchy sequences, $(f(x_n))_n$ is Cauchy. Since Y is complete, $(f(x_n))_n$ converges to some $y \in Y$. Thus $\lim_{n\to\infty} f(x_n) = y$ exists.

Let $(x_n)_n$ and $(x'_n)_n$ be sequences in X converging to x. Interleave the sequences to get a sequence $(\tilde{x}_n)_n$ converging to x. Then $\lim_{n\to\infty} f(\tilde{x}_n)$ exists, so $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x'_n)$.

Sequential characterization: Suppose f is not continuous at x. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exist $x_\delta \in B_X(x;\delta)$ such that $d_Y(f(x_\delta), f(x)) \geq \varepsilon$.
Consider the sequence $(x_{1/n})_n$. Then $(x_{1/n})_n$ converges to x, so $(f(x_{1/n}))_n$ converges to whatever $(f(x))_n$ converges to, which is $f(x)$. Contradiction. $\square$

Proof of proposition 7.1. Let $\varepsilon > 0$. Let $\delta_x > 0$ be such that $f(B_X(x;\delta_x)) \subseteq B_Y(f(x);\varepsilon/2)$ [which exists by continuity].

Let $\mathcal{U}$ be a finite subcover of $\{B_X(x;\delta_x/2)\}_{x\in X}$ [which exists by compactness]. Choose the minimum of these $\delta_x/2$ to get $\delta > 0$.

Let $x_1, x_2 \in X$ within δ of each other. Since $\mathcal{U}$ is a cover, there exists $x \in X$ such that $x_1 \in B_X(x;\delta_x/2) \in \mathcal{U}$, so $d(f(x_1), f(x)) < \varepsilon/2$. By the $\triangle$ inequality, $d(x_2, x) < \delta_x/2 + \delta < \delta_x$, so that $d(f(x_2), f(x)) < \varepsilon/2$. Then $d(f(x_1), f(x_2)) < \varepsilon$. $\square$

Proposition 7.3. *Completeness in proposition 7.1 is unnecessary.*

Proof. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}} \to x$. Interleave this with the constant sequence $(x)_{n\in\mathbb{N}}$ to get a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ converging to x. Then $(f(\tilde{x}_n))_{n\in\mathbb{N}}$ is Cauchy. But every other term is $f(x)$, so $(f(x_n))_{n\in\mathbb{N}} \to f(x)$. By sequential characterization, f is continuous.

The rest of the proof doesn't require completeness of Y. $\square$

Exercise 7.4. *The counterexample $x \mapsto x^2$ had both the domain and codomain unbounded.*

Construct a function $f\colon X \to Y$ between metric spaces X and Y such that X is closed, Y is compact, f maps Cauchy sequences to Cauchy sequences, but f is not uniformly continuous.

Solution. Take $f\colon \mathbb{R} \to [-1,1]$ to be $x \mapsto \sin x^2$. This is continuous, so it maps Cauchy sequences to Cauchy sequences.

But f is not uniformly continuous. Take $\varepsilon = 1$. For any $\delta > 0$, choose $n = \left\lceil \frac{1}{\delta^2} \right\rceil$ and $x^2 = n\pi$, $y^2 = (n + \frac{1}{2})\pi$. Then $y^2 - x^2 = (y-x)(x+y) = \frac{1}{2}\pi$. So $y - x < \frac{1}{4\sqrt{n}}\sqrt{\pi} < \frac{1}{\sqrt{n}} \leq \delta$. But $|f(x) - f(y)| = 1$. ■

Exercise 7.5. *Construct a function $f\colon X \to Y$ between metric spaces X and Y such that X is bounded, f maps Cauchy sequences to Cauchy sequences, but f is not uniformly continuous.*

Solution. Take $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ to be $x \mapsto \tan^{-1} x$. ■

Exercise 7.6. *Construct a function $f\colon X \to Y$ between metric spaces X and Y such that X is bounded, Y is compact, f maps Cauchy sequences to Cauchy sequences, but f is not uniformly continuous.*

Solution. Take $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to [-1,1]$ to be $x \mapsto \sin((\tan^{-1} x)^2)$. ■

Exercise 7.7. *Construct a function $f\colon X \to Y$ between metric spaces X and Y such that $f(X)$ is bounded but incomplete, f maps Cauchy sequences to Cauchy sequences, but f is not uniformly continuous.*

Solution. Take $f\colon [1,\infty) \to (0,2]$ to be $x \mapsto \cos^2(\pi x^2) + \frac{1}{x^2}$.

If boundedness of X is desired, compose f with $\tan^{-1}$ again. ■

Problem 7.4. *An F_σ set is a countable union of closed sets. Complete the following steps to prove that the discontinuity set D_f of any function $f\colon \mathbb{R} \to \mathbb{R}$ is an F_σ set.*

(a) Given $f\colon \mathbb{R} \to \mathbb{R}$ and $\alpha > 0$, f is said to be α-continuous at $x \in \mathbb{R}$ if there exists a $\delta > 0$ such that for all $y, z \in B(x;\delta)$, $|f(y) - f(z)| < \alpha$. Show that the set $D^\alpha = \{x \in \mathbb{R} \mid f \text{ is not } \alpha\text{-continuous at } x\}$ is closed, for each $\alpha > 0$.

(b) Show that $D^\alpha \subseteq D_f$ for any $\alpha > 0$.

(c) Show that

$$D_f = \bigcup_{n\in\mathbb{N}} D^{1/n}.$$

Solution. Let x be a limit point of D^α. Suppose f is α-continuous at x. Then there exists a δ-ball around x such that $|f(y) - f(z)| < \alpha$ for all $y, z \in B(x;\delta)$. But x is a limit point, so there exists a $y \in D^\alpha$ such that $y \in B(x;\delta/2)$. Then $B(y;\delta/2) \subseteq B(x;\delta)$, so f is α-continuous at y. Contradiction.

Suppose f is continuous at x. Then for any $\alpha > 0$, there exists a δ-ball around x such that the output of every point in the ball is within $\alpha/2$ of $f(x)$. Hence the output of any pair of points in the ball is within α of each other. Thus f is α-continuous at x. Thus $D_f^C \subseteq (D^\alpha)^C$ for any $\alpha > 0$.

Finally,

$$\begin{aligned} x \in D_f &\iff \exists\, \alpha > 0\, \forall\, \delta > 0\, \exists\, y \in B(x;\delta) \text{ such that } |f(x) - f(y)| \geq \alpha \\ &\implies \exists\, \alpha > 0\, \forall\, \delta > 0\, \exists\, z, y \in B(x;\delta) \text{ such that } |f(y) - f(z)| \geq \alpha \end{aligned}$$

by taking $z = x$

$$\begin{aligned} &\iff \exists\, \alpha > 0 (x \in D^\alpha) \\ &\iff \exists\, n \in \mathbb{N} (x \in D^{1/n}) \end{aligned}$$

since $D^a \subseteq D^b$ for $a < b$.

$$\iff x \in \bigcup_{n\in\mathbb{N}} D^{1/n}.$$

Conversely $\bigcup_{n\in\mathbb{N}} D^{1/n} \subseteq D_f$ by step (b). Thus, $D_f \in F_\sigma$. ∎