Lecture 28 – Questions

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Problem 28.1. *Construct a function* $f\colon (-1,1) \to \mathbb{R}$ *such that* f *is differentiable on* $(-1,1)$*,* $f'(0) > 0$*, but* f *is not monotone in any neighbourhood of* 0*.*

Solution.

$$f(x) = \begin{cases} \frac{1}{2}x + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then for $x \neq 0$,

$$f'(x) = \frac{1}{2} + 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

and $f'(0) = \frac{1}{2}$.

f is differentiable everywhere, so if it is monotone on an interval, f' must have the same sign on that interval.

But for each $x_n = \frac{1}{2n\pi} \xrightarrow{n\to\infty} 0$, $f'(x_n) < 0$. So no neighbourhood of 0 can contain only positive values of f'. ■

Problem 28.2. *Let* $f\colon (a,b) \to \mathbb{R}$ *be differentiable and monotonically increasing. Suppose*

$$Z = \{x \in (a,b) : f'(x) = 0\}$$

does not have any limit points in (a,b)*. Is* f *necessarily strictly increasing on* (a,b)*?*

Solution. Obviously? Suppose it is only weakly increasing, that is, there exist $x_1 < x_2$ such that $f(x_1) = f(x_2)$. Since it is increasing, f is constant on $[x_1, x_2]$ and f' is 0 on (x_1, x_2). Then x_1 is a limit point of Z. ■

Problem 28.3. *Is there a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that f' vanishes only on the rationals?*

Problem 28.4. *Construct three functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ such that*

(i) they are continuous on $[0,1]$,

(ii) they are differentiable on $(0,1)$, and

(iii) there is no $c \in (0,1)$ for which the vectors

$$(f'(c), g'(c), h'(c)) \quad \textit{and} \quad (f(1)-f(0), g(1)-g(0), h(1)-h(0))$$

are linearly dependent in $\mathbb{R}^3$.

Solution. Take the helix

$$f(x) = x, \quad g(x) = \cos 2\pi x, \quad h(x) = \sin 2\pi x.$$

Then

$$(f', g', h')(c) = (1, -2\pi \sin 2\pi c, 2\pi \cos 2\pi c)$$

and

$$(f,g,h)(1) - (f,g,h)(0) = (1,0,0).$$

But both $g'(c)$ and $h'(c)$ cannot be 0 simultaneously. ■