Assignment 3

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Problem 3.1.

Solution (A). Let $b > 1$ and $n > 0$. Let $B = \{t \in \mathbb{R} : t > 0, t^n < b\}$. Since $b > 1$, $b^n > b$. Moreover, since $x < y \implies x^n < y^n$, where x and y are positive, we have that b is an upper bound for B. By definition of B, $1 \in B$. Thus B has a supremum.

Now if $u < v$ are positive real numbers, then $u^n < v^n$. Thus there can be at most one positive t such that $t^n = b$.

We now show that $(\sup B)^n = b$. For this we show that B has no largest element. For any $u \in B$, let $\delta = \min\left\{\frac{b-u^n}{2^n}, 1\right\}$. Then

$$\begin{aligned}(u(1+\delta))^n &= u^n(1 + n\delta + \cdots + \delta^n) \\ &= u^n + \delta \sum_{j=1}^{n} \binom{n}{j} \\ &< u^n + 2^n\delta \\ &\leq u^n + b - u^n \\ &= b\end{aligned}$$

Thus $u(1+\delta)$ is an element of B greater than u.

This implies that $\sup B \notin B$. Now let $u = \sup B$ and suppose $u^n > b$. Let

$$\begin{aligned} v &= u + \frac{u^n - b}{nb + u^{n-1}} \\ &= \frac{nbu + b}{nb + u^{n-1}} \end{aligned}$$

as before. Then $0 < v < u$ and

$$\begin{aligned} v^n - b &= \frac{(nbu + b)^n - b(nb + u^{n-1})^n}{(nb + u^{n-1})^n} \\ &> \frac{(nbu + b)^n - (nbu + u^n)^n}{(nb + u^{n-1})^n} \\ &> 0 \end{aligned}$$

But since $x \mapsto x^n$ is