Assignment 2

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Problem 2.1. Let F and G be ordered fields with the LUB property. In Lecture 04, we defined $h\colon F \to G$ as

$$h(z) = \sup_G \{w \in \mathbb{Q} : w \leq z\}.$$

Show that h is a field isomorphism, *i.e.*,

(1) h is a bijection between F and G,

(2) $h(x + y) = h(x) + h(y)$ for all $x, y \in F$,

(3) $h(x \cdot y) = h(x) \cdot h(y)$ for all $x, y \in F$.

Proof. Lecture 4. $\square$

Problem 2.2. In this problem, you may assume the well-definedness, commutativity and associativty of addition of Dedekind cuts (as defined in Lecture 04). Let $O = \{z \in \mathbb{Q} : z < 0\}$. Verify that O is a Dedekind cut, and $A + O = A$ for all Dedekind cuts A. Let A be a Dedekind cut. Define a Dedekind cut B such that $A + B = O$. You must justify your answer.

Proof. Lecture 4. $\square$

Problem 2.3. Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ be sequences of rational numbers such that $b_n \neq 0$ for all $n \in \mathbb{N}$. Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

(i) Are a and b equivalent?

(ii) Are a and b equivalent if a is a $\mathbb{Q}$-bounded sequence?

Solution.

(i) No. $a_n = n + 1$ and $b_n = n$ gives a counterexample.

(ii) Yes.

Let a be bounded by M. Let n_0 be such that for all $n \geq n_0$, $\frac{1}{2} < \frac{a_n}{b_n}$. Then, for all $n \geq n_0$, $|b_n| < 2|a_n| \leq 2M$. Thus b is bounded.

Let $\varepsilon > 0$. Let N be such that for all $n \geq N$,

$$\left|\frac{a_n}{b_n} - 1\right| < \frac{\varepsilon}{2M}.$$

Then for all $n \geq N$,

$$\begin{aligned} |a_n - b_n| &= |b_n|\left|\frac{a_n}{b_n} - 1\right| \\ &< 2M\frac{\varepsilon}{2M} \\ &= \varepsilon. \end{aligned}$$

Problem 2.4. You cannot use the existence (or the LUB property) of the ordered field of real numbers in this problem, so you must work "within" $\mathbb{Q}$.