Assignment 2

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Problem 2.1. Let F and G be ordered fields with the LUB property. In Lecture 04, we defined $h\colon F \to G$ as

$$h(z) = \sup_G \{w \in \mathbb{Q} : w \leq z\}.$$

Show that h is a field isomorphism, *i.e.*,

(1) h is a bijection between F and G,

(2) $h(x+y) = h(x) + h(y)$ for all $x, y \in F$,

(3) $h(x \cdot y) = h(x) \cdot h(y)$ for all $x, y \in F$.

Proof. Lecture 4. □

Problem 2.2. In this problem, you may assume the well-definedness, commutativity and associativty of addition of Dedekind cuts (as defined in Lecture 04). Let $O = \{z \in \mathbb{Q} : z < 0\}$. Verify that O is a Dedekind cut, and $A + O = A$ for all Dedekind cuts A. Let A be a Dedekind cut. Define a Dedekind cut B such that $A + B = O$. You must justify your answer.

Proof. Lecture 4. □

Problem 2.3. Let $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ be sequences of rational numbers such that $b_n \neq 0$ for all $n \in \mathbb{N}$. Suppose

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

(i) Are a and b equivalent?

(ii) Are a and b equivalent if a is a $\mathbb{Q}$-bounded sequence?

Solution.

(i) No. $a_n = n + 1$ and $b_n = n$ gives a counterexample.

(ii) Yes.

Let a be bounded by M. Let n_0 be such that for all $n \geq n_0$, $\frac{1}{2} < \frac{a_n}{b_n}$. Then, for all $n \geq n_0$, $|b_n| < 2|a_n| \leq 2M$. Thus b is bounded.

Let $\varepsilon > 0$. Let N be such that for all $n \geq N$,

$$\left| \frac{a_n}{b_n} - 1 \right| < \frac{\varepsilon}{2M}.$$

Then for all $n \geq N$,

$$\begin{aligned} |a_n - b_n| &= |b_n| \left| \frac{a_n}{b_n} - 1 \right| \\ &< 2M \frac{\varepsilon}{2M} \\ &= \varepsilon. \end{aligned}$$

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Problem 2.4. You cannot use the existence (or the LUB property) of the ordered field of real numbers in this problem, so you must work "within" $\mathbb{Q}$.

(1) Show that every monotone $\mathbb{Q}$-bounded monotone sequence of rational numbers is $\mathbb{Q}$-Cauchy.

(2) Consider the following sequence:

$$x_n = \begin{cases} 2, & \text{if } n = 0, \\ x_{n-1} - \dfrac{x_{n-1}^2 - 2}{2x_{n-1}} & \text{if } n \geq 1. \end{cases}$$

Confirm that $(x_n)_{n\in\mathbb{N}}$ is well-defined, *i.e.*, $x_n \neq 0$ for all $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ is $\mathbb{Q}$-Cauchy, but not convergent in $\mathbb{Q}$.

Solution. (1) Let $(x_n)_{n\in\mathbb{N}}$ be a monotone $\mathbb{Q}$-bounded sequence. WLOG assume that it is increasing.

Let $\varepsilon > 0$. Let

$$N = \inf\{n \in \mathbb{Z} \mid n\varepsilon \text{ is an upper bound of } (x_n)_n\}.$$

Why is the set non-empty? There is an upper bound b for $\{x_n\}_n$. Archimedeanly, there is an $n_0 : n_0\varepsilon > b$. Then n_0 belongs to the set.

Why is it bounded below? Archimedeanly, there is an $n_1 : n_1\varepsilon < x_0$. Then n_1 is a lower bound of the set.

Thus $N\varepsilon$ is an upper bound, but $(N-1)\varepsilon$ is not. So there exists some M such that $x_M > (N-1)\varepsilon$. Then for all $n \geq M$, $(N-1)\varepsilon < x_n \leq N\varepsilon$. So for any $n \geq m \geq M$,

$$|x_n - x_m| = x_n - x_m < N\varepsilon - (N-1)\varepsilon = \varepsilon.$$

(2)

$$\begin{aligned} x_n &= \frac{x_{n-1}^2 + 2}{2x_{n-1}} \qquad (*) \\ \implies x_n^2 - 2 &= \frac{x_{n-1}^4 + 4x_{n-1}^2 + 4 - 8x_{n-1}^2}{4x_{n-1}^2} \\ &= \frac{(x_{n-1}^2 - 2)^2}{4x_{n-1}^2} \end{aligned}$$

This shows that $x_n^2 > 2$ for all $n \in \mathbb{N}$. From $(*)$, $x_{n-1} > 0$ implies $x_n > 0$, so $(x_n)_n > 0$. Thus

$$x_n - x_{n-1} = -\frac{x_{n-1}^2 - 2}{2x_{n-1}} < 0$$

and so $(x_n)_n$ is decreasing.

From the first part, $(x_n)_n$ is $\mathbb{Q}$-Cauchy. But suppose it had a limit $x \in \mathbb{Q}$. Note that $x \neq 0$, since $x_n^2 > 2$. Then

$$x = x - \frac{x^2 - 2}{2x} \implies x^2 = 2,$$

which is not possible. ■

Problem 2.5. A *digit* is any element of the set $S = \{0, 1, \dots, 9\}$. An *admissible sequence of digits* is a sequence $(a_n)_{n \in \mathbb{N}^*} \subseteq S$ satisfying the property: there is no $N \geq 1$ such that $a_n = 9$ for all $n \geq N$. Given $x \in [0, 1)$, we say that an admissible sequence of digits $(d_n)_{n \in \mathbb{N}^*}$ is a *decimal representation* of x if

$$\sup\left\{ D_n := \sum_{j=1}^{n} \frac{d_j}{10^j} \;\middle|\; n \in \mathbb{N} \right\} = x.$$

Show that every admissible sequence of digits is the decimal representation of a number $x \in [0, 1)$, and conversely, every $x \in [0, 1)$ admits a unique decimal representation defined as above.
Note: In this problem, you may freely use the standard properties of real numbers.

Solution. Let $(d_n)_n$ be an admissible sequence of digits. Note that the given set is non-empty, and bounded above by $\sum_{j=1}^{n} \frac{9}{10^j} = 1$. Thus the supremum exists. It is obviously non-negative.

Why is it less than 1? Let i be the first index where $d_i \neq 9$. Then for each $n \geq i$,

$$\begin{aligned}\sum_{j=1}^{n} \frac{d_j}{10^j} &= \sum_{j=1}^{i-1} \frac{d_j}{10^j} + \frac{d_i}{10^i} + \sum_{j=i+1}^{n} \frac{d_j}{10^j} \\ &\leq \sum_{j=1}^{i-1} \frac{d_j}{10^j} + \frac{d_i}{10^i} + \sum_{j=i+1}^{n} \frac{9}{10^j} \\ &= \sum_{j=1}^{i-1} \frac{d_j}{10^j} + \frac{d_i+1}{10^i} \\ &\leq \sum_{j=1}^{i-1} \frac{d_j}{10^j} + \frac{9}{10^i} < 1.\end{aligned}$$

Of course, for each $n < i$, the inequality still holds. Thus $\sum_{j=1}^{i} \frac{d_j}{10^j} + \frac{1}{10^i}$ is an upper bound less than 1.

For the converse, let $d_0 = 0$. For each $n \in \mathbb{N}^*$, let

$$d_n = \max\Big\{d \in S \;\Big|\; D_{n-1} + \frac{d}{10^n} \leq x\Big\},$$

where $D_{n-1} = \sum_{j=1}^{n-1} \frac{d_j}{10^j}$. This ensures that $D_n \leq x < D_n + 10^{-n}$ for each n. How? Holds trivially for $n = 0$. If it holds for $n-1$, then d_n is chosen such that

$$D_n \leq x < D_n + \frac{1}{10^n},$$

if $d_n \neq 9$. But if $d_n = 9$, then $D_n \leq x < D_{n-1} + 10^{-n+1} = D_n + 10^{-n}$. Thus it holds for all n.

Then x is an upper bound for $(D_n)_n$. Suppose there is a lesser upper bound s. Then there exists n such that $10^{-n} < x - s$. But then $x < D_n + 10^{-n} < D_n + x - s$ so $s < D_n$. Thus $\sup_n D_n = x$.

For uniqueness, let $(d_n)_n$ and $(d'_n)_n$ be two distinct admissible sequences of digits. Suppose they first differ at the index j, with $d_j < d'_j$, and that

$d_k \neq 9$ with $k \geq j$. Then for $n \geq k$,

$$\begin{aligned}
D_n &= D'_{j-1} + \frac{d_j}{10^j} + \sum_{i=j+1}^{n} \frac{d_i}{10^i} \\
&\leq D'_{j-1} + \frac{d'_j - 1}{10^j} + \sum_{i=j+1}^{n} \frac{d_i}{10^i} \\
&\leq D'_{j-1} + \frac{d'_j - 1}{10^j} + \sum_{i=j+1}^{n} \frac{9}{10^i} - \frac{1}{10^k} \\
&= D'_{j-1} + \frac{d'_j}{10^j} - \frac{1}{10^k} - \frac{1}{10^n} \\
&< D'_j - \frac{1}{10^k}.
\end{aligned}$$

Thus $\{D_n\}$ is bounded above by $\sup_n D'_n - 10^{-k}$. So $\sup_n D_n < \sup_n D'_n$. ■