UMA204: Introduction to Basic Analysis

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Contents

Lecture 12: Mon 29 Jan '24

0.1 Connected Sets

Definition 0.1.

(a) Let (X, d) be a metric space. A pair of sets $A, B \subseteq X$ are said to be *separated* in X if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.

(b) A set $E \subseteq X$ is said to be *disconnected* if it is the union of two separated sets in X.

(c) E is *connected* if it is not disconnected.

Examples.

- Sets $A = (-1, 0)$ and $B = (0, 1)$ are separated in $\mathbb{R}$. Note that sgn is continuous on $A \cup B$ but does not satisfy the intermediate value property.

  However, if $A = (-1, 0]$ instead, then all continuous functions on $A \cup B$ satisfy the intermediate value property.

- The empty set is connected.

- $\mathbb{Q}$ is disconnected in $\mathbb{R}$. The partition $\{\mathbb{Q} \cap (-\infty, \sqrt{2}), \mathbb{Q} \cap (\sqrt{2}, \infty)\}$ separates $\mathbb{Q}$.

- $\mathbb{Q}$ is disconnected even in $\mathbb{Q}$.

Exercise 0.2. Let $E \subseteq Y \subseteq (X, d)$. Then E is connected relative to Y iff E is cpnnected in X.

Theorem 0.3. Let $E \subseteq \mathbb{R}$. Then E is connected iff E is convex, *i.e.*, for all $x < y \in E$, $[x, y] \subseteq E$.

Proof. Suppose E is connected, but not convex, *i.e.*, there exist $x < y \in E$ and some $r \in (x, y)$ that is not in E. Then $A = (-\infty, r] \cap E$ and $B = [r, \infty) \cap E$ separate E.

Conversely, suppose E is convex but not connected. Then there exist $A, B \subseteq E$ that separate E. Let $x \in A$ and $y \in B$ and suppose WLOG that $x < y$. Note that $A \cap [x, y]$ is non-empty and bounded. Let $r = \sup(A \cap [x, y])$.

By the lemma below, $r \in \overline{A \cap [x, y]} \subseteq \overline{A} \cap [x, y]$ so $r \in \overline{A}$. Disconnectedness forces that $r \notin B \iff r \in A$ so $x \le r < y$.

But since r is the supremum of $A \cap [x, y]$, $(r, y) \subseteq B$. This gives $r \in \overline{B}$, violating the separation of A and B. □

0.2 The Cantor Set

Definition 0.4 (Perfect set)**.** A set $E \subseteq (X, d)$ is said to be *perfect* if every point of E is a limit point of E.

Note that $E = [0, 1]$ is perfect in $\mathbb{R}$. Can we produce a "sparse" perfect set? Throwing away isolated points makes the set open. Throwing away a finite number of open sets preserves perfectness, but there are still *intervals of positive length*.

Can we produce a perfect set such that

(i) it contains no intervals of positive length?

(ii) E is *nowhere dense*, *i.e.*, $(\overline{E})^\circ = \varnothing$?

Note that the second condition implies the first.

Lecture 13: Wed 31 Jan '24

Definition 0.5 (Ternary expansion)**.** Let $x \in [0,1]$. A *ternary expansion* of x is a sequence $(d_1, d_2, \dots) \subseteq \{0,1,2\}$ such that

$$x = \sup\left\{ D_k = \sum_{j=1}^{k-1} \frac{d_j}{3^j} : k \geq 1 \right\}$$

which is equivalent to

$$\sum_{j=1}^{\infty} \frac{d_j}{3^j} = x$$

We write $x = 0.d_1 d_2 d_3 \dots$ to denote this.

Example. For $x = \frac{1}{3}$, we have both $x = 0.1000\dots$ and $x = 0.0222\dots$, so ternary expansions are not unique.

Let $I_0 = [0, \frac{1}{3}]$, $I_1 = [\frac{1}{3}, \frac{2}{3}]$ and $I_2 = [\frac{2}{3}, 1]$. Let $x \in [0,1]$. Choose $d_1 = j$ such that $x \in I_j$ (in ambiguous cases, pick any one). Then

$$\begin{aligned} x &\in \left[\frac{d_1}{3}, \frac{d_1 + 1}{3}\right] \\ \implies 0 &\leq x - \frac{d_1}{3} \leq \frac{1}{3} \\ \implies D_1 &\leq x \leq D_1 + \frac{1}{3} \end{aligned}$$

Let I_{j0}, I_{j1}, I_{j2} be the subdivisions of I_j. Choose $d_2 = l$, where $x \in I_{jl}$ iff

$$\begin{aligned} x &\in \left[\frac{d_1}{3} + \frac{d_2}{9}, \frac{d_1}{3} + \frac{d_2 + 1}{9}\right] \\ &\implies D_2 \leq x \leq D_2 + \frac{1}{9} \end{aligned}$$

How do we break ties?

Scheme A If at the k^{th} state, $x \in [0,1)$ is an endpoint of 2 intervals, pick the right interval. This gives a unique expansion. That is, pick d_k such that $D_k \leq x < D_k + \frac{1}{3}$.

Scheme B For $x \in (0,1]$, always pick the left interval. That is, pick d_k such that $D_k < x \leq D_k + \frac{1}{3}$.

We make the following observations:

- Ambiguity only occurs at endpoints of "middle thirds".
- Say x is an endpoint of a middle third. Let k be the first stage where

	0	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{1}{2}$
A	$0.000\dots$		$0.100\dots$	$0.200\dots$	$0.1100\dots$	$0.111\dots$
B		$0.222\dots$	$0.022\dots$	$0.122\dots$	$0.1022\dots$	$0.111\dots$

Table 1: Scheme A vs Scheme B

ambiguity occurs. Then if x is the left endpoint, scheme A gives $x = 0.d_1d_2\dots d_{k-1}1000\dots$ and scheme B gives $x = 0.d_1d_2\dots d_{k-1}0222\dots$. If x is the right endpoint, scheme A gives $x = 0.d_1d_2\dots d_{k-1}2000\dots$ and scheme B gives $x = 0.d_1d_2\dots d_{k-1}1222\dots$.

Note that this ambiguity can be resolved by a scheme C, which picks the expansion which has no 1 starting from the point of ambiguity.

Theorem 0.6. There exists a non-empty $E \subseteq [0,1]$ such that

(i) E is compact.

(ii) $E = \{\text{limit points of } E\}$.

(iii) $E^\circ = \overline{E}^\circ = \emptyset$.

(iv) E is uncountable.

Proof.

$$E = \{x \in [0,1] : x \text{ admits at least one ternary expansion with only 0's and 2's}\}$$

We can construct this set by removing the middle thirds.

$$E_0 = [0,1]$$

$$E_1 = E_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

$$E_2 = E_1 \setminus \left[\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{4}{9}, \frac{5}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right]$$

$$E_m = E_{m-1} \setminus \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

We claim that $E = \bigcap_{m=1}^{\infty} E_m$ satisfies the conditions of the theorem. We have that E is non-empty.

Since E is the intersection of closed sets, E is closed. Since E is bounded, E is compact.

We have that $E^\circ = \varnothing$ since E does not contain any open intervals. *Formally*, we will show that for any interval (a, b), there exist k and m such that $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ is contained in (a, b).

Heuristically, we see that the length of the removed intervals is $\frac{1}{3} + \frac{1}{9} + \cdots = 1$, so that the remaining set cannot contain any interval of positive length.

Uncountability is by a diagonal argument. $\square$

Lecture 14: Thu 01 Feb '24

Chapter 1

Sequences & Series

1.1 Sequences & Subsequences

Definition 1.1. Let (X, d) be a metric space. A squence in X is a function $f \colon \mathbb{N} \to X$, more commonly written as $(f(k))_{k \in \mathbb{N}} \subseteq X$.

We say that a sequence $(x_n)_{n \in \mathbb{N}}$ *converges* in X if there exists an $x \in X$ such that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. In this case, we call x a limit of $(x_n)_{n \in \mathbb{N}}$ and write

$$\lim_{k \to \infty} x_k = x \quad \text{or} \quad x_k \to x \text{ as } k \to \infty.$$

If $(x_n)_{n \in \mathbb{N}}$ does not converge, we say that it *diverges*.

Examples.

- When $(X, d) = (\mathbb{R}, |\cdot|)$, this definition reduces to the definition in UMA101.
- Let $x_n = (\frac{1}{n}, \frac{2}{n^2}) \in (\mathbb{R}^2, \|\cdot\|)$ for each $n \geq 1$.
  We claim that $\lim_{n \to \infty} x_n = (0, 0)$.

  Proof. Let $\varepsilon > 0$. Choose an $N > \frac{\sqrt{5}}{\varepsilon}$. Then for all $n \geq N$,

$$\begin{aligned} \left\| \left(\frac{1}{n}, \frac{2}{n^2} \right) \right\|^2 &= \frac{1}{n^2} + \frac{4}{n^4} \\ &\leq \frac{5}{n^2} \\ &< \varepsilon. \end{aligned}$$

  □

- Let $x = \left(\frac{1}{n}, (-1)^n\right)_{n\in\mathbb{N}^*}$ with standard norm. Then $(x_n)_{n\in\mathbb{N}^*}$ diverges.

Theorem 1.2. Let (X, d) be a metric space.

(i) Let $(x_n)_{n\in\mathbb{N}} \subseteq X$. Then, $\lim_{n\to\infty} x_n = x$ iff every ε-ball centred at x contains all but finitely many terms of (x_n).

(ii) Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Then $x = y$.

(iii) If $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges, then $\{x_n : n \in \mathbb{N}\}$ is a bounded set in (X, d).

(iv) Let $E \subseteq X$. Then $x \in \overline{E}$ iff there exists a sequence $(x_n) \subseteq E$ such that $\lim_{n\to\infty} x_n = x$.

Proof.

(i) Let (x_n) be convergent to x. Then all terms except the first N lie inside the ε- neighborhood of x. The converse is similarly true.

(ii) Let x and y be distinct limits of (x_n). Choose $\varepsilon = \frac{d(x,y)}{2} > 0$. Then for large enough n,

$$\begin{aligned} d(x, y) &\le d(x, x_n) + d(x_n, y) \\ &< \varepsilon + \varepsilon \\ &= d(x, y). \end{aligned} \qquad \square$$

(iii) Let (x_n) be convergent to x. Let N be such that for all $n \ge N$, $d(x_n, x) < 1$. Then $\rho = \sum_{k=0}^{N} d(x_k, x) + 1$ works as a radius for $B(x, \rho) \supseteq \{x_n : n \in \mathbb{N}\}$.

(iv) Let $x \in \overline{E}$. Then every ε-neighborhood of x intersects E. By the axiom of choice, we can choose a sequence $(x_n) \subseteq E$ such that $d(x_n, x) < \frac{1}{n}$. This converges to x.

Conversely if there exists a sequence $(x_n) \to x$ within E, then every ε-neighborhood of x intersects E.

Definition 1.3. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly incresing sequence in $\mathbb{N}$. Then $(x_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of (x_n).

Any limit of a subsequence of (x_n) is called a *subsequential limit* of (x_n).