MA241: Ordinary Differential Equations

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Contents

Lecture 01: Wed 03 Jan '24

0 The Course

MS Teams Code: `ucwewgr`
Textbook: *Differential Equations and Dynamical Systems* by Lawrence Perko.
TA: Babhrubahan Bose (PhD student)
Prerequisites: Real analysis, specifically convergence of sequences and series of functions. Linear algebra. Some topology.

0.1 Introduction

We wish to solve systems of the form

$$\dot{x} = f(x)$$

where $x\colon (a,b) \subseteq \mathbb{R} \to \mathbb{R}^n$, $x(t) = (x_1(t), \ldots, x_n(t))$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ and

$$\dot{x}(t) \coloneqq (x_1'(t), \ldots, x_n'(t)).$$

So the system is shorthand for

$$\begin{aligned} x_1'(t) &= f_1(x_1(t), \ldots, x_n(t)) \\ &\vdots \\ x_n'(t) &= f_n(x_1(t), \ldots, x_n(t)). \end{aligned}$$

We will later show that higher order derivatives and equations like $\dot{x} = f(t, x)$ can be reduced to this form.

1 Linear Systems

Where f is linear, *i.e.*,

$$f(x) = Ax \text{ where } A \in M(n, \mathbb{R}).$$

$M(n, \mathbb{R})$ is the set of all $n \times n$ matrices with real entries.

For $n = 1$ this reduces to $\dot{x} = ax$, $a \in \mathbb{R}$. A (The) solution is $x(t) = ce^{at}$, $c \in \mathbb{R}$.

Theorem 1.1 (Uniqueness)**.** All solutions to $\dot{x} = ax$ are of the form $x(t) = ce^{at}$ for some $c \in \mathbb{R}$.

Proof. For any solution x, $(xe^{-at})' = \dot{x}e^{-at} - axe^{-at} = 0$. Thus $x = ce^{at}$ for some $c \in \mathbb{R}$. □

Consider $n = 2$.

Definition 1.2 (Uncoupled system)**.** A system $\dot{x} = f(x)$ is *uncoupled* if f_j does not depend on x_i for $i \neq j$.

Consider an uncoupled linear system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} x.$$

The unique solution is $x(t) = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{pmatrix}$, $c_1, c_2 \in \mathbb{R}$.

Phase Portrait of an ODE System

We can plot the solution curves of this system in $\mathbb{R}^2$ for various values of the parameters. This plot is called the *phase portrait* of the system. The paths of the solutions are called *orbits*.

We can also write the solution as

$$x(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Consider the map $\phi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\phi(t, c) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} c.$$

ϕ is then called the *flow* or the *dynamical system* associated with the ODE system.

An interpretation of the phase portrait requires viewing the tangent vectors at any point $x(t)$ in a path, as $\dot{x}(t)$ or $f(x)$. The system can be viewed through the lens of its vector field, which is given by $f(x)$. Any solution to the system is simply a curve which lies tangent to the vector field at every point.