MA 361: Probability Theory

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Contents

The course

Grading

Lecture 1.
Thursday
August 01

- Homework: 20%
- Two midterms: 15% each
- Final: 50%

Chapter 1

Review

1.1 Discrete probability

Definition 1.1 (Discrete probability space)**.** A discrete probability space is a pair (Ω, p) where Ω is a finite or countable set called *sample space* and $p : \Omega \to [0, 1]$ is a function giving the *elementary probabilities* of each $\omega \in \Omega$ such that

$$\sum_{\omega \in \Omega} p(\omega) = 1.$$

Examples.

- "Toss a fair n times" is modeled as
$$\Omega = \{0, 1\}^n$$
with
$$p(\omega) \equiv \frac{1}{2^n}.$$
- "Throw r balls randomly into m bins" is modeled as
$$\Omega = [m]^r$$
with p given by the multinomial distribution (assuming uniformity).
- "A box has N coupons, draw one of them."
$$\Omega = [N]$$
$$p = \omega \mapsto \frac{1}{N}.$$
- "Toss a fair coin countably many times." The set of outcomes is clear: $\Omega = \{0, 1\}^{\mathbb{N}}$. What about the elementary probabilities?

  Probabilities of some events are also fairly intuitive. For example, the event
$$A = \{\underline{\omega} \in \Omega \mid \omega_1 = 1, \omega_2 = 1, \omega_3 = 0\}$$

has probability 1/8. Similarly $B = \{\underline{\omega} \in \Omega \mid \omega_1 = 1, \omega_2 = 0\}$ has probability 1/4. Where does this come from?

What about this event:

$$C = \{\underline{\omega} \in \Omega \,\Big|\, \frac{1}{n}\sum_{i=1}^{n} \omega_i \to 0.6\}$$

What about:

$$D = \{\underline{\omega} \in \Omega \,\Big|\, \sum_{i=1}^{n} \omega_i = \frac{n}{2} \text{ for infinitely many } n\}$$[1]

- "Draw a number uniformly at random from $[0,1]$." Ω is obviously $[0,1]$. Again some events have obvious probabilities.

$$A = [0.1, 0.3] \implies \Pr(A) = 0.2$$

Similarly

$$B = [0.1, 0.2] \cup (0.7, 1) \implies \Pr(B) = 0.4$$

What about $C = \mathbb{Q} \cap [0,1]$? What about D, the $\frac{1}{3}$-Cantor set?

The $\frac{1}{3}$-Cantor set is given by the limit of the following sequence of sets.

$$\begin{aligned} K_0 &= [0,1] \\ K_1 &= [0,1/3] \cup [2/3,1] \\ K_2 &= [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1] \\ &\vdots \end{aligned}$$

where each K_{n+1} is obtained by removing the middle third of each interval in K_n.[2]

The resolution for the above examples is achieved by taking the 'obvious' cases as definitions.

What we agree on:

(#) $\Pr([a,b]) = b - a$ for all $0 \le a \le b \le 1$.

What we wish for:

($*$) If $A \cap B = \varnothing$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

($**$) If $A_n \downarrow A$, then $\Pr(A_n) \downarrow \Pr(A)$.

Question: Does there exist a $\Pr\colon 2^{[0,1]} \to [0,1]$ that satisfies (#), ($*$) and ($**$)? **No.**

[1] $\Pr(C) = 0$ and $\Pr(D) = 1$.

[2] $\Pr(C) = \Pr(D) = 0$.

Question: Does there exist a $\Pr\colon 2^{[0,1]} \to [0,1]$ that satisfies (#) and ($*$) and even *translational invariance*? **Yes.**
However, it is not unique.

What about the same for a probability measure on $[0,1]^2$ that is translation and rotation invariant?
What about $[0,1]^3$?[3]

Lack of uniqueness is a huge issue. The way out is the following: restrict the class of sets on which Pr is defined to a σ-algebra.

Definition 1.2 (σ-algebra)**.** Given a set Ω, a collection $\mathcal{F} \subseteq 2^{\Omega}$ is called a σ-algebra if

(i) $\varnothing \in \mathcal{F}$.

(ii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$.

(iii) If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

This gives us a modified question.
Question: Does there exist *any* σ-algebra $\mathcal{F}$ on $[0,1]$ and a function $\Pr\colon \mathcal{F} \to [0,1]$ that satisfies (#), ($*$) and ($**$)?

Answer: Yes, and it is sort-of unique.

Exercise 1.3. *Prove that ($**$) is equivalent to the following: if $(B_n)_{\mathbb{N}}$ are pairwise disjoint, then*

$$\Pr\Big(\bigcup B_n\Big) = \sum \Pr(B_n).$$

A σ-algebra that works for our case is the *smallest* one that contains all intervals.

Exercise 1.4. *If $\{\mathcal{F}_i\}_{i \in I}$ are σ-algebras on Ω, then $\bigcap_{i \in I} \mathcal{F}_i$ is also a σ-algebra.*

This allows us to make sense of the word 'smallest' above.

Definition 1.5. Let $S \subseteq 2^{\Omega}$. The *smallest* σ-algebra containing S is given by the intersection of all σ-algebras on Ω that contain S.

This will contain S since 2^{Ω} itself is a σ-algebra.

Definition 1.6 (Borel σ-algebra)**.** The *Borel σ-algebra* on $[0,1]$ is the smallest σ-algebra containing all intervals in $[0,1]$. It is denoted by $\mathcal{B}_{[0,1]}$.

[3] The Banach-Tarski paradox gives a "no" for the 3D case.