MA 361 HOMEWORK-6

11.09.2024

Problem 1. (1) Show that if X and Y are independent integer valued random variables then show that

$$\mathbf{P}(X+Y=n)=\sum_m \mathbf{P}(X=m)\mathbf{P}(Y=n-m).$$

(2) Use the above to show that if X has Poisson(λ) distribution and Y has Poisson(μ) distribution and X, Y are independent then $X+Y$ has Poisson $(\lambda+\mu)$ distribution.

Problem 2. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces. For $x \in \Omega_1$ and $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ let $E_x := \{y \in \Omega_2 : (x,y) \in E\}$. Prove that $E_x \in \mathcal{F}_2$ for all $x \in \Omega_1$. (Hint: Show that $\mathcal{R} = \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 : E_x \in \mathcal{F}_2 \text{ for all } x \in \Omega_1\}$ is a $\sigma-$ algebra containing the generating set of $\mathcal{F}_1 \otimes \mathcal{F}_2$.)

Problem 3. Suppose $\{\mu_\alpha : \alpha \in I\}$ and $\{\nu_\beta : \alpha \in J\}$ are two families of Borel probability measures on $\mathbb{R}$. If both these families are tight, show that the family $\{\mu_\alpha \otimes \nu_\beta : \alpha \in I,\ \beta \in J\}$ is also tight.

Problem 4. If $A \in \mathcal{B}(\mathbb{R}^2)$ has positive Lebesgue measure, show that for some $x \in \mathbb{R}$ the set $A_x := \{y \in \mathbb{R} : (x,y) \in A\}$ has positive Lebesgue measure in $\mathbb{R}$.

Problem 5. Let P and Q are real polynomials. What is the Lebesgue measure of the set $\{(x,y) \in \mathbb{R}^2 : P(x) = Q(y)\}$?

Problem 6. Suppose X and Y are independent random variables with density f and g respectively. Show that $X+Y$ has density h, where h is defined as

$$h(x) = \int f(x-y)g(y)dy.$$

Problem 7. If U, V are independent and have uniform([0,1]) distribution, find the distribution of $U+V$.

Problem 8. If X is independent of itself, then X is constant a.s.

Problem 9. If $X \sim \text{Exp}(1)$, show that $\lfloor X \rfloor$ and $X - \lfloor X \rfloor$ are independent.

Problem 10. Suppose $(X_1, \ldots, X_n)$ has a multivariate Normal distribution. Show that if X_i are pairwise independent, then they are independent.

Problem 11. Let $\Omega = \{1, 2, \ldots, n\}$ with the power set sigma-algebra and uniform probability measure. Let $X_p(k) = \mathbf{1}_{p \text{ divides } k}$. Are X_2 and X_3 independent? [*Note:* The answer may depend on n.]