Assignment 2

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Problem 1. *Let G be a finite group with $(3, |G|) = 1$ and $(ab)^3 = a^3b^3$ for all $a, b \in G$. Show that G is abelian.*

Proof. Ignore the trivial group. Let $1 \neq x, y \in G$. By Lagrange's theorem, $x = a^3, y = b^3$. Also $(ab)^2 = b^2a^2$ or $abab = bbaa$. Also $(ab)^{-2} = (a^{-1})^2(b^{-1})^2$ and $(ba)^{-2} = (b^{-1})^2(a^{-1})^2$.

$abab = bbaa$ and $baba = aabb$ together give $bbaaa = aaabb$ so $bbbaaab = baaabbb$. Similarly, $aabbb = bbbaa$ so $aaabbba = abbbaaa$. $a^3b^3a = ab^3a^3$ and $ba^3b^3 = b^3a^3b$. Thus $aba^3b^3 = a^3b^3ab$.

$(ab)^4 = a^3b^3ab = aba^3b^3 = ab^3a^3b$. $xy = b^{-1}yxb = ayxa^{-1}$.

$xyx^{-1}y^{-1} = a^3b^3a^{-3}b^{-3} = (ab)^3(a^{-1}b^{-1})^3 = (aba^{-1}b^{-1})^3$. ∎

Problem 2. *Let G be abelian and $a, b \in G$ have orders m, n respectively. Show that there exists an element of order* $\operatorname{lcm}(m, n)$.

Solution. Consider $S = \{a, ab, ab^2, \ldots, ab^{n-1}\}$. Let $\operatorname{ord}(ab^i) = k_i$.

∎

Problem 3. *Let $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R}, (a, b) \neq (0, 0) \right\}$. Show that G is a group under matrix multiplication. What is G isomorphic to?*

Solution. Let $X_1 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$. Then $X_1X_2 = \begin{pmatrix} a_1a_2 - b_1b_2 & a_1b_2 + a_2b_1 \\ -a_2b_1 - a_1b_2 & -b_1b_2 + a_1a_2 \end{pmatrix}$. Thus X_1X_2 is of the given form. We need to verify that $(a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) \neq (0, 0)$.

Notice that

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1). \tag{1}$$

Thus if $(a_1, b_1) \neq (0,0) \neq (a_2, b_2)$, then $(a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) \neq (0,0)$. Thus G is closed under multiplication.

Equation (1) also shows that G is isomorphic to $\mathbb{C}^\times$ under the map $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + ib$. ■