MA 231: Topology

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The course

Course website

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Timings

Lectures: MWF 3–4 pm
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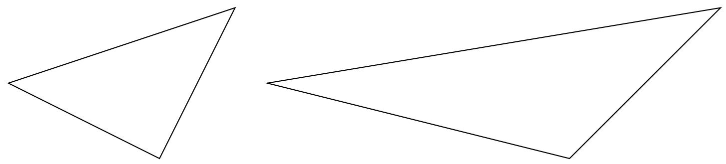
Chapter 1

Introduction

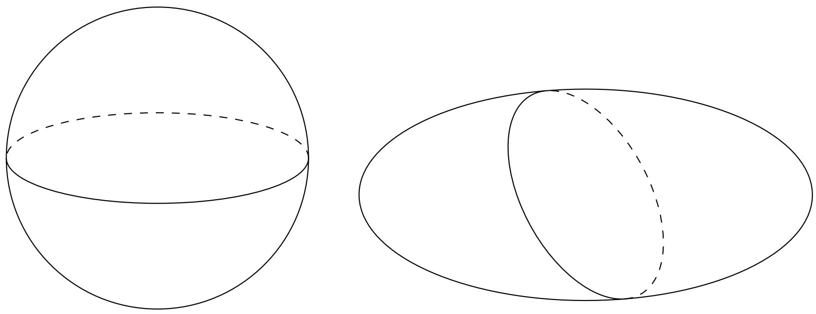
Many objects in mathematics can be described as either *numbers* (arithmetic, algebra) or *shapes* (geometry).

Topology is geometry made “flexible”. How so? Topology allows for deformations, that retain a notion of “proximity” between points.

The following two triangles are identical for the topologist.

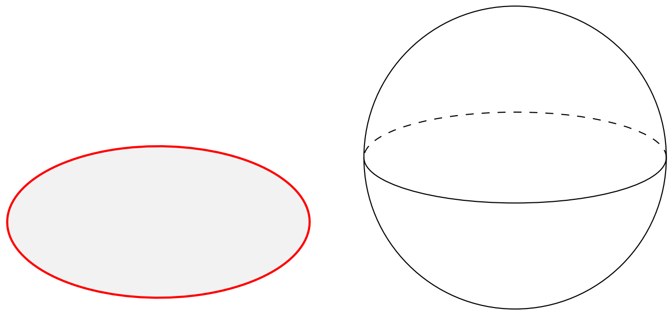


So are a sphere and an ellipsoid.



Why is this useful?

Often, we only need to understand shapes qualitatively. For example, a disk and a sphere look somewhat similar. However, a qualitative difference is that the disk has a boundary, while the sphere does not.



Dynamical systems

Shapes arise while modelling physical systems. A robotic arm may be described by some angles, and so might the solar system. Two friends playing hide-and-seek might be described by their positions.

Such parameters often lend themselves to a simple geometric description in a *phase space* or *configuration space*.

Poincaré discussed the stability of the solar system around 1890 by studying the topology of the phase space.

Knot theory

Whether a knot can be untied without cutting is a topological question. A qualitative understanding of the knot is sufficient.

Graphs

1.1 What can you expect?

The course will cover point-set topology for more that half the semester. Problems will bring out logical thinking, and sometimes your visual imagination will be of great use.

Later, we will cover algebraic invariants, which form a bridge between numbers and shapes.

Lecture 1

Chapter 2

Set theory

Definition 2.1 (Maps)**.** A *map* or a *function* between two sets A and B assigns to every element of A a unique element of B.

$$\begin{aligned} f\colon A &\to B \\ a &\mapsto f(a) \end{aligned}$$

An *injective* or *one-to-one* map is one where distinct elements of A are mapped to distinct elements of B.

$$f(x) = f(y) \implies x = y$$

A *surjective* or *onto* map is one where for every $b \in B$ there exists an $a \in A$ that maps to b.

$$f(A) = B$$

A *bijective* map is one that is both injective and surjective.

Proposition 2.2. *There is a bijection between* $\mathbb{R}$ *and* $\mathbb{R}^2$.

Proof. We will use theorem 2.3. $f = x \mapsto (x, 0)$ is an injective map from $\mathbb{R}$ to $\mathbb{R}^2$. For finding an injection, $g\colon \mathbb{R}^2 \to \mathbb{R}$, we will go via the map $\widehat{g}\colon (0,1)^2 \to (0,1)$.

This is sufficient because arctan provides a nice bijection between $(0,1)$ and $\mathbb{R}$ (and hence $(0,1)^2$ and $\mathbb{R}^2$).

Given $(x,y) \in (0,1)^2$, let $0.x_1x_2x_3\dots$ and $0.y_1y_2y_3\dots$ be the binary expansions of x and y respectively. Interleave the bits to get $0.x_1y_1x_2y_2x_3y_3\dots$. (If we declare the expansions to be non-terminating, they are unique.) This is a bijection. ■

Theorem 2.3 (Schröder-Bernstein)**.** *If A and B are sets such that there is an injective map $f\colon A \to B$ and a surjective map $g\colon A \to B$, then there is a bijection $h\colon A \leftrightarrow B$.*

Following McMullen's notes. WLOG assume A and B are disjoint. Let $F = f \cup g\colon A \cup B \to B \cup A$ be the union of the two maps (as sets). That is,

$$F(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

For any x, we say that x is the *parent* of $F(x)$, which is the *child*.

Now note the following.

(1) Each element of $A \cup B$ is a parent and has a single child.

(2) There may be elements in $A \cup B$ that are not children. Call them *godfathers*.

(3) Each child has a unique parent.

Consider any element $x \in A \cup B$. We have two possibilities.

(1) $\cdots \mapsto x'' \mapsto x' \mapsto x$.

(2) $y \mapsto \cdots \mapsto x' \mapsto x$, where y is a godfather.

We divide A into three mutually disjoint sets, A_0, A_A and A_B.

$$\begin{aligned} A_0 &= \{x \in A \mid \text{possibility (i) happens}\}, \\ A_A &= \{x \in A \mid \text{possibility (ii) happens and } y \in A\}, \\ A_B &= \{x \in A \mid \text{possibility (ii) happens and } y \in B\}. \end{aligned}$$

Similarly, define B_0, B_A and B_B.

Now $F|_{A_0} : A_0 \to B_0$ is a bijection, $F|_{A_A} : A_A \to B_A$ is a bijection, and $F|_{B_B} : B_B \to A_B$ is a bijection. Flipping the last bijection gives a bijection $F|_{B_B}^{-1} : A_B \to B_B$.

Now we can define a bijection $h\colon A \to B$ as $h = F|_{A_0} \cup F|_{A_A} \cup F|_{B_B}^{-1}$. ■

We saw that $|\mathbb{R}| = |\mathbb{R}^2|$, even though both behave very differently in a lot of ways. We need a notion of "proximity". **Lecture 2.** Monday August 5

Definition 2.4 (Topology)**.** Let X be a set. A *topology* on X is a collection $\mathcal{T} \subseteq 2^X$ that

(T1) contains $\varnothing$ and X,

(T2) is closed under finite intersections, and

(T3) is closed under arbitrary unions.

$(X, \mathcal{T})$ is called a *topological space*. Elements of $\mathcal{T}$ are called *open sets*.

Remark. A collection $\{U_\alpha\}_{\alpha \in J}$ is an *indexed* collection of sets. We will sometimes not mention the index set J.

Examples.

- $\mathcal{T} = \{\varnothing, X\}$ is called the trivial or *indiscrete* topology.
- $\mathcal{T} = 2^X$ is called the *discrete* topology.
- $X = \{a, b\}$ has topologies
  - $\mathcal{T} = \{\varnothing, \{a, b\}\}$,
  - $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}\}$ and $\mathcal{T} = \{\varnothing, \{b\}, \{a, b\}\}$,
  - $\mathcal{T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.
- $X = \{a, b, c\}$ has topologies
  - $\{\varnothing, \{a, b, c\}\}$,
    $\{\varnothing, \{a, b\}, \{a, b, c\}\}$, $\{\varnothing, \{b, c\}, \{a, b, c\}\}$, $\{\varnothing, \{c, a\}, \{a, b, c\}\}$,
  - $\{\varnothing, \{a\}, \{a, b, c\}\}$, $\{\varnothing, \{b\}, \{a, b, c\}\}$, $\{\varnothing, \{c\}, \{a, b, c\}\}$,
  - $\{\varnothing, \}$
- For any X, the set
  $$\mathcal{T} = \{A \in 2^X \mid X \setminus A \text{ is finite}\} \cup \{\varnothing\}$$
  is the *cofinite* topology. This is a topology because
  (T1) $\varnothing$ is by definition, and the complement of X is finite,
  (T2) Let $U, V \in \mathcal{T}$. If either is $\varnothing$, then $U \cap V = \varnothing$. Otherwise, $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is finite.
  (T3) Let $\{U_\alpha\} \subseteq \mathcal{T}$. Then $X \setminus (\bigcup_\alpha U_\alpha) = \bigcap_\alpha (X \setminus U_\alpha)$ is finite.
- The *cocountable* topology on X is what you would expect.
- Theorem 2.7.

Definition 2.5 (Metric)**.** A *metric* on a set X is a map $d \colon X \times X \to \mathbb{R}$ satisfying

(M1) (positive definiteness) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$,

(M2) (symmetry) $d(x, y) = d(y, x)$ for all x, y,

(M3) (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y, z.

(X, d) is called a *metric space*.

Examples.

- $(\mathbb{R}, |\cdot|)$.
- $(\mathbb{R}^n, \|\cdot\|_2)$ and in general $(\mathbb{R}^n, \|\cdot\|_p)$.
- $d(x, y) = \mathbf{1}_{x \neq y}$ on any set X.

Definition 2.6 (Open sets)**.** An *open ball* in a metric space (X, d) is a subset

$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$$

where $x_0 \in X$ and $r > 0$.

An *open set* is a subset $U \subseteq X$ if for any $x \in U$, there is an open ball $B(x, r) \subseteq U$.

Theorem 2.7. *Let (X, d) be a metric space. The collection of open sets*

$$\mathcal{T} = \{U \subseteq X \mid U \text{ is open}\}$$

is a topology on X.

Proof. $\varnothing$ is vacuously open, X is trivially so.
A ball here and a ball there, the smaller one is everywhere.
Going to a party, without your friends?
That attitude's not good.
QED, leaving no dead ends,
Take along your entire neighbourhood. ∎

Theorem 2.8. *The co-finite topology on an infinite set is not metrizable.*

Proof. A metric space is a Hausdorff space. But in the co-finite topology, any two (non-empty) open sets intersect. ∎

Question: When is a topology metrizable?

Lecture 3. Wednesday August 7

Definition 2.9 (Refinement)**.** A topology $\mathcal{T}'$ on a set X is said to be *finer* than another topology $\mathcal{T}$ on the same set if $\mathcal{T}' \supseteq \mathcal{T}$. It is said to be *strictly finer* if $\mathcal{T}' \supsetneq \mathcal{T}$.

Definition 2.10 (Basis)**.** A *basis* on a set X is a collection of subsets $\mathcal{B}$ such that

(B1) for any $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$;

(B2) for any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists a $B_\cap \in \mathcal{B}$ such that $x \in B_\cap \subseteq B_1 \cap B_2$.

Definition 2.11. The *topology generated by a basis $\mathcal{B}$* is

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X \mid \forall x \in U \ \exists B \in \mathcal{B}, x \in B \subseteq U\}.$$

Corollary 2.12.

$$\mathcal{T}_{\mathcal{B}} = \left\{\bigcup S \mid S \subseteq \mathcal{B}\right\}.$$

Proof. Let $U \in \mathcal{T}_{\mathcal{B}}$. For each $x \in U$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then $U = \bigcup\{B_x \mid x \in U\}$.

Conversely let $U = \bigcup S$ where $S \subseteq \mathcal{B}$. Then for each $x \in U$, there exists a $B_x \in S$ such that $x \in B_x \subseteq U$. So $U \in \mathcal{T}_{\mathcal{B}}$. ■

Proposition 2.13. *A topology $\mathcal{T}'$ generated by $\mathcal{B}'$ is finer that $\mathcal{T}$ generated by $\mathcal{B}$ iff for any $x \in B \in \mathcal{B}$, there exists a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Let $U \in \mathcal{T}$. For any $x \in U$ let $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then for any $x \in U$ there exists a $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subseteq B_x \subseteq U$. Taking the union gives $U \in \mathcal{T}'$.

Conversely, suppose $U \in \mathcal{T} \implies U \in \mathcal{T}'$. Let $B \in \mathcal{B}$ and $x \in B$. ■

Lecture 4.
Friday
August 9

Lemma 2.14. *The standard topology on $\mathbb{R}$ has a countable basis.*

Proof.

$$\mathcal{B} := \{(p, q) \in \mathbb{Q}^2 \mid p < q\}.$$

For any interval $(a, b) \subseteq \mathbb{R}$, with $a < b$, there exists a decreasing sequence $(p_n)_n \subseteq \mathbb{Q}$ and an increasing sequence $(q_n)_n \subseteq \mathbb{Q}$ such that $p_n \downarrow a$ and $q_n \uparrow b$. Then

$$(a, b) = \bigcup_{n \in \mathbb{N}} (p_n, q_n).$$ ■

Lemma 2.15. *Any basis of $\mathcal{T}'$ is uncountable.*

Proof. For each $x \in \mathbb{R}$ consider the open set $[x, x+1) \in \mathcal{T}'$. Since $\mathcal{B}'$ is a basis, there is a $B_x \in \mathcal{B}'$ such that $x \in B_x \subseteq [x, x+1)$. Note that $\inf B_x = x$. Thus $x \mapsto B_x$ is injective (B_x and B_y have different infimums for $x \neq y$). This shows that $|\mathcal{B}'| \geq |\mathbb{R}|$. ■

Corollary 2.16. *$\mathcal{T}'$ is strictly finer than $\mathcal{T}$.*

We will soon see that $\mathcal{T}'$ is not even metrizable.

Definition 2.17 (Hausdorff)**.** A topological space $(X, \mathcal{T})$ is called *Hausdorff* if for any distinct $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $U \ni x$ and $V \ni y$.

Lemma 2.18. *Every metric space is Hausdorff.*

Proof. Separate two points by open balls of radius half the distance between them. ■

Definition 2.19 (closed)**.** A subset $A \subseteq X$ is called *closed* if its complement $X \setminus A$ is open.

Examples.

- $\mathbb{R} \setminus (0,1) = (-\infty, 0] \cup [1, \infty)$ is closed.
- For any topological space $(X, \mathcal{T})$, $\varnothing$ and X are clopen.
- For the discrete topology, every subset is clopen.

Corollary 2.20. *Closed sets are closed under arbitrary intersections and finite unions.*

Proof. De Morgan on definition 2.4. ■

Definition 2.21. Let $A \subseteq (X, \mathcal{T})$.

- The *interior* of A, denoted A° is the union of all open sets contained in A.
- The *closure* of A, denoted $\bar{A}$, is the intersection of all closed sets containing A.

Remarks.

- $A^\circ \subseteq A \subseteq \bar{A}$.
- A° is open and $\bar{A}$ is closed.
- If A is open, $A^\circ = A$. If A is closed, $\bar{A} = A$.

Lemma 2.22. *$x \in \bar{A}$ iff any open set containg x intersects A.*

Proof.

$$\begin{aligned} x \in \bar{A} &\iff \forall U \in \mathcal{T}(A \subseteq X \setminus U \to x \in X \setminus U) \\ &\iff \forall U \in \mathcal{T}(x \in U \to A \cap U \neq \varnothing) \end{aligned}$$

More verbosely,

$$\begin{aligned} x \in \bar{A} &\iff \forall V \in 2^X (V \text{ is closed } \wedge A \subseteq V \to x \in V) \\ &\iff \forall V \in 2^X (X \setminus V \text{ is open} \wedge X \setminus V \subseteq X \setminus A \to x \notin (X \setminus V)) \\ &\iff \forall U \in \mathcal{T}(U \subseteq X \setminus A \to x \notin U) \\ &\iff \forall U \in \mathcal{T}(U \cap A = \varnothing \to x \notin U) \\ &\iff \forall U \in \mathcal{T}(x \in U \to U \cap A \neq \varnothing) \end{aligned}$$

■

Lecture 4

Definition 2.23. A point $x \in X$ is a *limit point* of A is any open set containing x intersects $A \setminus \{x\}$.

Lemma 2.24. *$\bar{A} = A \cup A'$, where A' is the set of all limit points of A.*

Proof.

$$\begin{aligned} x \in \bar{A} &\iff \forall U \in \mathcal{T}(x \in U \to U \cap A \neq \varnothing) \\ &\iff \forall U \in \mathcal{T}(x \in U \to (x \in A \vee U \cap (A \setminus \{x\}) \neq \varnothing)) \end{aligned}$$

■

Lecture 5. Monday August 12

2.0.1 Topological proof of the infinitude of primes

Lemma 2.25. *Arithmetic progressions form a basis of a topology on $\mathbb{Z}$.*

Proof. Let $S(a, b) = \{an + b \mid n \in \mathbb{Z}\}$.

$S(1, 0) = \mathbb{Z}$. Suppose $x \in S(a_1, x) \cap (a_2, x)$. Then $x \in S(a_1 a_2, x) \subseteq S(a_1, x) \cap S(a_2, x)$. ■

Theorem 2.26. *There are infinitely many primes.*

Proof. Notice that $\mathbb{Z} \setminus S(a, b) = \bigcup_{i=1}^{a-1} S(a, b + i)$. Thus $S(a, b)$ is clopen.

Now notice that $\bigcup_{p \text{ prime}} S(p, 0) = \mathbb{Z} \setminus \{\pm 1\}$.

The union of finitely many closed sets is closed, so if there were a finite number of primes, then $\mathbb{Z} \setminus \{\pm 1\}$ would be closed. Thus $\{\pm 1\}$ would be open, which is impossible, since each basis is infinite. ■

From lemma 2.24, we can say that a set is closed iff it contains all its limit points.

Example. In $\mathbb{R}_L$, the lower-limit topology on $\mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. We need to show that any basis element containing x intersects $\mathbb{Q}$. But basis elements are of the form $[a, b)$, so every basis element intersects $\mathbb{Q}$. Thus $x \in \overline{\mathbb{Q}}$.

Lecture 6. Wednesday August 14

Lecture 7. Monday August 19

Definition 2.27. Let X, Y be topological spaces. A map $f\colon X \to Y$ is *continuous* if for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.

Examples.

- The constant map $x \mapsto y_0$ for some fixed $y_0 \in Y$ is open. The preimage of *any* subset of Y is either $\varnothing$ or X.
- $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}_L$ is not continuous. $\mathrm{id}^{-1}[0, 1) = [0, 1)$ is not open in $\mathbb{R}$. Notice that the inverse map *is* continuous. In general, $\mathrm{id}\colon (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous iff $\mathcal{T}$ is finer than $\mathcal{T}'$.

Corollary 2.28. *Let $\mathcal{B}$ be a basis for Y. $f\colon X \to Y$ is continuous iff $f^{-1}(B)$ is open for each $B \in \mathcal{B}$.*

Proof. Suppose this property holds. Let $V = \bigcup_\alpha B_\alpha$ be open. Then $f^{-1}(V) = \bigcup_\alpha f^{-1}(B_\alpha)$ is a union of open sets and hence open.

The forward direction is simply because each basis element is open. ∎

Proposition 2.29. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Then f is continuous (in the standard topology) iff for every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.*

Proof. Suppose f is continuous. Let x_0 and ε be given. $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ is an open set in $\mathbb{R}$. Thus its preimage is open. But x_0 belongs to this preimage, so there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

For the converse, let V be open in $\mathbb{R}$. ∎

Proposition 2.30. *Let $f\colon X \to Y$. The following are equivalent.*

(1) f is continuous.

(2) For every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X.

(3) For every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof.

(1 $\iff$ 2) $f^{-1}(S^c) = (f^{-1}(S))^c$.

(2 $\implies$ 3) The preimage of $\overline{f(A)}$ is closed. But $A \subseteq f^{-1}(\overline{f(A)})$. So $\overline{A} \subseteq f^{-1}(\overline{f(A)})$.

(3 $\implies$ 2) Let $B \subseteq Y$ be closed. Let $A = f^{-1}(B)$. Then

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B.$$

Thus $\overline{A} \subseteq f^{-1}(B) = A$. So $\overline{A} = A$ is closed. ∎

Definition 2.31. A map $f\colon X \to Y$ is a *homeomorphism* if it is a bijection and f and f^{-1} are both continuous.

Two topological spaces X and Y are *equivalent* if there exists a homeomorphism between them.

Remarks.

- If f is a homeomorphism, then the image of any open set is open.

- The properties of being “Hausdorff”, “second countable”, etc. are “topological properties” because they are invariant under homeomorphisms.

Next quiz: Monday, 2024-08-26

Lecture 9. Friday August 23

Remark (A subtlety). There could be a strictly finer topology on a set X which is nevertheless homeomorphic.

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. Then $\mathcal{T}$ is finer that $\mathcal{T}'$ iff $\mathcal{T} \supseteq \mathcal{T}'$ iff $\mathrm{id}\colon (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous.

Thus in a case as above, the *identity map* is not a homeomorphism, but there is some other homeomorphism.

Example. Consider $\mathbb{R}$ with the topologies $\mathcal{T} = \{(-n, n) \mid n \in \mathbb{N}\} \cup \{\mathbb{R}\}$ and $\mathcal{T}' = \{(-2n, 2n) \mid n \in \mathbb{N}\} \cup \{\mathbb{R}\}$.

But $x \mapsto 2x$ is a homeomorphism from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T}')$.

Theorem 2.32. *Let $X = \bigcup_\alpha U_\alpha$ where each U_α is open. A map $f\colon X \to Y$ is continuous iff $f|_{U_\alpha}$ is continuous for each α, where we consider the subspace topology on each U_α.*

Proof. We have already seen that if f is continuous, then all restrictions are continuous.

Suppose the restrictions are continuous. Let $V \subseteq Y$ be open. Then

$$f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha) = \bigcup_\alpha f|_{U_\alpha}^{-1}(V)$$

is open, since $f|_{U_\alpha}^{-1}(V)$ is open in U_α and hence in X. ∎

We say that continuity is a “local” property. If f is continuous in neighborhoods of each point, then f is continuous by the above theorem.

Theorem 2.33 (Pasting lemma)**.** *Suppose $X = A \cup B$ where A and B are closed, and $f\colon A \to Y$ and $g\colon B \to Y$ are continuous with $f = g$ on $A \cap B$. Then $h\colon X \to Y$ defined by*

$$h(x) = (f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let $C \in Y$ be closed Then

$$(f \cup g)^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

is closed, since $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B respectively, and hence in X. ∎

2.1 Connectedness

Definition 2.34 (Connectedness)**.** X is *connected* if $\varnothing$ and X are the only clopen sets in X.

This is an awesome definition. The motivation is the following:

Definition 2.35 (Separation)**.** A *separation* of X is a pair of disjoint nonempty open sets U and V such that $X = U \cup V$.

Proposition 2.36. *X is connected iff there is no separation of X.*

Proof. Suppose there is a separation U and V. Then U and V are clopen, but neither is $\varnothing$ or X. Thus X is not connected.

Conversely, suppose there is no separation of X. Let $U \neq \varnothing, X$ be open. Then $V = X \setminus U$ cannot be open, since otherwise U and V would separate X. Thus U is not closed. ■

Proposition 2.37. *$[0, 1]$ is connected.*

Proof. Assume U and V separate $[0, 1]$. WLOG suppose $0 \in U$.

Let $a = \inf V$. Since V is closed, $a \in V$. But $[0, a) \subseteq U$, so $a \in U$ since U is closed. ■

Lecture 10. Monday August 26

Lemma 2.38. *If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.*

Proof. Suppose WLOG that f is surjective, *i.e.*, $f(X) = Y$.

Assume there exist two non-empty disjoint open sets U, V such that $U \cup V = Y$. Then $X = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ gives a separation of X. The two sets are open by continuity, disjoint by disjointedness in Y, and non-empty by surjectivity. ■

Proposition 2.39. *$\mathbb{R}$ is connected.*

Proof 1. Adapt the proof of $[0, 1]$ being connected.

Let U, V separate $\mathbb{R}$. Let $a = \sup U$. ■

Proof 2. We already know that $[0, 1]$ is connected. For each $n \in \mathbb{Z}^+$, the linear map $0 \mapsto -n$, $1 \mapsto n$ is continuous (in fact, a homeomorphism). Thus $[-n, n]$ is connected.

The union of all these intervals is $\mathbb{R}$, and they intersect in $\{0\}$. Thus $\mathbb{R}$ is connected. ■

Proof 3. ■

Lemma 2.40. *If X and*