Homework 9

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Problem 1. Determine whether the following statements are true or false with proper justification.

(1) Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables taking values in $\mathbb{R}$ and defined on the same probability space. Then $\frac{X_n}{n} \xrightarrow{\mathsf{P}} 0$.

(2) Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables, taking values in $\mathbb{R}$ and are defined on same probability space. Then $\frac{X_n}{n} \to 0$ almost surely.

(3) Let X be a random variable which is finite almost surely. Then $\frac{X}{n} \xrightarrow{\mathsf{P}} 0$.

(4) Let X be a random variable which is finite almost surely. Then $\frac{X}{n} \to 0$ almost surely.

Solution.

(1) **True.** Fix a $\delta > 0$. Then $\mathbf{P}\{|X_n/n| > \delta\} = \mathbf{P}\{|X_1| > n\delta\} \to 0$ as $n \to \infty$.

(2) **False.** Not necessarily. Let $X_1 = k$ with probability $B\frac{1}{k^2}$ for $k \geq 1$, where

$B = \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{-1}$ is the normalizing constant. Then

$$\begin{aligned}\mathbf{P}\left\{\frac{X_n}{n} \geq 1\right\} &= \mathbf{P}\{X_1 \geq n\} \\ &= B \sum_{k \geq n} \frac{1}{k^2} \\ &\geq B\frac{1}{n}\end{aligned}$$

using that $\sum_{k=N}^{\infty} \frac{1}{k^2} \geq \int_N^{\infty} \frac{1}{x^2}\,\mathrm{d}x$. Thus $\{X_n/n \geq 1\}$ are independent events with probabilities summing to infinity. By the second Borel-Cantelli lemma, $\mathbf{P}\{X_n/n > 1 \text{ i.o.}\} = 1$, so $\mathbf{P}\{X_n/n \to 0\} = 0$.

(3) **True.**

(4) **True.** If $X < \infty$, then $X/n \to 0$. Thus $X/n \xrightarrow{\text{a.s.}} 0$. ■

Problem 2. Let X_n and X be random variables on a common probability space. Show that if $X_n \xrightarrow{\mathsf{P}} X$ then there is a subsequence n_k such that $X_{n_k} \to X$ almost surely.

Solution. Let $n_1 = 1$ and for each $k \geq 2$, let $n_k \geq n_{k-1}$ be such that

$$\mathbf{P}\left\{|X_{n_k} - X| > \frac{1}{k}\right\} \leq \frac{1}{k^2}.$$

(Since $\mathbf{P}\{|X_n - X| > 1/k\} \to 0$ as $n \to \infty$.) Fix an $M \in \mathbb{N} \setminus \{0\}$ and observe

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbf{P}\left\{|X_{n_k} - X| > \frac{1}{M}\right\} &\leq \sum_{k=1}^{M} \mathbf{P}\left\{|X_{n_k} - X| > \frac{1}{M}\right\} \\ &\qquad + \sum_{k>M} \mathbf{P}\left\{|X_{n_k} - X| > \frac{1}{k}\right\} \\ &\leq M + \sum_{k>M} \frac{1}{k^2} \\ &< \infty.\end{aligned}$$

By the Borel-Cantelli lemma, $\mathbf{P}\left\{|X_{n_k} - X| > \frac{1}{M} \text{ i.o.}\right\} = 0$. In other words,

$$\mathbf{P}\left(\bigcup_{K=1}^{\infty} \bigcap_{k \geq K} \left\{|X_{n_k} - X| \leq \frac{1}{M}\right\}\right) = 1.$$

Since the intersection of countably many almost sure events is almost sure, we have

$$\mathbf{P}\{\lim_{k\to\infty} X_{n_k} = X\} = \mathbf{P}\left(\bigcap_{M=1}^{\infty}\bigcup_{K=1}^{\infty}\bigcap_{k\geq K}\left\{|X_{n_k} - X| \leq \frac{1}{M}\right\}\right) = 1. \qquad \blacksquare$$

Problem 3. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution $\mu \in \mathcal{P}(\mathbb{R})$. Recall that the support of μ is the smallest closed set K with $\mu(K) = 1$. Show that $\overline{\{X_1, X_2, \ldots\}} = K$ almost surely.

Problem 4. Let X_n be independent and $\mathbf{P}\{X_n = n - a\} = \mathbf{P}\{X_n = -n - a\} = 1/2$ where $a > 0$ is fixed. For what values of a does the series $\sum_{n=1}^{\infty} X_n$ converge absolutely, a.s.?

Problem 5. Suppose X_n are i.i.d. random variables with finite mean. Which of the following assumptions guarantee that $\sum_{n=1}^{\infty} X_n$ converges a.s.?

(1) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge 1] < \infty$.

(2) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge |X_n|] < \infty$.

Problem 6 (Large deviation for Bernoullis)**.** Let X_n be i.i.d. Ber(1/2). Fix $p > 1/2$.

(1) Show that $\mathbf{P}\{S_n > np\} \leq e^{-np\lambda}\left(\frac{e^{\lambda}+1}{2}\right)^n$ for any $\lambda > 0$.

(2) Optimize over λ to get $\mathbf{P}\{S_n > np\} \leq e^{-nI(p)}$ where $I(p) = -p\log p - (1-p)\log(1-p)$. (Observe that this is the entropy of the Ber(p) measure.)

Solution.

(1) Let $Y_n = X_n - p$. Then $\mathbf{E}[Y_n] = 0$ and $|Y_n| \leq p$. By Hoeffding's inequality,

$$\begin{aligned}\mathbf{P}\{S_n^X > np\} &= \mathbf{P}\{S_n^Y > 0\} \\ &= \end{aligned}$$

$\blacksquare$

Problem 7. Carry out the same program for i.i.d. Exp(1) random variables and deduce that $\mathbf{P}\{S_n > nt\} \le e^{-nI(t)}$ for $t > 1$ and $\mathbf{P}\{S_n < nt\} \le e^{-nI(t)}$ for $t < 1$ where $I(t) := t - 1 - \log t$.

Solution.

$$\mathbf{E}[e^{\lambda S_n}] = \prod_{k=1}^{n} \mathbf{E}[e^{\lambda X_k}] = \frac{1}{(1-\lambda)^n}.$$

Thus for $t > 1$,

$$\begin{aligned}
\mathbf{P}\{S_n > nt\} &= \mathbf{P}\{e^{\lambda S_n} > e^{n\lambda t}\} \\
&\le e^{-n\lambda t}\,\mathbf{E}[e^{\lambda S_n}] \\
&= \left(\frac{e^{-\lambda t}}{1-\lambda}\right)^n.
\end{aligned}$$

Optimizing over λ,

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{e^{-\lambda t}}{1-\lambda} \\
&= \frac{-te^{-\lambda t}(1-\lambda) + e^{-\lambda t}}{(1-\lambda)^2} \\
\implies 1 &= t(1-\lambda) \\
\implies \lambda &= 1 - \frac{1}{t}.
\end{aligned}$$

So $e^{-\lambda t} = e^{-t+1}$ and $\frac{1}{1-\lambda} = t$. Thus

$$\mathbf{P}\{S_n > nt\} \le (e^{-t+1+\log t})^n = e^{-nI(t)}.$$

Similarly for $t < 1$. ■

Problem 8. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

(1) Let X, Y be random variables on Ω. Define a function $d(X,Y) = \mathbf{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$. Show that d is a metric on the set of all random variables and $d(X_n, X) \to 0$ if and only if $X_n \xrightarrow{\mathsf{P}} X$.

(2) Show that if X_n, X are random variables such that any subsequence of X_n has a further subsequence that converges almost surely to X then $X_n \xrightarrow{\mathsf{P}} X$.

Solution.

(1) Let $f(x,y) = \frac{|x-y|}{1+|x-y|}$.

$$\begin{aligned}\mathbf{E}[f(X,Y)] &= \mathbf{E}[f(X,Y)\mathbf{1}_{|X-Y|<t}] + \mathbf{E}[f(X,Y)\mathbf{1}_{|X-Y|\geq t}] \\ &\leq t + \mathbf{P}\{|X-Y| \geq t\}.\end{aligned}$$

Thus for any $\delta > 0$,

$$\mathbf{E}[f(X_n, X)] \leq \delta + \mathbf{P}\{|X_n - X| \geq \delta\} \to \delta.$$

(2) Let $p_n = \mathbf{P}\{|X_n - X| > \delta\}$. Let $(p_{n_k})_k$ be a convergent subsequence. Then $(p_{n_{k_j}})_j$ is a further subsequence that converges to 0, since $X_{n_{k_j}} \xrightarrow{\text{a.s.}} X$ implies $X_{n_{k_j}} \xrightarrow{\mathsf{P}} X$. Thus $p_{n_k} \to 0$, so $\limsup p_n = 0$. ■

Problem 9. Let X_n, Y_n, X, Y be random variables on a common probability space. If $X_n \xrightarrow{\mathsf{P}} X$ and $Y_n \xrightarrow{\mathsf{P}} Y$ (all random variables on the same probability space), show that $f(X_n, Y_n) \xrightarrow{\mathsf{P}} f(X,Y)$ for any continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$. In particular, this implies if $X_n \xrightarrow{\mathsf{P}} X$ and $Y_n \xrightarrow{\mathsf{P}} Y$ then for any $a, b \in \mathbb{R}$, $aX_n + bY_n \xrightarrow{\mathsf{P}} aX + bY$.

Solution. Let $(n_k)_k$ be any subsequence of $\mathbb{N}$. Then $X_{n_k} \xrightarrow{\mathsf{P}} X$ and $Y_{n_k} \xrightarrow{\mathsf{P}} Y$. So there is a subsequence $(n_{k_j})_j$ such that $X_{n_{k_j}} \xrightarrow{\text{a.s.}} X$ and $Y_{n_{k_j}} \xrightarrow{\text{a.s.}} Y$. Then $f(X_{n_{k_j}}, Y_{n_{k_j}}) \xrightarrow{\text{a.s.}} f(X,Y)$ by continuity of f. Thus every subsequence of $f(X_n, Y_n)$ has a further subsequence that converges almost surely to $f(X,Y)$. By the previous problem, $f(X_n, Y_n) \xrightarrow{\mathsf{P}} f(X,Y)$. ■

Problem 10. Give examples of two sequences of random variables X_n and Y_n such that $X_n \xrightarrow{\mathrm{d}} X$ and $Y_n \xrightarrow{\mathrm{d}}$ but $X_n + Y_n$ does not converge in distribution to $X + Y$.

Solution. Let $X_n \sim \mathrm{Ber}(1/2)$ and $Y_n = 1 - X_n$. Choose X, Y i.i.d. $\mathrm{Ber}(1/2)$. Then $X_n \xrightarrow{\mathrm{d}} X$ and $Y_n \xrightarrow{\mathrm{d}} Y$, but $X_n + Y_n \sim \delta_1$. ∎