Homework 3

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Problem 1. Suppose μ_n, μ are discrete probability measures supported on $\mathbb{Z}$ having probability mass functions p_n, p respectively. Show that $\mu_n \xrightarrow{\mathrm{d}} \mu$ if and only if $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$.

Solution. Let F_n, F be the CDFs of μ_n, μ respectively.
($\Longrightarrow$) Suppose $\mu_n \xrightarrow{\mathrm{d}} \mu$. Fix a $k \in \mathbb{Z}$. Let $0 < \varepsilon < 1$. Then there exists an $N \in \mathbb{N}$ such that $d(\mu_n, \mu) \leq \varepsilon$ for all $n \geq N$.

For each $m \in \mathbb{Z}$, F_n and F are constant on $[m, m+\varepsilon]$. Thus for each $m \in \mathbb{N}$,

$$\begin{aligned} F_n(m+\varepsilon) + \varepsilon \geq F(m) &\implies F(m) - F_n(m) \leq \varepsilon, \\ F(m+\varepsilon) + \varepsilon \geq F_n(m) &\implies F_n(m) - F(m) \leq \varepsilon. \end{aligned}$$

Thus for $n \geq N$,

$$|F_n(k) - F(k)| \leq \varepsilon \quad \text{and} \quad |F_n(k-1) - F(k-1)| \leq \varepsilon.$$

This gives

$$\begin{aligned} |p_n(k) - p(k)| &= |F_n(k) - F_n(k-1) - F(k) + F(k-1)| \\ &\leq |F_n(k) - F(k)| + |F_n(k-1) - F(k-1)| \\ &\leq 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, $p_n(k) \to p(k)$.

($\Longleftarrow$) Conversely, suppose $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$. Fix an $x \in \mathbb{Z}$ and let $\varepsilon > 0$.

Since $\mu[-k, k] \to 1$ as $k \to \infty$, we can choose $M \in \mathbb{N}$ such that $[-M, M] \ni x$ and $\mu[-M, M] > 1 - \varepsilon$. This forces $\mu(-\infty, -M) < \varepsilon$.

For each $k \in [-M, M]$, there exists an $N_k \in \mathbb{N}$ such that $|p_n(k) - p(k)| < \frac{\varepsilon}{2M+1}$ for all $n \geq N_k$. Choosing N to be the maximum of these N_k's gives for each $n \geq N$,

$$\begin{aligned}\mu_n[-M, M] &= \sum_{k=-M}^{M} p_n(k) \\ &= \sum_{k=-M}^{M} p(k) + \sum_{k=-M}^{M} (p_n(k) - p(k)) \\ &> 1 - \varepsilon - \sum_{k=-M}^{M} \frac{\varepsilon}{2M+1} \\ &= 1 - 2\varepsilon.\end{aligned}$$

Thus $\mu_n(-\infty, -M) < 2\varepsilon$ for each $n \geq N$, so

$$\begin{aligned}F_n(x) - F(x) &= \mu_n(-\infty, x] - \mu(-\infty, x] \\ &= \mu_n(-\infty, -M) - \mu(-\infty, -M) + \mu_n[-M, x] - \mu[-M, x] \\ &= \mu_n(-\infty, -M) - \mu(-\infty, -M) + \sum_{k=-M}^{x} (p_n(k) - p(k)) \\ \implies |F_n(x) - F(x)| &< 3\varepsilon + \sum_{k=-M}^{x} \frac{\varepsilon}{2M+1} \\ &< 4\varepsilon.\end{aligned}$$

Thus $F_n(x) \to F(x)$ for each $x \in \mathbb{Z}$, and so $F_n(y) = F_n(\lfloor y \rfloor) \to F(\lfloor y \rfloor) = F(y)$ for each $y \in \mathbb{R}$. By a proposition done in class, we have $\mu_n \xrightarrow{\text{d}} \mu$. ■

Problem 2. For what $A \subseteq \mathbb{R}$ and $B \subseteq (0, \infty)$ is the restricted family $\{N(\mu, \sigma^2) \mid \mu \in A, \sigma^2 \in B\}$ tight?

We will throughout assume that $\sigma > 0$.

Solution. Call the given family $\mathcal{A}$. Denote the pdf of the measure $N_{\mu,\sigma^2} = N(\mu, \sigma^2)$ by f_{μ,σ^2}.

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

$\mathcal{A}$ is tight iff A and B are bounded, or vacuously so if one of them is empty. We will ignore the empty case hereon.

To show that $\mathcal{A}$ is not tight, it suffices to produce an $\varepsilon > 0$ such that for each compact $K = [a, b] \subseteq \mathbb{R}$, there exists a $\rho \in \mathcal{A}$ such that $\rho(K^c) \geq \varepsilon$.

- Suppose A is unbounded and let $\sigma^2 \in B$. Then for each set $K = [a,b] \subseteq \mathbb{R}$, there exists a $\mu \in A \setminus K$. If $\mu > b$, then
$$N_{\mu,\sigma^2}(K^c) \geq N_{\mu,\sigma^2}[\mu,\infty) = \frac{1}{2}.$$
Similarly, if $\mu < a$, then
$$N_{\mu,\sigma^2}(K^c) \geq N_{\mu,\sigma^2}(-\infty,\mu] = \frac{1}{2}.$$
Thus $\mathcal{A}$ is not tight.

- Suppose B is unbounded and let $\mu \in A$. Let $K = [a,b] \subseteq \mathbb{R}$. If $\mu \notin K$, then $N_{\mu,\sigma^2}(K^c) \geq \frac{1}{2}$ as in the previous case. If $\mu \in K$, choosing $\sigma^2 > (b-a)^2$ yields
$$N_{\mu,\sigma^2}(K) = \int_a^b f_{\mu,\sigma^2}(x)\,\mathrm{d}x \leq \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{d}x = \frac{b-a}{\sqrt{2\pi\sigma^2}} < \frac{1}{\sqrt{2\pi}}.$$
Thus $N_{\mu,\sigma^2}(K^c) > 1 - \frac{1}{\sqrt{2\pi}} > 0$. $\mathcal{A}$ is not tight.

To show that $\mathcal{A}$ is tight whenever A and B are bounded, we will need the following claim.

Claim. *For each $\mu \in \mathbb{R}$, $\sigma^2 \in (0,\infty)$ and $a, b \in \mathbb{R}$,*
$$N_{\mu,\sigma^2}[a,b] = N_{0,1}\left[\frac{a-\mu}{\sigma}, \frac{b-\mu}{\sigma}\right].$$

Proof.
$$\begin{aligned} N_{\mu,\sigma^2}[a,b] &= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,\mathrm{d}x \\ &= \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y && \left(\text{substitute } y = \tfrac{x-\mu}{\sigma}\right) \\ &= N_{0,1}\left[\frac{a-\mu}{\sigma}, \frac{b-\mu}{\sigma}\right]. \end{aligned}$$
□

Suppose A and B are bounded. Let M be such that $A \subseteq [-M, M]$ and $B \subseteq (0, M^2]$. Given $\varepsilon > 0$, let $\delta > 0$ be such that $N_{0,1}[-\delta,\delta] > 1 - \varepsilon$. This exists since $N_{0,1}[-x,x] \to 1$ as $x \to \infty$.

Choose $K_\varepsilon = [-M - M\delta, M + M\delta]$. Then for each $\rho = N(\mu, \sigma^2) \in \mathcal{A}$,

$$\begin{aligned}
\rho(K) &= N_{\mu,\sigma^2}[-M - M\delta, M + M\delta] \\
&= N_{0,1}\left[\frac{-M\delta - M - \mu}{\sigma}, \frac{M\delta + M - \mu}{\sigma}\right] \\
&\geq N_{0,1}\left[-\delta - \frac{M+\mu}{M}, \delta + \frac{M-\mu}{M}\right] \\
&\geq N_{0,1}[-\delta, \delta] \qquad (*) \\
&> 1 - \varepsilon
\end{aligned}$$

ans so $\rho(K_\varepsilon^c) < \varepsilon$. Equation $(*)$ is since $\mu \in [-M, M]$ implies $M \pm \mu \in [0, 2M]$ is positive and hence $\left[-\delta - \frac{M+\mu}{M}, \delta + \frac{M-\mu}{M}\right] \supseteq [-\delta, \delta]$.

Since ε was arbitrary, $\mathcal{A}$ is tight. ■