MA 200: Multivariable Calculus

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1 due August 12 16

The course

Grading

Lecture 1.
Friday
August 2

- Homework: 20%
- Quizzes: 20%
- Midterm: 20%
- Final: 40%

Textbooks

-

Chapter 1

Linear algebra

Definition 1.1 (homogeneous function)**.** Let V be a vector space over $\mathbb{R}$. A function $f\colon V \setminus \{0\} \to \mathbb{R}$ is called a *homogeneous function* of degree k if

$$f(rx) = r^k f(x)$$

for each $x \in V \setminus \{0\}$ and $r > 0$.

Remarks.

- If f and g are homogeneous functions of degree k and l respectively, then $f \cdot g$ is homogeneous of degree $k + l$ and f/g is homogeneous of degree $k - l$ (provided g is never zero).

- $f \equiv 0$ is homogeneous of any degree.

Definition 1.2 (norm)**.** Let V be a vector space over $\mathbb{R}$. A norm $\|\cdot\|$ on V is a function from V to $\mathbb{R}$ that satisfies

(N1) (Positivity) $\|x\| \geq 0$ for any $x \in V$.

(N2) (Definiteness) $\|x\| = 0$ iff $x = 0$.

(N3) (Homogeneity) $\|rx\| = |r|\|x\|$ for any $x \in V$ and $r \in \mathbb{R}$.

(N4) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in V$.

Definition 1.3 (normed linear space)**.** A vector space V equipped with a norm $\|\cdot\|$ is called a *normed linear space*.

Remark. Any normed linear space $(V, \|\cdot\|)$ can be given a metric space structure by defining the distance $d(x, y)$ between $x, y \in V$ as $\|x - y\|$.

The set $B(x, r) \coloneqq \{y \in V \mid \|x - y\| < r\}$ is called the open ball of radius r centered at x.

The set $S(x, r) \coloneqq \{y \in V \mid \|x - y\| = r\}$ is called the sphere of radius r centered at x.

Exercise 1.4 (reverse triangle inequality)**.** *Let V be a normed linear space. Show that*

$$\Big| \|x\| - \|y\| \Big| \leq \|x - y\| \tag{1.1}$$

for any $x, y \in V$.

Proof. First observe from homogeneity (N3) that $\|x\| = \|-x\|$ for any $x \in V$. Next, from the triangle inequality (N4) we have

$$\|x\| \leq \|x - y\| + \|y\|$$

so that

$$\|x\| - \|y\| \leq \|x - y\|.$$

Similarly,

$$\|y\| \leq \|y - x\| + \|x\|$$

so that

$$-\|x - y\| \leq \|x\| - \|y\|.$$

Combining these gives the result. ■

This shows that $f = x \mapsto \|x\|$ is a (Lipschitz) continuous function on V.

Definition 1.5 (metric space)**.** A *metric space* is a set X equipped with a function $d\colon X \times X \to \mathbb{R}$ called a *metric* that satisfies the following properties:

(M1) $d(x, y) \geq 0$ for any $x, y \in X$.

(M2) $d(x, y) = 0$ iff $x = y$.

(M3) $d(x, y) = d(y, x)$ for any $x, y \in X$.

(M4) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

Exercise 1.6 (self)**.** *Show that any normed linear space $(V, \|\cdot\|)$ is a metric space under the distance $d(x, y) = \|x - y\|$.*

Proof. (M1) and (M2) are immediate from (N1) and (N2). (N3) implies (M3) by scaling by -1. Triangle implies triangle. ■

Definition 1.7 (continuity)**.** Let (X, d) and (Y, ρ) be metric spaces. A function $f\colon X \to Y$ is called *continuous* at $a \in X$ iff

$$\begin{aligned} x_n \to a &\implies f(x_n) \to f(a), \text{ or} \\ d(x_n, a) \to 0 &\implies \rho(f(x_n), f(a)) \to 0 \end{aligned}$$

Exercise 1.8 (product metric spaces)**.** *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $d\colon X_1 \times X_2 \to \mathbb{R}$ be defined by*

$$d((x_1, x_2), (y_1, y_2)) \coloneqq d_1(x_1, y_1) + d_2(x_2, y_2).$$

Show that d is a metric on $X_1 \times X_2$.

Let $(z_n)_{n\in\mathbb{N}} = ((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence in $X_1 \times X_2$. Show that $z_n \to (x, y)$ iff $x_n \to x$ and $y_n \to y$.

Proof. Suppose $x_n \to x$ and $y_n \to y$. That is, $d_1(x_n, x) \to 0$ and $d_2(y_n, y) \to 0$. Thus $d_1(x_n, x) + d_2(y_n, y) \to 0$.

Conversely if $d_1(x_n, x) + d_2(y_n, y) \to 0$ and each is nonnegative, then $d_1(x_n, x) \to 0$ and $d_2(y_n, y) \to 0$. ■

Remark. $\widetilde{d}$ given by

$$\widetilde{d}((x_1, x_2), (y_1, y_2)) := \min\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

is *not* a metric on $X_1 \times X_2$ as it fails definiteness.

However, $\max\{d_1, d_2\}$ *is* a metric.

Exercise 1.9. *Let $(V, \|\cdot\|)$ be a normed linear space.*

- *The addition map $(x, y) \mapsto x + y$ is a continuous map from $V \times V$ to V.*
- *The scalar multiplication map $(\alpha, x) \mapsto \alpha x$ is continuous from $\mathbb{R} \times V$ to V.*

Solution.

- $\|x' + y' - (x + y)\| \le \|x' - x\| + \|y' - y\| = \|(x', y') - (x, y)\|$.
- $\|\alpha' x' - \alpha x\| \le \|\alpha' x' - \alpha x'\| + \|\alpha x' - \alpha x\| = |\alpha' - \alpha|\|x'\| + |\alpha|\|x' - x\|$.

  Thus choosing $\delta = \varepsilon / \max\{|\alpha|, \|x\|\}$ gives

  $$\|\alpha' x' - \alpha x\| \le \max\{|\alpha|, \|x\|\}(|\alpha' - \alpha| + \|x' - x\|) < \varepsilon$$

  whenever $|\alpha' - \alpha| + \|x' - x\| < \delta$.

Repeated in problem 1.1. ■

Examples.

- (ℓ^p norm) $\mathbb{R}^n$ with $p \in [1, \infty]$ and

  $$\|x\|_p := \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}$$

  where

  $$\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$$

  is the limit of the l^p norms as $p \to \infty$.

Exercise 1.10. *See problem 1.6.*

Definition 1.11 (norm equivalence)**.** Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on V. We say that $\|\cdot\|_a$ and $\|\cdot\|_b$ are *equivalent* if these exist $c_1, c_2 > 0$ such that

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a$$

for all $x \in V$. We write $\|\cdot\|_a \sim \|\cdot\|_b$.

Exercise 1.12. *Check that $\sim$ is an equivalence relation.*

Solution. Reflexivity is obvious. Symmetry is since

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a \implies \frac{1}{c_2}\|x\|_b \leq \|x\|_a \leq \frac{1}{c_1}\|x\|_b.$$

For transitivity, let

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a,$$
$$c_3\|x\|_b \leq \|x\|_c \leq c_4\|x\|_b.$$

Then

$$c_1c_3\|x\|_a \leq \|x\|_c \leq c_2c_4\|x\|_a. \qquad \blacksquare$$

Lecture 2. Monday August 5

Proposition 1.13. *Equivalent norms induce the same topology. That is, let $\|\cdot\|_a \sim \|\cdot\|_b$. Then a set is open (resp. compact) under $\|\cdot\|_a$ iff it is open (resp. compact) under $\|\cdot\|_b$.*

Proof. Suppose $c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a$.

Let $U \subseteq V$ be open under $\|\cdot\|_a$. Let $x \in U$. There exists $\varepsilon > 0$ such that $\|y - x\|_a < \varepsilon \implies y \in U$. But then $\|y - x\|_b < c_1\varepsilon \implies y \in U$. Thus U is open under $\|\cdot\|_b$.

Compactness follows from openness. $\blacksquare$

Proposition 1.14. *Every ℓ^p norm is equivalent to ℓ^∞.*

Proof. Let $x \in \mathbb{R}^n$. Then $\|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}}\|x\|_\infty$. $\blacksquare$

The usual topology on $\mathbb{R}^n$ is the one induced by the Euclidean norm. This norm itself is induced by the inner product $\langle x, y\rangle = \sum_{i=1}^n x_iy_i$. Using Cauchy-Schwarz, we can define the angle between two vectors $x, y \in \mathbb{R}^n$ to be

$$\cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right).$$

Proposition 1.15. *Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then the function $x \mapsto \|x\|$ is Lipschitz continuous with respect to the Euclidean topology.*

Proof.

$$\begin{aligned}\|x\| &= \left\|\sum x_i e_i\right\| \\ &\leq \sum |x_i| \|e_i\| \\ &\leq M\|x\|_2\end{aligned}$$

where $M = \sum \|e_i\|$.

The reverse triangle inequality gives

$$\begin{aligned}\Big|\|x\| - \|y\|\Big| &\leq \|x - y\| \\ &\leq M\|x - y\|_2.\end{aligned}$$ ■

Theorem 1.16. *Any two norms on $\mathbb{R}^n$ are equivalent.*

Proof. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then $x \mapsto \|x\|$ is continuous with respect to $\|\cdot\|_2$. Let

$$S(0,1)_{\|\cdot\|_2} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\} = S^{n-1}.$$

$\|\cdot\|$ attains a minimum and a maximum on S^{n-1} by compactness. Thus there exist positive constants c_1, c_2 such that

$$c_1 \leq \|x\| \leq c_2$$

for all $x \in S^{n-1}$.

Now for any $x \in \mathbb{R}^n \setminus \{0\}$, dividing by $\|x\|_2$ gives a point that lies on S^{n-1}. Thus

$$c_1 \leq \left\|\frac{x}{\|x\|_2}\right\| \leq c_2.$$

By homogeneity (N3),

$$c_1\|x\|_2 \leq \|x\| \leq c_2\|x\|_2.$$

This is also trivially true for $x = 0$.

Thus $\|\cdot\| \sim \|\cdot\|_2$. ■

Remark. The idea of the proof is as follows.

Any homogenous function is determined by its value on the unit sphere. A homogenous function of degree *zero* is essentially nothing but a function on the unit sphere ($f(v) = f(\widehat{v})$).

The function $x \mapsto \frac{\|x\|}{\|x\|_2}$ is a continuous homogenous function on degree 0. The unit sphere is known to be compact under the Euclidean norm (and every other, but not before we complete the proof). Thus

$$c_1 \leq \frac{\|x\|}{\|x\|_2} \leq c_2$$

for some positive constants c_1, c_2.

Definiteness and $\triangle$ are required for the ratio to be continuous. Homogeneity is required for it to be homogenous. Is positivity required?

Remark. We technically only need to show $c_1\|x\|_2 \leq \|x\|$, since the other inequality is proven in the previous proof. It is nonetheless clearer to show both inequalities.

Exercise 1.17 (Self)**.** *Show that (N1) follows from (N3) and (N4).*

Solution. Let $v \in V$. By triangle inequality, $\|v\| = \|-v + 2v\| \leq \|-v\| + \|2v\|$. By homogeneity, this is $3\|v\|$. Thus $\|v\| \leq 3\|v\|$, so $\|v\| \geq 0$. ■

Remarks (Finite-dimensional vector spaces).

- Let V be a vector space over $\mathbb{R}$ with dimension $n < \infty$. Using a basis for V, any norm on V induces a norm on $\mathbb{R}^n$, and vice versa. Norms on V are in a one-to-one correspondence with norms on $\mathbb{R}^n$.
- Thus any two norms on V are equivalent.
- Any two inner products on V will also be equivalent due to this.
- Any finite-dimensional vector space over $\mathbb{R}$ is complete.

Exercise 1.18. *Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be given by $f(x) = \frac{1}{x}$. Show that f is continuous. What is the key idea of your proof?*

Solution. Let $x_0 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon > 0$. Choose $\delta = \min\{\varepsilon \cdot \frac{1}{2}|x_0|^2, \frac{1}{2}|x_0|\}$. Then for any x in the δ-neighbourhood of x_0,

$$\begin{aligned} |f(x) - f(x_0)| &= \left|\frac{1}{x} - \frac{1}{x_0}\right| \\ &= \frac{|x_0 - x|}{|x||x_0|} \\ &< \frac{\delta}{|x||x_0|} \\ &< \frac{2\delta}{|x_0|^2} \\ &\leq \varepsilon. \end{aligned}$$

■

Remark. The proof works by bounding $\frac{1}{|x|}$. The rest goes to zero as $x \to a$. We will do a similar proof in proposition 1.38.

On $\mathbb{R}^n$, we will always fix the ℓ^2-norm

Lecture 3.
Wednesday
August 7

Notation.

$$L(\mathbb{R}^n, \mathbb{R}^m) = \{T\colon \mathbb{R}^n \to \mathbb{R}^m \mid T \text{ is linear}\}$$

and

$$M_{m\times n}(\mathbb{R}) \cong L(\mathbb{R}^n, \mathbb{R}^m)$$

using the isomorphism $A \mapsto T_A$ where

$$\begin{aligned} T_A\colon \mathbb{R}^n &\to \mathbb{R}^m \\ v &\mapsto Av, \end{aligned}$$

where v is interpreted as a column vector. We will also write $L(\mathbb{R}^n)$ for $L(\mathbb{R}^n, \mathbb{R}^n)$.

Definition 1.19 (liminf and limsup)**.** Let $f\colon X \to \mathbb{R}$ be a function on a topological space X. We define the *limit inferior* and *limit superior* of f as

$$\liminf_{x\to a} f(x) = \sup_V \inf_{x\in V} f(x)$$
$$\limsup_{x\to a} f(x) = \inf_V \sup_{x\in V} f(x)$$

where V ranges over all open neighbourhoods of a that contain at least one point other than a.

Definition 1.20 (O notation)**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^n \to \mathbb{R}^k$. We say that

(i) $f(x) = o(g(x))$ as $x \to a$ if

$$\lim_{x\to a} \frac{\|f(x)\|}{\|g(x)\|} = 0,$$

(ii) $f(x) = O(g(x))$ as $x \to a$ if

$$\limsup_{x\to a} \frac{\|f(x)\|}{\|g(x)\|} < \infty.$$

where the assumption is that g is non-zero in some neighbourhood of a.

Exercise 1.21. *Show that the definition of O is equivalent to the following:*

We say that $f(x) = O(g(x))$ as $x \to a$ if there exists an open neighbourhood V of a such that $\frac{\|f(x)\|}{\|g(x)\|}$ is bounded on V.

Solution. Call the ratio h.

$$\begin{aligned} \inf_V \sup_{x\in V} h(x) \le \infty &\iff \exists V (\sup_{x\in V} h(x) < \infty) \\ &\iff \exists V \exists M (\forall x \in V, h(x) \le M) \end{aligned}$$ ■

Exercise 1.22.

Lecture 3: Big-Oh and matrix norms

1.1 Matrix norms

Definition 1.23 (Hilbert-Schmidt norm)**.** For a matrix $A \in M_{m\times n}(\mathbb{R})$, we define the *Hilbert-Schmidt* or *Frobenius* norm by

$$\|A\|_{HS} = \Big(\sum_{i,j} a_{ij}^2\Big)^{1/2}$$

Exercise 1.24. *Show that* $\|A\|_{HS}^2 = \mathrm{Tr}(A^\top A) = \mathrm{Tr}(AA^\top)$.

Solution.

$$\begin{aligned}
(A^\top A)_{ii} &= \sum_k (A^\top)_{ik} A_{ki} \\
&= \sum_k a_{ki}^2 \\
\implies \mathrm{Tr}(A^\top A) &= \sum_i \sum_k a_{ki}^2 \\
&= \|A\|_{HS}^2.
\end{aligned}$$

Since $\|A\|_{HS} = \|A^\top\|_{HS}$, we also have $\mathrm{Tr}(AA^\top) = \|A\|_{HS}^2$. ∎

Proposition 1.25. *Any linear transformation is continuous.*

Proof. Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. Then

$$\begin{aligned}
\|Tx\| &= \Big\|T\Big(\sum x_i e_i\Big)\Big\| \\
&= \Big\|\sum x_i T e_i\Big\| \\
&\le \sum |x_i| \|Te_i\| \\
&\le \|x\| \sum \|Te_i\| \\
&= M\|x\| \qquad (1.2)
\end{aligned}$$

where $M = \|Te_1\| + \cdots + \|Te_n\|$.

Now

$$\|Tx - Ty\| = \|T(x-y)\| \le M\|x-y\|$$

says that T is Lipschitz continuous with Lipschitz constant M. ∎

We temporarity define two norms on $M_{m\times n}(\mathbb{R})$:

$$\begin{aligned}
\|T\|_S &= \sup_{\|x\|=1} \|Tx\| \\
\|T\|_B &= \sup_{\|x\|\le 1} \|Tx\|
\end{aligned}$$

Lemma 1.26. $\|T\|_S = \|T\|_B$.

Proof. From the definition it is obvious that $\|T\|_S \leq \|T\|_B$. Now for any $x \in \mathbb{R}^n \setminus \{0\}$, let $y = x/\|x\|$.

$$\begin{aligned} \|Ty\| &\leq \|T\|_S \\ \frac{\|Tx\|}{\|x\|} &\leq \|T\|_S \\ \implies \|Tx\| &\leq \|T\|_S\|x\| \end{aligned}$$

Thus for $\|x\| \leq 1$, we have $\|Tx\| \leq \|T\|_S$ (check 0 separately). So $\|T\|_B \leq \|T\|_S$. ■

Definition 1.27 (Operator norm)**.** For any $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, we define the *operator norm* by

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

From the previous lemma, we can also write

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}.$$

From equation (1.2), we have

$$\|T\| \leq \|Te_1\| + \cdots + \|Te_n\|.$$

So the operator norm is finite.

Proposition 1.28. *The operator norm is a norm on $L(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. Let $T, S \in L(\mathbb{R}^n, \mathbb{R}^m)$.

(N1) Positivity is by positivity of the vector norm.

(N2) Suppose T is not identically zero. Let $v \neq 0$ be such that $\|Tv\| \neq 0$. Then

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} \geq \frac{\|Tv\|}{\|v\|} > 0.$$

(N3) $\|\lambda T\| = \sup_{\|x\|=1} \|\lambda Tx\| = |\lambda| \sup_{\|x\|=1} \|Tx\| = |\lambda|\|T\|$.

(N4)

$$\begin{aligned} \|T + S\| &= \sup_{\|x\|=1} \|(T + S)x\| \\ &\leq \sup_{\|x\|=1} \|Tx\| + \|Sx\| \\ &\leq \sup_{\|x\|=1} \|Tx\| + \sup_{\|x\|=1} \|Sx\| \\ &= \|T\| + \|S\|. \end{aligned}$$

■

Proposition 1.29. *Let $T_2 \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $T_1 \in L(\mathbb{R}^n, \mathbb{R}^k)$. Then*

$$\|T_1 \circ T_2\| \leq \|T_1\|\|T_2\|$$

Proof. Let $x \in \mathbb{R}^m$ with $\|x\| = 1$. Then

$$\|T_1 T_2 x\| \leq \|T_1\|\|T_2 x\| \leq \|T_1\|\|T_2\|. \qquad \blacksquare$$

Since $M_{m\times n}(\mathbb{R}) \cong \mathbb{R}^{mn}$, we can conclude that the Hilbert-Schmidt norm and the operator norm are equivalent, as are any two norms on $M_{m\times n}(\mathbb{R})$. Thus we can talk about openness and continuity without specifying the norm. Problem 1.10 discusses their equivalence with specific bounds.

Proposition 1.30. $\mathrm{GL}_n(\mathbb{R})$ *is open in* $M_n(\mathbb{R})$.

Proof. $\det\colon M_n(\mathbb{R}) \to \mathbb{R}$ is continuous because it is a polynomial in the entries of the matrix. Note that $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$, so it is the preimage of an open set, which is open by proposition 1.36. $\blacksquare$

Determinants pose a problem in infinite dimensions. This also doesn't provide estimates on the size of the neighbourhood. We will go through Rudin's proof in proposition 1.38 which avoids determinants.

Meanwhile, here is a special case.

Lemma 1.31. *There is an open ball around I in $M_n(\mathbb{R})$ that is contained in* $\mathrm{GL}_n(\mathbb{R})$.

Proof. A reasonable guess for the radius is 1 (intuiting from the 1D case). Let $X \in M_n(\mathbb{R})$ with $\|X - I\| < 1$.

Let $v \in \mathbb{R}^n \setminus \{0\}$. Then $\|(X - I)v\| < \|v\|$ implies that $(X - I)v \neq v$. Thus $Xv \neq 0$ and so X is invertible. $\blacksquare$

This will be useful in proposition 1.38. This can also be proven by borrowing the following result from $\mathbb{C}$.

$$\frac{1}{1-z} = 1 + z + z^2 + \dots \quad \text{for} \quad |z| < 1.$$

(This was the first thought a student had when prompted.) We approach it this way as well, since this gives us an explicit inverse.

Lemma 1.32. *Let $Z \in M_n(\mathbb{R})$ be such that $\|Z\| < 1$. Then*

(i) $\sum_{n=0}^{\infty} Z^n$ *converges.*

(ii) $I - Z$ *is invertible.*

(iii) $(I - Z)^{-1} = \sum_{n=0}^{\infty} Z^n$.

Proof. By $\|AB\| \le \|A\|\|B\|$, $\|Z^k\| \le \|Z\|^k$.

It is easy to see that the series converges by the Cauchy criterion. For any $\varepsilon > 0$, there is some n such that

$$\left\|\sum_{k=n}^{m} Z^k\right\| \le \sum_{k=n}^{m} \|Z^k\| < \varepsilon$$

for all $m > n$.

To see that $I - Z$ is invertible, note that

$$\|I - Z\| \ge \Big|\|I\| - \|Z\|\Big| = 1 - \|Z\| > 0.$$

Finally, let $S_n = \sum_{k=0}^{n} Z^k$ and $S_\infty = \lim_{n\to\infty} S_n$. Then $(I-Z)S_n = I - Z^{n+1}$ and so $(I-Z)S_n \to I$ as $n \to \infty$. Since matrix multiplication is continuous, we can take the limit inside the product and get $(I-Z)S_\infty = I$. ■

Remark. For infinite-dimensional spaces, we also need to show $S_\infty(I-Z) = I$, which will be done in the exact same way.

Proposition 1.33. *$A \mapsto A^{-1}$ is continuous on* $\mathrm{GL}_n(\mathbb{R})$.

Proof. Let $A \in \mathrm{GL}_n(\mathbb{R})$. Then $A^{-1} = \frac{1}{\det A}\operatorname{adj} A$. Each entry of A^{-1} is a rational function in the entries of A, so $A \mapsto A^{-1}$ is continuous by exercise 1.34. ■

Lecture 4. Friday August 9

Exercise 1.34. *Let $U \subseteq \mathbb{R}^n$ be an open set. Let $f\colon U \to \mathbb{R}^m$ be such that*

$$f(x) := (f_1(x), f_2(x), \dots, f_n(x)), \quad x \in U$$

Show that f is continuous at $a \in U$ iff each f_i is continuous at a.

Solution. Consider the ℓ^1 norm on $\mathbb{R}^m$.

Suppose f is continuous. Since $|f_1(x) - f_1(y)| \le \|f(x) - f(y)\|$, so is each f_i.

Suppose each f_i is continuous at a. For any $\varepsilon > 0$, there exists $\delta_i > 0$ such that $|f_i(x) - f_i(a)| < \frac{1}{m}\varepsilon$ in a δ_i-neighbourhood of a. Let $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$. ■

Exercise 1.35. *Let $f(x) = o(g(x))$ and $g(x) = O(h(x))$. Then show that $f(x) = o(h(x))$.*

Solution.

$$\limsup_{x\to a} \frac{\|f(x)\|}{\|h(x)\|} < c \quad \text{and} \quad \lim_{x\to a} \frac{\|g(x)\|}{\|h(x)\|} = 0$$

Thus

$$\limsup_{x\to a} \frac{\|f(x)\|}{\|h(x)\|} = 0.$$ ■

Proposition 1.36. *Suppose X and Y are metric spaces. Then the following are equivalent.*

(i) f is continuous.

(ii) $f^{-1}(V)$ is open whenever V is open in Y.

Solution. Suppose f is continuous. Let $V \subseteq Y$ be open.

Let $x \in f^{-1}(V)$. Then $f(x) \in V$. There is some $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq V$. But by continuity, there is some $\delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq V$. Thus $B(x, \delta) \subseteq f^{-1}(V)$.

Conversely, suppose $f^{-1}(V)$ is open whenever V is open. Then for any $x \in X$ and $\varepsilon > 0$, we have that $f^{-1}(B(f(x), \varepsilon))$ is open. So some δ-neighbourhood of x in X that is contained in $f^{-1}(B(f(x), \varepsilon))$. ■

Assignment 1

up August 2
due August 12
quiz August 14

Problem 1.1. *Let $(V, \|\cdot\|)$ be a normed linear space.*

(i) Show that the addition map $(u, v) \mapsto u + v$ is continuous.

(ii) Show that the scalar multiplication map $(\alpha, u) \mapsto \alpha u$ is continuous.

Proof.

(i) $\|u_2 + v_2 - (u_1 + v_1)\| \leq \|u_2 - u_1\| + \|v_2 - v_1\|$.

(ii) $\|\alpha_2 u_2 - \alpha_1 u_1\| = \|\alpha_2 u_2 - \alpha_1 u_2 + \alpha_1 u_2 - \alpha_1 u_1\| = \|(\alpha_2 - \alpha_1)u_2 + \alpha_1(u_2 - u_1)\| \leq |\alpha_2 - \alpha_1|\|u_2\| + |\alpha_1|\|u_2 - u_1\|$. ■

Problem 1.2. *Let $(V, \|\cdot\|)$ be a normed linear space. Prove that*

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|$$

for all $x, y \in V$. Show that the function $x \mapsto \|x\|$ from V to $\mathbb{R}$ is continuous.

Proof. By the $\triangle$ inequality,

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \implies \|x\| - \|y\| \leq \|x - y\|.$$

Similarly

$$\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| \implies \|x\| - \|y\| \geq -\|x - y\|.$$

Thus

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|.$$

To show that $\|\cdot\|$ is continuous, do what exactly? Notice

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|?$$ ■

Problem 1.3. *For $x, y \in \mathbb{R}^n$, show that*

$$|\langle x,\, y\rangle| \leq \|x\|_2 \|y\|_2 \tag{1.3}$$

Also show that the two sides in equation (1.3) are equal if and only if x and y are linearly dependent over $\mathbb{R}$.

Proof. If either of x or y is 0, both sides are 0.

Suppose $x, y \neq 0$. Let $\widehat{x} = \dfrac{x}{\|x\|_2}$ and $\widehat{y} = \dfrac{y}{\|y\|_2}$. Then proving equation (1.3) amounts to proving

$$|\langle \widehat{x},\, \widehat{y}\rangle| \leq 1$$

because of homogeneity of the inner product.

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} (\widehat{x}_i - \widehat{y}_i)^2 \\ 0 &\leq \sum_{i=1}^{n} \widehat{x}_i^2 - 2\widehat{x}_i\widehat{y}_i + \widehat{y}_i^2 \\ 2\sum_{i=1}^{n} \widehat{x}_i\widehat{y}_i &\leq \sum_{i=1}^{n} \widehat{x}_i^2 + \sum_{i=1}^{n} \widehat{y}_i^2 \\ \langle \widehat{x},\, \widehat{y} \rangle &\leq 1. \end{aligned}$$

Similarly $\langle -\widehat{x},\, \widehat{y} \rangle \leq 1$, which gives $\langle \widehat{x},\, \widehat{y} \rangle \geq -1$. ■

Problem 1.4. *Let $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Show that $\{x_k\}_{k\in\mathbb{N}}$ converges to x if and only if $\{\langle x_k,\, y\rangle\}$ converges to $\langle x,\, y\rangle$ for all $y \in \mathbb{R}^n$.*

Proof. Suppose $x_k \to x$. Let $y \in \mathbb{R}^n$. Then

$$|\langle x_k,\, y\rangle - \langle x,\, y\rangle| = |\langle x_k - x,\, y\rangle| \leq \|x_k - x\|\|y\| \to 0.$$

Conversely, suppose $\langle x_k,\, y\rangle \to \langle x,\, y\rangle$ for all $y \in \mathbb{R}^n$. Then $\langle x_k,\, e_i\rangle \to \langle x,\, e_i\rangle$ for all i. Thus $x_k \to x$ componentwise. ■

Problem 1.5. *Let $1 < p, q < \infty$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Show that for any $a \geq 0$ and $x \geq 0$ the following holds:*

$$xa \leq \frac{a^p}{p} + \frac{x^q}{q}. \tag{1.4}$$

Show that in equation (1.4) equality holds if and only if $x^q = a^p$.

Proof. Let $a \geq 0$ be fixed. Define $f(x) = xa - \dfrac{a^p}{p} - \dfrac{x^q}{q}$. This is differentiable on $[0, \infty)$ since $q > 0$. $f'(x) = a - x^{q-1}$. Thus

$$\begin{aligned} f'(x) \leq 0 &\iff x^{q-1} \leq a \\ &\iff x^{q/p} \leq a \\ &\iff x^q \leq a^p. \end{aligned}$$

Thus f is decreasing on $[a^{p/q}, \infty)$ and increasing on $[0, a^{p/q}]$. Thus $f(x) \geq f(a^{p/q}) = 0$. Moreover, since $f'(x) \neq 0$ for $x^q \neq a^p$, we have $f(x) = 0 \iff x^q = a^p$.

Thus $xa \leq \dfrac{a^p}{p} + \dfrac{x^q}{q}$ with equality only if $x^q = a^p$. ■

Problem 1.6. *For $1 \leq p \leq \infty$ and $x = (x_1, x_2, \ldots, x_n)$, we define*

$$\|x\|_p = \begin{cases} \Big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\Big)^{\frac{1}{p}} & 1 \leq p < \infty \\ \max\limits_{1\leq i\leq n} |x_i| & p = \infty \end{cases}$$

(i) Let $1 \le q \le \infty$ *be such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$*. For any* $x, y \in \mathbb{R}^n$*, show that*

$$|\langle x,\, y\rangle| \le \|x\|_p \|y\|_q \text{ and } \|x+y\|_p \le \|x\|_p + \|y\|_p. \tag{1.5}$$

(ii) Show that $\|\cdot\|_p$ *defines a norm on* $\mathbb{R}^n$*.*

(iii) Show that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$ *for any* $x \in \mathbb{R}^n$*.*

Proof. We first deal with the case $p = \infty$ for parts (a) and (b).

(i) $q = 1$.

$$\begin{aligned}
|\langle x,\, y\rangle| &= |x_1 y_1 + x_2 y_2 + \cdots + x_n y_n| \\
&\le |x_1||y_1| + |x_2||y_2| + \cdots + |x_n||y_n| \\
&= \max_{1\le i\le n} |x_i|(|y_1| + |y_2| + \cdots + |y_n|) \\
&= \|x\|_\infty \|y\|_1
\end{aligned}$$

and

$$\begin{aligned}
\|x+y\|_\infty &= \max_{1\le i\le n} |x_i + y_i| \\
&\le \max_{1\le i\le n} (|x_i| + |y_i|) \\
&\le \max_{1\le i,j\le n} (|x_i| + |y_j|) \\
&= \max_{1\le i\le n} |x_i| + \max_{1\le j\le n} |y_j| \\
&= \|x\|_\infty + \|y\|_\infty.
\end{aligned}$$

(ii) We have positivity by definition. $\|x\|_p = 0 \iff \max_{1\le i\le n}|x_i| = 0 \iff |x_1| = |x_2| = \cdots = |x_n| = 0 \iff x = 0$, so definiteness holds. Homogeneity is since

$$\|\alpha x\|_\infty = \max_{1\le i\le n} |\alpha x_i| = |\alpha| \max_{1\le i\le n} |x_i| = |\alpha| \|x\|_\infty.$$

Triangle inequality is proven above.

Thus $\|\cdot\|_\infty$ is a norm.

Now we deal with the case $1 \le p < \infty$.

(i) For $|\langle x,\, y\rangle| \le \|x\|_p \|y\|_q$, we only concern ourselves with $1 < p, q < \infty$. The case $p = 1$ requires $q = \infty$, which is covered above with p and q interchanged. We will show that the ratio of the two sides is bounded by 1.

$$
\begin{aligned}
\frac{|\langle x, y\rangle|}{\|x\|_p\|y\|_q} &= \left|\frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{\|x\|_p\|y\|_q}\right| \\
&\leq \sum_{i=1}^{n} \frac{|x_i||y_i|}{\|x\|_p\|y\|_q} \\
&\leq \sum_{i=1}^{n} \left(\frac{1}{p}\frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y_i|^q}{\|y\|_q^q}\right) && \text{(by equation (1.4))} \\
&= \frac{1}{p}\frac{\sum_i |x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{\sum_i |y_i|^q}{\|y\|_q^q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}
$$

We use this result to prove the triangle inequality. (We did this in a UM 204 assignment last semester, with ample of hints and time to spare.)

$$
\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^{n} |x_i + y_i|^p \\
&= \sum_{i=1}^{n} |x_i + y_i||x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1}
\end{aligned}
$$

Let $X = (|x_1|, |x_2|, \ldots, |x_n|)$ and $Z = (|x_1+y_1|^{p-1}, |x_2+y_2|^{p-1}, \ldots, |x_n+y_n|^{p-1})$. Then by equation (1.4),

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} &= |\langle X, Z\rangle| \\
&\leq \|X\|_p\|Z\|_q
\end{aligned}
$$

where $q = \frac{p}{p-1}$

$$
\begin{aligned}
&\leq \|x\|_p(|x_1+y_1|^p + \cdots + |x_n+y_n|^p)^{\frac{p}{p-1}} \\
&= \|x\|_p\|x+y\|_p^{p-1}.
\end{aligned}
$$

Similarly,

$$
\sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \leq \|y\|_p\|x+y\|_p^{p-1}.
$$

This gives

$$\begin{aligned}\|x+y\|_p^p &\le (\|x\|_p + \|y\|_p)\|x+y\|_p^{p-1} \\ \|x+y\|_p &\le \|x\|_p + \|y\|_p.\end{aligned}$$

(ii) Positivity is again by definition. $\|x\|_p = 0 \iff |x_i|^p = 0$ for all i, which is iff $x = 0$. Homogeneity is trivial to check.

$$\begin{aligned}\|\alpha x\|_p &= (|\alpha x_1|^p + |\alpha x_2|^p + \cdots + |\alpha x_n|^p)^{\frac{1}{p}} \\ &= (|\alpha|^p|x_1|^p + |\alpha|^p|x_2|^p + \cdots + |\alpha|^p|x_n|^p)^{\frac{1}{p}} \\ &= |\alpha|\|x\|_p.\end{aligned}$$

Triangle inequality is proven above.

Thus $\|\cdot\|_p$ is a norm.

We now prove part (c). The case $x = 0$ is trivial since $\|x\|_p = \|x\|_\infty = 0$ for any p.

WLOG let $\|x\|_\infty = |x_1| > 0$. Then for $1 \le p < \infty$,

$$\begin{aligned}\|x\|_p &= |x_1|\left(1 + \frac{|x_2|^p}{|x_1|^p} + \cdots + \frac{|x_n|^p}{|x_1|^p}\right)^{\frac{1}{p}} \\ &\le |x_1| \cdot n^{\frac{1}{p}}\end{aligned}$$

Further,

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \ge (|x_1|^p)^{\frac{1}{p}} = |x_1|.$$

Thus

$$|x_1| \le \|x\|_p \le n^{\frac{1}{p}}|x_1|.$$

As $p \to \infty$, $n^{\frac{1}{p}} \to 1$. Thus by the squeeze theorem, $\|x\|_p \to |x_1| = \|x\|_\infty$. ■

Problem 1.7. *Let $C[a,b]$ be the set of all complex-valued continuous functions on $[a,b]$.*

(i) Let $f \in C[a,b]$ be such that f is non-negative and $\int_a^b f(x)\,\mathrm{d}x = 0$. Show that $f \equiv 0$.

(ii) For $f \in C[a,b]$, define

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|, \qquad \|f\|_1 := \int_a^b |f(x)|\,\mathrm{d}x.$$

Show that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are norms on $C[a,b]$.

(iii) Are the above two norms on $C[a,b]$ equivalent? Are they comparable?

Solution.

(i) Suppose f is non-zero at some point $c \in [a, b]$. By continuity, $f(x) \geq \frac{f(c)}{2}$ in some neighbourhood $[c - \delta, c + \delta]$. Then f is lower bounded by the step function

$$g(x) = \begin{cases} \frac{f(c)}{2} & x \in [c - \delta, c + \delta] \\ 0 & \text{otherwise} \end{cases}$$

which has positive integral. This would force $\int_a^b f(x)\,\mathrm{d}x > 0$. Contradiction! Such a c cannot exist.

(ii) Clearly both are non-negative. $\|f\|_\infty = 0 \iff |f(x)| \leq 0$ for all $x \in [a, b]$, which is iff $f \equiv 0$. Definiteness of $\|\cdot\|_1$ is by the previous part. Homogeneity is obvious. Triangle inequality is an extension of the triangle inequality for complex numbers.

(iii) They are *not* equivalent. Consider $[a, b] = [0, 1]$ and $f(x) = e^{-\lambda x}$. Then $\|f\|_\infty = 1$ and $\|f\|_1 = \frac{1-e^{-\lambda}}{\lambda}$. One can choose λ to make $\|f\|_1$ arbitrarily close to 0. Thus there are no constants $c_1, c_2 > 0$ such that

$$c_1\|f\|_\infty \leq \|f\|_1 \leq c_2\|f\|_\infty.$$

However, we *can* compare the norms as

$$\|f\|_1 \leq (b - a)\|f\|_\infty.$$

This is simply by noticing that the constant function $x \mapsto \|f\|_\infty$ upper bounds $|f(x)|$ and has integral $(b - a)\|f\|_\infty$ over $[a, b]$. ■

Problem 1.8. *For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, let $\|A\|$ denote the operator norm of A. Show that*

$$\|A\| = \inf\{M : \|Ax\| \leq M\|x\| \text{ for all } x \in \mathbb{R}^n\}.$$

Proof. $\|Ax\| \leq M\|x\|$ is trivially true for $x = 0$ no matter what M is. Thus

$$\begin{aligned} &\inf\{M : \|Ax\| \leq M\|x\| \text{ for all } x \in \mathbb{R}^n\} \\ &\qquad = \inf\{M : \|Ax\| \leq M\|x\| \text{ for all } x \in \mathbb{R}^n \setminus \{0\}\} \\ &\qquad = \inf\left\{M : \left\|A\frac{x}{\|x\|}\right\| \leq M \text{ for all } x \in \mathbb{R}^n \setminus \{0\}\right\} \\ &\qquad = \inf\{M : \|Ay\| \leq M \text{ for all } y \in S^{n-1}\} \\ &\qquad = \inf\{\text{upper bounds of } \{\|Ay\| : y \in S^{n-1}\}\} \\ &\qquad = \sup\{\|Ay\| : y \in S^{n-1}\} \\ &\qquad = \|A\|. \end{aligned}$$

■

Problem 1.9. *Let A be a real symmetric $n \times n$ matrix.*

(i) Show that all eigenvalues of A are real.

(ii) For $1 \le i \le n$, let λ_i denote the eigenvalues of A. Show that

$$\|A\| = \max_{1\le i\le n} |\lambda_i|.$$

Solution.

(i) View A as a linear operator on $\mathbb{C}^n$. Let λ be an eigenvalue of A and v be the corresponding eigenvector. Then

$$\lambda\langle v,\, v\rangle = \langle Av,\, v\rangle = \langle v,\, Av\rangle = \overline{\lambda}\langle v,\, v\rangle.$$

Thus $\lambda = \overline{\lambda}$ is real.

(ii) (assuming spectral theorem) WLOG let $\lambda_1 = \max_{1\le i\le n}|\lambda_i|$. Write any vector $x \in \mathbb{R}^n$ as a linear combination of orthonormal eigenvectors $x = \sum_{i=1}^n c_i v_i$, where v_i is the eigenvector corresponding to λ_i. Then $Ax = \sum_{i=1}^n c_i \lambda_i v_i$.

$$\begin{aligned} \|Ax\|^2 &= \sum_{i=1}^n c_i^2 \lambda_i^2 \\ &\le \lambda_1^2 \sum_{i=1}^n c_i^2 \\ &= \lambda_1^2 \|x\|^2. \end{aligned}$$

Thus $\|A\| \le \lambda_1$. Moreover, $\|Av_1\| = |\lambda_1|\|v_1\|$. Thus $\|A\| \ge \lambda_1$. ■

Problem 1.10. *Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^k, \mathbb{R}^n)$. Show that*

$$\|A\| \le \|A\|_{HS} \le \sqrt{n}\|A\| \quad \textit{and} \quad \|AB\|_{HS} \le \|A\|_{HS}\|B\|_{HS}.$$

Proof. $\|A\|_{HS} = \sqrt{\operatorname{Tr}(A^\top A)}$. Recall that the trace of a matrix is the sum of its eigenvalues.

Let $v_1, v_2, \ldots, v_n$ be orthonormal eigenvectors of $A^\top A$ with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ (spectral theorem). Each λ_i is non-negative, since $\langle A^\top Ax,\, x\rangle = \langle Ax,\, Ax\rangle \ge 0$.

Then for any $x = \sum_{i=1}^n c_i v_i$ with $\|x\| = 1$,

$$\|Ax\|^2 = \langle Ax,\, Ax\rangle = \langle A^\top Ax,\, x\rangle = \sum_{i=1}^n c_i^2 \lambda_i \le \lambda_1$$

where the equality holds for $x = v_1$. Thus $\|A\| = \sqrt{\lambda_1}$. Since $\|A\|_{HS}^2 = \sum_{i=1}^n \lambda_i$, we have $\lambda_1 \le \|A\|_{HS}^2 \le n\lambda_1$. This gives $\|A\| \le \|A\|_{HS} \le \sqrt{n}\|A\|$.

For $1 \le i \le m$ and $1 \le j \le k$ let

$$a_i = \begin{pmatrix} A_{i1} & A_{i2} & \cdots & A_{in} \end{pmatrix}^\top, \qquad b_j = \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} \langle a_1, b_1\rangle & \langle a_1, b_2\rangle & \cdots & \langle a_1, b_k\rangle \\ \langle a_2, b_1\rangle & \langle a_2, b_2\rangle & \cdots & \langle a_2, b_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_m, b_1\rangle & \langle a_m, b_2\rangle & \cdots & \langle a_m, b_k\rangle \end{pmatrix}$$

so by Cauchy-Schwarz,

$$\begin{aligned} \|AB\|_{HS}^2 &= \sum_{i=1}^m \sum_{j=1}^k \langle a_i, b_j\rangle^2 \\ &\le \sum_{i=1}^m \sum_{j=1}^k \|a_i\|^2 \|b_j\|^2 \\ &= \left(\sum_{i=1}^m \|a_i\|^2\right)\left(\sum_{j=1}^k \|b_j\|^2\right) \\ &= \|A\|_{HS}^2 \|B\|_{HS}^2. \end{aligned}$$

■

Remark. A far simpler proof that I missed is the following.

$$\begin{aligned} \|Ax\|^2 &\le \sum_i \langle a_i, x\rangle^2 \\ &\le \sum_i \|a_i\|^2 \|x\|^2 \\ &= \|A\|_{HS}^2 \|x\|^2 \end{aligned} \qquad \begin{aligned} \|A\|_{HS}^2 &= \sum_j \sum_i a_{ij}^2 \\ &= \sum_j \|Ae_j\|^2 \\ &\le \sum_j \|A\|^2 \\ &= n\|A\|^2. \end{aligned}$$

Quiz

Problem 1.11. *Recall the definition of a homogeneous function. Let $f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a continuous, non-vanishing homogenous function of degree k and $\|\cdot\|$ be a fixed norm on $\mathbb{R}^n$. Show that there exist positive constants $C_1, C_2 > 0$ such that*

$$C_1\|x\|^k \le |f(x)| \le C_2\|x\|^k,$$

for every $0 \ne x \in \mathbb{R}^n$.

Proof. Choose $C_1 = \min_{\|x\|=1}|f(x)|$ and $C_2 = \max_{\|x\|=1}|f(x)|$. They exist by compactness of the unit sphere, and are positive since f does not vanish.

Then for any $x \ne 0$,

$$|f(x)| = \|x\|^k \left| f\left(\frac{x}{\|x\|}\right)\right|$$

is bounded between $C_1\|x\|^k$ and $C_2\|x\|^k$. ■

Problem 1.12. *Let V be a vector space over $\mathbb{R}$. Let d be the discrete metric on V. Is d induced by a norm on V?*

Solution. No. Suppose $d(x,y) = \|x-y\|$ for some norm $\|\cdot\|$, for all $x, y \in V$. Let $x \neq y$. Then $d(x,y) = 1 = \|x-y\|$. But $d(2x, 2y) = 1 = \|2x-2y\| = 2\|x-y\| = 2$. Contradiction! ■

Problem 1.13. *For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, show that $\|A\| = \|A^\top\|$.*

Proof. Notice by Cauchy-Schwarz that for any vector v in a real inner product space,

$$\|v\| = \sup_{\|w\|=1} \langle w,\, v\rangle.$$

(The supremum is achieved at $v/\|v\|$ for $v \neq 0$.) Then

$$\begin{aligned}
\|A\| &= \sup_{x\in S^{n-1}} \|Ax\| \\
&= \sup_{x\in S^{n-1}} \sup_{y\in S^{m-1}} \langle y,\, Ax\rangle \\
&= \sup_{y\in S^{m-1}} \sup_{x\in S^{n-1}} \langle A^\top y,\, x\rangle \\
&= \sup_{y\in S^{m-1}} \|A^\top y\| \\
&= \|A^\top\|.
\end{aligned}$$

■

Problem 1.14. *Find maximum of $x + 2y + 3z$ subject to the condition $x^2 + y^2 + z^2 = 1$.*

Solution. The function is continuous and the constraint is compact. Thus a maximum exists.

Let $r = (x, y, z)$ and $n = (1, 2, 3)$. As discussed in the previous problem,

$$\max_{\|r\|=1} \langle n,\, r\rangle = \|n\|.$$

Thus the maximum is $\sqrt{14}$. ■

Problem 1.15. *See problem 1.9.*

Exercise 1.37. *Let Z be as in $(I-Z)^{-1} = I+Z+O(Z^2)$ and also $(I-Z)^{-1} = I+Z+o(Z^2)$.*

Lecture 5. Monday August 12

The proof of proposition 1.33 is nice and sweet. However, the proof in Rudin generalises better to infinite dimensions. We thus prove it again.

Proposition 1.38.

(i) Let $A \in M_n(\mathbb{R})$ be such that $\|I - A\| < 1$. Then $A \in \mathrm{GL}_n(\mathbb{R})$.

(ii) Let $A \in \mathrm{GL}_n(\mathbb{R})$ be fixed and let $B \in M_n(\mathbb{R})$ be such that

$$\|B - A\| < \|A^{-1}\|^{-1}.$$

Then $B \in \mathrm{GL}_n(\mathbb{R})$.

(iii) $A \mapsto A^{-1}$ is continuous on $\mathrm{GL}_n(\mathbb{R})$.

Remark. The second part shows that $\mathrm{GL}_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$.

Proof. We proved the first part earlier in lemma 1.31 and again in lemma 1.32 (let $Z = I - A$, then $I - Z = A$).

For the second part, let $A \in \mathrm{GL}_n(\mathbb{R})$ be fixed and let $\|B-A\| < \|A^{-1}\|^{-1}$. We can write $B - A$ as $A(A^{-1}B - I)$. Now

$$\begin{aligned}\|A^{-1}B - I\| &= \|A^{-1}(B - A)\| \\ &\le \|A^{-1}\|\|B - A\| \\ &< 1.\end{aligned}$$

Then by the first part, $A^{-1}B \in \mathrm{GL}_n(\mathbb{R})$, so that $B \in \mathrm{GL}_n(\mathbb{R})$.

For the last part, we want $B^{-1} \to A^{-1}$ as $B \to A$.

$$B^{-1} - A^{-1} = A^{-1}(A - B)B^{-1} \tag{1.6}$$

We need to bound $\|B^{-1}\|$. Let W be an open neighbourhood of A of radius $\frac{1}{2}\|A^{-1}\|^{-1}$. Then $W \subseteq \mathrm{GL}_n(\mathbb{R})$.

For any $B \in W$, $\|A - B\|\|A^{-1}\| < \frac{1}{2}$ and

$$\begin{aligned}\|B^{-1}\| - \|A^{-1}\| &\le \|B^{-1} - A^{-1}\| \\ &\le \|B^{-1}\|\|A - B\|\|A^{-1}\| && \text{(by equation (1.6))} \\ &\le \frac{1}{2}\|B^{-1}\|.\end{aligned}$$

This bounds $\|B^{-1}\|$ above by $2\|A^{-1}\|$. Using equation (1.6) again, we have

$$\begin{aligned}\|B^{-1} - A^{-1}\| &\le \|A^{-1}\|\|A - B\|\|B^{-1}\| \\ &\le 2\|A^{-1}\|^2 \cdot \|A - B\|.\end{aligned}$$

As $B \to A$, $B^{-1} \to A^{-1}$. ■

Idea. This is similar in spirit to exercise 1.18.

- Equation (1.6) is similar to taking the common denominator in $\frac{1}{x} - \frac{1}{a}$.
- The choice of W is similar to choosing $\delta \leq \frac{1}{2}|a|$, and leads to an identical bound.

Chapter 2

Differentiation

Definition 2.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. We say that f is *differentiable* at $a \in \mathbb{R}$ if
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$
exists. We denote this limit by $f'(a)$ and call it the *derivative* of f at a.

This doesn't make sense for $f\colon \mathbb{R}^n \to \mathbb{R}^m$ when $n > 2$ (for $n = 2$ we can identify $\mathbb{R}^2$ with $\mathbb{C}$).

Theorem 2.2 (Hurwitz' theorem)**.** *$\mathbb{R}^n$ is a*

We will redefine differentiability for real functions.

Proposition 2.3. *Let U be an open subset of $\mathbb{R}$ and $f\colon U \to \mathbb{R}$. Let $a \in U$. Then f is differentiable at a if and only if there exists a linear map $T \in L(\mathbb{R}, \mathbb{R})$ such that*
$$f(a+h) - f(a) = Th + o(h).$$

Proof. Suppose f is differentiable at $a \in U$.
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$
We can rewrite this as
$$\begin{aligned} &\lim_{h\to 0} \frac{f(a+h)-f(a)-f'(a)h}{h} = 0 \\ \implies &\lim_{h\to 0} \frac{|f(a+h)-f(a)-T_{f'(a)}h|}{|h|} = 0 \end{aligned}$$
where $T_\alpha \in L(\mathbb{R}, \mathbb{R})$ is the linear map $x \mapsto \alpha x$.

Conversely, suppose there exists a linear map T such that $f(a+h) -$

$f(a) - Th = o(h)$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{|f(a+h) - f(a) - Th|}{|h|} &= 0 \\
\implies \lim_{h\to 0} \left| \frac{f(a+h) - f(a)}{h} - T(1) \right| &= 0 \\
\implies \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= T(1).
\end{aligned}$$

■

Lecture 6. Monday August 19

Definition 2.4. Let $U \subseteq \mathbb{R}^n$ be an open set containing a. Let $f\colon U \to \mathbb{R}^m$. We say that f is *differentiable* at a if there exists a linear map $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - Th\|}{\|h\|} = 0.$$

We say that T is the *derivative* of f at a and write $f'(a) = T$.

If f is differentiable at every point in U, we say that f is differentiable on U.

Writing $f'(a)$ requires the derivative to be unique.

Proposition 2.5. *Let $T_1, T_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$ be satisftying the definition of differentiability at a for $f\colon U \to \mathbb{R}^m$. Then $T_1 = T_2$.*

Proof. Let $T = T_1 - T_2$. Then

$$\begin{aligned}
Th &= T_1 h - T_2 h \\
&= (f(a+h) - f(a) - T_2 h) - (f(a+h) - f(a) - T_1 h) \\
&= o(h) - o(h) = o(h).
\end{aligned}$$

We have $\lim\limits_{h\to 0} \frac{\|Th\|}{\|h\|} = 0$. Let $v \in \mathbb{R}^n \setminus \{0\}$. As $t \to 0$, $tv \to 0$. Thus

$$\begin{aligned}
\lim_{t\to 0} \frac{\|T(tv)\|}{\|tv\|} &= \lim_{t\to 0} \frac{|t|\|Tv\|}{|t|\|v\|} \\
&= \frac{\|Tv\|}{\|v\|} = 0.
\end{aligned}$$

Thus $Tv = 0$ for all $v \in \mathbb{R}^n$. ■

Proposition 2.6. *Differentiability at a point implies continuity at that point.*

Proof. Suppose f is differentiable at a with $f'(a) = T$. Let

$$q(h) = f(a+h) - f(a) - Th.$$

$$\begin{aligned}
\|f(a+h) - f(a)\| &= \|f(a+h) - f(a) - Th + Th\| \\
&\leq \|q(h)\| + \|Th\| \\
&\leq \frac{\|q(h)\|}{\|h\|}\|h\| + \|T\|\|h\|.
\end{aligned}$$

As $h \to 0$, each term goes to 0. ■

For *finding* the derivative, it is helpful to do the following:

- Use little-o notation.
- Identify the linear map T.
- Ignore the little-o terms.

If $f(a+h) = f(a) + Th + o(h)$, then $f'(a) = T$.

Examples.

- Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be given by $f(x) = c$ for some constant $c \in \mathbb{R}^m$. For any $a \in \mathbb{R}^n$, we can write $f(a+h) = f(a) + 0 + 0$. Thus $f'(a) = 0$.
- Let $f \in L(\mathbb{R}^n, \mathbb{R}^m)$. Then $f(a+h) = f(a) + f(h) + 0$. Thus $f'(a) = f$.
- Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$. This is a special case of the previous example. $f'(a) = \text{id}$.

Even though we are developing calculus on $\mathbb{R}^n$, it is trivially extended to all finite-dimensional normed linear spaces over $\mathbb{R}$ via the natural identification with $\mathbb{R}^n$.