PROBLEMS IN PROBABILITY THEORY

20.08.2024

Instruction: The homework is due on 11:30 am, 27.08.2024. You can either submit a hard copy in the beginning of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. Suppose μ_n, μ are discrete probability measures supported on $\mathbb{Z}$ having probability mass functions $(p_n(k))_{k\in\mathbb{Z}}$ and $(p(k))_{k\in\mathbb{Z}}$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $p_n(k) \to p(k)$ for each $k \in \mathbb{Z}$.

Problem 2. For what $A \subseteq \mathbb{R}$ and $B \subseteq (0,\infty)$ is the restricted family $\{N(\mu,\sigma^2) : \mu \in A \text{ and } \sigma^2 \in B\}$ tight?

Problem 3 (Lévy metric). Let $\mathcal{P}([-1,1]) \subseteq \mathcal{P}(\mathbb{R})$ be the set of all Borel probability measures μ such that $\mu([-1,1]) = 1$. For $\varepsilon > 0$, find a finite ε-net for $\mathcal{P}([-1,1])$. [*Note:* Recall that an ε-net means a subset such that every element of $\mathcal{P}([-1,1])$ is within ε distance of some element of the subset. Since $\mathcal{P}([-1,1])$ is compact, we know that a finite ε-net exists for all $\varepsilon > 0$.]

Problem 4. Fix $\mu \in \mathcal{P}(\mathbb{R})$. For $s \in \mathbb{R}$ and $r > 0$, let $\mu_{r,s} \in \mathcal{P}(\mathbb{R})$ be defined as $\mu_{r,s}(A) = \mu(rA+s)$ where $rA+s = \{rx+s : x \in A\}$. For which $R \subseteq (0,\infty)$ and $S \subseteq \mathbb{R}$ is it true that $\{\mu_{r,s} : r \in R,\ s \in S\}$ a tight family? [*Remark:* If not clear, just take μ to be the Lebesgue measure on $[0,1]$.]

Problem 5.
(1) Show that the family of exponential distributions $\{\text{Exp}(\lambda) : \lambda > 0\}$ is not tight.
(2) For what $A \subseteq \mathbb{R}$ is the restricted family $\{\text{Exp}(\lambda) : \lambda \in A\}$ tight?

Problem 6. Suppose $\mu_n, \mu \in \mathcal{P}(\mathbb{R})$ and that F_μ is continuous. If $\mu_n \xrightarrow{d} \mu$, show that $F_{\mu_n}(t) - F_\mu(t) \to 0$ uniformly over $t \in \mathbb{R}$. [*Restatement:* When F_μ is continuous, convergence to μ in Lévy metric also implies convergence in Kolmogorov-Smirnov metric.]

Problem 7. Let F be a CDF on $\mathbb{R}$. If $\sum_{x\in\mathbb{R}}(F(x) - F(x-))^2 = 1$, show that the measure is supported on a single point.

Problem 8. Let μ be a probability measure on $\mathbb{R}$. If $\mu(a,b) \leq C(b-a)$ for all $a < b$ for some $C < \infty$, show that μ is absolutely continuous and has a density that is bounded by C.

Problem 9. Let $\mu \in \mathcal{P}(\mathbb{R})$.

(1) For any $n \geq 1$, define a new probability measure by $\mu_n(A) = \mu(n.A)$ where $n.A = \{nx : x \in A\}$. Does μ_n converge as $n \to \infty$?

(2) Let μ_n be defined by its CDF

$$F_n(t) = \begin{cases} 0 & \text{if } t < -n, \\ F(t) & \text{if } -n \leq t < n, \\ 1 & \text{if } t \geq n. \end{cases}$$

Does μ_n converge as $n \to \infty$?

(3) In each of the cases, describe μ_n in terms of random variables. That is, if X has distribution μ, describe a transformation $T_n(X)$ that has the distribution μ_n.