Homework 10

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Problem 1. Let $X_1, X_2, \dots$ be i.i.d from μ. For each n, define the random probability measure $\mu_n = \frac{1}{n}(\delta_{X_1} + \cdots + \delta_{X_n})$. If F_n, F are the cumulative distribution functions of μ_n and μ, show that for any $x \in \mathbb{R}$, we have $F_n(x) \xrightarrow{\text{a.s.}} F(x)$.

Solution. Fix $x \in \mathbb{R}$. Then $F_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_i \le x\}}$. $\mathbf{P}\{X_i \le x\} = F(x)$ and X_i's are i.i.d, so $\mathbf{1}_{\{X_i \le x\}}$ are i.i.d $\mathrm{Ber}(F(x))$ random variables. By the strong law, $F_n(x) \xrightarrow{\text{a.s.}} \mathbf{E}[\mathbf{1}_{\{X_1 \le x\}}] = F(x)$. ∎

Problem 2. Let X_n be a sequence of random variables with zero means, unit variances. Assume that $|\mathrm{Cov}(X_n, X_m)| \le \delta(|n-m|)$ where $\delta(k) \to 0$ as $k \to \infty$. Show that $\frac{1}{n}S_n \xrightarrow{\mathsf{P}} 0$.

Solution. Compute the variance of S_n.

$$\begin{aligned}
\mathrm{Var}(S_n) &= \sum_{i=1}^n \mathrm{Var}(X_i) + \sum_{i \ne j} \mathrm{Cov}(X_i, X_j) \\
&\le n + 2[(n-1)\delta(1) + (n-2)\delta(2) + \cdots + \delta(n-1)] \\
\implies \mathrm{Var}\left(\frac{1}{n}S_n\right) &\le \frac{1}{n} + \frac{2}{n}\sum_{k=1}^{n-1} \delta(k).
\end{aligned}$$

Claim. *Let $T_n = \sum_{k=1}^n \delta(k)$. Then $\frac{1}{n}T_n \to 0$ as $n \to \infty$.*

Proof of Claim. Let $\varepsilon > 0$, and choose N such that $\delta(n) < \frac{\varepsilon}{2}$ for all $n \geq N$. Then for $n \geq N$,

$$\frac{1}{n}T_n = \frac{1}{n}T_N + \frac{1}{n}\sum_{k=N+1}^{n} \delta(k) \leq \frac{1}{n}T_N + \frac{\varepsilon}{2}.$$

For large enough n, we have $\frac{1}{n}T_N < \frac{\varepsilon}{2}$, so $\frac{1}{n}T_n < \varepsilon$. □

Thus $\mathrm{Var}(\frac{1}{n}S_n) \to 0$ as $n \to \infty$. By Chebyshev's inequality, we have $\frac{1}{n}S_n \xrightarrow{\mathsf{P}} 0$. ■