MA 361 HOMEWORK-1

06.08.2024

Instruction: The homework is due on 11:30 am, 13.08.2024. You can either submit a hard copy in the begining of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading.

Problem 1. Let X be an arbitrary set. Let $\mathcal{F} := \{A \subseteq X : A \text{ is countable or } A^c \text{ is countable}\}$. Prove that $\mathcal{F}$ is a sigma-algebra. Let S be the collection of all singletons in X. Prove that $\sigma(S) = \mathcal{F}$.

Problem 2. On $[0,1]$, let $\mathcal{A}$ be the algebra generated by finite unions of left-open, right-closed intervals and let $\mathcal{B}$ be the Borel sigma-algebra. Define $\mu : \mathcal{A} \to [0,1]$ by $\mu(A) = 1$ if $A \supseteq (0,\varepsilon)$ for some $\varepsilon > 0$ and $\mu(A) = 0$ otherwise.

(1) Show that μ is a finitely additive measure on $\mathcal{A}$.

(2) Show that μ can not be extended to a measure on $\mathcal{B}$.

(3) Why does this not contradict the Carathèodory extension theorem?

Problem 3. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω.

(1) Show that $\mathcal{F}$ is closed under countable intersections ($\bigcap_n A_n$), under set differences ($A \setminus B$), under symmetric differences ($A \Delta B$).

(2) If A_n is a countable sequence of subsets of Ω, the set $\limsup A_n$ (respectively $\liminf A_n$) is defined as the set of all $\omega \in \Omega$ that belongs to infinitely many (respectively, all but finitely many) of the sets A_n.

If $A_n \in \mathcal{F}$ for all n, show that $\limsup A_n \in \mathcal{F}$ and $\liminf A_n \in \mathcal{F}$. [**Hint:** First express $\limsup A_n$ and $\liminf A_n$ in terms of A_ns and basic set operations].

(3) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, what are $\limsup A_n$ and $\liminf A_n$?

Problem 4. Let $(\Omega, \mathcal{F})$ be a set with a σ-algebra.

(1) Suppose $\mathbf{P}$ is a probability measure on $\mathcal{F}$. If $A_n \in \mathcal{F}$ and A_n increase to A (respectively, decrease to A), show that $\mathbf{P}(A_n)$ increases to (respectively, decreases to) $\mathbf{P}(A)$.

(2) Suppose $\mathbf{P} : \mathcal{F} \to [0,1]$ is a function such that (a) $\mathbf{P}(\Omega) = 1$, (b) $\mathbf{P}$ is finitely additive, (c) if $A_n, A \in \mathcal{F}$ and A_ns increase to A, then $\mathbf{P}(A_n) \uparrow \mathbf{P}(A)$. Then, show that $\mathbf{P}$ is a probability measure on $\mathcal{F}$.

Problem 5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbf{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

Problem 6. Let $\mathcal{F}$ be a sigma-algebra on $\mathbb{N}$ that is strictly smaller than the power set. Show that there exist $m \neq n$ such that elements of $\mathcal{F}$ do not separate m and n (i.e., the following holds: any $A \in \mathcal{F}$ either contains both m, n or neither). Is the same conclusion valid if $\mathbb{N}$ is replaced by any set Ω?

Problem 7. Let X be an arbitrary set.

(1) Suppose S is a collection of subsets of X and a, b are two elements of X such that any set in S either contains a and b both, or contains neither. Let $\mathcal{F} = \sigma(S)$. Show that any set in $\mathcal{F}$ has the same property (either contains both a and b or contains neither).

(2) Let $S = \{(a, b] \cup [-b, -a) : a < b \text{ are real numbers}\}$. Show that $\sigma(S)$ is strictly smaller than the Borel σ-algebra of $\mathbb{R}$.

Problem 8. Show that each of the following collection of subsets of $\mathbb{R}$ generate the same sigma-algebra (which we call the Borel sigma-algebra).

(1) $\{[a, b] : a \leq b \text{ and } a, b \in \mathbf{Q}\}$.

(2) The collection of all compact sets.

Problem 9. Let Ω be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega : A \text{ is finite or } A^c \text{ is finite }\}$. Define $\mu : \mathcal{A} \to \mathbb{R}_+$ by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite.

(1) Show that $\mathcal{A}$ is an algebra and that μ is finitely additive on $\mathcal{A}$.

(2) Show that if Ω is countable μ does not extend to a measure on $\mathcal{F} = \sigma(\mathcal{A})$.

(3) Under what conditions does μ extend to a probability measure on $\mathcal{F}$?

Problem 10. On $\mathbb{N} = \{1, 2, \ldots\}$, let A_p denote the subset of numbers divisible by p. Describe $\sigma(\{A_p : p \text{ is prime}\})$ as explicitly as possible.