MA 361 HOMEWORK 7

25.09.2024

Instruction: The homework is due on 11:30 am, 01.10.2024. You can either submit a hard copy in the beginning of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. Let μ and ν be Borel probability measures on $\mathbb{R}$. Suppose there exists a probability measure θ on $\mathbb{R}^2$ having marginals $\theta \circ \Pi_1^{-1} = \mu$ and $\theta \circ \Pi_2^{-1} = \nu$ such that $\theta\{(x,x) : x \in \mathbb{R}\} > 0$. Then show that μ and ν cannot be singular.

[In the language of random variables, the hypothesis says that we can couple $X \sim \mu$ and $Y \sim \nu$ such that $X = Y$ with positive probability.]

Problem 2. Place r_m balls in m bins at random and count the number of empty bins Z_m. Fix $\delta > 0$. If $r_m > (1+\delta)m \log m$, show that $\mathbf{P}(Z_m > 0) \to 0$ while if $r_m < (1-\delta)m \log m$, show that $\mathbf{P}(Z_m > 0) \to 1$.

Problem 3. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ and let $\theta = \frac{1}{2}\mu + \frac{1}{2}\nu$.

(1) Show that $\mu \ll \theta$ and $\nu \ll \theta$.

(2) If $\mu \perp \nu$, describe the Radon Nikodym derivative of μ w.r.t. θ.

Problem 4. (Requires product measure) Decide true or false and justify. Take μ_i, ν_i to be probability measures on $(\Omega_i, \mathcal{F}_i)$.

(1) If $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$, then $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$.

(2) If $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$, then $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$.

Problem 5. (1) If $\mu_n \ll \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \ll \nu$? If $\mu_n \perp \nu$ for each n and $\mu_n \xrightarrow{d} \mu$, then is it necessarily true that $\mu \perp \nu$? In either case, justify or give a counterexample.

(2) Suppose X, Y are independent (real-valued) random variables with distribution μ and ν respectively. If μ and ν are absolutely continuous w.r.t Lebesgue measure, show that the distribution of $X + Y$ is also absolutely continuous w.r.t Lebesgue measure.

Problem 6. In each case, decide if $\mu \ll \nu$ and if so, compute the Radon-Nikodym derivative.

(1) $\mu = \text{Bin}(n, p)$ and $\nu = \text{Bin}(n', p')$.

(2) $\mu = \text{Pois}(\lambda)$ and $\nu = \text{Pois}(\lambda')$.

(3) $\mu = N(\mu, \sigma^2)$ and $\nu = N(0, 1)$.

(4) $\mu = \text{Exp}(1)$ and $\nu = N(0, 1)$.

Problem 7. (Chung-Erdös inequality). Let A_i be events in a probability space. Show that

$$\mathbf{P}\left\{\bigcup_{k=1}^{n} A_k\right\} \geq \frac{(\sum_{k=1}^{n} \mathbf{P}(A_k))^2}{\sum_{k,\ell=1}^{n} \mathbf{P}(A_k \cap A_\ell)}.$$

Problem 8. Suppose each of $r = \lambda n$ balls are put into n boxes at random (more than one ball can go into a box). If N_n denotes the number of empty boxes, show that for any $\delta > 0$, as $n \to \infty$,

$$\mathbf{P}\left(\left|\frac{N_n}{n} - e^{-\lambda}\right| > \delta\right) \to 0$$

Problem 9. Let $A_1, A_2, \ldots$ be i.i.d. uniform random subsets of $[n]$ (i.e., $\mathbf{P}(A_1 = S) = 2^{-n}$ for each $S \subseteq [n]$). Imagine sampling $A_1, A_2, \ldots$ suvvessively and let T_n be the first time when we have two subsets that are disjoint from each other. Show that

$$\mathbf{P}\left\{T_n \geq \left(\frac{2}{\sqrt{3}}\right)^n h_n\right\} \to 1 \text{ if } h_n \to 0.$$

[Hint: Use the random variable $\sum_{A_i, A_j, i<j} \mathbf{1}_{\{A_i \cap A_j = \emptyset\}}$.]

Problem 10. Let $X_1, X_2, \ldots$ be i.i.d. fair coin tosses. Let L_n be the length of the longest run of heads in $X_1, \ldots, X_n$ (a run is a segment of consecutive tosses). Show that for any $\varepsilon > 0$,

$$\mathbf{P}\{(1 - \varepsilon)\log_2 n \leq L_n \leq (1 + \varepsilon)\log_2 n\} \to 1.$$