MA 231: Topology

Naman Mishra

August 2024

Contents

The course

Course website

Lecture 1.
Friday
August 02

Timings

Lectures: MWF 3–4 pm
Tutorial: Tue 5–6 pm

Grading

- Quizzes:
- Midterm:
- Final:

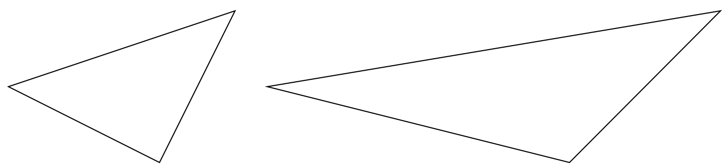
Chapter 1

Introduction

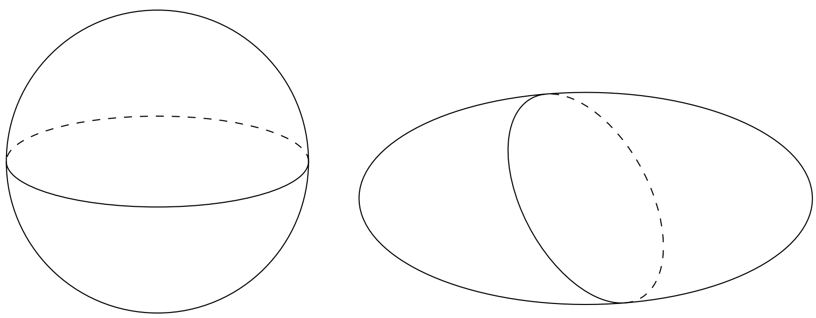
Many objects in mathematics can be described as either *numbers* (arithmetic, algebra) or *shapes* (geometry).

Topology is geometry made “flexible”. How so? Topology allows for deformations, that retain a notion of “proximity” between points.

The following two triangles are identical for the topologist.

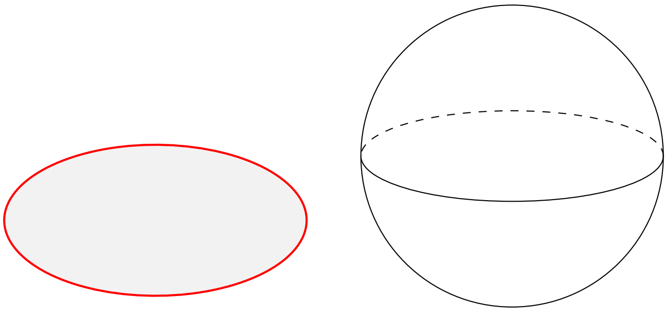


So are a sphere and an ellipsoid.



Why is this useful?

Often, we only need to understand shapes qualitatively. For example, a disk and a sphere look somewhat similar. However, a qualitative difference is that the disk has a boundary, while the sphere does not.



Dynamical systems

Shapes arise while modelling physical systems. A robotic arm may be described by some angles, and so might the solar system. Two friends playing hide-and-seek might be described by their positions.

Such parameters often lend themselves to a simple geometric description in a *phase space* or *configuration space*.

Poincaré discussed the stability of the solar system around 1890 by studying the topology of the phase space.

Knot theory

Whether a knot can be untied without cutting is a topological question. A qualitative understanding of the knot is sufficient.

Graphs

1.1 What can you expect?

The course will cover point-set topology for more that half the semester. Problems will bring out logical thinking, and sometimes your visual imagination will be of great use.

Later, we will cover algebraic invariants, which form a bridge between numbers and shapes.

Lecture 1

Chapter 2

Set theory

Definition 2.1 (Maps)**.** A *map* or a *function* between two sets A and B assigns to every element of A a unique element of B.

$$\begin{aligned} f\colon A &\to B \\ a &\mapsto f(a) \end{aligned}$$

An *injective* or *one-to-one* map is one where distinct elements of A are mapped to distinct elements of B.

$$f(x) = f(y) \implies x = y$$

A *surjective* or *onto* map is one where for every $b \in B$ there exists an $a \in A$ that maps to b.

$$f(A) = B$$

A *bijective* map is one that is both injective and surjective.

Proposition 2.2. *There is a bijection between $\mathbb{R}$ and $\mathbb{R}^2$.*

Proof. We will use theorem 2.3. $f = x \mapsto (x, 0)$ is an injective map from $\mathbb{R}$ to $\mathbb{R}^2$. For finding an injection, $g\colon \mathbb{R}^2 \to \mathbb{R}$, we will go via the map $\widehat{g}\colon (0,1)^2 \to (0,1)$.

This is sufficient because arctan provides a nice bijection between $(0,1)$ and $\mathbb{R}$ (and hence $(0,1)^2$ and $\mathbb{R}^2$).

Given $(x, y) \in (0,1)^2$, let $0.x_1x_2x_3\ldots$ and $0.y_1y_2y_3\ldots$ be the binary expansions of x and y respectively. Interleave the bits to get $0.x_1y_1x_2y_2x_3y_3\ldots$. (If we declare the expansions to be non-terminating, they are unique.) This is a bijection. ■

Theorem 2.3 (Schröder-Bernstein)**.** *If A and B are sets such that there is an injective map $f\colon A \to B$ and a surjective map $g\colon A \to B$, then there is a bijection $h\colon A \leftrightarrow B$.*

Following McMullen's notes. WLOG assume A and B are disjoint. Let $F = f \cup g \colon A \cup B \to B \cup A$ be the union of the two maps (as sets). That is,

$$F(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

For any x, we say that x is the *parent* of $F(x)$, which is the *child*.

Now note the following.

(i) Each element of $A \cup B$ is a parent and has a single child.

(ii) There may be elements in $A \cup B$ that are not children. Call them *godfathers*.

(iii) Each child has a unique parent.

Consider any element $x \in A \cup B$. We have two possibilities.

(i) $\cdots \mapsto x'' \mapsto x' \mapsto x$.

(ii) $y \mapsto \cdots \mapsto x' \mapsto x$, where y is a godfather.

We divide A into three mutually disjoint sets, A_0, A_A and A_B.

$$\begin{aligned} A_0 &= \{x \in A \mid \text{possibility (i) happens}\}, \\ A_A &= \{x \in A \mid \text{possibility (ii) happens and } y \in A\}, \\ A_B &= \{x \in A \mid \text{possibility (ii) happens and } y \in B\}. \end{aligned}$$

Similarly, define B_0, B_A and B_B.

Now $F|_{A_0} : A_0 \to B_0$ is a bijection, $F|_{A_A} : A_A \to B_A$ is a bijection, and $F|_{B_B} : B_B \to A_B$ is a bijection. Flipping the last bijection gives a bijection $F|_{B_B}^{-1} : A_B \to B_B$.

Now we can define a bijection $h \colon A \to B$ as $h = F|_{A_0} \cup F|_{A_A} \cup F|_{B_B}^{-1}$. ■