Homework 2

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Problem 1.

(1) Let Ω be a set and $A \subseteq \Omega$. Define a function $\mathbf{1}_A \colon \Omega \to \mathbb{R}$ as follows.

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

What is the smallest σ-algebra on Ω with respect to which $\mathbf{1}_A$ becomes a random variable?

(2) Assume that $A \in \mathcal{F}$. Give an explicit description of the push-forward measure $\mathbf{P} \circ (\mathbf{1}_A)^{-1}$ on $\mathbb{R}$.

(3) Define $T \colon \Omega \to \mathbb{R}^n$ by $T(\omega) = (\mathbf{1}_{A_1}(\omega), \ldots, \mathbf{1}_{A_n}(\omega))$ where $A_1, \ldots, A_n$ are given subsets of Ω. What is the smallest σ-algebra on Ω such that T becomes a random variable?

(4) Assume $A_1, A_2, \ldots, A_k \in \mathcal{F}$. Describe the push-forward measure $\mathbf{P} \circ T^{-1}$ on $\mathbb{R}^n$.

Solution.

(1) We need $\mathbf{1}_A^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{B}(\mathbb{R})$.

$$(\mathbf{1}_A)^{-1}(B) = \begin{cases} \varnothing & \text{if } 0, 1 \notin B, \\ A & \text{if } 1 \in B, 0 \notin B, \\ A^c & \text{if } 0 \in B, 1 \notin B, \\ \Omega & \text{if } 0, 1 \in B. \end{cases}$$

Thus $\mathcal{F}$ must contain $\varnothing, A, A^c, \Omega$. $\mathcal{F} = \{\varnothing, A, A^c, \Omega\}$ is itself a σ-algebra, hence the smallest one that works. This is in fact the σ-algebra generated by $\{A\}$.

(2) Let $B \in \mathcal{B}(\mathbb{R})$. Then

$$(\mathbf{P} \circ (\mathbf{1}_A)^{-1})(B) = \begin{cases} 0 & \text{if } 0, 1 \notin B, \\ \mathbf{P}(A) & \text{if } 1 \in B, 0 \notin B, \\ \mathbf{P}(A^c) & \text{if } 0 \in B, 1 \notin B, \\ 1 & \text{if } 0, 1 \in B. \end{cases} \qquad \blacksquare$$

(3) Since A_i is the inverse image of

$$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R} \times}_{i-1} \{1\} \underbrace{\times \mathbb{R} \times \cdots \times \mathbb{R}}_{k-i-1},$$

we have that $\mathcal{F}$ contains all A_i. Since $\mathcal{F}$ is a σ-algebra, it must contain all countable unions and intersections of the A_i. Let $\mathcal{F} = \sigma\{A_i\}$.

$T(\omega) \in \{0, 1\}^k$ for all ω. Let $A^{(1)} = A$ and $A^{(0)} = A^c$ for any event A. Then

$$T^{-1}(B) = \bigcup_{v \in B \cap 2^k} \bigcap_{i=1}^{n} A_i^{(v_i)} \in \mathcal{F}$$

Thus the preimage of any Borel set is in $\mathcal{F}$.

(4) For any $B \subseteq \mathbb{R}^n$,

$$T^{-1}(B) = \bigcup_{v \in B \cap 2^k} \bigcap_{i=1}^{k} A^{(v_i)}.$$

This is a disjoint union of sets in $\mathcal{F}$, so

$$(\mathbf{P} \circ T^{-1})(B) = \sum_{v \in B \cap 2^k} \mathbf{P}\left(\bigcap_{i=1}^{k} A^{(v_i)}\right).$$

Problem 2. Recall the Lévy metric d defined in class. Show the following.

(1) Let a_n be a sequence of real numbers converging to a. For any $x \in \mathbb{R}$, δ_x is the measure defined as follows: for $A \subseteq \mathbb{R}$,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Using the definition of the metric show that

$$d(\delta_{a_n}, \delta_a) \to 0 \text{ as } n \to \infty.$$

(2) Consider the sequence of measures $\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{i/n}$ and μ is the uniform measure on $[0, 1]$. Using the definition show that

$$d(\mu_n, \mu) \to 0 \text{ as } n \to \infty.$$

Solution.

(1) The CDF of δ_x is $F_x(\omega) = [\omega \geq x]$. If $|x - y| = \varepsilon$, then

$$F_x(\omega + \varepsilon) = [\omega + \varepsilon \geq x] = [\omega \geq x - \varepsilon] \geq [\omega \geq y] = F_y(\omega)$$

since $\omega \geq y \implies \omega \geq x - |x - y|$. Thus $d(\delta_x, \delta_y) \leq |x - y|$. As $a_n \to a$, $d(\delta_{a_n}, \delta_a) \to 0$.

(2) The CDF of μ is $F(x) = x$ for $x \in [0, 1]$. The CDF of μ_n is

$$F_n(x) = \frac{\lfloor nx \rfloor}{n} \quad \text{for } x \in [0, 1].$$

($\lfloor nx \rfloor$ counts the number of points $i/n \leq x$, and each of those has weight $1/n$.) We claim that $d(\mu_n, \mu) \leq 1/n$.

Let $x \in [0, 1]$. Then

$$\begin{aligned} F\left(x + \frac{1}{n}\right) + \frac{1}{n} &= x + \frac{2}{n} \\ &= \frac{nx + 2}{n} \\ &> \frac{\lfloor nx \rfloor}{n} = F_n(x). \end{aligned}$$

and

$$\begin{aligned} F_n\left(x + \frac{1}{n}\right) + \frac{1}{n} &= \frac{\lfloor n(x + 1/n) \rfloor + 1}{n} \\ &= \frac{\lfloor nx \rfloor + 2}{n} \\ &> \frac{nx}{n} \\ &= x = F(x). \end{aligned}$$

Thus

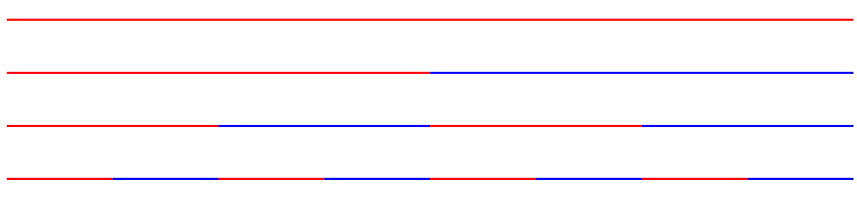
$$\begin{aligned} \frac{1}{n} \in \{\varepsilon > 0 : F_n(x+\varepsilon) + \varepsilon \geq F(x) \text{ and } \\ F(x+\varepsilon) + \varepsilon \geq F_n(x) \text{ for all } x \in [0,1]\} \end{aligned}$$

and so $d(\mu_n, \mu)$, which is the infimum of all such ε, is at most $1/n$. It follows that $\lim_{n\to\infty} d(\mu_n, \mu) = 0$ by the squeeze theorem. ∎

Problem 3. For $k \geq 0$, define the functions $r_k \colon [0,1) \to \mathbb{R}$ by writing $[0,1) = \bigsqcup_{0 \leq j < 2^k} I_j^{(k)}$ where $I_j^{(k)}$ is the dyadic interval $[j2^{-k}, (j+1)2^{-k})$ and setting

$$r_k(x) = \begin{cases} -1 & \text{if } x \in I_j^{(k)} \text{ for odd } j, \\ 1 & \text{if } x \in I_j^{(k)} \text{ for even } j. \end{cases}$$

Fix $n \geq 1$ and define $T_n \colon [0,1) \to \{-1,1\}^n$ by $T_n(x) = (r_0(x), \ldots, r_{n-1}(x))$. Find the push-forward of the Lebesgue measure on $[0,1)$ under T_n.



Solution. ∎

Problem 4. If $T \colon \mathbb{R} \to \mathbb{R}$, show that T is Borel measurable if it is (1) continuous or (2) increasing.

Solution.

(1) The preimage of any open set is open.

(2) For any $x \in \mathbb{R}$, $T^{-1}(-\infty, x]$ is the set of all $y \in \mathbb{R}$ such that $T(y) \leq x$. Since T is increasing, $T(y) \leq x \implies T(y') \leq x$ for all $y' \leq y$. Thus $T^{-1}(-\infty, x] = (-\infty, y_0]$ or $(-\infty, y_0)$ for some $y_0 \in \mathbb{R}$. ■

Problem 5 (Change of variable for densities)**.** Let μ be a probability measure on $\mathbb{R}$ with density f, by which we mean that its CDF $F_\mu(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ (you may assume that f is continuous, non-negative and the Riemann integral $\int_{\mathbb{R}} f = 1$). Then, find the (density of the) push-forward measure of μ under

(1) $T(x) = x + a$,

(2) $T(x) = bx$,

(3) T is any increasing and differentiable function.

Solution.

(1) The CDF of the push-forward measure is

$$\begin{aligned} F_{\mu T^{-1}}(x) &= \mu(T^{-1}(-\infty, x]) \\ &= \mu(-\infty, x - a] \\ &= F_\mu(x - a) \\ &= \int_{-\infty}^{x-a} f(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{x} f(t - a)\,\mathrm{d}t. \end{aligned}$$

Thus the density of the push-forward measure is $x \mapsto f(x - a)$.

(2)

$$\begin{aligned}
F_{\mu T^{-1}}(x) &= \mu(T^{-1}(-\infty, x]) \\
&= \mu(-\infty, x/b] \\
&= F_\mu(x/b) \\
&= \int_{-\infty}^{x/b} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{x} \frac{1}{b} f(t/b)\,\mathrm{d}t.
\end{aligned}$$

The density is $x \mapsto \frac{1}{b} f(x/b)$.

(3) Let $U(y) = \sup\{x \mid T(x) \leq y\}$. Since T is continuous and increasing, we have that if $U(y)$ is finite, then $T(U(y)) = x$. We note that

$$T(x) \leq y \iff x \leq U(y),$$

so that

$$\mu T^{-1}(-\infty, y] = \mu(-\infty, U(y)].$$

If T is strictly increasing, then $U(y)$ is continuous and the true inverse of T. If further $T' > 0$, then U is differentiable, and

$$\begin{aligned}
\mu T^{-1}(-\infty, y] &= \mu(-\infty, U(y)] \\
&= F_\mu(U(y)) \\
&= \int_{-\infty}^{U(y)} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{y} f(U(y))U'(y)\,\mathrm{d}y.
\end{aligned}$$

The density of the push-forward measure is $f \circ T^{-1} \cdot (T^{-1})'$.

$$f_{\mu T^{-1}}(y) = f(T^{-1}(y)) \cdot \frac{1}{T'(T^{-1}(y))}$$

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