MA 361 HOMEWORK 10

17.10.2024

Instruction: The homework is due on 11:30 am, 24.10.2024. You can either submit a hard copy in the beginning of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. Let $X_1, X_2, \dots$ be i.i.d from μ. For each n, define the random probability measure $\mu_n = \frac{1}{n}(\delta_{X_1} + \dots + \delta_{X_n})$. If F_n, F are the cumulative distribution functions of μ_n and μ, show that for any $x \in \mathbb{R}$, we have $F_n(x) \stackrel{a.s.}{\to} F(x)$.

Problem 2. Let X_n be a sequence of random variables with zero means, unit variances. Assume that $|\mathrm{Cov}(X_n, X_m)| \leq \delta(|n-m|)$ where $\delta(k) \to 0$ as $k \to \infty$. Show that $\frac{1}{n}S_n \stackrel{P}{\to} 0$.

Problem 3. Show that the the following are equivalent conditions for uniform integrability of a sequence $\{X_n\}$.

(1) $c_n X_n \stackrel{L^1}{\to} 0$ whenever $c_n \to 0$.

(2) There exists M such that $\mathbf{E}|X_n| \leq M$ for all n and the following holds. For any $\varepsilon > 0$, there exists $\delta > 0$ such that any measurable set A with $\mathbf{P}(A) < \delta$ satisfies $\int_A X_n dP < \varepsilon$ for all n.

Problem 4. Let $\{X_i\}_{i \in I}$ be a family of r.v on $(\Omega, \mathcal{F}, \mathbf{P})$. If $\{X_i\}_{i \in I}$ is uniformly integrable, then show that $\sup_i \mathbf{E}|X_i| < \infty$. Give a counterexample to the converse statement.

Problem 5. Let $X_i \sim \mu$ be i.i.d. If S_n converge a.s., show that $\mu = \delta_0$.

Problem 6. Let $X_1, X_2, \dots$ be i.i.d. random variables with finite expectation m. Show that

$$\mathbf{E}\left[\left| \frac{S_n}{n} - m \right| \right] \to 0.$$

Problem 7. Let $X_1, X_2, \dots$ be i.i.d. non negative random variables. Let G_n be the geometric mean of $X_1, \dots, X_n$. In each of the following cases, show that G_n converges almost surely to a constant and find the constant. (a) $X_1 \sim \mathrm{Unif}[0,1]$, (b) $X_1 \sim \mathrm{Exp}(1)$.

Problem 8. Suppose $0 \leq X_1 \leq X_2 \leq \ldots$. Assume that $\frac{\mathbf{E}[X_n]}{n^\alpha} \to A$ and $\mathrm{Var}(X_n) \leq Bn^{2\beta}$ for some $0 < A, B < \infty$ and $0 < \beta < \alpha < \infty$. Show that $\frac{X_n}{n^\alpha} \stackrel{a.s.}{\to} A$.

Problem 9. Let X_k be independent random variables with $\mathbf{P}\{X_k = k^a\} = \mathbf{P}\{X_k = -k^a\} = \frac{1}{2}$ for some $a \geq 0$. Show that $\frac{S_n}{n} \stackrel{P}{\to} 0$ if and only if $a < \frac{1}{2}$.