# Assignment 1

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**Problem 1.** *Let $n \geq 3$. Let $r, s$ be the usual generators of $D_{2n}$. Prove that the map $\varphi\colon D_{2n} \to \mathrm{GL}_2(\mathbb{R})$ defined by*

$$\varphi(r) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \varphi(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*where $\theta = \frac{2\pi}{n}$ extends to an injective homomorphism.*

**Problem 2.** *Show that $(\mathbb{Z}, +) \ncong (\mathbb{Q}, +)$, $S_m \ncong S_n$ for $m \neq n$ and $(\mathbb{R}, +) \ncong (\mathbb{Q}, +)$.*

*Solution.* Let $\varphi\colon (\mathbb{Z}, +) \to (\mathbb{Q}, +)$ be a homomorphism. Let $\varphi(1) = a$. Then $\varphi(n) = na$ for $n \in \mathbb{N}$. $\varphi$ cannot be surjective, as no integer multiple of $a$ can equal $a/2$.

For $m \neq n$, $S_m$ and $S_n$ have different cardinalities. So do $\mathbb{R}$ and $\mathbb{Q}$. ■

**Problem 3.** *For a commutative ring $A$, let*

$$A^\times = \{a \in A \mid ab = 1 \text{ for some } b \in A\}$$

*be the unit group of $A$. Show that $A^\times$ is a group (under multiplication).*

*Are $\mathbb{Z}[x]^\times$ and $\mathbb{Q}[x]^\times$ isomorphic? Are $(\mathbb{Z}[x], +)$ and $(\mathbb{Q}[x], +)$ isomorphic?*

*Solution.* $1 \in A^\times$ is the identity. If $a_1, a_2 \in A^\times$, then $a_1 a_2 \in A^\times$, since there exist $b_1, b_2 \in A$ such that $(a_1 a_2)(b_2 b_1) = a_1 b_1 = 1$. Associativity is borrowed from $A$. Inverses exist by definition.

$\mathbb{Z}[x]^\times = \{1, -1\}$, while $\mathbb{Q}[x]^\times = \mathbb{Q} \setminus \{0\}$. They are not isomorphic.

Assume $\varphi\colon \mathbb{Q}[x] \to \mathbb{Z}[x]$ is an homomorphism. Then $\varphi(p) \neq 0$ for each non-zero polynomial $p$.

Let $\varphi(1) = a_0 + a_1 x + \cdots + a_n x^n$, with $a_n \neq 0$. Now

$$\varphi(1) = \varphi\left(k \cdot \frac{1}{k}\right) = \varphi(k)\varphi\left(\frac{1}{k}\right) = k\varphi(1)\varphi\left(\frac{1}{k}\right)$$

for each $k \in \mathbb{Z}$. Thus $a_n$ is a multiple of $k$ for each $k \in \mathbb{Z}$. Absurd. ■

**Problem 4.** *Prove that a finitely generated subgroup of* $(\mathbb{Q}, +)$ *is cyclic.*

*Solution.* Let $H = \left\langle \frac{m_1}{n_1}, \ldots, \frac{m_k}{n_k} \right\rangle$. Write this as $\left\langle \frac{m_1'}{N}, \ldots, \frac{m_k'}{N} \right\rangle$, where $N = \operatorname{lcm}(n_1, \ldots, n_k)$. Then $H = \sum_{i=1}^k a_i \frac{m_i'}{N}$ for some $a_i \in \mathbb{Z}$. Thus

$$H = \frac{1}{N}\langle m_1', \ldots, m_k' \rangle = \frac{1}{N}\langle \gcd(m_1', \ldots, m_k') \rangle.$$ ■

**Problem 5.** *Show that* $(\mathbb{Z}^m, +) \ncong (\mathbb{Z}^n, +)$ *for* $m \neq n$.

**Problem 6.** *If* $\tau_1 = (a_1\ b_1)$ *and* $\tau_2 = (a_2\ b_2)$ *are transpositions in* $S_n$ *with* $\{a_1, b_1\} \cap \{a_2, b_2\} = \varnothing$, *show that* $\tau_1\tau_2 = \tau_2\tau_1$.

*Solution.* We have

$$\tau_1(x) = \begin{cases} b_1 & \text{if } x = a_1, \\ a_1 & \text{if } x = b_1, \\ x & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tau_2(x) = \begin{cases} b_2 & \text{if } x = a_2, \\ a_2 & \text{if } x = b_2, \\ x & \text{otherwise.} \end{cases}$$

Now

$$\tau_1\tau_2(x) = \begin{cases} b_2 & \text{if } x = a_2, \\ a_2 & \text{if } x = b_2, \\ b_1 & \text{if } x = a_1, \\ a_1 & \text{if } x = b_1, \\ x & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tau_2\tau_1(x) = \begin{cases} b_1 & \text{if } x = a_1, \\ a_1 & \text{if } x = b_1, \\ b_2 & \text{if } x = a_2, \\ a_2 & \text{if } x = b_2, \\ x & \text{otherwise.} \end{cases}$$

Thus $\tau_1\tau_2 = \tau_2\tau_1 = (a_1\ b_1)(a_2\ b_2)$. ■

**Problem 7.** *Let* $X_{2n} = \langle x, y \mid x^n = y^2 = 1, xy = yx^2 \rangle$. *What is* $X_{2n}$?

*Solution.*

$$x^4 = y^2x^4 = y(yx^2)x^2 = yx(yx^2) = (yxx)y = xyy = x.$$

Thus $x^3 = 1$. If $3 \nmid n$, then $x = 1$ and $X_{2n} = \{1, y\} \cong (\mathbb{Z}_2, +)$.

If $3 \mid n$, then $X_{2n} = D_6$, since we can simplify the relations to

$$X_{2n} = \langle x, y \mid x^3 = y^2 = 1, xy = yx^{-1} \rangle.$$

This is the presentation of $D_6$ with $r \mapsto x$ and $s \mapsto y$. ∎

**Problem 8.** *Let* $G = \left\{ \begin{pmatrix} \pm 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{Z}/n\mathbb{Z} \right\}$. *Show that* $G \cong D_{2n}$ $(n \geq 3)$.

*Solution.* Put $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = R^c \qquad \text{and} \qquad \begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix} = SR^c.$$

Note that $R^n = 1$, $S^2 = 1$ and

$$SRS = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = R^{-1}.$$

Thus $\varphi \colon D_{2n} \to G$ defined by

$$\varphi(r) = R \quad \text{and} \quad \varphi(s) = S$$

extends to an isomorphism. ∎

**Problem 9.** *Let* $H_{\mathbb{R}} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$. *Show that* $H_{\mathbb{R}}$ *is a group and that every non-identity element has infinite order.*

*Solution.* Let $X_1 = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}$ be elements of $H_{\mathbb{R}}$. Then

$$X_1 X_2 = \begin{pmatrix} 1 & a_1 + a_2 & b_1 + b_2 + a_1 c_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{pmatrix} \in H_{\mathbb{R}}.$$

Note that the identity matrix is in $H_{\mathbb{R}}$. Associativity is inherited from $M_3(\mathbb{R})$.

$a_2 = -a_1$, $c_2 = -c_1$ and $b_2 = -b_1 - a_1 c_2 = a_1 c_1 - b_1$ gives $X_1 X_2 = I$.

Let $X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$. It is easy to observe that

$$X^n = \begin{pmatrix} 1 & na & * \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}$$

If either of $a$ or $c$ is non-zero, then $X^n$ is not the identity matrix. If $a = c = 0$, then

$$X^n = \begin{pmatrix} 1 & 0 & nb \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $X^n = I$ iff $X = I$. ■

**Problem 10.** *Define the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the multiplication laws $i^2 = j^2 = k^2 = -1$, $ij = (-j)i = k$, $jk = (-k)j = i$ and $ki = (-i)k = j$. Also assume that $-1$ commutes with and flips the sign of each element. Show that $Q_8$ is a group. Show that every subgroup of $Q_8$ is normal, yet it is not abelian.*

*Solution.* I refuse to show it is a group. Let's show that every subgroup is normal. A subgroup $N \leq Q_8$ is normal iff $gng^{-1} \in N$ for all $g \in Q_8$ and $n \in N$.

1 and $-1$ commute with all elements, so we only need to check $n, g \in \{\pm i, \pm j, \pm k\}$.

WLOG let $n = i$. $n$ commutes with $\pm i$, so $gng^{-1} = n \in N$ if $g = \pm i$. $n$ anti-commutes with $\pm j$ and $\pm k$. So if $g \in \{\pm j, \pm k\}$, then $gng^{-1} = -ngg^{-1} = -n = n^{-1} \in N$. ■