Homework 5

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Problem 1. If X is a random variable distributed as $N(0,1)$ then show that

(1) $\mathbf{E}[e^{\theta x}] = \exp(\frac{\theta^2}{2})$.

(2) Use the above to show that for all $t \geq 0$, $\mathbf{P}(X \geq t) \leq \exp(-\frac{t^2}{2})$.

Solution.

(1) Let $k = \frac{1}{\sqrt{2\pi}}$.

$$\begin{aligned}
\mathbf{E}[e^{\theta X}] &= \int_{-\infty}^{\infty} e^{\theta x} k e^{-\frac{x^2}{2}} \,\mathrm{d}x \\
&= e^{\frac{\theta^2}{2}} \int_{-\infty}^{\infty} k e^{-\frac{(x-\theta)^2}{2}} \,\mathrm{d}x \\
&= e^{\frac{\theta^2}{2}} \int_{-\infty}^{\infty} k e^{-\frac{y^2}{2}} \,\mathrm{d}y \\
&= \exp\left(\frac{\theta^2}{2}\right)
\end{aligned}$$

(2)

$$\begin{aligned}
\mathbf{P}(X \geq t) &= \mathbf{E}[\mathbf{1}_{X \geq t}] \\
&= \mathbf{E}\left[\frac{e^{tX}}{e^{tX}} \mathbf{1}_{X \geq t}\right] \\
&\leq \mathbf{E}\left[\frac{e^{tX}}{e^{t^2}} \mathbf{1}_{X \geq t}\right] && (*) \\
&\leq \frac{1}{e^{t^2}} \mathbf{E}\left[e^{tX}\right] \\
&= \frac{1}{e^{t^2}} \exp\left(\frac{t^2}{2}\right) \\
&= \exp\left(-\frac{t^2}{2}\right)
\end{aligned}$$

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($*$) holds since $X \geq t$ on the support.

Problem 2. If $\mathbf{E}|X| < \infty$, then $\mathbf{E}\Big[|X|\mathbf{1}_{\{|X|>A\}}\Big] \to 0$ as $A \to \infty$.

Solution. Consider the sequence of random variables $Y_n = |X|\mathbf{1}_{\{|X|\leq n\}}$. As $n \to \infty$, $Y_n \uparrow |X|$ pointwise. By the monotone convergence theorem, $\mathbf{E}\, Y_n \uparrow \mathbf{E}|X|$. But

$$\mathbf{E}|X| = \mathbf{E}\Big[|X|\mathbf{1}_{\{|X|\leq n\}} + |X|\mathbf{1}_{\{|X|>n\}}\Big] = \mathbf{E}\, Y_n + \mathbf{E}\Big[|X|\mathbf{1}_{\{|X|>n\}}\Big].$$

Taking limits, we get

$$\mathbf{E}\Big[|X|\mathbf{1}_{\{|X|>n\}}\Big] \to 0 \quad \text{as } n \to \infty$$

For any arbitrary A, $\mathbf{E}\Big[|X|\mathbf{1}_{\{|X|>A\}}\Big]$ is bounded above by $\mathbf{E}\Big[|X|\mathbf{1}_{\{|X|>\lfloor A\rfloor\}}\Big]$. Thus the result holds. ■