MA 361 HOMEWORK-2

13.08.2024

Instruction: The homework is due on 11:30 am, 20.08.2024. You can either submit a hard copy in the begining of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. (1) Let Ω be a set and $A \subseteq \Omega$. Define a function $1_A : \Omega \to \mathbb{R}$ as follows.

$$1_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A. \end{cases}$$

What is the smallest σ-algebra on Ω with respect to which 1_A becomes a random variable?

(2) Assume that $A \in \mathcal{F}$. Give an explicit description of the push-forward measure $P \circ (1_A)^{-1}$ on $\mathbb{R}$.

(3) Define $T : \Omega \to \mathbb{R}^n$ by $T(\omega) = (\mathbf{1}_{A_1}(\omega), \ldots, \mathbf{1}_{A_n}(\omega))$ where $A_1, \ldots, A_n$ are given subsets of Ω. What is the smallest σ-algebra on Ω such that T becomes a random variable?

(4) Assume $A_1, A_2, \cdots A_k \in \mathcal{F}$. Describe the push-forward measure $P \circ T^{-1}$ on $\mathbb{R}^n$.

Problem 2. Recall the Lévy metric d defined in the class. Show the followings.

(1) Let a_n be a sequence of real numbers converging to a. For any $x \in R$, δ_x is a measure defined as follows: for $A \subseteq \mathbb{R}$

$$\delta_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Using the definition of the metric show that

$$d(\delta_{a_n}, \delta_a) \to 0 \text{ as } n \to \infty.$$

(2) Consider the sequence of measures $\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{i/n}$ (this means that μ_n are discrete probability measures on R that put weight $1/n$ to each of the points i/n and put 0 weight everywhere else) and μ is the uniform measure on $[0, 1]$. Using the definition show that

$$d(\mu_n, \mu) \to 0 \text{ as } n \to \infty.$$

Problem 3. For $k \geq 0$, define the functions $r_k : [0,1) \to \mathbb{R}$ by writing $[0,1) = \bigsqcup_{0 \leq j < 2^k} I_j^{(k)}$ where $I_j^{(k)}$ is the dyadic interval $[j2^{-k}, (j+1)2^{-k})$ and setting

$$r_k(x) = \begin{cases} -1 & \text{if } x \in I_j^{(k)} \text{ for odd } j, \\ +1 & \text{if } x \in I_j^{(k)} \text{ for even } j. \end{cases}$$

Fix $n \geq 1$ and define $T_n : [0,1) \to \{-1,1\}^n$ by $T_n(x) = (r_0(x), \ldots, r_{n-1}(x))$. Find the push-forward of the Lebesgue measure on $[0,1)$ under T_n

Problem 4. If $T : \mathbb{R} \to \mathbb{R}$, show that T is Borel measurable if it is (1) continuous or (2) increasing.

Problem 5 (Change of variable for densities). (1) Let μ be a p.m. on $\mathbb{R}$ with density f by which we mean that its CDF $F_\mu(x) = \int_{-\infty}^{x} f(t)dt$ (you may assume that f is continuous, non-negative and the Riemann integral $\int_{\mathbb{R}} f = 1$). Then, find the (density of the) push forward measure of μ under (a) $T(x) = x + a$ (b) $T(x) = bx$ (c) T is any increasing and differentiable function.