Homework 8

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Problem 1. Let X_n be independent random variables with $X_n \sim \mathrm{Ber}(p_n)$. For $k \geq 1$, find a sequence (p_n) so that almost surely, the sequence $X_1, X_2, \dots$ has infinitely many segments of ones of length k but only finitely many segments of ones of length $k+1$. By a segment of length k we mean a consecutive sequence $X_i, X_{i+1}, \dots, X_{i+k-1}$.

Solution. Let $p_n = 0$ if n is a multiple of $k+1$ and 1 otherwise. That is, $(p_1, p_2, \dots, p_k, p_{k+1}) = (1, 1, \dots, 1, 0)$ and (p_n) is periodic with period $k+1$. Any $k+1$ consecutive indices $i, i+1, \dots, i+k$ must contain a multiple of $k+1$, so

$$\mathbf{P}\{X_i = X_{i+1} = \cdots = X_{i+k} = 1\} = 0.$$

By the union bound, the probability that there is *any* segment of ones of length $k+1$ is 0.

The event that there are infinitely many segments of ones of length k is a subset of the event

$$A \coloneqq \{X_n = 1 \text{ whenever } (k+1) \nmid n\}.$$

This has probability 1, since

$$A = \bigcap_{(k+1)\nmid n} \{X_n = 1\}$$

is the intersection of almost sure events.[1] ■

[1] In case this cannot be assumed, $\mathbf{P}(A) = 1$ since A^c is a countable union of zero-probability events.

Problem 2. Let $A_1, A_2, \dots$ be a sequence of M dependent events (i.e., A_i and A_j are independent iff $|i - j| > M$). Prove that $\mathbf{P}(A_n \text{ i.o.}) = 0$ or 1.

Solution. Let $N = M + 1$ for convenience. Consider the N subsequences

$$\begin{gathered} A_1, A_{1+N}, A_{1+2N}, \dots \\ A_2, A_{2+N}, A_{2+2N}, \dots \\ \vdots \\ A_N, A_{N+N}, A_{N+2N}, \dots \end{gathered}$$

Then A_n i.o. iff infinitely many events from at least one of these subsequences occur. But $A_k, A_{k+N}, A_{k+2N}, \dots$ are independent for any k, so by Kolmogorov's zero-one law, $\mathbf{P}(A_{k+nN} \text{ i.o.}) = 0$ or 1 (where k is fixed and n varies).

$$\{A_n \text{ i.o.}\} = \bigcup_{k=1}^{N} \{A_{k+nN} \text{ i.o.}\}.$$

- If $\mathbf{P}(A_{k+nN} \text{ i.o.}) = 0$ for all k, then $\mathbf{P}(A_n \text{ i.o.}) = 0$ by the union bound.
- If $\mathbf{P}(A_{k+nN} \text{ i.o.}) = 1$ for some k, then $\mathbf{P}(A_n \text{ i.o.}) \geq \mathbf{P}(A_{k+nN} \text{ i.o.}) = 1$.

Thus $\mathbf{P}(A_n \text{ i.o.}) = 0$ or 1. ■

Problem 3. Let $A_1, A_2, \dots$ be a sequence of events. Let $(n_k)_k$ be any sequence and $C_k \coloneqq \bigcup_{n_k \leq n < n_{k+1}} A_n$. Show that if $\sum_n \mathbf{P}(C_n) < \infty$ then $\mathbf{P}(A_n \text{ i.o.}) = 0$.

Solution. It is assumed that n_k are increasing. $\{A_n \text{ i.o.}\} = \{C_n \text{ i.o.}\}$, so by the first Borel-Cantelli lemma, $\mathbf{P}\{A_n \text{ i.o.}\} = 0$. ■

Problem 4. Let $A_1, A_2, \dots$ be a sequence of events in $(\Omega, \mathcal{F}, \mathbf{P})$. Let p_k be the probability that at least one of the events $A_k, A_{k+1}, \dots$ occcurs.

(1) If $\inf p_k > 0$, then show that A_n occurs infinitely often, w.p.p (with positive probability).

(2) If $p_k \to 0$, then show that only finitely many A_n occur, w.p.1 (with probability 1).

Solution. Let B_k be the event that no A_n occurs for $n \geq k$, and B be the event that only finitely many A_n occur. Then $B_1 \subseteq B_2 \subseteq \dots$ and $B = \bigcup_k B_k$. Thus $\mathbf{P}(B) = \lim_k \mathbf{P}(B_k) = \lim_k (1 - p_k)$.

(1) $\mathbf{P}(B) < 1$.

(2) $\mathbf{P}(B) = 1$. ∎

Problem 5. Let A_n be a sequence of independent events with $\mathbf{P}(A_n) < 1$ for all n. Show that $\mathbf{P}(\bigcup_n A_n) = 1$ implies $\sum_n \mathbf{P}(A_n) = \infty$.

Solution. Let $B_n = \bigcup_{k \geq n} A_k$. We know that $A_1, \dots, A_{n-1}, B_n$ are independent. Thus

$$\mathbf{P}(A_1^c \cap \dots \cap A_{n-1}^c \cap B_n^c) = \prod_{k=1}^{n-1} (1 - p_k)(1 - \mathbf{P}(B_n))$$

If this were positive, then $\mathbf{P}(\bigcup_n A_n) < 1$. Thus $\mathbf{P}(B_n) = 1$. Then $\{A_n \text{ i.o.}\} = \cap_n B_n$ almost surely, so $\mathbf{P}(A_n \text{ i.o.}) = 1$. By the first Borel-Cantelli lemma, $\sum_n p_n = \infty$. ∎

Problem 6. Let X_n and X be random variables such that for all $\delta > 0$, $\sum_n \mathbf{P}(|X_n - X| > \delta) < \infty$. Show that $X_n \xrightarrow{\text{a.s.}} X$.

Solution. Let $A_n = \{|X_n - X| > \frac{1}{n} \text{ finitely often}\}$. By the first Borel-Cantelli lemma, $\mathbf{P}(A_n) = 1$. Then $\mathbf{P}(\cap_n A_n) = 1$ gives that for each $\frac{1}{n}$, there exists N such that $|X_n - X| \leq \frac{1}{n}$ for all $n \geq N$, almost surely. Thus $X_n \xrightarrow{\text{a.s.}} X$. ∎

Problem 7. Let X_n be any sequence of random variables such that $X_n < \infty$ almost surely. Show that there are constants $c_n \to \infty$ such that $X_n / c_n \to 0$ almost surely.

Solution. Let c_n be such that $\mathbf{P}\{nX_n \geq c_n\} < \frac{1}{2^n}$. Then $\sum_n \mathbf{P}\{X_n/c_n \geq \frac{1}{n}\} < \infty$, so $X_n / c_n \xrightarrow{\text{a.s.}} 0$. ∎

Problem 8. Let $X_1, X_2, \ldots$ be i.i.d. Exp(1) random variables and $M_n = \max_{1 \le m \le n} X_m$. Show that

(1) $\limsup_{n\to\infty} X_n/\log n = 1$ almost surely.

(2) $M_n/\log n \to 1$ almost surely.

Solution.

(1) $\mathbf{P}\{X_i \ge t\} = e^{-t}$. Thus $\mathbf{P}\{X_n \ge k\log n\} = \frac{1}{n^k}$ is summable iff $k > 1$. Thus $X_n/\log n \ge 1$ infinitely often a.s., so $\limsup X_n/\log n \ge 1$ almost surely. For any $\varepsilon > 0$, $X_n/\log n \ge 1+\varepsilon$ only finitely often a.s., so $\limsup X_n/\log n \le 1$ almost surely.

(2) $M_n/\log n = \max_{1\le m\le n} X_m/\log n$. If $X_m/\log m \ge 1+\varepsilon$ only finitely often, then $M_n/\log n < 1+\varepsilon$ for large enough n. (when $X_k/\log k$ is bounded by $1+\varepsilon$ and $\log n$ is large enough to make $X_m/\log n$ small even when $X_m/\log m$ is large).

Now $M_n \le 1-\varepsilon$ iff $X_k \le (1-\varepsilon)\log n$ for $k \in [n]$.

$$\begin{aligned}
\mathbf{P}\{M_n \le 1-\varepsilon\} &= \prod_{k=1}^{n}\left(1 - e^{-(1-\varepsilon)\log n}\right) \\
&= \left(1 - \frac{1}{n^{1-\varepsilon}}\right)^n \\
&\le e^{-\frac{1}{n^{1-\varepsilon}}n} \\
&= e^{-n^{\varepsilon}}.
\end{aligned}$$

Let $\varepsilon = \frac{1}{m}$. Then $\lim_{n\to\infty}\frac{n^2}{e^{n^{\varepsilon}}} = \lim_{u\to\infty}\frac{u^{2m}}{e^2} = 0$. Thus $e^{-n^{\varepsilon}}$ is summable. $M_n \le 1-\varepsilon$ finitely often a.s. for each ε. Intersecting these events gives $M_n/\log n \to 1$ almost surely. ■

Problem 9. Let $X_1, X_2, \ldots$ be independent. Show that $\sup X_n < \infty$ almost surely iff $\sum_n \mathbf{P}(X_n > A) < \infty$ for some A.

Solution. If $X_n > A$ only finitely often, say only for some $n \le N$, then $\sup X_n \le \max\{X_1, \ldots, X_N\} \vee A$.

Conversely, suppose for all A, $X_n > A$ infinitely often. Then $\sup X_n = \infty$.

Thus $\sup X_n < \infty$ iff there is some A such that $X_n > A$ only finitely often. The two Borel-Cantelli lemmas give the result. ■

Problem 10. Let ξ, ξ_n be i.i.d. random variables with $\mathbf{E}[\log_+ \xi] < \infty$ and $\mathbf{P}(\xi = 0) < 1$.

(1) Show that the radius of convergence of the random power series $\sum_{n=0}^{\infty} \xi_n z^n$ is almost surely a constant.

(2) Show that the radius of convergence is 1 almost surely by showing $\limsup_{n\to\infty} |\xi_n|^{\frac{1}{n}} = 1$ almost surely.

Solution.

(1) Kolmogorov.

(2)

■

Problem 11. Consider the Bernoulli bond percolation on $\mathbb{Z}^d$ as defined in class. Determine which of the following events are tail events: there exists a unique infinite cluster, 0 belongs to a infinite cluster, there are infinitely many infinite clusters, there are finitely many infinite clusters.

Solution.

- Whether there exists a unique infinite cluster is *not* a tail event. Even in $\mathbb{Z}^1$, assume all edges except possibly $(-1, 0)$ and $(0, 1)$ are retained. Then there is either 1 infinite cluster or 2, depending on these 2 edges.
- Whether 0 belongs to an infinite cluster is *not* a tail event. Same counterexample as before.
- Whether there are infinitely many infinite clusters *is* a tail event. Finitely many edges can only affect finitely many clusters.
- Whether there are finitely many infinite clusters is the complement of the previous one, so also a tail event. ■

Problem 12. Consider the Bernoulli bond percolation on $\mathbb{Z}^d$ as defined in class and consider the critical probability $p_c(\mathbb{Z}^d)$. Show that $p_c(\mathbb{Z}^d) > 0$ for all $d \geq 1$.

Solution. We need to show that there is some $p > 0$ such that G_p doesn't have an infinite cluster almost surely.

For $x \in \mathbb{Z}^d$, let C_x be the event that 0 is connected to x via a path contained entirely in the region $\{y \in \mathbb{Z}^d \mid \|y\|_1 \leq \|x\|_1\}$. Let $p_n = \max_{\|x\|_1 = n} \mathbf{P}(C_x)$. Then $p_n \leq d p_{n-1} p$, so that $p_n \leq d^n p^n$. If $A_n = \bigcup_{\|x\|_1 = n} C_x$, then $\mathbf{P}(A_n) \leq K n^d d^n p^n$ for some constant K. By the ratio test, this is summable for $p < \frac{1}{d}$. The Borel-Cantelli lemma gives $\mathbf{P}(A_n \text{ i.o.}) = 0$.

But for there to be an infinite cluster containing 0, 0 must be connected to some point in each $\{\|x\|_1 = n\}$. For each of these sets, at least one point will be reachable from 0 without passing through any point with a larger Manhattan norm. Thus $\mathbf{P}(A_n \text{ i.o.}) = 0$ gives that $\mathbf{P}(0 \text{ is in an infinite cluster}) = 0$. By the union bound, $\mathbf{P}(\text{there is an infinite cluster}) = 0$. ∎

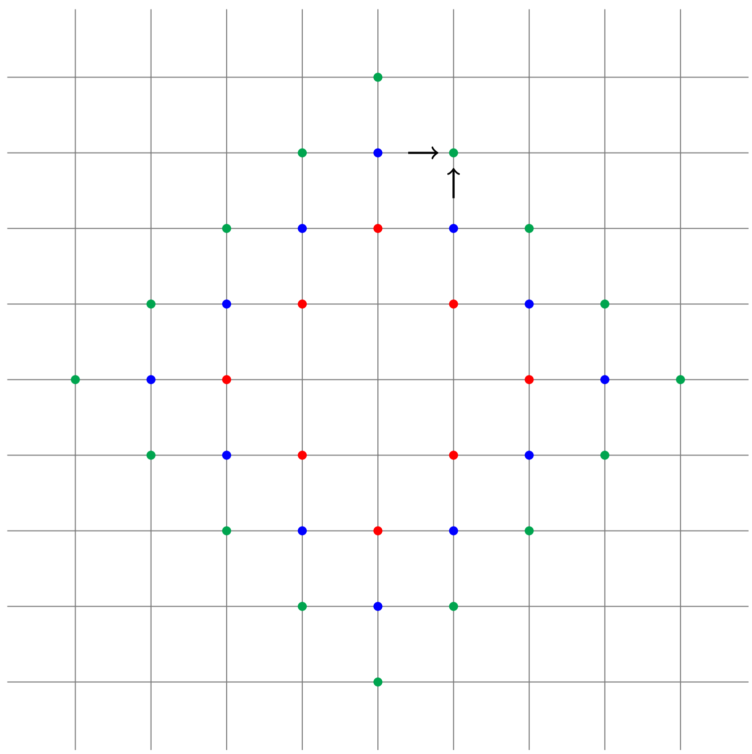

Figure 1: The green region must be reached first via the blue, from one of the d possible directions.