HOMEWORK 8

02.10.2024

Instruction: The homework is due on 11:30 am, 08.10.2024. You can either submit a hard copy in the beginning of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. Let X_n be independent random variables with $X_n \sim \text{Ber}(p_n)$. For $k \geq 1$, find a sequence (p_n) so that almost surely, the sequence $X_1, X_2, \ldots$ has infinitely many segments of ones of length k but only finitely many segments of ones of length $k+1$. By a segment of length k we mean a consecutive sequence $X_i, X_{i+1}, \ldots, X_{i+k-1}$.

Problem 2. Let $A_1, A_2, \ldots$ be a sequence of M dependent events (i.e., A_i and A_j are independent iff $|i-j| > M$). Prove that $\mathbf{P}(A_n \text{ i.o.}) = 0$ or 1.

Problem 3. Let $A_1, A_2, \ldots$ be a sequence of events. Let n_k be any sequence and $C_k := \bigcup_{n_k \leq n < n_{k+1}} A_n$. Show that if $\sum_n \mathbf{P}(C_n) < \infty$ then $\mathbf{P}(A_n \text{ i.o.}) = 0$.

Problem 4. Let $A_1, A_2, \ldots$ be a sequence of events in $(\Omega, \mathcal{F}, \mathcal{P})$. Let p_k be the probability that at least one of the events $A_k, A_{k+1}, \ldots$ occurs.

(1) If $\inf_k p_k > 0$, then show that A_n occurs infinitely often, w.p.p (with positive probability).

(2) If $p_k \to 0$, then show that only finitely many A_n occur, w.p.1 (with probability 1).

Problem 5. Let A_n be a sequence of independent events with $\mathbf{P}(A_n) < 1$ for all n. Show that $P(\cup_n A_n) = 1$ implies $\sum_n \mathbf{P}(A_n) = \infty$.

Problem 6. Let X_n and X are random variables and for all $\delta > 0, \sum_n \mathbf{P}(|X_n - X| > \delta) < \infty$. Show that $X_n \to X$ almost surely.

Problem 7. Let X_n be any sequence of random variables such that $X_n < \infty$ almost surely. Show that there are constants $c_n \to \infty$ such that $X_n / c_n \to 0$ almost surely.

Problem 8. Let $X_1, X_2, \ldots$ be i.i.d. Exp(1) random variables and $M_n = \max_{1 \le m \le n} X_m$. Show that

(1) $\limsup_{n\to\infty} X_n/\log n = 1$ almost surely.

(2) $M_n/\log n \to 1$ almost surely.

Problem 9. Let $X_1, X_2, \ldots$ be independent. Show that $\sup X_n < \infty$ almost surely iff $\sum_n \mathbf{P}(X_n > A) < \infty$ for some A.

Problem 10. Let ξ, ξ_n be i.i.d. random variables with $\mathbf{E}[\log_+ \xi] < \infty$ and $\mathbf{P}(\xi = 0) < 1$.

(1) Show that the radius of convergence of the random power series $\sum_{n=0}^{\infty} \xi_n z^n$ is almost surely a constant.

(2) Show that the radius of convergence is 1 almost surely by showing $\limsup\limits_{n\to\infty} |\xi_n|^{\frac{1}{n}} = 1$ *a.s.*

Problem 11. Consider the Bernoulli bond percolation on $\mathbb{Z}^d$ as defined in the class. Determine which of the following events are tail events: there exists a unique infinite cluster, 0 belongs to a infinite cluster, there are infinitely many infinite clusters, there are finitely many infinite clusters.

Problem 12. Consider the Bernoulli bond percolation on $\mathbb{Z}^d$ as defined in the class and consider the critical probability $p_c(\mathbb{Z}^d)$. Show that $p_c(\mathbb{Z}^d) > 0$ for all $d \ge 1$.