MA 212: Algebra I

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Contents

The course

Grading

This is tentative.

- Quizzes: 30%
- Midterm: 30%
- Final: 40%

Lecture 1.
Friday
August 02

Chapter 1

Groups

Definition 1.1 (Binary operation)**.** A *binary operation* $\cdot$ on a set A is any map from $A \times A \to A$, written $(a, b) \mapsto a \cdot b$.

We say that $\cdot$ is *associative* if for all $a, b, c \in A$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

and *commutative* if for all $a, b \in A$,

$$a \cdot b = b \cdot a.$$

Examples.

- Addition and multiplication are associative and commutative binary operations on $\mathbb{R}$.
- Subtraction, division and exponentiation are non-associative and non-commutative binary operations.
- Composition is an associative but non-commutative binary operation on X^X.

Definition 1.2 (Group)**.** A *group* is a set G equipped with a binary operation $\cdot$ satisfying the following properties:

(G1) **associativity:** $\cdot$ is associative;

(G2) **identity:** there exists an element $1_G = e \in G$ such that $1_G \cdot x = x \cdot 1_G = x$ for all $x \in G$;

(G3) **inverse:** for every $x \in G$, there exists an element $y \in G$ such that $x \cdot y = y \cdot x = 1_G$. We write y as x^{-1}.

If $\cdot$ is also commutative, we say that G is an *abelian group*.

A subset $H \subseteq G$ is a *subgroup* of G if H is a group with respect to the same binary operation $\cdot$. We write $H \leq G$.

Examples.

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are abelian groups.
- $(\mathbb{R}^\times, \cdot)$ is a group but $(\mathbb{R}, \cdot)$ is not.
- $(\mathrm{GL}_n(\mathbb{R}), \cdot)$ is a non-abelian group, where
$$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}.$$
- For any $n \in \mathbb{N}^+$, $(S_n, \circ)$ is a group, where
$$S_n = \{\sigma \colon [n] \to [n] \mid \sigma \text{ is bijective}\}.$$
$$\begin{aligned} S_1 &= \{1\}, \\ S_2 &= \{1, (12)\}, \\ S_3 &= \{1, (12), (13), (23), (123), (132)\}. \end{aligned}$$
S_1 and S_2 are abelian, but S_3 is not. Let $x = (12)$ and $y = (13)$, then
$$(x \circ y)(1) = x(3) = 3, \quad (x \circ y)(2) = x(2) = 1, \quad (x \circ y)(3) = x(1) = 2,$$
but
$$(y \circ x)(1) = y(2) = 2, \quad (y \circ x)(2) = y(1) = 3, \quad (y \circ x)(3) = y(3) = 1.$$
- Let $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$. Then H is an abelian subgroup of the non-abelian $\mathrm{GL}_2(\mathbb{R})$.

Remarks (New groups from old).

- Let $(A, \cdot)$ and $(B, *)$ be groups. The cartesian product $A \times B$ is a group with respect to the operation
$$(a_1, b_1) \star (a_2, b_2) = (a_1 \cdot a_2, b_1 * b_2).$$
defined componentwise.
- Let X be a set and $S = \mathbb{R}^X$. Then S is an abelian group under addition (pointwise). In fact, if $(G, \cdot)$ is a group, then G^X is a group under the operation
$$(f \cdot g)(x) = f(x) \cdot g(x).$$
If G is abelian, then so is G^X.
- Given any set A, we can form the group $S(A)$ of all bijections from A to itself, under composition.

Proposition 1.3. *Let $(G, \cdot)$ be a group. Then* *(i) the identity element* 1_G *is unique;*

(ii) the inverse of each element $x \in G$ *is unique;*

(iii) $(x^{-1})^{-1} = x$ *for all* $x \in G$*;*

(iv) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ *for all* $x, y \in G$*;*

(v) The product $a_1 a_2 \dots a_n$ *does not depend on bracketing.*

Proof.

(i) Suppose e and f are both identities of G. Then

$$e = e \cdot f = f.$$

(ii) Suppose y and y' are both inverses of x. Then

$$xy = 1_G \implies y'xy = y' \implies y' = y.$$

(iii) We have

$$x \cdot x^{-1} = 1_G = x^{-1} \cdot x.$$

reinterpreted in the context of x^{-1}.

(iv) Checking

$$(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xx^{-1} = 1_G.$$

Alternatively, let $z = (xy)^{-1}$. Then

$$\begin{aligned} (xy)z &= 1_G \\ (x^{-1}x)yz &= x^{-1} \\ yz &= x^{-1} \\ (y^{-1}y)z &= y^{-1}x^{-1} \\ z &= y^{-1}x^{-1}. \end{aligned}$$

(v) Induct on n. Look at the rightmost left bracket

$$a_1 \dots a_n = (a_1 \dots a_k) \cdot (a_{k+1} \dots a_n).$$ ■

Corollary 1.4 (Cancellation law)**.** *Let* $(G, \cdot)$ *be a group. If* $x, y, z \in G$ *and* $xy = xz$*, then* $y = z$*.*

Proof. Multiply by x^{-1} on the left. ■