MA 200: Multivariable Calculus

Naman Mishra

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The course

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Chapter 1

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Definition 1.1 (homogeneous function)**.** Let V be a vector space over $\mathbb{R}$. A function $f\colon V \setminus \{0\} \to \mathbb{R}$ is called a *homogeneous function* of degree k if

$$f(rx) = r^k f(x)$$

for each $x \in V \setminus \{0\}$ and $r > 0$.

Remarks.

- If f and g are homogeneous functions of degree k and l respectively, then $f \cdot g$ is homogeneous of degree $k + l$ and f/g is homogeneous of degree $k - l$ (provided g is never zero).
- $f \equiv 0$ is homogeneous of any degree.

Definition 1.2 (norm)**.** Let V be a vector space over $\mathbb{R}$. A norm $\|\cdot\|$ on V is a function from V to $\mathbb{R}$ that satisfies

(N1) (Positivity) $\|x\| \geq 0$ for any $x \in V$.

(N2) (Definiteness) $\|x\| = 0$ iff $x = 0$.

(N3) (Homogeneity) $\|rx\| = |r|\|x\|$ for any $x \in V$ and $r \in \mathbb{R}$.

(N4) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in V$.

Definition 1.3 (normed linear space)**.** A vector space V equipped with a norm $\|\cdot\|$ is called a *normed linear space*.

Remark. Any normed linear space $(V, \|\cdot\|)$ can be given a metric space structure by defining the distance $d(x, y)$ between $x, y \in V$ as $\|x - y\|$.

The set $B(x, r) \coloneqq \{y \in V \mid \|x - y\| < r\}$ is called the open ball of radius r centered at x.

The set $S(x, r) \coloneqq \{y \in V \mid \|x - y\| = r\}$ is called the sphere of radius r centered at x.

Exercise 1.4 (reverse triangle inequality)**.** *Let V be a normed linear space. Show that*

$$\Big|\|x\| - \|y\|\Big| \le \|x - y\|$$

for any $x, y \in V$.

This shows that $f = x \mapsto \|x\|$ is a (Lipschitz) continuous function on V.

Definition 1.5 (continuity)**.** Let (X, d) and (Y, ρ) be metric spaces. A function $f\colon X \to Y$ is called *continuous* at $a \in X$ iff

$$\begin{aligned} x_n \to a &\implies f(x_n) \to f(a), \text{ or} \\ d(x_n, a) \to 0 &\implies \rho(f(x_n), f(a)) \to 0 \end{aligned}$$

Exercise 1.6 (product metric spaces)**.** *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $d\colon X_1 \times X_2 \to \mathbb{R}$ be defined by*

$$d((x_1, x_2), (y_1, y_2)) \coloneqq d_1(x_1, y_1) + d_2(x_2, y_2).$$

Show that d is a metric on $X_1 \times X_2$.

Let $(z_n)_{n\in\mathbb{N}} = ((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence in $X_1 \times X_2$. Show that $z_n \to (x, y)$ iff $x_n \to x$ and $y_n \to y$.

Remark. $\widetilde{d}$ given by

$$\widetilde{d}((x_1, x_2), (y_1, y_2)) \coloneqq \min\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

is *not* a metric on $X_1 \times X_2$.

It disobeys definiteness.

Exercise 1.7. *Let $(V, \|\cdot\|)$ be a normed linear space.*

- *The addition map $(x, y) \mapsto x + y$ is a continuous map from $V \times V$ to V.*
- *The scalar multiplication map $(\alpha, x) \mapsto \alpha x$ is continuous from $\mathbb{R} \times V$ to V.*

Examples.

- (ℓ^p norm) $\mathbb{R}^n$ with $p \in [1, \infty]$ and

  $$\|x\|_p \coloneqq \Big(|x_1|^p + \cdots + |x_n|^p\Big)^{1/p}$$

  where

  $$\|x\|_\infty \coloneqq \max\{|x_1|, \ldots, |x_n|\}$$

  is the limit of the l^p norms as $p \to \infty$.

Exercise 1.8. *Check that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$, and that its limit as $p \to \infty$ is $\|\cdot\|_\infty$.*

Definition 1.9 (norm equivalence)**.** Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on V. We say that $\|\cdot\|_a$ and $\|\cdot\|_b$ are *equivalent* if these exist $c_1, c_2 > 0$ such that

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a$$

for all $x \in V$. We write $\|\cdot\|_a \sim \|\cdot\|_b$.

Exercise 1.10. *Check that* $\sim$ *is an equivalence relation.*