Homework 4

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Problem 1. Find integrable random variables X_n, X for each of the following situations.

- $X_n \to X$ a.s. but $\mathbf{E}[X_n] \not\to \mathbf{E}[X]$.
- $X_n \to X$ a.s. and $\mathbf{E}[X_n] \to \mathbf{E}[X]$ but there is no dominating integrable random variable Y for the sequence $\{X_n\}$.

Solution. Fix the space $([0,1], \mathcal{B}_{[0,1]}, \lambda)$.

- Define
$$X_n(\omega) = \begin{cases} n & \text{if } \omega < \frac{1}{n}, \\ 0 & \text{if } \omega \geq \frac{1}{n}. \end{cases}$$
Then for all $\omega \in (0,1]$, $X_n(\omega) \to 0$, but
$$\mathbf{E}[X_n] = n \times \frac{1}{n} + 0 \times \left(1 - \frac{1}{n}\right) = 1$$
does not converge to $\mathbf{E}[0] = 0$.

- Define
$$X_n(\omega) = \begin{cases} n+1 & \text{if } \frac{1}{n+1} < \omega \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$
Then for all $\omega \in (0,1]$, $X_n(\omega) \to 0$. Futhermore,
$$\mathbf{E}[X_n] = (n+1)\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n} \to 0 = \mathbf{E}[0].$$

However, any dominating integrable random variable Y must satisfy

$$Y \geq \sum_{n=1}^{\infty} X_n \text{ a.s.}$$

since the X_i's have disjoint support. That is, $\sup_n X_n = \sum_{n=1}^{\infty} X_n$. In other words,

$$Y(x) \geq n+1 \quad \text{for } \frac{1}{n+1} < x \leq \frac{1}{n}$$

almost everywhere.

But then (by MCT or via integration)

$$\mathbf{E}[Y] \geq \sum_{n=1}^{\infty} (n+1)\left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

No dominating integrable random variable exists. ■

Problem 2. Let X be a non-negative random variable. Show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\}\,\mathrm{d}t$ (in particular, if X is a non-negative integer valued, then $\mathbf{E}[X] = \sum_{n=0}^{\infty} \mathbf{P}(X \geq n)$) by showing the following steps.

(1) Prove the equality for $X = \mathbf{1}_A$.

(2) Prove the equality for simple functions.

(3) Use Monotone Convergence Theorem to conclude the equality.

Solution.

(1) Let $X = \mathbf{1}_A$ for some $A \in \mathcal{B}_{\mathbb{R}}$. Then

$$\begin{aligned}
\mathbf{E}[X] &= \mathbf{P}(A) \\
&= \mathbf{P}\{\mathbf{1}_A = 1\} \\
&= \int_0^1 \mathbf{P}\{\mathbf{1}_A > 0\}\,\mathrm{d}t \\
&= \int_0^1 \mathbf{P}\{\mathbf{1}_A > t\}\,\mathrm{d}t \\
&= \int_0^\infty \mathbf{P}\{\mathbf{1}_A > t\}\,\mathrm{d}t.
\end{aligned}$$

since $\mathbf{1}_A$ only takes the values 0 and 1.

(2) Let $X = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ be a simple function where A_i's are pairwise disjoint. Suppose the equality holds for $Y = X - a_n \mathbf{1}_{A_n}$. Then

$$\begin{aligned}
\int_0^\infty \mathbf{P}\{X > t\}\,\mathrm{d}t &= \int_0^\infty \mathbf{P}\{\mathbf{1}_{A_n} = 0, Y > t\} + \mathbf{P}\{\mathbf{1}_{A_n} = 1, Y + a_n > t\}\,\mathrm{d}t \\
&= \int_0^\infty \mathbf{P}\{Y > t\}\,\mathrm{d}t + \int_0^\infty \mathbf{P}\{\mathbf{1}_{A_n} = 1, a_n > t\}\,\mathrm{d}t \\
&= \int_0^\infty \mathbf{P}\{Y > t\}\,\mathrm{d}t + \int_0^{a_n} \mathbf{P}(A)\,\mathrm{d}t \\
&= \mathbf{E}[Y] + a_n\,\mathbf{P}(A_n) \\
&= \mathbf{E}[X].
\end{aligned}$$

The second equality is since A_i's are disjoint, so that $Y > 0 \implies \mathbf{1}_{A_n} = 0$ and (contrapositive) $\mathbf{1}_{A_n} = 1 \implies Y = 0$. By induction, the equality holds for all simple functions.

(3) Let X_n be a sequence of simple functions such that $X_n \uparrow X$ a.s. (this can always be done as discussed in the tutorial). Then $\mathbf{E}[X_n] \uparrow \mathbf{E}[X]$ by Monotone Convergence Theorem.

Claim. *For any $t > 0$, $\mathbf{P}\{X_n > t\} \uparrow \mathbf{P}\{X > t\}$.*

Proof of claim. Since $X_n \uparrow X$ a.s., we immediately have that $\mathbf{1}_{\{X_n > t\}}$ form an increasing sequence almost surely.

Let $\omega \in \Omega$ be such that $X_n(\omega) \uparrow X(\omega)$. Suppose $X(\omega) > t$. Then for large enough n, $X_n(\omega) > t$. Thus $\mathbf{1}_{\{X_n > t\}}(\omega) \uparrow 1 = \mathbf{1}_{\{X > t\}}(\omega)$. If $X(\omega) \le t$, then $X_n(\omega) \le t$ for all n and $\mathbf{1}_{\{X_n > t\}}(\omega) \uparrow 0 = \mathbf{1}_{\{X > t\}}(\omega)$.

Generalizing, we have $\mathbf{1}_{\{X_n > t\}} \uparrow \mathbf{1}_{\{X > t\}}$ almost surely. MCT gives the result. □

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\int_0^\infty \mathbf{P}\{X > t\}\,\mathrm{d}t &\ge \int_0^\infty \mathbf{P}\{X_n > t\}\,\mathrm{d}t \\
\implies \int_0^\infty \mathbf{P}\{X > t\}\,\mathrm{d}t &\ge \lim_{n\to\infty} \int_0^\infty \mathbf{P}\{X > t\}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \mathbf{E}[X_n] \\
&= \mathbf{E}[X].
\end{aligned}$$

However, Fatou's lemma gives

$$\begin{aligned}
\int_0^\infty \mathbf{P}\{X > t\}\,\mathrm{d}t &= \int_0^\infty \lim_{n\to\infty} \mathbf{P}\{X_n > t\}\,\mathrm{d}t \\
&\le \liminf_{n\to\infty} \int_0^\infty \mathbf{P}\{X_n > t\}\,\mathrm{d}t \\
&= \liminf_{n\to\infty} \mathbf{E}[X_n] \\
&= \mathbf{E}[X].
\end{aligned}$$

Together they yield the desired equality. ■