MA 361 HOMEWORK-5

03.09.2024

Instruction: The homework is due on 11:30 am, 10.09.2024. You can either submit a hard copy in the beginning of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. (1) (5 points): If X is a random variable distributed as $N(0,1)$ then show that $\mathbf{E}(e^{\theta X}) = \exp\left(\frac{\theta^2}{2}\right)$.

(2) (5 points): Use the above to show that for all $t > 0, \mathbf{P}\left(X \geq t\right) \leq \exp\left(-\frac{t^2}{2}\right)$.

Problem 2. (10 points): If $\mathbf{E}[|X|] < \infty$, then $\mathbf{E}[|X|\mathbf{1}_{\{|X|>A\}}] \to 0$ as $A \to \infty$.

Problem 3. Show that if X is a non-negative integrable random variable then

$$\lim_{t\to\infty} t\mathbf{P}(X \geq t) \to 0.$$

(Remark: Note that this is stronger than Markov's inequality. Markov's inequality just gives an upper bound of the left hand side.)

Problem 4. Suppose (X, Y) has the *standard bivariate normal distribution* with density $f(x, y) = (2\pi)^{-1}\exp\{-\frac{x^2+y^2}{2}\}$. Find the density of X/Y.

Problem 5. For any integrable random variable X having CDF F, show that

$$\mathbf{E}[X] = \int_0^\infty (1 - F(x) - F(-x))dx.$$

Problem 6. Let X be a non-negative random variable. If $\mathbf{E}[X]$ is finite, show that $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ is finite for any $a > 0$. Conversely, if $\sum_{n=1}^\infty \mathbf{P}\{X \geq an\}$ is finite for some $a > 0$, show that $\mathbf{E}[X]$ is finite.

Problem 7. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$. Show that $\mu_n \xrightarrow{d} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for every $f \in C_b(\mathbb{R})$. [Hint: for the non-trivial direction approximate the indicator functions $1_{\{X_n \leq x\}}$ both from above and below by continuous functions which are bounded and piece-wise linear.]

Problem 8. (1) If f_n be measurable functions such that $f_n \leq 1$ for all n the the prove that

$$\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n.$$

(2) Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^n)$ having densities f_n, f with respect to Lebesgue measure. If $f_n \to f$ a.e. (w.r.t. Lebesgue measure), show that $\mu_n(A) \xrightarrow{d} \mu(A)$ for all Borel set A.

Problem 9 (Moment matrices). Let $\mu \in \mathcal{P}(\mathbb{R})$ and let $\alpha_k = \int x^k d\mu(x)$ (assume that all moments exist). Then, for any $n \geq 1$, show that the matrix $(\alpha_{i+j})_{0\leq i,j\leq n}$ is non-negative definite. [**Suggestion:** First solve $n = 1$].

Problem 10. Let X be a non-negative random variable. If $\mathbf{E}[X] \leq 1$, then show that $\mathbf{E}[X^{-1}] \geq 1$.