Homework 2

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Problem 1.

(1) *Let Ω be a set and $A \subseteq \Omega$. Define a function $\mathbf{1}_A \colon \Omega \to \mathbb{R}$ as follows.*

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

What is the smallest σ-algebra on Ω with respect to which $\mathbf{1}_A$ becomes a random variable?

(2) *Assume that $A \in \mathcal{F}$. Give an explicit description of the push-forward measure $P \circ (\mathbf{1}_A)^{-1}$ on $\mathbb{R}$.*

Solution.

(1) We need $\mathbf{1}_A^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{B}(\mathbb{R})$.

$$(\mathbf{1}_A)^{-1}(B) = \begin{cases} \varnothing & \text{if } 0, 1 \notin B, \\ A & \text{if } 1 \in B, 0 \notin B, \\ A^c & \text{if } 0 \in B, 1 \notin B, \\ \Omega & \text{if } 0, 1 \in B. \end{cases}$$

Thus $\mathcal{F}$ must contain $\varnothing, A, A^c, \Omega$. $\mathcal{F} = \{\varnothing, A, A^c, \Omega\}$ is itself a σ-algebra, hence the smallest one that works.

(2) Let $B \in \mathcal{B}(\mathbb{R})$. Then

$$(P \circ (\mathbf{1}_A)^{-1})(B) = \begin{cases} 0 & \text{if } 0, 1 \notin B, \\ P(A) & \text{if } 1 \in B, 0 \notin B, \\ P(A^c) & \text{if } 0 \in B, 1 \notin B, \\ 1 & \text{if } 0, 1 \in B. \end{cases}$$ ■

Problem 2. *Recall the Lévy metric d defined in class. Show the following.*

(1) *Let a_n be a sequence of real numbers converging to a. For any $x \in \mathbb{R}$, δ_x is the measure define as follows: for $A \subseteq \mathbb{R}$,*

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Using the definition of the metric show that

$$d(\delta_{a_n}, \delta_a) \to 0 \text{ as } n \to \infty.$$

(2) *Consider the sequence of measures $\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{i/n}$ and μ is the uniform measure on $[0,1]$. Using the definition show that*

$$d(\mu_n, \mu) \to 0 \text{ as } n \to \infty.$$

Solution.

(1) The CDF of δ_x is $F_x(\omega) = [\omega \geq x]$. If $|x - y| = \varepsilon$, then

$$F_x(\omega + \varepsilon) = [\omega + \varepsilon \geq x] = [\omega \geq x - \varepsilon] \geq [\omega \geq y] = F_y(\omega)$$

since $\omega \geq y \implies \omega \geq x - |x - y|$. Thus $d(\delta_x, \delta_y) \leq |x - y|$. As $a_n \to a$, $d(\delta_{a_n}, \delta_a) \to 0$.

(2) The CDF of μ is $F(x) = x$ for $x \in [0,1]$. The CDF of μ_n is

$$F_n(x) = \frac{\lfloor nx \rfloor}{n} \quad \text{for } x \in [0,1].$$

($\lfloor nx \rfloor$ counts the number of points $i/n \leq x$, and each of those has weight $1/n$.) We claim that $d(\mu_n, \mu) \leq 1/n$.

Let $x \in [0,1]$. Then

$$\begin{aligned} F\left(x + \frac{1}{n}\right) + \frac{1}{n} &= x + \frac{2}{n} \\ &= \frac{nx + 2}{n} \\ &> \frac{\lfloor nx \rfloor}{n} = F_n(x). \end{aligned}$$

and

$$\begin{aligned} F_n\left(x + \frac{1}{n}\right) + \frac{1}{n} &= \frac{\lfloor n(x+1/n) \rfloor + 1}{n} \\ &= \frac{\lfloor nx \rfloor + 2}{n} \\ &> \frac{nx}{n} \\ &= x = F(x). \end{aligned}$$

Thus

$$\frac{1}{n} \in \{\varepsilon > 0 : F_n(x+\varepsilon) + \varepsilon \geq F(x) \text{ and} \\ F(x+\varepsilon) + \varepsilon \geq F_n(x) \text{ for all } x \in [0,1]\}$$

and so $d(\mu_n, \mu)$, which is the infimum of all such ε, is at most $1/n$. It follows that $\lim\limits_{n\to\infty} d(\mu_n, \mu) = 0$ by the squeeze theorem. ■

Problem 3. *For $k \geq 0$, define the functions $r_k \colon [0,1) \to \mathbb{R}$ by writing $[0,1) = \bigsqcup\limits_{0 \leq j < 2^k} I_j^{(k)}$ where $I_j^{(k)}$ is the dyadic interval $[j2^{-k}, (j+1)2^{-k})$ and setting*

$$r_k(x) = \begin{cases} -1 & \text{if } x \in I_j^{(k)} \text{ for odd } j, \\ 1 & \text{if } x \in I_j^{(k)} \text{ for even } j. \end{cases}$$

Fix $n \geq 1$ and define $T_n \colon [0,1) \to \{-1,1\}^n$ by $T_n(x) = (r_0(x), \ldots, r_{n-1}(x))$. Find the push-forward of the Lebesgue measure on $[0,1)$ under T_n.