MA 361 HOMEWORK-4

27.08.2024

Instruction: The homework is due on 11:30 am, 03.09.2024. You can either submit a hard copy in the beginning of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. Find integrable random variables X_n, X for each of the following situations.

(1) $X_n \to X$ a.s. but $\mathbf{E}[X_n] \not\to \mathbf{E}[X]$.

(2) $X_n \to X$ a.s. and $\mathbf{E}[X_n] \to \mathbf{E}[X]$ but there is no dominating integrable random variable Y for the sequence $\{X_n\}$ (i.e., there is no random variable Y such that Y is integrable and $|X_n| \leq Y$ for all n.)

Problem 2. Let X be a non-negative random variable. Show that $\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\}dt$ (in particular, if X is a non-negative integer valued, then $\mathbf{E}[X] = \sum_{n=1}^\infty \mathbf{P}(X \geq n)$) by showing the following steps.

(1) Prove the equality for $X = 1_A$.

(2) Prove the equality for simple functions.

(3) Use Monotone Convergence Theorem to conclude the equality.

Problem 3. Let X, Y be real valued random variables, X_n be a sequence of random variables taking values in extended real line and $c \in \mathbb{R}$. Show the followings:

(1) Composition of random variables is a random variable.

(2) Show that cX and $X + Y, XY$ are random variables.

(3) $\sup_{n\geq 1} X_n$ and $\inf_{n\geq 1} X_n$ are random variables.

(4) $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to} X_n$ are random variables.

Problem 4. Let X and Y are two random variables taking values in extended real line from $(\Omega, \mathcal{F})$. Show that the set $E = \{\omega \in \Omega : X(\omega) = Y(\omega)\}$ is a measurable set.

Problem 5. Let $0 < p < q$.

(1) If $X \in L^q$, show that $X \in L^p$ by showing that $\mathbf{E}[|X|^p] \leq \mathbf{E}[|X|^q]$.

(2) Equality holds iff X is a constant random variable.

Problem 6. Compute mean, variance and moments (as many as possible!) of the Normal(0,1), exponential(1), Beta(p,q) distributions.

Problem 7. If μ and ν are probability measures on a finite set $\mathcal{A}$, then the *relative entropy* of μ w.r.t. ν is defined as $D(\mu\|\nu) = \sum\limits_{a\in\mathcal{A}} \mu(a)\log\frac{\mu(a)}{\nu(a)}$. The quantity $H(\mu) := \sum\limits_{a\in\mathcal{A}} \mu(a)\log\frac{1}{\mu(a)}$ is called the *entropy* of μ.

(1) Show that $D(\mu\|\nu) \geq 0$ with equality if and only if $\mu = \nu$ (hint: use Jensen's inequality).

(2) Show that $0 \leq H(\mu) \leq \log|\mathcal{A}|$. When are the inequalities attained?

[*Clarification:* When $\mu(a) = 0$ the sumand is taken to be 0 but when $\mu(a) > 0$ but $\nu(a) = 0$, it is taken to be $+\infty$.].

Problem 8. Let X be a non negative integrable random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ with $\int X dP > 0$. For $A \in \mathcal{F}$ define $\int_A X dP := \int X \mathbf{1}_A dP$. Prove that $\mu(A) = \frac{1}{\int f}\int_A f dP$ is a probability measure on Ω.