MA 361 HOMEWORK 9

08.10.2024

Instruction: The homework is due on 11:30 am, 15.10.2024. You can either submit a hard copy in the beginning of the class or send a soft copy to sudeshnab@iisc.ac.in. Only submit the blue coloured problems for grading. No submission after the deadline will be accepted.

Problem 1. Determine whether the following statements are true or false with proper justification

(1) Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables taking values in $\mathbb{R}$ and defined on same probability space. Then $\frac{X_n}{n} \xrightarrow{P} 0$.

(2) Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables, taking values in $\mathbb{R}$ and are defined on same probability space. Then $\frac{X_n}{n} \to 0$ almost surely.

(3) Let X be a random variable which is finite almost surely. Then $\frac{X}{n} \xrightarrow{P} 0$.

(4) Let X be a random variable which is finite almost surely. Then $\frac{X}{n} \to 0$ almost surely.

Problem 2. Let X_n and X be random variables on a common probability space. Show that if $X_n \xrightarrow{P} X$ then there is a subsequence n_k such that $X_{n_k} \to X$ almost surely.

Problem 3. Let $X_1, X_2, \ldots$ be i.i.d. with distribution $\mu \in \mathcal{P}(\mathbb{R})$. Recall that the support of μ is the smallest closed set K with $\mu(K) = 1$. Show that $\overline{\{X_1, X_2, \ldots\}} = K$ a.s. (the left side is the closure of the set $\{X_n\}$)

Problem 4. Let X_n be independent and $\mathbf{P}(X_n = n^{-a}) = \frac{1}{2} = \mathbf{P}(X_n = -n^{-a})$ where $a > 0$ is fixed. For what values of a does the series $\sum X_n$ converge a.s.? For which values of a does the series converge absolutely, a.s.?

Problem 5. Suppose X_n are i.i.d random variables with finite mean. Which of the following assumptions guarantee that $\sum X_n$ converges a.s.?

(1) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge 1] < \infty$.

(2) (i) $\mathbf{E}[X_n] = 0$ for all n and (ii) $\sum \mathbf{E}[X_n^2 \wedge |X_n|] < \infty$.

Problem 6. (**Large deviation for Bernoullis**). Let X_n be i.i.d Ber(1/2). Fix $p > \frac{1}{2}$.

(1) Show that $\mathbf{P}(S_n > np) \le e^{-np\lambda}\left(\frac{e^\lambda+1}{2}\right)^n$ for any $\lambda > 0$.

(2) Optimize over λ to get $\mathbf{P}(S_n > np) \le e^{-nI(p)}$ where $I(p) = -p\log p - (1-p)\log(1-p)$. (Observe that this is the *entropy* of the $\mathrm{Ber}(p)$ measure).

Problem 7. Carry out the same program for i.i.d exponential(1) random variables and deduce that $\mathbf{P}(S_n > nt) \le e^{-nI(t)}$ for $t > 1$ and $\mathbf{P}(S_n < nt) \le e^{-nI(t)}$ for $t < 1$ where $I(t) := t - 1 - \log t$.

Problem 8. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

(1) Let X, Y be random variables on Ω. Define a function $d(X,Y) = \mathbf{E}\left(\frac{|X-Y|}{1+|X-Y|}\right)$. Show that d is a metric on the set of all random variables and $d(X_n, X) \to 0$ iff $X_n \stackrel{P}{\to} X$.

(2) Show that if X_n, X are random variables such that any subsequence of X_n has a further subsequence that converge almost surely to X then $X_n \stackrel{P}{\to} X$.

Problem 9. Let X_n, Y_n, X, Y be random variables on a common probability space. If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$ (all r.v.s on the same probability space), show that $f(X_n, Y_n) \to f(X,Y)$ for any continuous $f : \mathbb{R}^2 \to \mathbb{R}$. In particular, this implies if $X_n \stackrel{P}{\to} X, Y_n \stackrel{P}{\to} Y$ then for any $a, b \in \mathbb{R}, aX_n + bY_n \stackrel{P}{\to} aX + bY$.

Problem 10. Give examples of two sequence of random variables X_n and Y_n such that $X_n \stackrel{d}{\to} X$ and $Y_n \stackrel{d}{\to} Y$ but $X_n + Y_n$ does not converge in distribution to $X + Y$.