Homework 1

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Problem 1. *Let X be an arbitrary set. Let*

$$\mathcal{F} \coloneqq \{A \subseteq X \mid A \textit{ is countable or } A^c \textit{ is countable}\}.$$

Prove that $\mathcal{F}$ is a sigma-algebra. Let S be the collection of all singletons in X. Prove that $\sigma(S) = \mathcal{F}$.

Solution. Since $\varnothing$ is countable, $\varnothing \in \mathcal{F}$. For any $A \subseteq X$,

$$\begin{aligned} A \in \mathcal{F} &\iff A \text{ is countable or } A^c \text{ is countable} \\ &\iff A^c \text{ is countable or } (A^c)^c \text{ is countable} \\ &\iff A^c \in \mathcal{F}. \end{aligned}$$

Let $A_1, A_2, \dots \in \mathcal{F}$. If every A_i is countable, then $\bigcup_{i=1}^{\infty} A_i$ is a countable union of countable sets and hence countable.

If some A_i is uncountable, then A_i^c is countable (since $A_i \in \mathcal{F}$) so

$$\left(\bigcup_{j=1}^{\infty} A_j \right)^c = \bigcap_{j=1}^{\infty} A_j^c \subseteq A_i^c$$

is countable. In either case, the union is in $\mathcal{F}$.

Any countable set A can be written as a countable union of singletons $A = \bigcup_{a \in A} \{a\}$. Thus every countable set is in $\sigma(S)$. By closure under complements, every set whose complement is countable is also in $\sigma(S)$. Thus $\mathcal{F} \subseteq \sigma(S)$.

But $\mathcal{F}$ is a σ-algebra containing S, so $\sigma(S) \subseteq \mathcal{F}$. ∎

Problem 2. *On $[0, 1]$, let $\mathcal{A}$ be the algebra generated by finite unions of left-open, right-closed intervals and let $\mathcal{B}$ be the Borel sigma-algebra. Define $\mu \colon \mathcal{A} \to [0, 1]$ by $\mu(A) = 1$ if $A \supseteq (0, \varepsilon)$ for some $\varepsilon > 0$ and $\mu(A) = 0$ otherwise.*

(1) Show that μ is a finitely additive measure on $\mathcal{A}$.

(2) Show that μ can not be extended to a measure on $\mathcal{B}$.

(3) Why does this not contradict the Carathéodory extension theorem?

Solution.

(1) Let $A_1, A_2 \in \mathcal{A}$ be disjoint. First notice that $\mu(A_1) = \mu(A_2) = 1$ is not possible. This would imply that there exist $0 < \varepsilon_1, \varepsilon_2 \leq 1$ such that $(0, \varepsilon_1) \subseteq A_1$ and $(0, \varepsilon_2) \subseteq A_2$, which would force $A_1 \cap A_2 \supseteq (0, \varepsilon_1\varepsilon_2) \neq \varnothing$.

Suppose $\mu(A_1) = \mu(A_2) = 0$. Since A_1 and A_2 are finite unions of intervals, write $A_1 = \bigcup_{i=1}^n (a_i, b_i]$, where each a_i must be positive for $\mu(A_1) = 0$.
Then $\inf A_1 = \inf\limits_{1\leq i\leq n} a_i > 0$. Similarly $\inf A_2 > 0$. Thus $\inf(A_1 \cup A_2) > 0$ and so there is no $\varepsilon > 0$ for which $(0, \varepsilon) \subseteq A_1 \cup A_2$. That is, $\mu(A_1 \cup A_2) = 0$.

The remaining cases are $\mu(A_1) = 1, \mu(A_2) = 0$ and $\mu(A_1) = 0, \mu(A_2) = 1$. WLOG suppose $\mu(A_1) = 1$ and $\mu(A_2) = 0$. There exists an $\varepsilon > 0$ such that $(0, \varepsilon) \in A_1 \subseteq A_1 \cup A_2$. Thus $\mu(A_1 \cup A_2) = 1$.

Finite additivity holds in each case.

(2) μ can not be extended to a measure on $\mathcal{B}$ because it is not countably additive. Let $A_n = (\frac{1}{n+1}, \frac{1}{n}]$ for $n \geq 1$. $\{A_n\}$ are pairwise disjoint and $\mu(A_n) = 0$ for all n. However, $\bigcup_{n=1}^\infty A_n = (0, 1]$, whose measure is 1. Thus countable additivity fails and μ can not be extended to a measure on $\mathcal{B}$.

(3) This does not contradict the Carathèodory extension theorem because μ does not satisfy the criterion of being *countably additive*, which is a necessary condition for the theorem to apply. ■

Problem 3. *Let $\mathcal{F}$ be a σ-algebra of subsets of Ω.*

(1) Show that $\mathcal{F}$ is closed under countable intersections ($\bigcap_n A_n$), under set differences ($A \setminus B$), and under symmetric differences ($A\Delta B$).

(2) If A_n is a countable sequence of subsets of Ω, the set $\limsup A_n$ *(resp.* $\liminf A_n$*) is defined as the set of all $\omega \in \Omega$ that belong to infinitely many (resp. all but finitely many) of the sets A_n.*

If $A_n \in \mathcal{F}$ for all n, show that $\limsup A_n \in \mathcal{F}$ *and* $\liminf A_n \in \mathcal{F}$*.*

(3) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$, what are $\limsup A_n$ and $\liminf A_n$?

Solution.

(1) $\bigcap_n A_n = (\bigcup_n A_n^c)^c$ is in $\mathcal{F}$ by properties of σ-algebras. $A \setminus B$ is simply $A \cap B^c$, and the symmetric difference is simply $A \setminus B \cup B \setminus A$.

(2) Write $\limsup A_n = \bigcap_i \bigcup_{j \geq i} A_j$ and $\liminf A_n = \bigcup_i \bigcap_{j \geq i} A_j$.

(3) Let $A = \bigcup A_n$. For each i,
$$\bigcup_{j \geq i} A_j = A \quad \text{and} \quad \bigcap_{j \geq i} A_j = A_i.$$
Thus $\limsup A_n = \bigcap_i A = A$ and $\liminf A_n = \bigcup_i A_i = A$. ■

Problem 4. *Let $(\Omega, \mathcal{F})$ be a set with a σ-algebra.*

(1) Suppose $\mathbf{P}$ is a probability measure on $\mathcal{F}$. If $A_n \in \mathcal{F}$ and A_n increase to A (respectively, decrease to A), show that $\mathbf{P}(A_n)$ increases to (respectively, decreases to) $\mathbf{P}(A)$.

(2) Suppose $\mathbf{P}\colon \mathcal{F} \to [0,1]$ is a function such that (a) $\mathbf{P}(\Omega) = 1$, (b) $\mathbf{P}$ is finitely additive, (c) if $A_n, A \in \mathcal{F}$ and A_ns increase to A, then $\mathbf{P}(A_n) \uparrow \mathbf{P}(A)$. Then show that $\mathbf{P}$ is a probability measure on $\mathcal{F}$.

Solution.

(1) First notice that $X \subseteq Y \implies \mathbf{P}(X) \leq \mathbf{P}(Y)$ and $\mathbf{P}(X) + \mathbf{P}(X^c) = \mathbf{P}(\Omega) = 1$ by additivity.

Let $A_n \uparrow A$. Let $\Delta_n = A_n \setminus A_{n-1}$ for $n \geq 2$. By additivity,
$$\mathbf{P}(A_n) = \mathbf{P}(A_{n-1}) + \mathbf{P}(\Delta_n) = \mathbf{P}(A_1) + \sum_{i=2}^{n} \mathbf{P}(\Delta_n).$$
Thus $\mathbf{P}(A_n)$ increases to $\mathbf{P}(A_1) + \sum_{i=2}^{\infty} \mathbf{P}(\Delta_n)$. By countable additivity, $\mathbf{P}(A)$ is precisely this.

Now suppose $A_n \downarrow A$. Then $A_n^c \uparrow A^c$ (which are all in $\mathcal{F}$). Then $\mathbf{P}(A_n) = 1 - \mathbf{P}(A_n^c)$ decreases to $1 - \mathbf{P}(A^c) = \mathbf{P}(A)$.

(2) Let $\Delta_1, \Delta_2, \dots \in \mathcal{F}$ be disjoint. Let $A_n = \bigsqcup_{i=1}^{n} \Delta_i$ and $A = \bigsqcup_{i=1}^{\infty} \Delta_i$. By finite additivity, $\mathbf{P}(A_n) = \sum_{i=1}^{n} \mathbf{P}(\Delta_i)$. Thus $\mathbf{P}(A_n)$ increases to $\sum_{i=1}^{\infty} \mathbf{P}(\Delta_i)$. By (c), this is $\mathbf{P}(A)$, so countable additivity holds. ■

Problem 5. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} \mid \mathbf{P}(A) = 0$ or $1\}$. Show that $\mathcal{G}$ is a σ-algebra.*

Proof. Since $\mathbf{P}(\varnothing) = 0$ and $\mathbf{P}(\Omega) = 1$, $\varnothing, \Omega \in \mathcal{G}$.

Let $A \in \mathcal{G}$. Since $\mathbf{P}(A^c) = 1 - \mathbf{P}(A)$, $A^c \in \mathcal{G}$.

Let $A_1, A_2, \cdots \in \mathcal{G}$. If $\mathbf{P}(A_i) = 0$ for all i, then $\mathbf{P}(\bigcup A_i) = 0$ and $\bigcup A_i \in \mathcal{G}$. If $\mathbf{P}(A_i) = 1$ for some i, then $\mathbf{P}(\bigcup A_i) = 1$ and $\bigcup A_i \in \mathcal{G}$. ∎

Problem 6. *Let $\mathcal{F}$ be a σ-algebra on $\mathbb{N}$ that is strictly smaller than the power set. Show that there exist $m \neq n$ such that elements of $\mathcal{F}$ do not separate m and n (i.e., the following holds: any $A \in \mathcal{F}$ either contains both m and n, or neither). Is the same conclusion valid if $\mathbb{N}$ is replaced by any set Ω?*

Solution. Let $\mathcal{F}$ be a σ-algebra that separates any two natural numbers. Fix an $n \in \mathbb{N}$. Let $B_m \in \mathcal{F}$ contain n but not m for each $m \neq n$. Then $\bigcap_{m \neq n} B_m = \{n\} \in \mathcal{F}$. Thus $\mathcal{F} = 2^{\mathbb{N}}$.

This argument would not work for any arbitrary set, since the intersection may not be countable. ∎

Problem 7. *Let X be an arbitrary set.*

(1) Suppose S is a collection of subsets of X and a, b are two elements of X that are not separated by any element of S. Let $\mathcal{F} = \sigma(S)$. Show that no set in $\mathcal{F}$ separates a and b.

(2) Let $S = \{(a, b] \cup [-b, -a) : a < b \in \mathbb{R}\}$. Show that $\sigma(S)$ is strictly smaller than the Borel σ-algebra on $\mathbb{R}$.

Solution.

(1) Consider the collection

$$\mathcal{G} = \{A \in 2^{\mathcal{F}} : a \in A \iff b \in A\}.$$

This is a σ-algebra containing S, since

(ς1) $a \in \varnothing \iff b \in \varnothing$;

(ς2) if $A \in \mathcal{G}$, then $a \in A \iff b \in A$, so $a \in A^c \iff b \in A^c$;

(ς3) if $A_1, A_2, \cdots \in \mathcal{G}$, then in the case that some A_j contains both a and b, the union contains both a and b, and in the case that no A_j contains either a or b, the union similarly contains neither.

Since $\mathcal{F} = \sigma(S)$ is the intersection of all σ-algebras containing S, $\mathcal{F} \subseteq \mathcal{G}$. Thus no set in $\mathcal{F}$ separates a and b.

(2) $S \in \mathcal{B}(\mathbb{R})$, so $\sigma(S) \subseteq \mathcal{B}(\mathbb{R})$.

No set in S separates 1 and -1. However, the $[0,1] \in \mathcal{B}(\mathbb{R})$ separates them. Thus $\sigma(S) \neq \mathcal{B}(\mathbb{R})$. ■

Problem 8. *Show that each of the following collection of subsets of $\mathbb{R}$ generate the same σ-algebra (which we call the Borel σ-algebra).*

(1) $\{[a,b] : a \leq b$ *and* $a, b \in \mathbb{Q}\}$.

(2) The collection of all compact sets.

Solution. Call the σ algebras generated by the two given subsets σ_1 and σ_2 respectively. Note that $\sigma_1 \subseteq \sigma_2$.

From the exercise following the definition of Borel σ-algebra, we have that

$$\mathcal{B}(\mathbb{R}) = \sigma\{\text{open sets in } \mathbb{R}\} = \sigma\{\text{open intervals in } \mathbb{R}\}.$$

Since closed sets are complements of open sets, $\sigma_1 \subseteq \sigma_2 \subseteq \mathcal{B}(\mathbb{R})$. We need to show that $\mathcal{B}(\mathbb{R}) \subseteq \sigma_1$.

Let $U \subseteq \mathbb{R}$ be open. For each $x \in U$, there exists a compact interval $[a_x, b_x]$ with rational endpoints such that $x \in [a_x, b_x] \subseteq U$. Then

$$U = \bigcup_{x \in U} [a_x, b_x] \ \in \sigma_1,$$

since there are only countably many such $[a_x, b_x]$. Thus $\mathcal{B}(\mathbb{R}) \subseteq \sigma_1 \subseteq \sigma_2$. We get $\sigma_1 = \sigma_2 = \mathcal{B}(\mathbb{R})$ as desired. ■

Problem 9. *Let Ω be an infinite set and let $\mathcal{A} = \{A \subseteq \Omega \mid A$ or A^c is finite$\}$. Define $\mu \colon \mathcal{A} \to \mathbb{R}_+$ by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite.*

(1) Show that $\mathcal{A}$ is an algebra and that μ is finitely additive on $\mathcal{A}$.

(2) Show that if Ω is countable μ does not extend to a measure on $\mathcal{F} = \sigma(\mathcal{A})$.

(3) Under what conditions does μ extend to a probability measure on $\mathcal{F}$?

Solution.

(i) Obviously $\mathcal{A}$ is closed under complements. A finite union of finite sets is finite, and any union where at least one set is cofinite is cofinite. Thus $\mathcal{A}$ is closed under finite unions.

Let $A, B \in \mathcal{A}$ be disjoint. If both are finite, then $A \cup B$ is finite and $\mu(A \cup B) = 0 = \mu(A) + \mu(B)$. At most one of these can be cofinite, in which case $\mu(A \cup B) = 1 = \mu(A) + \mu(B)$.

(ii) WLOG suppose $\Omega = \mathbb{N}$. Then $\{n\}, \mathbb{N} \in \mathcal{F}$ and $\bigsqcup_{n=1}^{\infty} \{n\} = \mathbb{N}$. But

$$\mu(\mathbb{N}) = 1 \neq \sum_{n=1}^{\infty} \mu(\{n\}) = 0.$$

Thus μ can not be extended to a measure on $\mathcal{F}$.

(iii) If Ω is uncountable, then μ is countably additive. Let $A_1, A_2, \dots \in \mathcal{A}$ be disjoint. If any of these is cofinite, then every other is finite, and

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = 1 = \sum_{n=1}^{\infty} \mu(A_n).$$

If all of them are finite, then the union is at most countable. The union is in $\mathcal{A}$ only if it is finite, since the complement of a countable set would be uncountable. In that case,

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = 0 = \sum_{n=1}^{\infty} \mu(A_n).$$

Thus by Carathéodory's extension theorem, μ extends to a measure on $\mathcal{F} = \sigma(\mathcal{A})$. ■

Problem 10. *On* $\mathbb{N} = \{1, 2, \dots\}$*, let* A_p *denote the subset of numbers divisible by* p*. Describe* $\sigma(\{A_p : p \text{ is prime}\})$ *as explicitly as possible.*

Call $\sigma(\{A_p : p \text{ is prime}\})$ simply σ. The idea is that

- $\mathbb{N}$ is countable, as is $\{A_p : p \text{ is prime}\}$. It would make sense for σ to be explicitly generated by countable unions/intersections/complements of these sets.

- Any $m, n \in \mathbb{N}$ having the same prime factors are not separated by any A_p. By problem 7, σ does not separate them either. Thus we can consider the partition of $\mathbb{N}$ into sets of numbers with the same prime factors.

- This too is countable, so if we σ contains each such set, it is essentially the power set of this partition.

Solution. Let $\mathbb{P}$ be the set of primes. Denote the set of finite subsets of $\mathbb{P}$ as $\mathrm{Fin}(\mathbb{P})$. For each $P \in \mathrm{Fin}(\mathbb{P})$, let $[P]$ denote the set of all $n \in \mathbb{N}$ whose set of prime factors is precisely P. That is,

$$[P] = \bigcap_{p \in P} A_p \cap \bigcap_{q \in P^c} A_q^c.$$

By the fundamental theorem of arithmetic,

$$\mathbb{N} = \bigsqcup_{P \in \mathrm{Fin}(\mathbb{P})} [P].$$

Let

$$\Sigma = \left\{ \bigcup_{P \in S} [P] \,\middle|\, S \subseteq \mathrm{Fin}(\mathbb{P}) \right\}.$$

(ς1) When $S = \varnothing$, $\bigcup_{P \in S}[P] = \varnothing$.

(ς2) Since $[P]$ form a partition of $\mathbb{N}$, the complement of $\bigcup_{P \in S}[P]$ is $\bigcup_{P \in S^c}[P]$. Σ is closed under complements.

(ς3) $\mathrm{Fin}(\mathbb{P})$ is countable, so Σ is closed under countable unions.

Thus Σ is a σ-algebra. Since

$$A_p = \bigcup_{P \in \mathrm{Fin}(\mathbb{P}) : p \in P} [P],$$

$\sigma \subseteq \Sigma$.

But each $[P]$ is in σ, as is every countable union of them. Thus $\Sigma \subseteq \sigma$. ■