MA 368A: Exclusion Processes

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Contents

The course

Course webpage

Lecture 1.
Thursday
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Grading

- Class participation: 5%
- Homework: 15%
- Midterm: 30%
- Final: 50%

Chapter 1

Review

Definition 1.0.1 (Stochastic process)**.** A *stochastic process* is a family of random variables $\{X(t)\}_{t \in T}$, where T is the index set (usually thought of as time).

$X(t)$ is said to be the *state* at time t. The set where $X(t)$ takes its values is called the *state space*, usually denoted S or Ω.

Examples.

- (Boring) $X(t)$'s are independent, as in the case of coin flips.
- (Gambler's ruin) A gambler walks into a casino with k coins. and wants to go home with $n > k$ coins. In each round, they bet a coin and either loses it or gains one. Their current bank balance is $X(t)$. The process stops when $X(t) = 0$ or $X(t) = n$. Suppose $T = \mathbb{N}$. Then we write $(X_n)_{n \in \mathbb{N}}$ to denote the process.

1.1 Discrete time Markov chains

Definition 1.1.1 (DTMC)**.** A *discrete time Markov chain* (DTMC) is a stochastic process $(X_n)_{n \in \mathbb{N}}$ with the Markov property:

$$\Pr(X_{n+1} = x \mid X_0 = x_0, \ldots, X_n = x_n) = \Pr(X_{n+1} = x \mid X_n = x_n)$$

for each $n \in \mathbb{N}$ and $x_0, \ldots, x_n, x \in S$.

That is, the future depends on the past only through the present. Equivalently, the future and the past are independent when conditioning on the present.

We will assume S is countable.

Definition 1.1.2 (Transition probabilities)**.** For any DTMC, we define for each $i, j \in S$ the quantity

$$p_{ij} = \Pr(X_{n+1} = j \mid X_n = i)$$

as the *one-step transition probability* from i to j.

The matrix

$$P = (p_{ij})_{i,j \in S}$$

is called the *transition matrix* of the DTMC.

We write $P^n_{ij} = \Pr(X_n = j \mid X_0 = i)$ for the n-step transition probability from i to j.

The transition matrix obeys the following properties:

(i) $p_{ij} \geq 0$ for all $i, j \in S$.

(ii) (row stochastic property) $\sum_{j \in S} p_{ij} = 1$ for all $i \in S$.

Remark. In principle, the transition probabilities can depend on n. For the most part, we will keep p_{ij}'s free of any n's. This makes the process *time-homogeneous.*

Theorem 1.1.3 (Chapman-Kolmogorov equation)**.** *For any DTMC, we have*

$$P^{n+m}_{ij} = \sum_{k \in S} P^n_{ik} P^m_{kj}$$

for all $i, j \in S$ *and* $n, m \in \mathbb{N}$.

Corollary 1.1.4. *For any DTMC with transition matrix* P*, the* n*-step transition matrix is given by* P^n*.*

This justifies the notation P^n_{ij} instead of p^n_{ij}.

1.1.1 Classification of states

Definition 1.1.5.

- We say that j is *accessible* from i if there exists $n \in \mathbb{N}$ such that $P^n_{ij} > 0$, and write $i \to j$. In particular, i is always accessible from itself.
- We say that i and j *communicate* if $i \to j$ and $j \to i$, and write $i \leftrightarrow j$.

Exercise 1.1.6. *Commincation is an equivalence relation.*

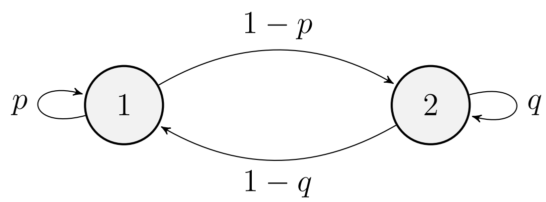
Thus the state space S can be partitioned into *communication classes*.

Definition 1.1.7.

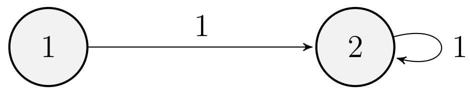
- If there is a single communication class, the chain is said to be *irreducible*.
- We define T_i to be the first return time of state i and $f_i = \Pr(T_i < \infty)$.
- A state i is *recurrent* if $f_i = 1$, and *transient* if $f_i < 1$.

Examples.

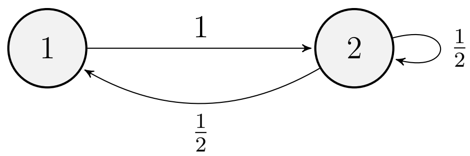
- Consider the following DTMC.



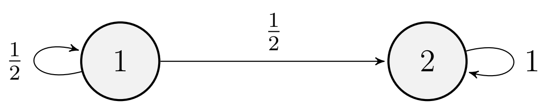
  - $p = 0, q = 1$ makes 1 transient and 2 recurrent.



  - $p = 0, q = \frac{1}{2}$ makes both states recurrent.



  - $p = \frac{1}{2}, q = 1$ makes 1 transient and 2 recurrent.



Exercise 1.1.8. *State i is recurrent iff $\sum_{n=1}^{\infty} P_{ii}^n = \infty$.*

Exercise 1.1.9. *Recurrence is a class property.*

Example (Simple random walk on $\mathbb{Z}$). For each i, let $p_{i,i+1} = p$ and $p_{i,i-1} = 1 - p$. Suppose $p \in (\frac{1}{2}, 1)$.

- Every state is accessible from every other state. Thus there is a single communication class.
- Every state is transient.

If $p = \frac{1}{2}$, then every state is recurrent.

Exercise 1.1.10. *For a finite DTMC, there is at least one recurrent state.*

Definition 1.1.11. Let $m_i = \mathrm{E}[T_i]$ be the expected return time of state i. A recurrent state i is said to be *positive recurrent* if $m_i < \infty$, and *null recurrent* if $m_i = \infty$.

Exercise 1.1.12. *Positive recurrence is a class property.*

Exercise 1.1.13 (SSRW on $\mathbb{Z}$)**.** *For the simple random walk on $\mathbb{Z}$ with $p = \frac{1}{2}$, all states are null recurrent.*

Solution. Suppose each state is positive recurrent. Using translational invariance with theorem 1.1.16, the stationary distribution of the chain is uniform. This is not possible, since it would force $\sum_{i \in \mathbb{Z}} \pi_i = \infty$. ∎

Definition 1.1.14 (Stationarity)**.** We say that a probability distribution π on S is a *stationary* or *steady state* distribution if

$$\sum_{i \in S} \pi_i p_{ij} = \pi_j$$

for all $j \in S$. Equivalently, $\pi P = \pi$ (where π is treated as a row vector).

Exercise 1.1.15. *Let π_0 be the initial distribution of the chain. Then the distribution at time n is given by $\pi_0 P^n$.*

Thus, the definition of stationarity says that if the chain starts with distribution π, it will always stay in distribution π.

Theorem 1.1.16. *Suppose a DTMC $(X_n)_{n\in\mathbb{N}}$ is irreducible and positive (that is, each state is positive recurrent). Then*

$$\pi_i = \frac{1}{m_i} \quad \text{for all } i \in S$$

is the unique stationary distribution.

Remarks.

- A stationary distribution exists and is unique if the chain is irreducible and positive.
- If i is positive recurrent, then $\pi_i > 0$. If i is null recurrent, then $\pi_i = 0$.