MA 212: Algebra I

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Contents

The course

Grading

This is tentative.

- Quizzes: 30%
- Midterm: 30%
- Final: 40%

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Chapter 1

Groups

Definition 1.1 (Binary operation)**.** A *binary operation* $\cdot$ on a set A is any map from $A \times A \to A$, written $(a, b) \mapsto a \cdot b$.

We say that $\cdot$ is *associative* if for all $a, b, c \in A$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

and *commutative* if for all $a, b \in A$,

$$a \cdot b = b \cdot a.$$

Examples.

- Addition and multiplication are associative and commutative binary operations on $\mathbb{R}$.
- Subtraction, division and exponentiation are non-associative and non-commutative binary operations.
- Composition is an associative but non-commutative binary operation on X^X.

Definition 1.2 (Group)**.** A *group* is a set G equipped with a binary operation $\cdot$ satisfying the following properties:

(G1) **associativity:** $\cdot$ is associative;

(G2) **identity:** there exists an element $1_G = e \in G$ such that $1_G \cdot x = x \cdot 1_G = x$ for all $x \in G$;

(G3) **inverse:** for every $x \in G$, there exists an element $y \in G$ such that $x \cdot y = y \cdot x = 1_G$. We write y as x^{-1}.

If $\cdot$ is also commutative, we say that G is an *abelian group*.

A subset $H \subseteq G$ is a *subgroup* of G if H is a group with respect to the same binary operation $\cdot$. We write $H \leq G$.

Examples.

- $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$ and $(\mathbb{C},+)$ are abelian groups.
- $(\mathbb{R}^\times,\cdot)$ is a group but $(\mathbb{R},\cdot)$ is not.
- $(\mathrm{GL}_n(\mathbb{R}),\cdot)$ is a non-abelian group, where
$$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}.$$
- For any $n \in \mathbb{N}^+$, $(S_n,\circ)$ is a group, where
$$S_n = \{\sigma\colon [n] \to [n] \mid \sigma \text{ is bijective}\}.$$
$$\begin{aligned} &S_1 = \{1\}, \\ &S_2 = \{1,(12)\}, \\ &S_3 = \{1,(12),(13),(23),(123),(132)\}. \end{aligned}$$
S_1 and S_2 are abelian, but S_3 is not. Let $x=(12)$ and $y=(13)$, then
$$(x\circ y)(1) = x(3) = 3, \quad (x\circ y)(2) = x(2) = 1, \quad (x\circ y)(3) = x(1) = 2,$$
but
$$(y\circ x)(1) = y(2) = 2, \quad (y\circ x)(2) = y(1) = 3, \quad (y\circ x)(3) = y(3) = 1.$$
- Let $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$. Then H is an abelian subgroup of the non-abelian $\mathrm{GL}_2(\mathbb{R})$.

Remarks (New groups from old).

- Let $(A,\cdot)$ and $(B,*)$ be groups. The cartesian product $A\times B$ is a group with respect to the operation
$$(a_1,b_1) \star (a_2,b_2) = (a_1\cdot a_2, b_1 * b_2).$$
defined componentwise.
- Let X be a set and $S = \mathbb{R}^X$. Then S is an abelian group under addition (pointwise). In fact, if $(G,\cdot)$ is a group, then G^X is a group under the operation
$$(f\cdot g)(x) = f(x)\cdot g(x).$$
If G is abelian, then so is G^X.
- Given any set A, we can form the group $S(A)$ of all bijections from A to itself, under composition.

Proposition 1.3. *Let $(G,\cdot)$ be a group. Then* *(i) the identity element 1_G is unique;*

(ii) the inverse of each element $x \in G$ is unique;

(iii) $(x^{-1})^{-1} = x$ for all $x \in G$;

(iv) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ for all $x, y \in G$;

(v) The product $a_1 a_2 \dots a_n$ does not depend on bracketing.

Proof.

(i) Suppose e and f are both identities of G. Then

$$e = e \cdot f = f.$$

(ii) Suppose y and y' are both inverses of x. Then

$$xy = 1_G \implies y'xy = y' \implies y' = y.$$

(iii) We have

$$x \cdot x^{-1} = 1_G = x^{-1} \cdot x.$$

reinterpreted in the context of x^{-1}.

(iv) Checking

$$(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xx^{-1} = 1_G.$$

Alternatively, let $z = (xy)^{-1}$. Then

$$\begin{aligned} (xy)z &= 1_G \\ (x^{-1}x)yz &= x^{-1} \\ yz &= x^{-1} \\ (y^{-1}y)z &= y^{-1}x^{-1} \\ z &= y^{-1}x^{-1}. \end{aligned}$$

(v) Induct on n. Look at the rightmost left bracket

$$a_1 \dots a_n = (a_1 \dots a_k) \cdot (a_{k+1} \dots a_n).$$ ■

Corollary 1.4 (Cancellation law)**.** *Let $(G, \cdot)$ be a group. If $x, y, z \in G$ and $xy = xz$, then $y = z$.*

Proof. Multiply by x^{-1} on the left. ■

Lecture 2. Monday August 05

Definition 1.5 (Order)**.** The order of an element $x \in G$ is the smallest $n \in \mathbb{N}$ such that $x^n = 1_G$ if it exists, and ∞ otherwise.

Examples.

- $G = \mathbb{Z}/n\mathbb{Z} = \{\bar{a} \mid 0 \le a < n\}$ where $\bar{a} = \{a + kn \mid k \in \mathbb{Z}\}$ under the operation $\bar{a} + \bar{b} = \overline{a+b}$.
- $G = \mathbb{C}^\times$. All roots of unity have finite order.
- $G = \mathrm{GL}_2(\mathbb{R})$. The matrix
$$\alpha_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
has order n if $\theta = \frac{2\pi}{n}$. [This is a homomorphism from $(\mathbb{R}, +)$ to $(G, \cdot)$]
- $G = \mathrm{GL}_2(R)$ where R is a ring. Elements of the following set may have finite order.
$$\{g \in M_2(R) \mid \det(g) \text{ is a unit in } R\}$$

Proposition 1.6 (Crystallographic restriction)**.** *Let* $x \in \mathrm{GL}_2(\mathbb{Z})$. *Then* $\operatorname{ord} x \in \{1, 2, 3, 4, 6, \infty\}$.

Definition 1.7 (Subgroup)**.** A set $H \subseteq G$ is a *subgroup* of G if it is a group under the same operation. We write $H \le G$.

Examples.

- $G = \mathbb{Z}$. Then $H \le G \iff H = n\mathbb{Z}$ for some $n \in \mathbb{N}$.

  Proof. Ignore the trivial case $H = \{0\}$. Let n be the smallest positive element of H. Then $n\mathbb{Z} \subseteq H$ by closure under addition. For any $m \in H$, write $m = qn + r$ with $0 \le r < n$. Then $r = m - qn \in H$. Since n is the smallest positive element of H, $r = 0$. Thus $H \subseteq n\mathbb{Z}$. ■

- Let $|G| = 2k < \infty$. Then G has an element of order 2.

  Proof. Suppose not. Then for any $x \in G \setminus \{1\}$, $x^{-1} \ne x$. Thus $G \setminus \{1\}$ is a disjoint union of pairs $\{x, x^{-1}\}$. This would imply $|G|$ is odd. ■

- Let G be a group such that $x^2 = 1$ for all $x \in G$. Then G is abelian.

  Proof. Let $x, y \in G$. Then
$$\begin{aligned} & (xy)^2 = 1 \\ \implies & xyxy = 1 \\ \implies & xy = y^{-1}x^{-1} = yx \end{aligned}$$
■

- Let G be a finite group where each element is its own inverse. What can be said about $|G|$?

  $(G, \cdot)$ can be viewed as a vector space over $(\mathbb{F}_2, +, \cdot)$ with the scalar product of $x \in G$ and $c \in \mathbb{F}_2$ given by x^c. Let $n = \dim_{\mathbb{F}_2} G$ (possibly zero). Then $(G, \cdot) \cong (\mathbb{F}_2^n, +)$ and thus $|G| = 2^n$. (ref. structure theorem for finitely generated abelian groups)

  Furthermore, $\mathbb{F}_2^n$ is a group of this form for all n. Thus the groups of this form are precisely $\{\mathbb{F}_2^n \mid n \in \mathbb{N}\}$ (up to isomorphism).

Proposition 1.8. *Let $H \subseteq G$. Then $H \leq G$ iff $H \neq \varnothing$ and H is closed under the operation $(x, y) \mapsto xy^{-1}$.*

Proof. The "only if" direction is trivial.

Suppose $H \neq \varnothing$ and H is closed under the operation. Let x be any element of H. Then $1 = xx^{-1} \in H$. Now for any $y \in H$, $y^{-1} = 1y^{-1} \in H$. Now for any $x, y \in H$, $xy = x(y^{-1})^{-1} \in H$. ■

Proposition 1.9. *Let $H \subseteq G$ be finite. Then $H \leq G$ iff $H \neq \varnothing$ and H is closed under multiplication.*

Proof. Let ■

1.1 Cyclic groups

Given $x \in G$, look at the set $\langle x \rangle = \{x^n \mid n \in \mathbb{Z}\}$. This is a cyclic subgroup of G.

We wish to classify all cyclic subgroups (up to isomorphism).

Example. Let $H \leq G$ with $|G| = n > 2$, $|H| = n - 1$. Is this possible?

No. Let $G \setminus H = \{x\}$. Then $x^{-1} = x$. Let $h \neq 1 \in H$. Then $xh = h' \in H$, so $x \in H$ (closure).

Generalising gives the following proposition.

Proposition 1.10. *No group can be the union of two proper subgroups.*

Proof. Suppose $G = H_1 \cup H_2$ where $H_1, H_2 \leq G$. Pick an $x \in H_1 \setminus H_2$ and $y \in H_2 \setminus H_1$. WLOG assume $xy \in H_1$. Then $y \in H_1$. This means at least one of $H_1 \setminus H_2$ and $H_2 \setminus H_1$ is empty. ■

Definition 1.11 (Homomorphism)**.** Let G and H be groups. A map $\varphi\colon G \to H$ is a *homomorphism* from G to H if it respects the group operation. That is,
$$\varphi(xy) = \varphi(x)\varphi(y)$$
for all $x, y \in G$.

- If φ is bijective, it is called an *isomorphism*.
- If $H = G$, it is an *automorphism*.

G and H are *isomorphic* ($G \cong H$) if there exists an isomorphism from G to H.

Definition 1.12 (Kernel)**.** The *kernel* of a homomorphism $\varphi\colon G \to H$ is the set
$$\ker\varphi = \{x \in G \mid \varphi(x) = 1_H\}.$$
The *image* of φ is the set
$$\operatorname{Im}\varphi = \{\varphi(x) \mid x \in G\}.$$

Examples.

- $\det\colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ is a homomorphism.
- $\mu\colon \mathbb{Z}/n\mathbb{Z} \to \mu_n$ given by
$$\mu\left(\bar{k}\right) = \exp\left(\frac{2\pi k}{n}\right)$$
is an isomorphism, where
$$\mu_n = \{n\text{th roots of unity}\} \subseteq \mathbb{C}.$$
- φ is injective iff $\ker\varphi = \{1_G\}$.
- $\exp\colon (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ is an isomorphism.
- $\mathbb{R}^\times \ncong \mathbb{C}^\times$.
- Let A, B be nonempty sets. Then $S_A \cong S_B$ iff A and B are in bijection.

  Proof. Suppose τ is a bijection from A to B. Then $\sigma \mapsto \tau\sigma\tau^{-1}$ is an isomorphism from S_A to S_B. ■

If two groups are isomorphic, they are essentially the same group. An isomorphism $\varphi\colon G \to H$ is only a "re-parameterization" of G in terms of H. **Lecture 3.** Wednesday August 07

Lecture 3

Lemma 1.13. $|\langle x\rangle| = \operatorname{ord} x$.

Proof. If $\operatorname{ord} x = \infty$, then $x^n \neq x^m$ for $n \neq m$. Thus $|\langle x\rangle| = \infty$.

If $\operatorname{ord} x = n < \infty$, then $x^0, x^1, \dots, x^{n-1}$ are distinct. Let $x^m \in \langle x\rangle$. Write $x^m = x^{qn+r} = x^r$ with $0 \leq r < n$. Thus these n elements are the only ones in $\langle x\rangle$. ∎

Proposition 1.14. *Let G be a cyclic group. Then*

(i) if $|G| = \infty$, then $G \cong \mathbb{Z}$;

(ii) if $|G| = n < \infty$, then $G \cong \mathbb{Z}/n\mathbb{Z}$.

Proof. Let $G = \langle x\rangle$. We want an isomorphism $\varphi\colon G \to \mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{N} \cup \{\infty\}$. It suffices to define $\varphi(x)$ and extend it to all of G.

If $|G| = \infty$, define $\varphi(x) = 1$. Then $\varphi(x^n) = n$ for all $n \in \mathbb{Z}$. This is a bijection and $\varphi(ab) = \varphi(a) + \varphi(b)$ holds.

If $|G| = \{1, x, \dots, x^{n-1}\}$, define $\varphi(x) = \overline{1} \in \mathbb{Z}/n\mathbb{Z}$. Then $\varphi(x^m) = \overline{m}$ for all $m \in \mathbb{Z}$. It is clearly a surjection. The kernel is $\{x^m \in G : n \mid m\} = \{1\}$, so it is injective. Finally, $\varphi(x^m x^k) = \varphi(x^{m+k}) = \overline{m+k} = \overline{m} + \overline{k}$. ∎

Cyclic groups are generated by a single element. What about groups generated by multiple elements?

Let $S \subseteq G$. Define two sets

$$\begin{aligned}
\langle S\rangle_1 &= \{s_1^{\varepsilon_1} \dots s_k^{\varepsilon_k} \mid s_i \in S, \varepsilon_i \in \{\pm 1\}\} \\
&= \{s_1^{\alpha_1} \dots s_k^{\alpha_k} \mid s_i \in S, \alpha_i \in \mathbb{Z}\} \\
\langle S\rangle_2 &= \bigcap_{S \subseteq H \leq G} H.
\end{aligned}$$

Lemma 1.15. $\langle S\rangle_1 = \langle S\rangle_2 =: \langle S\rangle$.

Proof. $\langle S\rangle_2 \leq G$ since the intersection of subgroups is a subgroup. We first check that $\langle S\rangle_1 \leq G$ under multiplication (which is essentially concatenation). Inverses are given by $s_1^{\varepsilon_1} \dots s_k^{\varepsilon_k} \mapsto s_k^{-\varepsilon_k} \dots s_1^{-\varepsilon_1}$.

Moreover, $S \subseteq \langle S\rangle_1$. Thus $\langle S\rangle_2 \subseteq \langle S\rangle_1$.

Since $\langle S\rangle_2$ is a group containing S, closure under products and inverses implies $\langle S\rangle_1 \subseteq \langle S\rangle_2$. ∎

Examples.

- S_n is generated by transpositions.
- $\mathrm{GL}_n(\mathbb{R})$ is generated by the elementary matrices

  $$E_{ij}(\lambda) = I_n + \lambda e_{ij},$$

  where $e_{pq} = (\delta_{ip}\delta_{jq})_{i,j=1}^n$, taken together with the diagonal matrices. [swapping is done by $(a, b) \mapsto (a, a+b) \mapsto (-a, a+b) \mapsto (b, a+b) \mapsto (b, a)$]

- $\mathbb{Q}^\times$ is not finitely generated. Take any finite set $S \subseteq \mathbb{Q}^\times$ and look at the numerators. There are finitely many primes in the numerators of S, so any prime not in the numerators of S is not in $\langle S \rangle$.

- $\mathrm{SL}_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) \mid \det M = 1\}$ is generated by
$$E_{ij}(\lambda) = I_n + \lambda e_{ij}, \quad \text{with } i \neq j.$$

- Let F be any infinite field. Then $(F^\times, \cdot)$ is not finitely generated. If $\operatorname{char} F = p$, then p is prime and

  Suppose $\operatorname{char} F = 0$. Then F contains (an isomorphic copy of) $\mathbb{Q}$. For $F^\times$ to be finitely generated, $Q^\times$ would have to be finitely generated. We will later see that subgroups of finitely generated groups are finitely generated. We will also see that $\mathbb{Q}^\times$ is not finitely generated. Thus $F^\times$ is not finitely generated.

Lecture 4.
Friday
August 09

- $\mathrm{GL}_n(F)$ is not finitely generated for any infinite field F.

  - There is an isomorphic copy of $F^\times$ in $\mathrm{GL}_n(F)$. If $\mathrm{GL}_n(F)$ were finitely generated, so would $F^\times$.
  - $\det \colon \mathrm{GL}_n(F) \to F^\times$ is a surjective homomorphism. If $\mathrm{GL}_n(F)$ were finitely generated, so would $F^\times$.

  However, $\mathbb{F}^\times$ is not finitely generated since it contains $\mathbb{Q}^\times$.

- In the non-abelian setting, a subgroup of a finitely generated group is not necessarily finitely generated. Let
$$G = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{R}).$$
Let
$$H = \left\{ g \in G \;\middle|\; g = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \leq G.$$
Check that
$$H = \left\{ \begin{pmatrix} 1 & n/2^m \\ 0 & 1 \end{pmatrix} \;\middle|\; n, m \in \mathbb{Z} \right\}.$$
This is not finitely generated. H is isomorphic to the additive group of rationals with power-of-2 denominators. The span of any finite set
$$S = \{\frac{n_1}{2^{m_1}}, \dots, \frac{n_k}{2^{m_k}}\},$$
cannot contain any rational with a denominator larger than $2^{\max m_i}$.

Exercise 1.16. *Can any non-empty finite set S be given the structure of a group? What if S is countable? What if it is any set?*

Solution. In the case $|S| = n$, there is an obvious isomorphism to $\mathbb{Z}/n\mathbb{Z}$. If $|S| = \aleph_0$, there is an obvious isomorphism to $\mathbb{Z}$.

If S is a set of sets, the symmetric difference $A \Delta B = (A \setminus B) \cup (B \setminus A)$ gives a group structure. Thus in pure set theory, any set can be given the structure of a group.

What if the elements of S are not sets? ■

1.2 Orders of Elements

Lemma 1.17. *Let G be a group. If $x^m = x^n = 1$, then $x^{(m,n)} = 1$.*

Proof. Bezout's identity. ■

Corollary 1.18. *If $x^\alpha = 1$, then $\operatorname{ord} x \mid \alpha$.*

Proof. $(\operatorname{ord} x, \alpha) \leq \operatorname{ord} x$ by elementary number theory. But $x^{(\operatorname{ord} x, \alpha)} = 1$ (by the previous lemma) gives $(\operatorname{ord} x, \alpha) \geq \operatorname{ord} x$ by minimality of $\operatorname{ord} x$. Thus $(\operatorname{ord} x, \alpha) = \operatorname{ord} x$ so $\operatorname{ord} x \mid \alpha$. ■

Lemma 1.19. *Let G be a group.*

(i) If $\operatorname{ord} x = \infty$, then $\operatorname{ord} x^k = \infty$ for every $k \in \mathbb{Z}^\times$.

(ii) If $\operatorname{ord} x = n < \infty$, then $\operatorname{ord} x^k = n/(n,k)$.

Proof. It suffices to prove the second statement. Let $y = x^k$ and $d = (n,k)$. Write $n = \widetilde{n} d$ and $k = \widetilde{k} d$. Suppose $y^m = 1$. Then by the previous corollary, $n \mid mk$ and so $\widetilde{n} \mid m\widetilde{k} \implies \widetilde{n} \mid m$.

Thus $m \geq \widetilde{n}$. But $y^{\widetilde{n}} = x^{k\widetilde{n}} = x^{n\widetilde{k}} = 1$. Thus $\operatorname{ord} y = \widetilde{n}$. ■

Lemma 1.20. *Let $H = \langle x \rangle$.*

(i) If $\operatorname{ord} x = \infty$, then H is generated by x^a iff $a = \pm 1$.

(ii) If $\operatorname{ord} x = n$, then H is generated by x^a iff $(a,n) = 1$.

Proof. For the first case, assume $H = \mathbb{Z}$ by isomorphism. $\mathbb{Z} = a\mathbb{Z} \implies \exists n \in \mathbb{Z}$ s.t. $an = 1$. Then $|a| = 1$. The converse is by inspection.

For the second, assume $H = \mathbb{Z}/n\mathbb{Z}$ by isomorphism. Let $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$ be a generator. Then $\operatorname{ord} \bar{a} = n$. By the previous lemma, $\operatorname{ord} \bar{a} = n/(n,a)$ (since $\operatorname{ord} \bar{1} = n$). ■

Lecture 5.
Monday
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1.3 Generation of groups

Lemma 1.21. *Let G be a group and let $a, b \in G$ commute. Let* $\operatorname{ord} a = m$, $\operatorname{ord} b = n$, $\operatorname{lcm}(m, n) = \ell$. *Then* $\operatorname{ord} ab \mid \ell$. *If* $(m, n) = 1$, *then* $\operatorname{ord} ab = \ell$.

Proof. $(ab)^\ell = a^\ell b^\ell = 1$.

Now suppose that $(m, n) = 1$. Let $d = \operatorname{ord} ab \implies d \mid \ell$. Now

$$\begin{aligned}(ab)^d = 1 &\implies a^d b^d = 1 \\ &\implies a^d = b^{-d}.\end{aligned}$$

Raising to the power m gives $a^{dm} = 1 = b^{-dm}$. Thus $n \mid md \implies n \mid d$ (coprime). Similarly $m \mid d$. Thus $nm = \ell \mid d$. Together with $d \mid \ell$, we get $d = \ell$. ∎

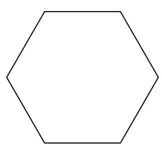
Examples.

- If $(a, b) \neq 1$, we can't say anything. For example, $b = a^{-1}$ gives $\operatorname{ord} ab = 1$.
- If $ab \neq ba$, things can go crazy. For example, $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1/2 \\ 2 & 0 \end{pmatrix}$. Then $a^2 = b^2 = 1$ but $ab = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ has infinite order.

Definition 1.22 (Presentation)**.**

Definition 1.23 (the dihedral group)**.** For $n \geq 3$, the dihedral group D_{2n} is the group of rigid motions of a regular n-gon R_n in $\mathbb{R}^2$.

Remark. A "rigid motion" is an isoemtry: a distance preserving bijection. For example, reflections and rotations. Note how rigid motions being a bijection (when restricted to the n-gon) implies that only those isometries that preserve the n-gon are allowed.



Rigid motions in $\mathbb{R}^n$ are given by $x \mapsto Ax + b$ where $A \in O_n$, the set of orthogonal matrices in M_n.

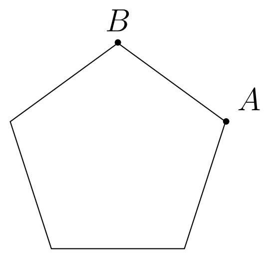
$$(A_1, b_1) \circ (A_2, b_2) = (A_1 A_2, A_1 b_2 + b_1).$$

$A_1 A_2 \in O_n$ so the product is closed. Associativity is inherited from function composition. The identity is $(1, 0)$ and the inverse of (A, b) is $(A^\top, -A^\top b)$.

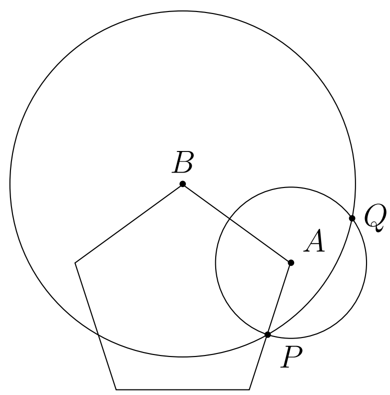
Lemma 1.24. *Every point P on R_n is determined, among all other points on R_n, by its distance from* any *two fixed adjacent vertices of R_n.*

That is, let A and B be adjacent vertices of R_n. Then for any $d_A, d_B \in \mathbb{R}^+$, there is at most one point P on R_n such that $d(P, A) = d_A$ and $d(P, B) = d_B$.

Proof. Look at the edge $\overline{AB}$.



Draw a circle of radius d_A around A and a circle of radius d_B around B. They intersect in at most two points, but they are on opposite sides of $\overline{AB}$. R_n is convex, so every point on R_n lies on one of only one side of $\overline{AB}$. Thus only one of these two points can lie on R_n.



■

Proposition 1.25. $|D_{2n}| = 2n$.

Proof. We first show that $|D_{2n}| \leq 2n$. Start with any two vertices A and B of R_n. Let $g \in D_{2n}$.

Claim. g takes vertices to vertices.

To see this, note that the vertices are special in that they are distinguised from all other points on R_n as follows:

> Let $P \in \mathbb{R}_n$ and $r > 0$ be small. We can find two points P'_r and P''_r on R_n such that $d(P, P'_r) = d(P, P''_r) = r$. If P is *not* a vertex, then $d(P'_r, P''_r) = 2r$. If P *is* a vertex, then $d(P'_r, P''_r) < 2r$.

Thus we can distinguish between P being a vertex or not solely by the distance function. Since g is an isometry (even Lipschitz), this property is preserved. Thus g takes vertices to vertices.

Claim. g preserves adjacency of vertices.

Fix a vertex A on R_n. Then $d(P, A)$ for a vertex distinct from A is minimized when P is adjacent to A. Since g preserves distances, g must take adjacent vertices to adjacent vertices.

Combining these two claims, we have proven that for any $P \in R_n$, $g(P)$ is uniquely determined by its distance from $g(A)$ and $g(B)$, where A and B are any two adjacent vertices. Thus g is determined by $g(A)$ and $g(B)$.

By the first claim, there are n possible choices for $g(A)$. By the second claim, there are 2 possible choices for $g(B)$. Thus there are at most $2n$ possible g's.

Finally, we can produce $2n$ distinct elements as follows.

- Consider the n rotations: rotate by $2\pi k/n$ for $k \in n$.
- The n reflections:
  - For odd n, reflect over the line through a vertex and the midpoint of the opposite edge.
  - For even n, reflect over the line through two opposite vertices or through two opposite midpoints.

Each reflection fixes exactly two points. Any non-trivial rotation fixes no points. Thus the $2n$ elements are distinct. ■

Notation. Let r denote the counter-clockwise rotation by $2\pi/n$ and let s denote the reflection over the line through some fixed vertex V_0.

Then $r^n = s^2 = 1$.

Observe that $\{1, r, r^2, \dots, r^{n-1}\}$ gives all the rotations in D_{2n}.

Lemma 1.26. *All reflections in D_{2n} are given by $\{s, rs, r^2s, \dots, r^{n-1}s\}$.*

Proof. All of these elements are distinct, since $r^k \neq 1$ for $0 < k < n$. None of these elements are rotations, since if $r^k s = r^m$ for some $k, m \in n$, then $s = r^{m-k}$, which is a contradiction. ■

Theorem 1.27. $|D_{2n}| = 2n$ *and* $D_{2n} = \{1, r, \dots, r^{n-1}, s, rs, \dots, r^{n-1}s\}$.

Proposition 1.28. *In D_{2n}, $sr = r^{-1}s$.*

Proof. From theorem 1.26, we know that rs is a reflection. Thus $(rs)(rs) = 1$, which immediately gives $sr = r^{-1}s$. ■

Next lecture: $D_{2n} = \langle r, s \mid r^n = s^2 = 1, sr = r^{-1}s \rangle$.

Lecture 5