E0 206: The Theorist's Toolkit

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Contents

Lectures

The course

Instructor: Anand Louis

Lecture 1. Wednesday August 6

Evaluation:

(35%) Roughly one homework every two weeks

(25%) Study project

(40%) Final exam

Tools which are generally useful in theoretical computer science

Chapter I

Probabilistic methods

Proposition I.1. *In any graph $G = (V, E)$, there is a cut $S \subseteq V$ such that $|E(S, V \setminus S)| \geq \frac{1}{2}|E|$.*

I will call such cuts half-cuts.

Proof. Pick a uniform random set $S \subseteq V$. Each vertex is in S with probability $1/2$. Thus each edge is in $E(S, V \setminus S)$ with probability $1/2$. The expected value of $E(S, V \setminus S)$ is thus $\frac{1}{2}|E|$. Therefore, there exists a subset of vertices for which this quantity is at least $\frac{1}{2}|E|$. ∎

We've used the following.

Lemma I.2. *For any random variable X,*

$$\Pr(X \geq \mathbf{E}\, X) > 0.$$

Proof. Assume the contrary. Then $\mathbf{E}[X] - X > 0$ almost surely. By positivity, the equality in $\mathbf{E}[\mathbf{E}[X] - X] \geq 0$ holds iff $\mathbf{E}[X] - X = 0$ almost surely. Contradiction. ∎

We can do slightly better than proposition I.1.

Proposition I.3. *Let S be a random cut of $G = (V, E)$ and $X = |E(S, V \setminus S)|$. Then $\Pr(X \geq (1-\delta)\,\mathbf{E}[X]) \geq \frac{\delta}{1+\delta}$.*

Proof. Let $Y = |E| - X \geq 0$. Now $X \leq (1-\delta)\,\mathbf{E}[X]$ is equivalent to $Y \geq |E| - \frac{1-\delta}{2}|E| = \frac{1+\delta}{2}|E|$. Since $\mathbf{E}[Y] = \frac{1}{2}|E|$, Markov's inequality gives

$$\Pr\left\{Y \geq \frac{1+\delta}{2}|E|\right\} \leq \frac{1}{1+\delta}.$$

Thus $\Pr(X > (1-\delta)\,\mathbf{E}\, X) \geq \frac{\delta}{1+\delta}$. ∎

Thus, for any $\delta > 0$ and $\varepsilon > 0$, we can sample a random cut $\log_{1+\delta} \varepsilon^{-1}$ times so that the largest of these cuts has size at least $(1-\delta)\frac{|E|}{2}$ with probability at least $1-\varepsilon$. If we choose δ such that $(1-\delta)\frac{|E|}{2} > \frac{|E|-1}{2}$, we get a half-cut.

I.1 Derandomization

We can derandomize the algorithm discussed previously. Label the vertices 1 through n.

$$\frac{|E|}{2} = \mathbf{E}[X] = \frac{1}{2}\,\mathbf{E}[X \mid 1 \in S] + \frac{1}{2}\,\mathbf{E}[X \mid 1 \notin S]$$

At least one of these has to be at least $\mathbf{E}\,X$. In this case, by symmetry, both are equal.

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{E}[X \mid 1 \in S] \\ &= \frac{1}{2}\,\mathbf{E}[X \mid 1 \in S, 2 \in S] + \frac{1}{2}\,\mathbf{E}[X \mid 1 \in S, 2 \notin S]. \end{aligned}$$

Let X_i denote whether $i \in S$ or $i \notin S$. If we could compare $\mathbf{E}[X \mid X_1, \ldots, X_k, k+1 \in S]$ and $\mathbf{E}[X \mid X_1, \ldots, X_k, k+1 \notin S]$, we would get an algorithm. Writing X as $\sum_{u,v} \mathbf{1}_{u,v \text{ cross the cut}}$. Taking the difference of both conditional expectations shows that adding $k+1$ locally greedily to one of S and $V \setminus S$ has the greater expectation of X. This gives a linear time algorithm.

Exercise I.4 (Local search for half-cut)**.** *Start with an arbitrary cut $S_0 \subseteq V$. If there is a vertex $v \in V$ such that moving it to the other side increases the (edge-)size of the cut, do it. This process terminates and yields a half-cut.*