MA 231: Topology

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August 2025

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The course

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Course Details

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- **Timing:** Tue, Thu 8:30 – 10:00 am.
- **Venue:** LH-1.
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- **Tutorial timing:** TBA
- **Microsoft Teams Code:** `tbjeh8w`

Examinations and Grades. Grades will be based on the assignments, midterm examination and final examinations with the following weightages.

(10%) Assignments, posted roughly once a week.

(40%) Midterm

(50%) Final exam

Suggested books:

(1) Armstrong, M. A., Basic Topology, Springer (India), 2004.

(2) Munkres, J. R., Topology, Pearson Education, 2005.

(3) Viro, O. Ya., Ivanov, O. A., Netsvetaev, N., and Kharlamov, V. M., Elementary Topology: Problem Textbook, AMS, 2008.

We will mostly follow the book **Elementary Topology: Problem Textbook** by **Viro et al.**, which is available online.

Additional Resources. This course was taught online in 2021. The lectures are online as are the whiteboards. If you want to printout the whiteboards, please use the compact version.

Note that students are responsible for all the material covered in the lectures this semester, which is likely to be more than that in the above resources. Experience also suggests that offline lectures are more effective. Thus, it is wise to use the above as supplements, not substitutes, for the lectures.

Syllabus. Open and closed sets, Continuous functions, Metric topology, Product topology, Connectedness and path-connectedness, Compactness, Countability axioms, Separation axioms, Complete metric spaces, Quotient topology, Topological groups, Orbit spaces, Urysohn's lemma, Metrizability, Baire Category theorem.

Study of properties associated with continuity and limits.

Examples.

- Any continuous function $f\colon [0,1] \to \{0,1\}$ is constant, by the intermediate value theorem.
- Any continuous function $f\colon [0,1] \to \mathbb{R}$ is bounded.

The first example illustrates connectivity, and the second compactness. We study continuous functions $f\colon X \to Y$ between topological spaces. Topological spaces must come with a notion of continuity and limits. They must coincide with our usual understanding in familiar settings such as $\mathbb{R}^n$, convergence of sequences, and so on. We should be able to construct new topological spaces from old ones.

Chapter I

Topological spaces

Our first attempt is *metric spaces*.

Definition I.1 (Metric space)**.** A *metric* on a set X is a function $d\colon X \times X \to [0, \infty)$ such that

(M1) $d(x, x) = 0$ for all $x \in X$,

(M2) [Symmetry] $d(x_1, x_2) = d(x_2, x_1)$ for all $x_1, x_2 \in X$, and

(M3) [Triangle inequality] $d(a, c) \leq d(a, b) + d(b, c)$ for all $a, b, c \in X$.

(M4) [Positivity] $d(x_1, x_2) = 0$ implies that $x_1 = x_2$.

The pair (X, d) is called a *metric space*.

Remark. If positivity does not hold, then we can define $a \sim b$ on X iff $d(a, b) = 0$. d then induces a metric on the quotient space $X/{\sim}$.

In the early 1900's, it wasn't clear whether the notion of a topological space is even required. In fact, Hausdorff introduced topological spaces in the second edition of his textbook, but went back to metric spaces in the third edition.

Here is a non-example of a metric space. Consider $X = \mathbb{R}^{\mathbb{R}}$ and say that $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$.

Claim. *There is no metric on $\mathbb{R}^{\mathbb{R}}$ such that $f_n \to f$ iff $d(f_n, f) \to 0$.*

We will prove this much later, once we study product topologies and first countability.

Suppose (X, d) is a metric space and $\sim$ an equivalence relation on X. We want that any continuous function $f\colon X \to Y$ for which $a \sim b$ implies $f(a) = f(b)$, the induced function $\widetilde{f}\colon X/{\sim} \to Y$ is continuous. This is similar to how group homomorphisms induce a homomorphism on quotient groups. In general, such quotient metrics do not exist.

We will come back to this with Hausdorff things.

Examples.

- Consider $\mathbb{R}$ with $a \sim b$ iff $a - b \in \mathbb{Q}$.
- $\mathbb{R}^2$ with equivalence classes as Gadgil drew.

The standard approach is to abstract properties of *open sets*. In the metric space (X, d), a set $U \subseteq X$ is open if for every $x \in U$ there exists an $\varepsilon > 0$ such that the ε-neighborhood of x is contained in U.

Definition I.2 (Topological space)**.** Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X such that

(T1) $\varnothing, X \in \mathcal{T}$;

(T2) if $\{U_\alpha\}_{\alpha \in I}$ is a subcollection of $\mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$;

(T3) if $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

Sets in $\mathcal{T}$ are called *open sets* and the pair $(X, \mathcal{T})$ is called a *topological space*. Complements of open sets are called *closed sets*.

Examples.

- [Discrete topology] For any set X, 2^X is a topology on X.
- [Indiscrete topology] For any set X, $\{\varnothing, X\}$ is a topology on X.
- [Standard topology on $\mathbb{R}$] The collection of all subsets that can be expressed as a union of open intervals is a topology on $\mathbb{R}$. Alternatively and more simply, the metric space definition also yields the same topology.

We will prove these examples one-by-one. We'll start skipping such proofs later in the course.

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Proposition I.3 (Discrete topology)**.** *For any set X, 2^X is a topology on X.*

Proof. $\varnothing$ and X are in 2^X by definition.

If $V_1, \ldots, V_n \subseteq X$, then their intersection is also a subset of X.

If V_α are subsets of X, so is their union. ■

Proposition I.4 (Indiscrete topology)**.** *For any set X, $\{\varnothing, X\}$ is a topology on X.*

Proof. $\varnothing, X \in \{\varnothing, X\}$ by definition.

If $\{V_\alpha\}_{\alpha \in I} \subseteq \{\varnothing, X\}$, then the union is similarly either $\varnothing$ (if all V_α are $\varnothing$) or X (if any V_α is X).

If $V_1, \ldots, V_n$ are in $\{\varnothing, X\}$, then we have two cases.

- If all V_i are X, then the intersection is X.
- Otherwise, one of them is $\varnothing$, so the intersection is $\varnothing$.

In either case, we have $V_1 \cap \cdots \cap V_n \in \{\varnothing, X\}$. ■

Proposition I.5 (Standard topology on $\mathbb{R}$)**.** *Let $\mathcal{T}$ be the collection of all subsets V or $\mathbb{R}$ such that for each $x \in V$, there exists $a, b \in \mathbb{R}$ such that $a < x < b$ and $(a, b) \subseteq V$.*

Proof. $\varnothing \in \mathcal{T}$ vacuously. $\mathbb{R} \in \mathcal{T}$ because for each x, we may choose $a = x - 1$ and $b = x + 1$.

Suppose $\{V_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}$ and $x \in \bigcup_{\alpha \in I} V_\alpha$. Then $x \in V$ for some $V \in \{V_\alpha\}_\alpha$. This gives a corresponding $a < x < b$ such that $(a, b) \in V$, so $(a, b) \in \bigcup_\alpha V_\alpha$.

For $V_1, \ldots, V_n \in \mathcal{T}$, let $x \in V_1 \cap \cdots \cap V_n$. There for each $i \in [n]$, we have $a_i < x < b_i$ such that $(a_i, b_i) \subseteq V_i$. Let $a = \max_i a_i$ and $b = \min_i b_i$. Then $a < x < b$ and

Claim. *For $1 \leq i \leq n$, $(a, b) \in V_i$.*

Proof. Since $a_i \leq a$ and $b \leq b_i$, we have $(a, b) \subseteq (a_i, b_i) \subseteq V_i$. □

Thus (a, b) is in the intersection. ■

Question I.6. *What is the cardinality of the set of all topologies on $\mathbb{R}$?*

The conjecture is that it should be the same as $2^{2^{\mathbb{R}}}$. This MSE post says that is true.

We continue with more examples.

Examples.

- [Cofinite topology] For any set X,

$$\mathcal{T} = \{V \subseteq X : X \setminus V \text{ is finite}\} \cup \{\varnothing\}$$

  is a topology on X. In contrast, the "finite topology", consisting of all finite subsets of X together with X itself is *not* a topology if X is infinite. If X is finite, it simply reduces to the discrete topology.

  The proof is mundane.

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- Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ (or $\omega + 1$) and
$$\mathcal{T} = 2^{\mathbb{N}} \cup \{A \cup \{\infty\} : A \subseteq \mathbb{N} \text{ is cofinite}\}.$$
This is a topology, and the set of convergent This is non-trivial. what?

Proposition I.7. *$\mathcal{T}$ is a topology on $\overline{\mathbb{N}}$.*

Proof. $\varnothing$ is finite. $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, and $\mathbb{N}$ is cofinite.

Let $\{V_\alpha\}_\alpha \subseteq \mathcal{T}$.

- If all of them are subsets of $\mathbb{N}$, so is the union.
- Otherwise, the collection contains a set $A \cup \{\infty\}$ for some cofinite set A. The union's complement is a subset of A's complement, hence cofinite.

Let $V_1, \ldots, V_n \in \mathcal{T}$.

- If any of them is a subset of $\mathbb{N}$, so is the intersection.
- Otherwise, $V_i = A_i \cup \{\infty\}$ for all i where A_i is cofinite. The intersection is $(\bigcap_i A_i) \cup \{\infty\}$, and the intersection of the A_i's is cofinite. ■

Theorem I.8 (Cotopology)**.** *A collection of subsets S of X form the closed sets of a topological space on X if the following hold:*

(1) $\varnothing, X \in S$,

(2) S is closed under finite unions, and

(3) S is closed under arbitrary intersections.

In proposition I.5 we had seen the standard topology on $\mathbb{R}$ defined as

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$$\mathcal{T} = \{V \subseteq \mathbb{R} : \forall x \in V, \text{ there exist } a < x < b \text{ such that } (a, b) \in V\}.$$

Proposition I.9. *Each interval (a, b) where $-\infty \leq a < b \leq \infty$ is open.*

Proof. Trivial. ■

Theorem I.10. *$\mathcal{T}$ is the set of unions of open intervals. That is,*

$$\mathcal{T} = \left\{ \bigcup_{\alpha \in A} (a_\alpha, b_\alpha) : \{(a_\alpha, b_\alpha)\}_{\alpha \in A} \text{ where } a_\alpha, b_\alpha \in [-\infty, \infty] \right\}$$

Proof. Suppose $S \in \mathcal{T}$. Then for each $x \in S$, we have $a_x < x < b_x$ such that $(a_x, b_x) \in S$. Thus $S = \bigcup_{x \in S} (a_x, b_x)$.

Conversely, if $S = \bigcup_{\alpha \in A} (a_\alpha, b_\alpha)$, then for every $x \in S$ there is an α such that $x \in (a_\alpha, b_\alpha) \subseteq S$. Thus $S \in \mathcal{T}$. ■

Theorem I.11. *Any open set $V \subseteq \mathbb{R}$ is the union of* disjoint *open intervals (allowing $\pm\infty$ as endpoints).*

We will cover multiple proofs for this.

Lemma I.12. *The class of equivalence relations on a set S is closed under arbitrary intersections. Thus, for any relation $R \subseteq S \times S$, there exists a minimal equivalence relation $E \subseteq S \times S$ containing R.*

Proof. All properties of equivalence relations are of the form "if aRb"-ish. Thus, defining $\mathcal{E}$ to be the set of equivalence relations on S containing R, their intersection gives what we desired.

Here is a concrete description for the minimal equivalence relation. Declare $a \sim b$ iff there exists a finite sequence $a = c_1, \ldots, c_n = b$ such that for all $i \in [n]$, either $c_1 R c_2$ or $c_2 R c_1$. Then $\sim$ is the required minimal equivalence relation.

- This is trivially an equivalence relation containing R.
- For every equivalence relation E containing R, any two elements connected by such a (finite) path must be related. ■

Proof of theorem I.11. Let $V = \bigcup_{\alpha \in A} (a_\alpha, b_\alpha)$. Define the relation R on A by $(\alpha, \beta) \in R \iff (a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) \neq \varnothing$. Call its symmetric, reflexive closure $\sim$. Let $\mathcal{I} = \{\bigcup_{\alpha \in B} (a_\alpha, b_\alpha) : B \in A/{\sim}\}$.

Each element of $\mathcal{I}$ is an interval since $\bigcup_{\alpha \in B} (a_\alpha, b_\alpha) = (a, b)$ where $a = \inf_{\alpha \in B} a_\alpha$ and $b = \inf_{\alpha \in B} b_\alpha$. For any $a < x < b$, there exists $a_\alpha \in (a, x)$ and $b_\beta \in (x, b)$. Since there is a path from α to β, we get that $x \in (a, b)$.

We need to show that elements of $\mathcal{I}$ are disjoint open intervals. Suppose $I_1, I_2 \in \mathcal{I}$ have a common point x. Let $x \in (a_\alpha, b_\alpha) \subseteq I_1$ and also $x \in (a_\beta, b_\beta) \subseteq I_2$. Then $(a_\alpha, b_\alpha) \sim (a_\beta, b_\beta)$, so $I_1 = I_2$. ■

A simpler description of this equivalence relation is $(a_1, b_1) \sim (a_2, b_2)$ iff the line joining these intervals is contained in V.

Definition I.13 (Cantor set)**.** The *Cantor set* is defined to be the set

$$C = \left\{ \sum_{i=1}^{\infty} \frac{2a_i}{3^i} : a \in 0, 1^{\mathbb{N}} \right\}$$

It is easy to see that this is a subset of $[0, 1]$. It is also easy to show that $(\frac{1}{3}, \frac{2}{3}) \notin X$.

- If $a_1 = 0$, then the rest of the summation is at most $\frac{1}{3}$.

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- If $a_1 = 1$, then the first term is already $\frac{2}{3}$.

For $\alpha_1, \ldots, \alpha_k \in \{0,1\}$, define

$$C_{\alpha_1,\ldots,\alpha_k} := \left[\sum_{i=1}^{k} \frac{2\alpha_i}{3^i}, \sum_{i=1}^{k} \frac{2\alpha_i}{3^i} + \frac{1}{3^k}\right]$$

In particular,

$$\begin{aligned} C_0 &= \left\{\sum_{i=1}^{\infty} \frac{2a_i}{3^i} : a \in \{0, .5, 1\}^{\mathbb{N}}, a_1 = 0\right\}, \\ C_1 &= \left\{\sum_{i=1}^{\infty} \frac{2a_i}{3^i} : a \in \{0, .5, 1\}^{\mathbb{N}}, a_1 = 1\right\}, \\ C_{0,0} &= \left\{\sum_{i=1}^{\infty} \frac{2a_i}{3^i} : a \in \{0, .5, 1\}^{\mathbb{N}}, a_1 = 0, a_2 = 0\right\}, \\ C_{0,1} &= \left\{\sum_{i=1}^{\infty} \frac{2a_i}{3^i} : a \in \{0, .5, 1\}^{\mathbb{N}}, a_1 = 0, a_2 = 1\right\}, \end{aligned}$$

and so on. Note that if $x \in C$, then

$$x = \sum_{i=1}^{\infty} \frac{2a_i}{3^i} \tag{I.1}$$

for some $(a_i)_i \in \{0,1\}^{\mathbb{N}}$. Then for each k, $x \in C_{a_1,\ldots,a_k}$.

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Lemma I.14. *For each $\alpha_1, \ldots, \alpha_k, \alpha_{k+1} \in \{0,1\}$,*

(1) $C_{\alpha_1,\ldots,\alpha_k,\alpha_{k+1}} \in C_{\alpha_1,\ldots,\alpha_k}$, and

(2) $C_{\alpha_1,\ldots,\alpha_k,0} \cap C_{\alpha_1,\ldots,\alpha_k,1} = \varnothing$.

Proof. Trivial. ∎

Let $C^{(k)} = \bigcup_{\alpha \in \{0,1\}^{[k]}} C_{\alpha_1,\ldots,\alpha_k}$. This is a finite union of closed intervals, hence closed. However, we also saw that $C \subseteq C^{(k)}$ for each k via equation (I.1). Thus $C \subseteq \bigcap_{k=1}^{\infty} C^{(k)}$.

Proposition I.15. *$C = \bigcap_{k=1}^{\infty} C^{(k)}$.*

Proof. Let $x \in \bigcap_{k=1}^{\infty} C^{(k)}$. That is, for each k, there exists $\alpha \in 2^{[k]}$ such that $x \in C_{\alpha_1,\ldots,\alpha_k}$.

Claim. *For $l \geq k$, if $x \in C_{\alpha_1,\ldots,\alpha_k}$ and $x \in C_{\beta_1,\ldots,\beta_l}$, then $\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k$.*

Proof. Lemma I.14 implies that $C_{\beta_1,\ldots,\beta_l} \subseteq C_{\beta_1,\ldots,\beta_k}$. Its second part implies $\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k$ by looking at the first point of difference, should it exist. $\square$

Thus, there exists a unique sequence $\alpha_1, \alpha_2, \ldots$ such that $x \in C_{\alpha_1,\ldots,\alpha_k}$ for all k.

Claim. $x = \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i}$.

Proof. For $k \geq 1$, $\sum_{i=1}^{k} \frac{2\alpha_i}{3^i} \in C_{\alpha_1,\ldots,\alpha_k}$. But so is x. Thus

$$\left| x - \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i} \right| \leq \frac{1}{3^k}.$$

Since this holds for all k, these two nubers are the same. $\square$

Since x was arbitrary, $\bigcap_{k=1}^{\infty} C^{(k)} \subseteq C$. $\blacksquare$

Corollary I.16. *As an intersection of closed sets, C is closed.*

I.1 Bases

We say that the natural topology $\mathcal{T}$ on $\mathbb{R}$ is the collection of unions of open intervals. We say that open intervals are a *base* for $\mathcal{T}$.

Definition I.17 (Base)**.** A collection of subsets $\mathcal{B}$ of X is said to be a base for a topology $\mathcal{T}$ on X if

$$\mathcal{T} = \{\bigcup_{\alpha \in I} B_\alpha : \{B_\alpha\} \subseteq \mathcal{B}\}.$$

Elements of $\mathcal{B}$ are called *basic open sets.*

Question I.18.

- *Is there a more convenient characterization of the bases for a topology?*
- *When is a collection $\mathcal{B} \subseteq 2^X$ a base for* some *topology?*
- *When are two collections $\mathcal{B}_1$ and $\mathcal{B}_2$ bases for the same topology?*

Examples.

- A base for the discrete topology on X is $\mathcal{B} = \binom{X}{1}$. This is the minimum (not just minimal) base for the discrete topology.

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- A base for the indiscrete topology on X is $\mathcal{B} = \{X\}$. This is one of the only two bases, the other being $\{\varnothing, X\}$.
- A base for the cofinite topology on any infinite set X is
$$\mathcal{B} = \{X \setminus F : 2^{2^{100}} < |F| < \infty\}.$$
- A base for the topology on $\mathbb{R}$ is
$$\{(a, b) : a, b \in \mathbb{Q}\}.$$
This is because any interval with real (or infinite) endpoints can be written as the union of intervals with rational endpoints tending towards those. This depends on theorem I.19(2) that follows.

Theorem I.19. *Let $\mathcal{T}$ be a topology on X and $\widehat{\mathcal{B}} \subseteq 2^X$.*

(1) $\widehat{\mathcal{B}}$ is a base for the topology $\mathcal{T}$ if every $V \in \widehat{\mathcal{B}}$ is open and for every $V \in \mathcal{T}$ and $x \in V$, there exists a $W \in \widehat{\mathcal{B}}$ such that $x \in W \subseteq V$.

(2) Let $\mathcal{B}$ be a given base for $\mathcal{T}$. Then $\widehat{\mathcal{B}}$ is a base for $\mathcal{T}$ if every $V \in \widehat{\mathcal{B}}$ is open and for all $V \in \mathcal{B}$ and $x \in V$, there exists a $W \in \widehat{\mathcal{B}}$ such that $x \in W \subseteq V$.

Proof. Assume that every $V \in \widehat{\mathcal{B}}$ is open. Since each $V \in \widehat{\mathcal{B}}$ is open, their unions are in $\mathcal{T}$.

(1) Assume that for all $x \in V \in \mathcal{T}$, there is a neighborhood $W \in \widehat{\mathcal{B}}$ of x that is contained within V.

For any $V \in \mathcal{T}$, simply write $V = \bigcup_{x \in V} W_x$ where W_x is gotten from the hypothesis.

(2) Assume the requisite hypothesis once more. For any $V \in \mathcal{T}$, write it as $\bigcup_{\alpha \in I} B_\alpha$ for a $\{B_\alpha\} \subseteq \mathcal{B}$. Further write each B_α as $B_\alpha = \bigcup_{x \in B_\alpha} W_x$. ■

A converse of theorem I.19 is also true.

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Theorem I.20. *Let $\mathcal{B}$ be a base for a topology $\mathcal{T}$ on X. Then a set $V \subseteq X$ is open iff for all $x \in V$, there exists a $W \in \mathcal{B}$ such that $x \in W \subseteq V$.*

Proof. Let V be open. Then, since $\mathcal{B}$ is a basis, we simply write V as a union of open sets, and thus obtain a W for every $x \in V$.

Now suppose that for every $x \in V$ there exists a $W_x \in \mathcal{B}$ with $x \in W_x \subseteq V$. Then $V = \bigcup_{x \in V} W_x$ is open as each basic open set is open. ∎

Question I.21. *Let $\mathcal{B} \subseteq 2^X$. When is $\mathcal{B}$ a base for* some *topology on X? That is, when is*

$$\mathcal{T} = \{\bigcup_{\alpha \in I} B_\alpha : \{B_\alpha\}_\alpha \subseteq \mathcal{B}\}$$

a topology?

It is easy to see that the resultant $\mathcal{T}$ is closed under arbitrary unions. It also contains the null set. We only require that it is closed under finite intersections, and contains X. Thus we need:

- $\bigcup \mathcal{B} = X$.
- For $V, W \in \mathcal{T}$, $V \cap W$ "must" belong to $\mathcal{T}$. Write $V = \bigcup_{\alpha \in I} V_\alpha$ and $W = \bigcup_{\beta \in J} W_\beta$ where V_α and W_β are in $\mathcal{B}$.

  Now

  $$V \cap W = \bigcup_{(\alpha,\beta) \in I \times J} V_\alpha \cap W_\beta.$$

  Theorem I.20 shows that, since $V_\alpha \cap W_\beta$ must be open, we "must" have for every $x \in V_\alpha \cap W_\beta$ the existence of a U_x such that $x \in U_x \subseteq V_\alpha \cap W_\beta$. In that case,

  $$V \cap W = \bigcup_{(\alpha,\beta) \in I \times J} \left(\bigcup_{x \in V_\alpha \cap W_\beta} U_x \right)$$

  would be in $\mathcal{T}$.

This yields the following criterion.

Theorem I.22 (Base criterion)**.** *$\mathcal{B} \subseteq 2^X$ is a base for a topology on X iff*

(1) $\bigcup \mathcal{B} = X$, and

(2) for any $V, W \in \mathcal{B}$ and $x \in V \cap W$, there exists a $U \in \mathcal{B}$ such that $x \in U \subseteq V \cap W$.

Proof. Let $\mathcal{T}$ be the set of unions of elements in $\mathcal{B}$. If $\mathcal{B}$ is a base, both the conditions hold easily by theorem I.20.

For the converse, assume that the conditions hold. $\varnothing, X \in \mathcal{T}$ trivially. Closure under arbitrary unions holds trivially. Write $V = \bigcup_{\alpha \in I} V_\alpha$ and $W = \bigcup_{\beta \in J} W_\beta$ where V_α and W_β are in $\mathcal{B}$. Now fix $x \in V \cap W$. Then $x \in V_{\alpha^*} \cap W_{\beta^*}$ for some $(\alpha^*, \beta^*) \in I \times J$. Thus, by the second hypothesis, there exists a $U \in \mathcal{B}$ such that $x \in U \subseteq V_{\alpha^*} \cap W_{\beta^*} \subseteq V \cap W$. By theorem I.19, $\mathcal{B}$ is a base. ■

Examples.

- On $\mathbb{R}^2$, let $\mathcal{B} = \{(a,b) \times (c,d) : a,b,c,d \in \mathbb{R}\}$. The union of these is $\mathbb{R}^2$. Let $(a_1,b_1) \times (c_1,d_1)$ and $(a_2,b_2) \times (c_2,d_2)$ be in $\mathcal{B}$. Their intersection is

  $$(\max\{a_1,a_2\}, \min\{b_1,b_2\}) \times (\max\{c_1,c_2\}, \min\{d_1,d_2\}) \in \mathcal{B}.$$

  Thus by theorem I.22, $\mathcal{B}$ is a base on $\mathbb{R}^2$.

- [Metric spaces] Let (X,d) be a metric space and $\mathcal{B} = \{B(x;r) : x \in X, r \in (0,\infty)\}$.

  Theorem I.23. *$\mathcal{B}$ is a base for a topology on X.*

  Proof. $\bigcup \mathcal{B} = X$ as each $x \in B(x;1)$. Let $V_1 = B(x_1;r_1)$ and $V_2 = B(x_2;r_2)$. For any $x \in V_1 \cap V_2$, let $r = \min\{r_1 - d(x,x_1), r_2 - d(x,x_2)\} > 0$. Then for any $y \in B(x;r)$, we have that

  $$d(y,x_1) \leq d(y,x) + d(x,x_1) < r_1 - d(x,x_1) + d(x,x_1) = r_1.$$

  Similarly $d(x_2,y) < r_2$. Thus $y \in V_1 \cap V_2$. Since y was arbitrary, $B(x;r_1) \subseteq V_1 \cap V_2$. ■

  We did not use positivity (distinct points having a positive distance) in this proof. Thus, such pseudometrics still generate a nice topology.

"Closeness" in arithmetic/algebra vs analysis/geometry.

- For $r \in (0,\infty)$, $x,y \in \mathbb{R}$ are r-close if $|x-y| < r$. That is, one cannot tell apart x and y at resolution r. But in an algebraic or purely arithmetic context, we do not have inequalities.

- For $n,m \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, we say $n,m \in \mathbb{Z}$ are d-close if $d \mid (n-m)$, that is, $[n] = [m] \in \mathbb{Z}/d\mathbb{Z}$. d-closeness is transitive in this case. This motivates the arithmetic progression topology.

Proposition I.24 (Arithmetic progression topology)**.** *Let $\mathcal{B} = \{n + d\mathbb{Z} : n \in \mathbb{Z}, d \in \mathbb{Z}^+\}$. This is a base on $\mathbb{Z}$.*

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Proof. $\mathbb{Z} = 0 + 1\mathbb{Z} \in \mathcal{B}$.

For $n_1 + d_1\mathbb{Z}$ and $n_2 + d_2\mathbb{Z}$ in $\mathcal{B}$ and x in their intersection, note that $x \in x + d_1 d_2\mathbb{Z}$. ∎

Note that any set $n + d\mathbb{Z}$ is also closed. That is, each basic open set is also a closed set. But

$$\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{p \text{ prime}} p\mathbb{Z}$$

is open. It cannot be closed, since $\{\pm 1\}$ cannot be open! (Any non-empty open set is the union of infinite sets.) If there were finitely many primes, this would be a finite intersection of closed sets. Hence, there are infinitely many primes (this is Furstenberg's proof of the infinitude of primes).