MA 231: Topology

Naman Mishra

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Contents

Lectures

The course

Textbook:

Lecture 1. Tuesday August 5

(1) Viro, O.Ya., Ivanov, O.A., Netsvetaev, N., and Kharlamov, V.M., Elementary Topology: Problem Textbook, AMS, 2008.

(2) Armstrong, M. A., Basic Topology, Springer (India), 2004.

(3) Munkres, J. R., Topology, Pearson Education, 2005.

Teams Code:

Study of properties associated with continuity and limits.

Examples.

- Any continuous function $f\colon [0,1] \to \{0,1\}$ is constant, by the intermediate value theorem.
- Any continuous function $f\colon [0,1] \to \mathbb{R}$ is bounded.

The first example illustrates connectivity, and the second compactness. We study continuous functions $f\colon X \to Y$ between topological spaces. Topological spaces must come with a notion of continuity and limits. They must coincide with our usual understanding in familiar settings such as $\mathbb{R}^n$, convergence of sequences, and so on. We should be able to construct new topological spaces from old ones.

Our first attempt is *metric spaces*.

Definition .1 (Metric space)**.** A *metric* on a set X is a function $d\colon X \times X \to [0,\infty)$ such that

(1) $d(x,x) = 0$ for all $x \in X$,

(2) [Symmetry] $d(x_1,x_2) = d(x_2,x_1)$ for all $x_1,x_2 \in X$, and

(3) [Triangle inequality] $d(a,c) \leq d(a,b) + d(b,c)$ for all $a,b,c \in X$.

(4) [Positivity] $d(x_1, x_2) = 0$ implies that $x_1 = x_2$.

The pair (X, d) is called a *metric space*.

Remark. If positivity does not hold, then we can define $a \sim b$ on X iff $d(a, b) = 0$. d then induces a metric on the quotient space $X/\sim$.

In the early 1900's, it wasn't clear whether the notion of a topological space is even required. In fact, Hausdorff introduced topological spaces in the second edition of his textbook, but went back to metric spaces in the third edition.

Here is a non-example of a metric space. Consider $X = \mathbb{R}^{\mathbb{R}}$ and say that $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$.

Claim. *There is no metric on* $\mathbb{R}^{\mathbb{R}}$ *such that* $f_n \to f$ *iff* $d(f_n, f) \to 0$.

We will prove this much later, once we study product topologies and first countability.

Suppose (X, d) is a metric space and $\sim$ an equivalence relation on X. We want that any continuous function $f \colon X \to Y$ for which $a \sim b$ implies $f(a) = f(b)$, the induced function $\widetilde{f} \colon X/\sim \to Y$ is continuous. This is similar to how group homomorphisms induce a homomorphism on quotient groups. In general, such quotient metrics do not exist.

We will come back to this with Hausdorff things.

Examples.

- Consider $\mathbb{R}$ with $a \sim b$ iff $a - b \in \mathbb{Q}$.
- $\mathbb{R}^2$ with equivalence classes as Jay drew.

The standard approach is to abstract properties of *open sets*. In the metric space (X, d), a set $U \subseteq X$ is open if for every $x \in U$ there exists an $\varepsilon > 0$ such that the ε-neighborhood of x is contained in U.

Definition .2 (Topological space)**.** Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X such that

(1) $\varnothing, X \in \mathcal{T}$;

(2) if $\{U_\alpha\}_{\alpha \in I}$ is a subcollection of $\mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$;

(3) if $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

Sets in $\mathcal{T}$ are called *open sets* and the pair $(X, \mathcal{T})$ is called a *topological space*.

Examples.

- [Discrete topology] For any set X, 2^X is a topology on X.

- [Indiscrete topology] For any set X, $\{\varnothing, X\}$ is a topology on X.
- [Standard topology on $\mathbb{R}$] The collection of all subsets that can be expressed as a union of open intervals is a topology on $\mathbb{R}$. Alternatively and more simply, the metric space definition also yields the same topology.