# MA 223: Functional Analysis

Naman Mishra

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# Contents

# Lectures

# The Course

**Teams Code:** `xr80svy`

**Lecture 1.** Monday August 4

The course name should be "Linear Functional Analysis".

$X$ and $Y$ are vector spaces. $T: X \to Y$ is a linear map. Given $y \in Y$, we want to "solve", that is, find $x \in X$ such that $Tx = y$.

All that changes from the linear algebra course is that the vector spaces need not be finite.

*Examples.*

- $C(\mathbb{R})$ may be an interesting infinite dimensional vector space.
- $C([0,1])$ and $C((0,1))$ are different spaces. For example, $\frac{1}{x}$.
- $C^k([0,1])$ and $C^k((0,1))$ are similarly different. The first is the set of functions on $[0,1]$ which can be extended to a $k$-times differentiable function on a slightly larger open interval.

The map $f \mapsto f'$ is a linear map between two infinite-dimensional vector spaces. "Solving" in this context is solving linear PDEs.

For determining whether a solution exists for all images, the determinant suffices in the finite-dimensional case. With infinite dimentsions, there's no hope.

We could instead use the rank-nullity theorem.

**Theorem .1** (Rank-Nullity)**.** *Let $X$ and $Y$ be finite dimensional vector spaces, and let $T: X \to Y$ be a linear map. Then*

$$\dim X = \mathcal{N}(T) + \dim \mathcal{R}(T),$$

*where $\mathcal{N}(T)$ and $\dim \mathcal{R}(T)$ are the rank and nullity of $T$, respectively.*

Using this, we can do a lot.

**Lemma .2.** *$T$ is injective iff $\ker(T) = \{0\}$.*

*Proof.* If $T$ is injective, then $\ker T$ contains only the zero vector. Conversely, if $\ker T = \{0\}$, then for any $x \in X$, $Tx = 0$ implies $x = 0$. ■

**Proposition .3.** *Let $X$, $Y$ be vector spaces of dimension $n < \infty$. Then a linear map $T\colon X \to Y$ is surjective iff it is injective.*

*Proof.* Using rank-nullity, $\dim \mathcal{R}(T) = n$, and the only $n$-dimensional subspace of $Y$ is $Y$ itself.

**Claim.** *The only $n$-dimensional subspace of an $n$-dimensional vector space is itself.*

*Proof.* Let $V$ be an $n$-dimensional vector space with an $n$-dimensional subspace $W$. Assume there exists a $v \in V \setminus W$. There exists a basis $\mathcal{B}$ of $V$ containing $v$. Then $\mathcal{B} \setminus \{v\}$ is a basis of $W$, contradicting that it is $n$-dimensional. □

■

But for infinite-dimensional vector spaces, **??** 1 does not hold. In fact, neither does proposition .3. The map $p \mapsto xp$ is an injective map over the space of polynomials, but is not surjective.

**Definition .4** ($\ell^p$ spaces)**.** For any $1 \leq p < \infty$, we define

$$\ell^p := \{x \in \mathbb{R}^{\mathbb{N}} : \sum |x_n|^p < \infty\}.$$

We also define

$$\begin{aligned}
\ell^\infty &:= \{x \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}, \\
C &:= \{x \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n \text{ exists}\}, \\
C_0 &:= \{x \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}, \\
C_{00} &:= \{x \in \mathbb{R}^{\mathbb{N}} : x_n = 0 \text{ for all but finitely many } n\}.
\end{aligned}$$

On the set of sequences, we have natural shift operators

$$\begin{aligned}
S_R(x_1, x_2, \dots) &:= (0, x_1, x_2, \dots), \\
S_L(x_1, x_2, \dots) &:= (x_2, x_3, \dots).
\end{aligned}$$

These are both linear maps. The first is injective but not surjective, while the second is surjective but not injective. Moreover, $S_L \circ S_R = \mathrm{id}$, but

$$(S_R \circ S_L)(x_1, x_2, \dots) = (0, x_2, \dots).$$

We are fucked.

**Theorem .5.** *Let $X = Y = \mathbb{R}^d$ and $T \colon X \to Y$ be a linear map. Suppose*

*(H1) [Approximate solvability] For every $y \in Y$, there exists a sequence $(x_n)_n$ in $X$ such that $Tx_n \to y$ in the Euclidean sense.*

*(H2) [Quantitative injectivity] There exists a constant $C > 0$ such that $\|x\| \leq C\|Tx\|$.*

*Then $T$ is surjective.*

Note that the second hypothesis immediately provides injectivity, since if $Tx = 0$, then $\|x\| \leq C\|Tx\| = 0$.

*Proof 1 (local compactness).* Let $y \in Y$. Then there is a sequence $(x_n)_n$ such that $Tx_n \to y$. By part (H2), we have that $\|x_n\|$ are uniformly bounded. Thus the sequence $(x_n)_n$ admits a subsequential limit $x = \lim_{k\to\infty} x_{n_k}$. Then

$$\|Tx - y\| \leq \|Tx - Tx_{n_k}\| + \|Tx_{n_k} - y\| \to 0.$$

Thus $Tx = y$. ■