UM101: Short Notes

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February 15, 2023

1 Set theory & the real number system

Definition 1.1. The set A along with a successor function S is called a Peano set if it obeys the Peano axioms.

(P1) There is an element called 0 in A.

(P2) For every $a \in A$, its successor $S(a)$ is also in A.

(P3) $\forall\, a \in A, S(a) \neq 0$.

(P4) For any $m, n \in A$, $S(m) = S(n)$ only if $m = n$.

(P5) (principle of mathematical induction) For any set $B \subseteq A$, if $0 \in B$ and $a \in B \Rightarrow S(a) \in B$, then $B = A$.

1.1 The ZFC Axioms

Definition 1.2. A **set** is a well-defined collection of (mathematical) objects, called the *elements* of that set. To say that a is an element of set A, we write $a \in A$. Otherwise, we write $a \notin A$.

Given two sets A and B, we say that:

($A \subseteq B$) A is a subset of B, *i.e.*, every element of A is an element of B.

($A \nsubseteq B$) A is not a subset of B, *i.e.*, there is some element in A which is not an element of B.

($A \subsetneq B$) A is a proper subset of B, *i.e.*, $A \subseteq B$ but $\exists\, b \in B$ such that $b \notin A$.

Axiom 1.1 (the basic axiom)**.** Every object is a set.

Axiom 1.2 (axiom of extension)**.** Two sets A, B are equal if they have exactly the same elements. In other words, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

Axiom 1.3 (axiom of existence)**.** There is a set with no elements called the empty set, denoted by the symbol $\varnothing$.

Axiom 1.4 (axiom of specification)**.** Let A be a set. Let $P(a)$ denote a property that applies to every element in A, i.e., for each $a \in A$, either $P(a)$ is true or it is false. Then there exists a subset

$$B = \{a \in A : P(a) \text{ is true}\}$$

Axiom 1.5 (axiom of pairing)**.** Given two sets A, B, there exists a set which contains precisely A, B as its elements, which we denote by $\{A, B\}$.

Axiom 1.6 (axiom of unions)**.** Given a set $\mathscr{F}$ of sets, there exists a set called the union of the sets in $\mathscr{F}$, denoted by $\bigcup_{A \in \mathscr{F}} A$, whose elements are precisely the elements of the elements of $\mathscr{F}$.

$$a \in \bigcup_{A \in \mathscr{F}} A \Leftrightarrow a \in A \text{ for some } A \in \mathscr{F}$$

Axiom 1.7 (axiom of powers)**.** Given a set A, there exists a set called power set of A denoted $\mathscr{P}(A)$, whose elements are precisely all the subsets of A.

Definition 1.3. A relation from set A to set B is a subset R of $A \times B$. For any $a \in A, b \in B$ we say $a\ R\ b$ iff $(a, b) \in R$.

- The *domain* of R is the set
$$\operatorname{dom}(R) = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$
- The *range* of R is the set
$$\operatorname{ran}(R) = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$$
- R is called a *function* from A to B, denoted as $R : A \to B$ iff
  - $\operatorname{dom}(R) = A$
  - for each $a \in A$ there is (at most) one $b \in B$ such that $(a, b) \in R$.

Axiom 1.8 (axiom of regularity)**.** Read up

Axiom 1.9 (axiom of replacement)**.** Read up

Axiom 1.10 (axiom of choice)**.** Read up

Definition 1.4. Given a set A, its *successor* is the set

$$A^+ = A \cup \{A\}.$$

A set A is said to be *inductive* if $\varnothing \in A$ and for every $a \in A$, we have $a^+ \in A$.

Axiom 1.11 (axiom of infinity)**.** There exists an inductive set.

Lemma 1.5. Let $\mathscr{F}$ be a nonempty set of inductive sets. (This exists by axiom of infinity and axiom of pairing). Then

$$\bigcap_{A \in \mathscr{F}} A \text{ is inductive.}$$

Theorem 1.6. There exists a *unique*, *minimal* inductive set ω, *i.e.*, for any inductive set S,

$$\omega \subseteq S$$

and if ω' is any other inductive set satisfying this property,

$$\omega = \omega'$$

Theorem 1.7. The ω in theorem 1.6 is a Peano set with successor function $a \mapsto a^+$.

Theorem 1.8 (principle of recursion)**.** Let A be a set, and $f : A \to A$ be a function. Let $a \in A$. Then, there exists a function $F : \omega \to A$ such that

(a) $F(\varnothing) = a$

(b) For some $b \in \omega$, we have $F(b^+) = f(F(b))$

1.2 Natural Numbers

Definition 1.9 (Peano addition)**.** Given a fixed $m \in \mathbb{N}$, the principle of recursion gives a unique function

$$\mathrm{sum}_m : \mathbb{N} \to \mathbb{N}$$

(A1) $\mathrm{sum}_m(0) = m$

(A2) $\mathrm{sum}_m(n^+) = (\mathrm{sum}_m(n))^+$.

Define

$$m + n := \mathrm{sum}_m(n)$$

Proposition 1.10. $2 + 3 = 5$

Definition 1.11 (Peano multiplication)**.** Let $m \in \mathbb{N}$. By the recursion principle, $\exists$ a unique function

$$\mathrm{prod}_m : \mathbb{N} \to \mathbb{N}$$

such that

(a) $\mathrm{prod}_m(0) = 0$

(b) $\mathrm{prod}_m(n^+) = m + \mathrm{prod}_m(n)$.

Theorem 1.12. The following hold:

(a) (Commutativity)

$$m + n = n + m$$
$$m \cdot n = n \cdot m$$

for all natural numbers m and n.

(b) (Associativity)

$$m + (n + k) = (m + n) + k$$
$$m \cdot (n \cdot k) = (m \cdot n) \cdot k$$

for all natural numbers m, n, k.

(c) (Distributivity)

$$m \cdot (n + k) = (m \cdot n) + (m \cdot k)$$

for all natural numbers m, n, k.

(d) $m + n = 0 \Leftrightarrow m = n = 0$ for any $m, n \in \mathbb{N}$

(e) $m \cdot n = 0 \Leftrightarrow m = 0$ or $n = 0$ for any $m, n \in \mathbb{N}$

(f) (Cancellation) $m + k = n + k \Leftrightarrow m = n$ for any $m, n, k \in \mathbb{N}$ and if $m \cdot k = n \cdot k$ and $k \neq 0$, then $m = n$.

1.2.1 Tao

Lemma 1.13. For any natural number n, $n + 0 = n$.

Lemma 1.14. For any natural numbers n and m, $n + m_{++} = (n + m)_{++}$

Corollary 1.15. $n_{++} = n + 1$.

Proposition 1.16. *(Addition is commutative)* For any natural numbers n and m, $n + m = m + n$

1.3 Fields, Ordered Sets and Ordered Fields

Definition 1.17. A field is a set F with 2 operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ such that

(F1) $+$ & $\cdot$ are commutative on F.

(F2) $+$ & $\cdot$ are associative on F.

(F3) $+$ & $\cdot$ satisfy distributivity on F, *i.e.*, $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$.

(F4) There exist 2 *distinct* elements, called 0 (additive identity) and 1 (multiplicative identity) such that

$$\begin{aligned} x + 0 &= x \\ x \cdot 1 &= x \end{aligned}$$

for all $x \in F$

(F5) For every $x \in F$, $\exists\, y \in F$ such that

$$x + y = 0$$

(F6) For every $x \in F \setminus \{0\}$, $\exists\, z \in F$ such that

$$x \cdot z = 1$$

Theorem 1.18. $(F, +, \cdot)$ is a field. Then for all x,

$$0 \cdot x = x \cdot 0 = 0$$

Definition 1.19. A set A with a relation $<$ is called an *ordered set* if

(O1) (Trichotomy) For every $x, y \in A$, exactly one of the following holds.

$$x < y, \quad x = y, \quad y < x$$

(O2) (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Notation: $x < y$ is read as "x is less than y"
$x \leq y$ means $x < y$ or $x = y$, read as "x is less that or equal to y".
$x > y$ is read as "x is greater that y" and equivalent to $y < x$.

Definition 1.20. An *ordered field* is a set that admits two operations $+$ and $\cdot$ and relation $<$ so that $(F, +, \cdot)$ is a field and $(F, <)$ is an ordered set and:

(O3) For $x, y, z \in F$, if $x < y$ then $x + z < y + z$.

(O4) For $x, y \in F$, if $0 < x$ and $0 < y$ then $0 < x \cdot y$.

Lemma 1.21. Given a field $(F, +, \cdot)$: For any element a in a field F, there exists ony one b such that $a + b = 0$. We will denote this b as $-a$. Similarly for any a in $F \setminus \{0\}$ there exists only one $b \in F$ such that $ab = 1$. We will denote this b as $\frac{1}{a}$ or a^{-1}.

Lemma 1.22. $-(-a) = a = (a^{-1})^{-1}$

Lemma 1.23. For any field $(F, +, \cdot)$, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.

Theorem 1.24. For any field $(F, +, \cdot)$, $0 < 1$.

1.4 Upper bounds & least upper bounds

Definition 1.25. A non-empty subset $S \subseteq F$ is said to be *bounded above* in F if there exists a $b \in F$ such that

$$a \leq b \; \forall \, a \in S$$

Here, b is called an *upper bound* of S. If $b \in S$, then b is a *maximum* of S.

Definition 1.26. Let $S \subseteq F$ be bounded above. An element $b \in F$ is said to be a *least upper bound* of S or a *supremum* of S if:

(a) b is an upper bound of S.

(b) If for $c \in F$, $c < b$, then c is not an upper bound of S. In other words, for any $c < b$, $\exists\, s_c \in S$ such that $c < s_c$.
Contrapositive: If c is an upper bound of S, then c is not less than b, *i.e.*, $b \leq c$.

1.5 The Real Numbers

Theorem 1.27 (Archimedean property of $\mathbb{R}$)**.** Let $x, y \in \mathbb{R}$ and $x > 0$, then $\exists\, n \in \mathbb{N}$ such that

$$n \cdot x > y.$$

2 Sequences & Series

2.1 Sequences

Definition 2.1. A sequence in $\mathbb{R}$ is a function $f : \mathbb{N} \to \mathbb{R}$. We denote this sequence by $\{a_n\}_{n \in \mathbb{N}}$, where

$$a_n = f(n) \quad \forall\, n \in \mathbb{N}$$

and a_n is called the n^{th} term of $\{a_n\}_{n \in \mathbb{N}}$.

Definition 2.2. We say that a sequence $\{a_n\} \subseteq \mathbb{R}$ is *convergent* (in $\mathbb{R}$) if $\exists\, L \in \mathbb{R}$ such that for each $\varepsilon > 0$, $\exists\, N_{\varepsilon,L} \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \forall\, n \geq N_{\varepsilon,L}$$

We will call L *a* limit of $\{a_n\}$ and we write:

$$a_n \to L \text{ as } n \to \infty$$

A sequence $\{a_n\}$ is said to be *divergent* if it is not convergent, *i.e.*, $\forall\, L \in \mathbb{R}$ and $N_L \in \mathbb{N}$, $\exists\, \varepsilon > 0$ and $N \geq N_L$ such that

$$|a_N - L| > \varepsilon$$

Theorem 2.3 (Uniqueness of limits)**.** Suppose L_1 and L_2 are limits of a (convergent) sequence $\{a_n\} \in \mathbb{R}$. Then $L_1 = L_2$.

Definition 2.4. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be *bounded* if $\exists\, M > 0$ such that $|a_n| < M \,\forall\, n \in \mathbb{N}$.

Theorem 2.5. Every convergent sequence is bounded.

Definition 2.6. A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotonically increasing* if $a_n \leq a_{n+1} \,\forall\, n \in \mathbb{N}$.

A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotonically decreasing* if $a_n \geq a_{n+1} \,\forall\, n \in \mathbb{N}$.

A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotone* if it is either monotonically increasing or monotonically decreasing.

Theorem 2.7 (Monotone convergence theorem)**.** A monotone sequence is convergent iff it is bounded.

Definition 2.8. We say that a sequence diverges to $+\infty$ if $\forall\, R \in \mathbb{R}$, $\exists\, N_R \in \mathbb{N}$ such that $a_n > R \,\forall\, n \geq N_R$.
We say that a sequence diverges to $-\infty$ if $\forall\, R \in \mathbb{R}$, $\exists\, N_R \in \mathbb{N}$ such that $a_n < R \,\forall\, n \geq N_R$.
We write $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, but this is purely notational and does not mean "$\{a_n\}$ has a limit".

Theorem 2.9 (Tao Theorem 6.1.19)**.** Suppose $\{b_n\}$ converges to $b \neq 0$ (and $\exists\, M \in \mathbb{N}$ such that $b_n \neq 0 \,\forall\, n \geq M$.)
Then $\left\{\frac{1}{b}\right\}_{n\geq M} \to \frac{1}{b}$ as $n \to \infty$.

2.2 Infinite series

Definition 2.10. An infinite series is a *formal expression* of the form

$$a_0 + a_1 + a_2 + \dots, \text{ or,} \sum_{n=0}^{\infty} a_n$$

Given $\sum_{n=0}^{\infty} a_n$, its sequence of partial sums (sops) is $\{s_n\}_{n=0}^{\infty}$ where

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ &\vdots \\ s_n &= a_0 + a_1 + \dots a_n \end{aligned}$$

We say that $\sum a_n$ is *convergent* with sum s if $\lim_{n\to\infty} s_n = s$. Otherwise, we say that $\sum a_n$ is divergent.

Theorem 2.11. Suppose $\sum a_n$ is convergent. Then

$$\lim_{n\to\infty} a_n = 0$$

Theorem 2.12 (Comparison test)**.** Suppose there exist constants $M \in \mathbb{N}$ and $0 < C$ such that

$$0 \leq a_n \leq C b_n \quad \forall\, n \geq M$$

If $\sum b_n$ converges, then $\sum a_n$ converges. In other words, If $\sum a_n$ diverges, $\sum b_n$ diverges.

Theorem 2.13 (Ratio test)**.** Let $\sum a_n$ be a series of positive terms. Suppose

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$$

Then,

(a) If $L < 1$, the series converges.

(b) If $L > 1$, the series diverges.

(c) If $L = 1$, the test is inconclusive.

Theorem 2.14. Suppose $\sum a_n$ and $\sum b_n$ converge with sums a and b respectively. Then, for constants l and m, $\sum la_n + mb_n$ converges to $la + mb$. Suppose $\sum |a_n|$ and $\sum |b_n|$ converge. Then, so does $\sum |la_n + mb_n|$ for any choice of l and m in $\mathbb{R}$.

Corollary 2.15. Suppose $\sum a_n$ converges and $\sum b_n$ diverges. Let $m \in \mathbb{R} \setminus \{0\}$. Then, $\sum(a_n + b_n)$ diverges, and $\sum mb_n$ diverges.

Definition 2.16. A series $\sum a_n$ of real numbers is said to *converge absolutely* if $\sum|a_n|$ converges. A series $\sum a_n$ of real numbers is said to *converge conditionally* if $\sum|a_n|$ diverges but $\sum a_n$ converges.

Theorem 2.17. If $\sum a_n$ converges absolutely, it must converge. Moreover, $|\sum a_n| \leq \sum|a_n|$.

Theorem 2.18 (Alternating series test)**.** Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive numbers going to 0. Then, $\sum(-1)^n a_n$ converges. Denoting the sum by S, we have that

$$0 < (-1)^n(S - s_n) < a_{n+1}.$$

Also called the Leibniz test.

3 Limits & Continuity

3.1 Limit of a function

Definition 3.1 (Neighborhood)**.** Given a real number p and an $\varepsilon > 0$, the *ε-neighborhood of* p is the open interval

$$N_\varepsilon(p) = (p - \varepsilon, p + \varepsilon) = \{x \in \mathbb{R} : |x - p| < \varepsilon\}.$$

Definition 3.2 (Limit of a function)**.** Given a function f that is defined on some $I = (a, p) \cup (p, b)$ with $a < b$, we say that f has a *limit* L as it approaches p iff for every $\varepsilon > 0 \; \exists \; \delta > 0$ such that

(a) $0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$ OR

(b) $x \in N_\delta(p) \setminus \{p\} \Rightarrow f(x) \in N_\varepsilon(L)$.

This is denoted as

$$\lim_{x \to p} f(x) = L.$$

Theorem 3.3 (Limit laws for functions)**.** Suppose f and g are functions such that

$$\lim_{x \to p} f(x) = a, \qquad \lim_{x \to p} g(x) = b.$$

Then,

$$\lim_{x \to p} (f \pm g)(x) = a \pm b \tag{1}$$

$$\lim_{x \to p} (f \cdot g)(x) = a \cdot b \tag{2}$$

$$\lim_{x \to p} (f/g)(x) = a/b \tag{3}$$

3.2 Continuity

Definition 3.4. Let $S \subseteq \mathbb{R}$ be a (nonempty) subset, $f : S \to \mathbb{R}$ and $p \in S$. We say that f is continuous at p iff:
for every $\varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ such that

$$|x - p| < \delta_\varepsilon \ \wedge \ x \in S \Rightarrow |f(x) - f(p)| < \varepsilon$$

We say that f is continuous on S iff f is continuous at each $p \in S$.

Theorem 3.5 (Algebraic laws for continuity)**.** Suppose f and g are continuous at $p \in S$. Then so are $f \pm g, fg$ and if $g(p) \neq 0, f/g$.

Theorem 3.6. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be continuous functions such that $f(A) := range(f) \subseteq B$. Then,

$$g \circ f : x \in A \mapsto g(f(x)) \in \mathbb{R}$$

is continuous.

Theorem 3.7 (intermediate value theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $y \in \mathbb{R}$ is a number between $f(a)$ and $f(b)$, *i.e.*, $y \in [f(a), f(b)]$. Then $\exists\, c \in [a, b]$ such that

$$f(c) = y$$

Corollary 3.8 (Bolzano's theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)$ and $f(b)$ take opposite signs. Then $\exists\, c \in (a, b)$ such that $f(c) = 0$.

Theorem 3.9 (the Borsuk-Ulam theorem)**.** Let S^n be the unit n-sphere, *i.e.*, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Let $f : S^n \to \mathbb{R}^n$ be a continuous function. Then f maps some pair of antipodal points to the same point.

$$\exists\, x \text{ such that } f(x) = f(-x)$$

Lemma 3.10. Let a_n, b_n be convergent sequences such that $a_n \leq b_n$ for all n (large enough). Then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Definition 3.11. A function $f : S \to \mathbb{R}$ is said to be *bounded above* on S if $\exists\, U \in \mathbb{R}$ such that $f(x) \leq U\ \forall\, x \in S$.

f is said to be *bounded* if $\exists\, M > 0$ such that $|f(x)| < M\ \forall\, x \in S$.

Theorem 3.12 (Continuous functions on compact intervals are bounded)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is a bounded function.

Definition 3.13. A function $f : S \to \mathbb{R}$ is said to have a *global maximum* on S at a point $p \in S$ if $f(x) \leq f(p)\ \forall\, x \in S$.

A function $f : S \to \mathbb{R}$ is said to have a *global minimum* on S at a point $p \in S$ if $f(x) \geq f(p)\ \forall\, x \in S$.

Theorem 3.14 (Extreme value theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains both a global maximum and a global minimum in $[a, b]$.

Corollary 3.15. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then (using IVT),

$$f([a, b]) = [\min_{[a,b]} f, \max_{[a,b]} f].$$

4 Differentiation

Definition 4.1. Let $f : (a, b) \to \mathbb{R}$ be a function and $p \in (a, b)$. We say that f is differentiable in (a, b) if

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{h} = \lim_{x\to p} \frac{f(x) - f(p)}{x - p}$$

exists, and the limit is called the derivative of f at p, denoted $f'(p)$.

If f is differentiable on each p in (a, b), it is said to be differentiable on (a, b) and $f' : (a, b) \to \mathbb{R}$ is called the derivative of f on (a, b).

We define two more functions:

(a) For any function $f : (a, b) \to \mathbb{R}$ and any $p \in (a, b)$, define

$$f^p_\delta : h \in (a - p, b - p) \setminus \{0\} \mapsto \frac{f(p+h) - f(p)}{h} \in \mathbb{R}.$$

(b) For any differentiable function $f : (a, b) \to \mathbb{R}$ and any $p \in (a, b)$, define

$$f^p_\Delta : h \in (a - p, b - p) \mapsto \begin{cases} \dfrac{f(p+h) - f(p)}{h} & h \neq 0 \\ f'(p) & h = 0 \end{cases} \in \mathbb{R}.$$

Theorem 4.2 (Differentiability $\Rightarrow$ continuity)**.** Let $f : (a, b) \to \mathbb{R}$ be differentiable at $p \in (a, b)$. Then f is continuous at p.

Theorem 4.3 (Algebra of derivatives)**.** Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at $p \in (a, b)$. Then

(a) $f + g$ is differentiable at p and $(f + g)' = f' + g'$.

(b) $f - g$ is differentiable at p and $(f - g)' = f' - g'$.

(c) $f \cdot g$ is differentiable at p and $(f \cdot g)' = f' \cdot g + f \cdot g'$.

(d) f/g is differentiable at p if $g \neq 0$ and $(f/g)' = \frac{f' \cdot g - f \cdot g'}{g^2}$.

Definition 4.4 (Inverse function)**.** Let $f : A \to B$ be bijective. Then for any $y \in B$, there exists (unique) $x_y \in A$ such that $f(x_y) = y$. We define the inverse function $f^{-1} : B \to A$ as

$$f^{-1}(y) = x_y.$$

and say that f is invertible on A.

Note that $(f \circ f^{-1})$ and $(f^{-1} \circ f)$ are the identity functions on B and A respectively.

For example, the function $f(x) = x^2$ is invertible on $\mathbb{R}^+$ and its inverse is $f^{-1}(x) = \sqrt{x}$.

Theorem 4.5 (inverse function properties)**.** Let $f : [a, b] \to \mathbb{R}$ be an invertible function on $[a, b]$ with range J.

(i) If f is (strictly) increasing, then so is f^{-1}.

(ii) If f is continuous, then $f : [a, b] \to J$ is strictly monotone and $f^{-1} : J \to [a, b]$ is continuous.

(iii) If f is differentiable at $p \in (a, b)$ with $f'(p) \neq 0$ *and continuous in some neighborhood around* p, then f^{-1} is differentiable at $f(p) = q \in J$ and $(f^{-1})'(q) = \dfrac{1}{f'(p)}$.

Theorem 4.6 (chain rule)**.** Let $f : (a, b) \to \mathbb{R}$ and $g : (c, d) \to \mathbb{R}$ with $f((a, b)) \subseteq (c, d)$ and f differentiable in (a, b). Let g be differentiable at $f(p) := q$. Then $g \circ f : (a, b) \to \mathbb{R}$ is differentiable at p and $(g \circ f)' = g' \circ f \cdot f'$ at p.

4.1 Local Extrema

Definition 4.7 (Local Extrema)**.** Let $f : A \to \mathbb{R}$. We say that f attains a local maximum at $a \in A$ iff $\exists\, \delta > 0$ such that

$$f(x) \leq f(a)\ \forall\, x \in N_\delta(a) \cap A.$$

We say that f attains a local minimum at $a \in A$ iff $\exists\, \delta > 0$ such that

$$f(x) \geq f(a)\ \forall\, x \in N_\delta(a) \cap A.$$

Theorem 4.8 (Extremum $\Rightarrow$ Stationary)**.** Let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b)$ such that f is differentiable at c. If f attains a local extremum at c, then $f'(c) = 0$. Points at which the derivative vanishes are called 'stationary points' and sometimes 'critical points'.

Theorem 4.9 (Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Definition 4.10 (Taylor Polynomial)**.** Let $f : (a, b) \to \mathbb{R}$ be k times differentiable at some $x_0 \in (a, b)$. The k^{th} Taylor polynomial at x_0 is defined as

$$P_k^{x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Theorem 4.11 (Taylor's Theorem)**.** Let $f : (a, b) \to \mathbb{R}$ be an $(n + 1)$ times differentiable function on (a, b). Note that this implies $f, f', f'', \ldots f^{(n)}$ are continuous. Let $x_0 \in (a, b)$. Then $\forall\ x \in (a, b)\ \exists\ c_x$ between x and x_0 such that

$$f(x) = P_n^{x_0}(x) + f^{(n+1)}(c_x)\frac{(x - x_0)^{n+1}}{(n + 1)!}.$$

5 Integration

Definition 5.1 (Partition)**.** A *partition* of $[a, b]$ is a finite subset

$$P = \{x_0, x_1, \ldots x_n\} \subseteq [a, b]$$

such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We write

$$P = \{x_0 < x_1 < \cdots < x_n\}.$$

Definition 5.2 (Refinement)**.** Given two partitions P and Q of $[a, b]$, Q is said to be a *refinement* of P if $P \subseteq Q$.

Definition 5.3 (Common refinement)**.** Given two partitions P and Q of $[a, b]$, the *common refinement* of P and Q is the smallest refinement of both P and Q simultaneously. Thus,

$$R = P \cup Q$$

is the common refinement of P and Q.

Definition 5.4 (Step function)**.** Given an interval $[a, b]$, a function $S : [a, b] \to \mathbb{R}$ is called a *step function* is there is some partition $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$ such that for each $j \in [1..n]$, $\exists\, s_j \in \mathbb{R}$ such that

$$s(x) = s_j \quad \forall\, x \in (x_{j-1}, x_j).$$

Definition 5.5 (Step Integration)**.** Given a step function $s : [a, b] \to \mathbb{R}$ corresponding to $P = \{x_0 < x_1 < \cdots < x_n\}$, define

$$\int_a^b s(x)\,\mathrm{d}x = \sum_{j=1}^{n} s_j(x_j - x_{j-1}).$$

We also define

$$\int_b^a s(x)\,\mathrm{d}x = -\int_a^b s(x)\,\mathrm{d}x.$$

Theorem 5.6 (Properties)**.**

(a) $\int_a^b (c_1 s(x) + c_2 t(x))\,\mathrm{d}x = c_1 \int_a^b s(x)\,\mathrm{d}x + c_2 \int_a^b t(x)\,\mathrm{d}x$.

(b) If $s \le t$ on $[a, b]$, then $\int_a^b s(x)\,\mathrm{d}x \le \int_a^b t(x)\,\mathrm{d}x$.

(c) $\int_{ka}^{kb} s(x/k)\,\mathrm{d}x = k \int_a^b s(x)\,\mathrm{d}x$.

Definition 5.7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let

$$S_f = \{s : [a, b] \to \mathbb{R} : s \text{ is a step function and } s \le f \text{ on } [a, b]\}$$

and

$$T_f = \{t : [a, b] \to \mathbb{R} : t \text{ is a step function and } t \ge f \text{ on } [a, b]\}.$$

Lemma 5.8. Let $f : [a, b] \to \mathbb{R}$ be bounded, *i.e.*, $\exists\, M > 0$ such that

$$-M \leq f(x) \leq M \;\forall\, x \in [a, b].$$

Then $\sup s_f$ and $\inf t_f$ exist and

$$-M(b-a) \leq \sup s_f \leq \inf t_f \leq M(b-a)$$

where $s_f = \left\{\int_a^b s(x)\,\mathrm{d}x : s \in S_f\right\}$ and $t_f = \left\{\int_a^b t(x)\,\mathrm{d}x : t \in T_f\right\}$.

Definition 5.9. Given a bounded function $f : [a, b] \to \mathbb{R}$, its *lower integral* is

$$\underline{I}(f) = \sup\left\{\int_a^b s(x)\,\mathrm{d}x : s \in S_f\right\}$$

and its *upper integral* is

$$\bar{I}(f) = \inf\left\{\int_a^b s(x)\,\mathrm{d}x : s \in T_f\right\}.$$

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* (not really) if $\underline{I}(f) = \bar{I}(f)$ and we call this quantity the integral of f over $[a, b]$, denoted by

$$\int_a^b f(x)\,\mathrm{d}x.$$

We also define

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x.$$

Theorem 5.10 (Monotone Integrable)**.** Every bounded monotone function on $[a, b]$ is Riemann integrable on $[a, b]$.

Definition 5.11 (Uniform Continuity)**.** A function $f : A \to \mathbb{R}$ is said to be *uniformly continuous* if for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that whenever $x, y \in A$ and $|x - y| < \delta_\varepsilon$, then $|f(x) - f(y)| < \varepsilon$.

Theorem 5.12 (Closed continuous $\Rightarrow$ uniformly continuous)**.** Every continuous function on a closed, bounded interval is uniformly continuous on $[a, b]$.

Theorem 5.13 (Continuity $\Rightarrow$ Riemann Integrability)**.** Let f be a continuous function on $[a, b]$. Then f is Riemann integrable on $[a, b]$.

Theorem 5.14 (Mean Value – Integrals)**.** Let f be a continuous function on $[a, b]$. Then there exists a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

Theorem 5.15 (The First Fundamental Theorem of Calculus)**.** Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Let

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \qquad \forall\, x \in [a, b].$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at some $p \in (a, b)$, then F is differentiable at p with $F'(p) = f(p)$.

Theorem 5.16 (Integral Triangle Inequality)**.** Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then $|f| : [a, b] \to \mathbb{R}$ is Riemann integrable and

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \le \int_a^b |f(x)|\,\mathrm{d}x.$$

Definition 5.17 (Primitive)**.** Given a function $f : (a, b) \to \mathbb{R}$, with $a < b$ and $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, a *primitive* or *antiderivative* of f on (a, b) is a differentiable function $F : (a, b) \to \mathbb{R}$ such that

$$F'(x) = f(x)\ \forall\, x \in (a, b).$$

Theorem 5.18 (The Second Fundamental Theorem of Calculus)**.** Let $f : (c, d) \to \mathbb{R}$ be a function such that $f_{[a,b]}$ ($[a, b] \subset (c, d)$) is Riemann integrable. Let F be a primitive of f on (c, d). Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

5.1 Logarithms & Exponentiation

Definition 5.19 (Natural Logarithm)**.** Let $x > 0$. The *natural logarithm* of x is the quantity

$$\ln(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

Theorem 5.20. The function $\ln : \mathbb{R}^+ \to \mathbb{R}$ has the following properties:

(a) $\ln(1) = 0$.

(b) $\ln(x) + \ln(y) = \ln(xy) \ \forall\, x, y \in \mathbb{R}^+$.

(c) ln is continuous and strictly increasing.

(d) ln is differentiable and
$$\ln'(x) = \frac{1}{x}.$$

(e) (Leibniz)
$$\int \frac{1}{t}\,\mathrm{d}t = \ln|t| + C.$$

(f) (Leibniz)
$$\int \ln x\,\mathrm{d}x = x\ln x - x + C.$$

(g) ln is bijective.

Definition 5.21 (e & Exponentiation)**.** Let e be the unique number that satisfies
$$\ln(e) = 1.$$
Given any $x \in \mathbb{R}$, let $\exp(x)$ be the unique positive y such that
$$\ln(y) = x.$$
That is, exp is the inverse function of ln.

Theorem 5.22. $\exp : \mathbb{R} \to \mathbb{R}^+$ has the following properties:

(a) $\exp(0) = 1$.

(b) $\exp(x + y) = \exp(x)\exp(y)$.

(c) exp is continuous and strictly increasing.

(d) exp is differentiable and
$$\exp'(x) = \exp(x) \ \forall\, x \in \mathbb{R}.$$

(e)
$$\int \exp(x)\,\mathrm{d}x = \exp(x) + C.$$

(f) exp is bijective.

(g) $\exp(r) = e^r \ \forall\, r \in \mathbb{Q}$.

6 Vector Spaces

Definition 6.1. Let $(F, \oplus, \odot)$ be a field. A *vector space over* F is a set V such that:

(a) Given any two elements $v, w \in V$, there exists a unique element $v + w \in V$ called its sum. (This $+$ may not be the same as $\oplus$).

(b) Given an $a \in F$ and a $v \in V$, there is a unique element $av = a \cdot v \in V$ called the scalar product of a and v.

satisfying the following axioms:

(V1) $v + w = w + v$ for all $v, w \in V$.

(V2) $(v + w) + u = v + (w + u)$ for all $v, w, u \in V$.

(V3) There is an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

(V4) For all $v \in V$, there is a unique element $-v \in V$ called the additive inverse of v such that $v + (-v) = 0$.

(V5) For all $a, b \in F$ and $v \in V$, we have

$$(a \odot b) \cdot v = a \cdot (b \cdot v)$$

Note that this implies $a \cdot (b \cdot v) = b \cdot (a \cdot v)$ by the commutativity of $\odot$.

(V6) Let 1_F be the multiplicative identity of F. Then,

$$1_F \cdot v = v \quad \text{for all} \quad v \in V$$

(V7) For all $a, b \in F$ and $v \in V$, we have

$$(a \oplus b) \cdot v = a \cdot v + b \cdot v$$

(V8) For all $a \in F$ and $v, w \in V$, we have

$$a \cdot (v + w) = a \cdot v + a \cdot w$$

We call the elements of F *scalars* and the elements of V *vectors*.

Proposition 6.2 (Vector properties)**.** Let V be a vector space over F. Then the following hold:

(a) V has a unique additive identity.

(b) $0_F v = 0_V$ for all $v \in V$.

(c) $a 0_V = 0_V$ for all $a \in F$.

(d) Each $v \in V$ has a unique additive inverse given by $(-1_F)v$.

(e) If $av = aw$ for some $a \in F \setminus \{0\}$ and $v, w \in V$, then $v = w$.

Definition 6.3 (Subspace)**.** Let V be a vector space over some field F. A subset $S \subseteq V$ is a (linear) *subspace* of V if the following hold:

(a) $0_V \in S$.

(b) If $v, w \in S$, then $v + w \in S$.

(c) If $a \in F$ and $v \in S$, then $av \in S$.

These properties together imply that S is also a vector space over F.

S is said to be a *proper* subspace of V if $S \neq V$ but also $S \neq \{0_V\}$.

Definition 6.4 (Span of finite sequences)**.** Let $v_1, v_2, \dots, v_m \in V$ be a finite sequence of vectors. A linear combination of $v_1, v_2, \dots, v_m$ is any vector of the form

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m,$$

where $a_1, a_2, \dots, a_m \in F$. The *span* of the finite sequence $v_1, v_2, \dots, v_m$ is the set of all linear combinations of $v_1, v_2, \dots, v_m$. That is,

$$\operatorname{span}(v_1, v_2, \dots, v_m) = \left\{ \sum_{j=1}^{m} a_j v_j : a_j \in F \right\}.$$

Definition 6.5 (Span of sets)**.** Let $S \subseteq V$ be a nonempty set. The *span* of the S is the set

$$\begin{aligned} \operatorname{span} S = \{v \in V :\ & \exists\, a_1, \dots, a_n \in F \\ & \text{and distinct } v_1, \dots, v_n \in S \\ & \text{such that } v = a_1 v_1 + \cdots + a_n v_n\}. \end{aligned}$$

Or

$$\operatorname{span} S = \bigcup_{\emptyset \neq \Lambda \subseteq^{\text{finite}} S} \operatorname{span} \Lambda.$$

$\operatorname{span} \emptyset$ is defined to be $\{0\}$.

Definition 6.6 (Basis)**.** Given a vector space V over F, a *basis* is a subset $B \subseteq V$ such that

(a) B is a spanning set, *i.e.*, $V = \text{span}(B)$.

(b) B is linearly independent.

Corollary 6.7. Let V be a finite dimensional vector space over a field F. Let S be a finite spanning set of V. Then S contains as a subset a basis of V.

Corollary 6.8. Every finite dimensional vector space has a basis.

Proposition 6.9. Let $L \subseteq V$ be linearly independent. Then for some $v \in V$, $L \cup \{v\}$ is linearly independent iff $v \notin \text{span}(L)$.

Corollary 6.10. Let V be a finite dimensional vector space. Let $L \subseteq V$ be a finite linearly independent set. Then there exist finitely many vectors $w_1, \ldots, w_m \in V$ such that $L \cup \{w_1, \ldots, w_m\}$ is a basis.

Theorem 6.11. Let V be a finite dimensional vector space. Let $S, L \subseteq V$ be such that S is a spanning set and L is linearly independent. Then

$$\#L \leq \#S.$$

Corollary 6.12. Every (finite) basis of a finite dimensional vector space has the same size.

Corollary 6.13. Let S, L, V be as in the previous theorem. If $\#L = \#S$, then both are bases of V.

Corollary 6.14 (Finite Basis of FDVS)**.** Every basis of a finite dimensional vector space is finite.

Definition 6.15 (Dimension)**.** Let V be a finite dimensional vector space. We define the length of any of its bases to be the *dimension* of V.

Proposition 6.16. Let V be a finite dimensional vector space. Let W be a subspace of V. Then W is finite dimensional and $\dim(W) \leq \dim(V)$.

Proposition 6.17. Let $T \in \mathscr{L}(V, W)$. Then $N(T) = \{0\}$ iff T is an injective transformation.

Theorem 6.18 (Rank-Nullity Theorem)**.** Let $T \in \mathscr{L}(V, W)$, where V is a finite-dimensional vector space. Then

$$\dim(N(T)) + \dim(R(T)) = \dim(V).$$

Corollary 6.19. If $\dim W < \dim V$, then there is no injective linear transformation from V to W.

Corollary 6.20. Let $T \in \mathscr{L}(V, W)$ where $\dim V = \dim W$. Then the following are equivalent:

(a) T is surjective.

(b) T is injective.

(c) T is invertible *as a linear transformation.* That is, there exists $T^{-1} \in \mathscr{L}(W, V)$ such that $T^{-1}T = I_W$ and $TT^{-1} = I_V$.