Assignment 01

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Problem 0.1. Let A be a Peano set and S be the successor function on A (as defined in the first lecture). Show, using only the axioms of Peano, that the range of S is $A \setminus \{0\}$. For this question, please interpret the words "function" and "range" in the way you did in school, and not in the set-theoretic way introduced in class.

Proof. Let $P(n)$ be the property that n is in the range of S, or $n = 0$. $P(0)$ is trivially true. If $P(n)$ is true, then $P(S(n))$ is trivially true, as $S(n)$ is the successor of n.

Thus by the principle of mathematical induction, $P(n)$ is true for all natural n.

This means that for all $n \neq 0 \in A$, n is in the range of S. Since 0 is not the successor of any natural number, it is not in the range of S.

Thus, the range of S is precisely $A \setminus \{0\}$. □

Problem 0.2. We mentioned in class that when listing the ZFC axioms, we do not need to addadditional axioms for the existence of the intersection or the set-difference of two sets. Using the ZFC axioms, prove the following statements.

(a) Given two sets A and B, show that $A \cap B$ exists as a set.

(b) Given two sets A and B, show that $A \setminus B$ exists as a set.

Proof. Using the axiom of specification, we have:

(a)
$$A \cap B = \{a \in A : a \in B\}$$

(b)
$$A \setminus B = \{a \in A : a \notin B\}$$
□

Problem 0.3. Given two objects a, b, let (a, b) denote the set $\{\{a\}, \{a, b\}\}$. First argue why the ZFC axioms guarantee the existence of this set. Then show that $(a, b) = (c, d)$ (as sets) if and only if $a = c$ and $b = d$.

Proof. By the basic axiom, a and b are sets.
By the axiom of pairing, set $\{a, a\} = \{a\}$ (by axiom of equality) exists.
By the axiom of pairing, set $\{a, b\}$ exists.
Applying the axiom of pairing on these two sets, the set $(a, b) = \{\{a\}, \{a, b\}\}$ exists.

If $a = c$ and $b = d$, $(a, b) = (c, d)$. Do I even need to compare the two sets manually? Invoke any of ZFC? Doesn't this follow directly from the notion of "equality"?

If $(a, b) = (c, d)$, then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. By the axiom of equality, $\{a\} \in (c, d) \Rightarrow \{a\} = \{c\}$ or $\{a\} = \{c, d\}$. In either case, the axiom of equality again implies that $c \in \{a\} \Rightarrow c = a$.

By the axiom of equality again, $\{c, d\} = \{a\}$ implying $d = a$, or $\{c, d\} = \{a, b\}$ implying $d \in \{a, b\}$. Thus d is either equal to a or to b. Similarly b is equal to either c or d.

If $b \neq d$, we must have $d = a$ and $b = c = a = d$, a contradiction.
Thus d is necessarily equal to b. $\square$

Problem 0.4. Prove lemma 1.5 (probably), *i.e.*, show that if $\mathscr{F}$ is a non-empty set of inductive sets, then

$$\bigcap_{A \in \mathscr{F}} A$$

is an inductive set.

Proof. I shall assume $I = \bigcap_{A \in \mathscr{F}} A$ to be defined as

$$\bigcap_{A \in \mathscr{F}} A = \left\{ a \in \bigcup_{A \in \mathscr{F}} A : a \in A \; \forall \; A \in \mathscr{F} \right\}$$

Since $\varnothing \in A$ for every $A \in \mathscr{F}$, and $\varnothing$ is also in the union of the sets contained in $\mathscr{F}$, $\varnothing \in I$

If an element $a \in I$, then a is in every $A \in \mathscr{F}$. Since $A \in \mathscr{F}$, A is an inductive set, and so $a \in A \Rightarrow a^+ \in A$. Thus $a^+ \in A$ for every $A \in \mathscr{F}$, which implies $a^+ \in I$.

These conditions together imply that I is an inductive set. $\square$

Problem 0.5. Let A, B, C, D be sets. Some of the following statements are always true, and the others are sometimes wrong. Decide which is which. For the ones you declare "always true", provide a proof. For the others, provide one counterexample each.

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$

(c) $C \cap (A \setminus B) = A \cap (C \setminus B)$

(d) $C \cup (A \setminus B) = A \cup (C \setminus B)$

Proof. (a) *True.* Call the LHS set P and the RHS set Q. If $(a, x) \in P$, then $a \in A$ and $x \in B \cup C \Leftrightarrow x \in B$ or $x \in C$. Thus $a \in A$ and $x \in B$, or $a \in A$ and $x \in C$. Thus $(a, x) \in A \times B$ or $(a, x) \in A \times C$. Thus $(a, x) \in (A \times B) \cup (A \times C) = Q$.

If $(a, x) \in Q$, then $(a, x) \in A \times B$ or $(a, x) \in A \times C$. Thus $a \in A$ and $x \in B$, or $a \in A$ and $x \in C$. Thus $a \in A$, and $x \in B$ or $x \in C$. Thus $a \in A$ and $x \in B \cup C$. Thus $(a, x) \in A \times (B \cup C) = P$.

Therefore by the axiom of equality, $P = Q$.

(b) *False.* Consider the sets $A = B = C = \{\varnothing\}, D = \varnothing$. Then $A \times B = \{(\varnothing, \varnothing)\}$ and $C \times D = \varnothing$. Also, $A \setminus C = \varnothing$ and $B \setminus D = \{\varnothing\}$.

LHS $= \{(\varnothing, \varnothing)\}$
RHS $= \varnothing \neq$ LHS

(c) *True.* Call the LHS set P and the RHS set Q.

$$\begin{aligned} P &= \{c \in C : c \in (A \setminus B)\} \\ &= \{c \in C : c \in \{a \in A : a \notin B\}\} \\ &= \{c \in C : c \in A, c \notin B\} \end{aligned}$$

Similarly $Q = \{a \in A : a \in C, a \notin B\}$.

Clearly, every element of P is in Q and vice versa. Thus the two sets are equal.

(d) *False.* Consider the sets $A = \varnothing, B = C = \{\varnothing\}$. Then LHS $= C \cup \varnothing = C = \{\varnothing\}$, but RHS $= \varnothing \cup \varnothing = \varnothing \neq$ LHS. $\square$

Problem 0.6. Let A be a set. Define a relation $\mathbf{R}$ such that for any subsets B and C of A,

$$B\mathbf{R}C \Leftrightarrow B \subseteq C$$

Remember that a relation $\mathbf{R}$ is a subset of a Cartesian product of sets. Is the relation that you've defined a function?

Proof. Since B and C are subsets of A, they can be precisely the elements of the power set of A. So $\mathbf{R}$ is a subset of the $P(A) \times P(A)$.

If $A = \varnothing$, the only subset of A is $\varnothing$. So B and C can only take one value, $\varnothing$, and indeed $\varnothing \subseteq \varnothing \Leftrightarrow \varnothing \mathbf{R} \varnothing$.

Thus, in this case, $\mathbf{R}$ is a function if it is taken to be a relation from $\{\varnothing\}$ to $\{\varnothing\}$. However, $\mathbf{R}$ could also be taken to be a relation from $\{\varnothing, \{\varnothing\}\}$ to $\{\varnothing\}$, in which case it is no longer a function as $\{\varnothing\}\mathbf{R}x$ is false for all $x \in \{\varnothing\}$.

If $A \neq \varnothing$, we have $\varnothing \in P(A)$ and $A \in P(A)$. Thus $\varnothing \mathbf{R} \varnothing$ and $\varnothing \mathbf{R} A$, with $\varnothing \neq A$. Thus for $\varnothing \in P(A)$ there are at least two x such that $(\varnothing, x) \in \mathbf{R}$. Therefore $\mathbf{R}$ is not a function. $\square$