UM 101: Lecture 10 (piece missing from the board shots)

Purvi Gupta

Example 12 (Geometric Series)**.** Claim. Let $-1 < x < 1$. Then, $\sum_{n=0}^{\infty} x^n$ converges and its sum is $\frac{1}{1-x}$. For $|x| \geq 1$, $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$ diverges.

Proof. Observe that for $x \neq 1$,

$$s_n = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x}{1-x}x^n.$$

We inspect the behavior of $\{x_n\}$ as $n \to \infty$. In the case when $|x| < 1$, we use the fact that $(1+y)^n \geq ny$ for any $y > 0$ and $n \in \mathbb{N}$, to observe that

$$(1 + (1/|x|) - 1)^n > n\left(\frac{1}{|x|} - 1\right) = nc.$$

Thus, $|x|^n < 1/(nc)$. By the “squeeze theorem" stated in HW04, we have that $\lim_{n\to\infty} x^n = 0$.

In the case, when $|x| > 1$, $(1 + |x| - 1)^n > n(|x| - 1)$. Thus, for any $R \in \mathbb{R}$, by the Archimedean principle, there exists an $N \in \mathbb{N}$ such that $|x|^N > R$. Thus, $\{x^n\}$ is unbounded, and therefore, divergent.

Returning to the series $\sum_{n=0}^{]} inftyx^n$. When $|x| < 1$, we use the limit laws of convergent sequences to say that

$$\lim_{n\to\infty} s_n = \frac{1}{1-x}.$$

When $|x| > 1$, $x \neq 1$, we use the fact that the sum of a convergent and divergent sequence is, in fact, divergent. Thus, $\{s_n\}$ is divergent.

For $x = 1$, observe that $s_n = n + 1 \to \infty$ as $n \to \infty$. Thus, $\sum_{n=0}^{\infty}(1)^n$ diverges. □

UM 101: Lecture 11

Purvi Gupta

Theorem 2.13 (Ratio Test)**.** *Let $\sum a_n$ be a series of non-negative terms. Suppose*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

(1) *If $L < 1$, the series converges.*
(2) *If $L > 1$, the series diverges.*
(3) *If $L = 1$, then the test is inconclusive.*

Proof. **Case 1.** $L < 1$**.** Choose an r such that $L < r < 1$. Choosing $\varepsilon = r - L > 0$, we obtain an $N \in \mathbb{N}$ such that $\dfrac{a_{n+1}}{a_n} < L + \varepsilon = r$ for all $n \geq N$. Thus,

$$\begin{aligned} a_{N+1} &< a_N r \\ a_{N+2} &< a_{N+1} r < a_N r^2 \\ a_{N+3} &< a_{N+2} r < a_N r^3 \\ &\vdots \\ a_{N+k} &< a_N r^k. \end{aligned}$$

In other words, for $n \geq N$, $a_n \leq \frac{a_N}{r^N} r^n = cr^n$. By the Comparison Test, and the convergence of the geometric series $\sum r^n$, $r < 1$, we have the convergence of $\sum a_n$.

Case 2. $L > 1$**.** Choose R such that $1 < R < L$. Then, choosing $\varepsilon = L - R$, we have that for some $N \in \mathbb{N}$, $\dfrac{a_{n+1}}{a_n} > L - \varepsilon = R > 1$ for all $n \geq N$. Thus, $a_{n+1} > a_n$ for all $n \geq N$. The sequence $\{a_n\}$ cannot converge to 0. Thus, $\sum a_n$ diverges.

Case 3. $L = 1$**.** $\sum \dfrac{1}{n^2}$ converges while $\lim_{n\to\infty} \dfrac{n+1}{n} = 1$. On the other hand, $\sum \dfrac{1}{n}$ diverges while $\lim_{n\to\infty} \dfrac{n^2+1}{n^2} = 1$. $\square$

Theorem 2.14 (Limit Laws for Series)**.** *Suppose $\sum a_n$ and $\sum b_n$ converge with sums a and b respectively. Then, for constants ℓ and m, $\sum \ell a_n + mb_n$ converges to $la + mb$. Suppose $\sum |a_n|$ and $\sum |b_n|$ converge. Then, so does $\sum |\ell a_n + mb_n|$ for any choice of ℓ and m in $\mathbb{R}$.*

Proof. Exercies! $\square$

Corollary 2.15. *Suppose $\sum a_n$ converges and $\sum b_n$ diverges. Let $m \in \mathbb{R} \setminus \{0\}$. Then, $\sum (a_n + b_n)$ diverges, and $\sum mb_n$ diverges.*

Definition 2.16. A series $\sum a_n$ of real numbers is said to converge absolutely if $\sum |a_n|$ converges. A series $\sum a_n$ of real numbers is said to converge conditionally if $\sum |a_n|$ diverges but $\sum a_n$ converges.

Theorem 2.17. *If $\sum a_n$ converges absolutely, it must converge. Moreover, $|\sum a_n| \leq \sum a_n$*

Proof. We will construct a new series as follows:

$$b_n = a_n + |a_n|.$$

Observe that $0 \leq b_n \leq 2|a_n|$. Thus, by the comparison test, $\sum b_n$ converges. Now, by the limit laws for convergent series, $\sum a_n = \sum (b_n - |a_n|)$ converges. $\square$

Example 13. Claim. $\sum \frac{(-1)^n}{n}$ is convergent.

Proof. Note that

$$\begin{aligned}
s_1 &= -1 \\
s_3 &= -1 + \frac{1}{2} - \frac{1}{3} > s_1 \\
s_5 &= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} > s_3 \\
&\vdots \\
s_{2k+1} &= \left(-1 + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{4}\right) + \cdots \left(-\frac{1}{2k+1}\right) < 0.
\end{aligned}$$

Thus, $\{s_{2k+1}\}$ being a bounded increasing sequence, converges to some limit, say ℓ.

$$\begin{aligned}
s_2 &= -\frac{1}{2} \\
s_4 &= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} < s_2 \\
&\vdots \\
s_{2k} &= -1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2k}\right) \geq -1.
\end{aligned}$$

Thus, $\{s_{2k}\}$ being a bounded decreasing sequence, converges to some limit, say m. Moreover, $s_{2k+1} = s_{2k} + \frac{1}{2k+1}$. So, by limit laws for sequences, $\ell = m$. Exercise: why does this suffice to claim that $\{s_n\}$ converges? □

Theorem 2.18 (Alternating Series Test/Leibniz Test)**.** *Suppose $\{a_n\}$ is an decreasing sequence of positive numbers going to* 0*. Then, $\sum(-1)^n a_n$ converges. Denoting the sum by S, we have that*

$$0 < (-1)^n(S - s_n) < a_{n+1}.$$

Proof. Same principle as the example of $\sum \frac{(-1)^n}{n}$. □

Remark. The estimate in AST allows us to estimate sums of alternating series within any prescribed error. For instance, to know $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ up to an error of 0.01. I need to find n so that

$$|S - s_n| < \frac{1}{100}.$$

Take $n = 99$, or the sum of the first 99 terms.