UM 101: Lecture 08
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Example 8. (b) $a_n = (-1)^n$, $n \in \mathbb{N}$. Then, $\{a_n\}_{n\in\mathbb{N}}$ is divergent.

Proof. Suppose $\{a_n\}_{n\in\mathbb{N}}$ is convergent and admits a limit $L \in \mathbb{R}$. Let $\varepsilon = 1$. By definition, there exists an $N \in \mathbb{N}$ such that

$$|a_n - L| < 1 \quad \forall n \geq N.$$

Thus, by the triangle inequality,

$$|a_{2N} - a_{2N+1}| \leq |a_{2N} - L| + |a_{2N+1} - L| < 1 + 1 = 2.$$

But the left-hand side is 2. This is a contradiction.

□

Definition 2.4. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be bounded if there exists an $M > 0$ such that

$$|a_n| < M$$

for all $n \in \mathbb{N}$.

Definition 2.5. A sequence $\{a_n\}$ is said to be (montonically) increasing if $a_n \leq a_{n+1}$ for all n. A sequence $\{a_n\}$ is said to be (montonically) decreasing if $a_n \geq a_{n+1}$ for all n. A sequence is said to be monotone if it is either increasing or decreasing.

Theorem 2.6. *A monotone sequence is convergent if and only if it is bounded.*

Proof. (=>) **Case 1.** Let $\{a_n\}$ be an increasing and bounded sequence. Then, there exists an $M > 0$ such that

$$|a_n| < M \quad \forall n.$$

In other words $-M < a_n < M$ for all n. Let $S = \{a_n : n \in \mathbb{N}\}$. S is nonempty and bounded above. By LUB, $b = \sup S$ exists in $\mathbb{R}$.

Let $\varepsilon > 0$. By the definition of the supremum and monotonicity, $\exists N \in \mathbb{N}$ such that

$$b - \varepsilon < a_N \leq a_n \quad \forall n \geq N.$$

On the other hand, $a_n \leq b < b + \varepsilon$ for all $n \in \mathbb{N}$. Thus, for all $n \geq N$,

$$|a_n - b| < \varepsilon.$$

But ε was arbitrary.

Case 2. What if $\{a_n\}$ is decreasing?

(<=) In homework 03, you will show that *every* convergent sequence is bounded! □

⚠ Divergent sequences may diverge for different reasons!

Definition 2.7. We say that a sequence $\{a_n\}$ diverges to ∞ or $\lim_{n\to\infty} a_n = +\infty$ if for every $R \in \mathbb{R}$, there exists an $N_R \in \mathbb{N}$ such that $a_n > R$ for every $n \geq N_R$. One can similarly give meaning to $\lim_{n\to\infty} a_n = -\infty$.

Example 9. We show that $\lim_{n\to\infty} n = +\infty$. Let $R \in \mathbb{R}$. If $R \leq 0$, then $n \geq R$ for all $n \geq 1$. So we may choose $N_R = 1$. If $R > 0$, then by the Archimedean property, there is a positive natural number N such that $N > R$. Choose $N_R = N$ in this case, we get that $n > R$ for all $n \geq N_R$. Thus, we have shown that for any $R \in \mathbb{R}$, there is an $N_R \in \mathbb{N}$ such that $|a_n| > R$ for all $n \geq N_R$.

Theorem 2.8 (Limit Laws)**.** *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences in $\mathbb{R}$ with limits a and b, respectively.*

(1) *For any $c \in \mathbb{R}$, $\{a_n + c\}$ converges to $a + c$ and $\{ca_n\}$ converges to ca.*
(2) *The sequence $\{a_n + b_n\}$ converges to $a + b$.*
(3) *The sequence $\{a_n b_n\}$ converges to ab.*
(4) *Suppose $b \neq 0$ and $\exists M \in \mathbb{N}$ such that $b_n \neq 0$ $\forall n \geq M$ then $\{1/b_n\}_{n\geq M}$ converges to $1/b$.*
(5) *Suppose $b \neq 0$ and $\exists M \in \mathbb{N}$ such that $b_n \neq 0$ $\forall n \geq M$ then $\{a_n/b_n\}_{n\geq M}$ converges to a/b.*

Proof. We will prove (4). Scrapwork: want:

$$\left|\frac{b-b_n}{bb_n}\right| < \varepsilon.$$

If we could show that

$$\left|\frac{b-b_n}{bb_n}\right| < M|b-b_n|$$

for some M independent of n, we would be done. For this, we need an (eventual) lower bound on $|b_n|$.

Let $\varepsilon_1 = |b|/2$. By convergence of $\{b_n\}$ to b, there exists an $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$,

$$|b_n - b| < \varepsilon_1 = |b|/2.$$

By the reverse triangle inequality,

$$\Big||b_n| - |b|\Big| < |b_n - b| < |b|/2.$$

Thus,

$$|b_n| > |b| - |b|/2 = |b|/2 \qquad \forall n \geq N_1. \tag{2.1}$$

This gives that for all $n \geq N_1$

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b_n - b}{b_n b}\right| < \frac{2}{|b|^2}|b_n - b|.$$

Let $\varepsilon > 0$.

Set $\varepsilon_2 = \varepsilon/M$, where $M = \dfrac{2}{|b|^2}$. Then, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$|b_n - b| < \varepsilon/M. \tag{2.2}$$

Let $N = \max\{N_1, N_2\}$. Then, by (2.1) and (2.2), for all $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < M|b_n - b| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done.

$\square$

Remark. The converse statements generally do not hold in the above theorem, mostly because the convergence of sequences such as $\{a_n + b_n\}$, $\{a_n b_n\}$ and $\{a_n/b_n\}$ may not in general imply the convergence of the individual sequences $\{a_n\}$ and $\{b_n\}$.

END OF LECTURE 8