UM 101 - A (Real) Analysis

complex reals rationals integers naturals

$\mathbb{C} \hookleftarrow \mathbb{R} \hookleftarrow \mathbb{Q} \hookleftarrow \mathbb{Z} \hookleftarrow \mathbb{N}$

$\mathbb{Q} \cap \mathbb{R}$

221 204, 221

$\mathbb{Q}$ x^2+2 — $\overline{\mathbb{Q}}$ algebraic numbers $(\sqrt{2})$

1. Set theory & the real number system

said to be

Definition 1.1 A set A is "s.t.b" a Peano set if it satisfies the following axioms:

(P1) there is a (distinguished) element, which we call 0, in A.

(P2) there is a function S from A to A called the successor function

(P3) $S(a) \neq 0 \;\forall a \in A$.

(P4) If $S(a) = S(b)$, then $a = b \;\forall a, b \in A$ (1-1, injective)

(P5) (Principle of mathematical induction) Suppose B is a subset of A s.t. $0 \in B$, and if $a \in B$, then $S(a) \in B$. Then, $B = A$.

Examples: ⓐ $A = \{0, 1, 2, 3, 4\}$ (P4) fails.

ⓑ $A = \{0, 0.5, 1, 1.5, 2, \ldots\}$, $S(x) = x+1$. (P5) fails.

$B = \{0, 1, 2, \ldots\}$ satisfies the hypothesis, but not the conclusion.

ⓒ $A = \{0, 1, 2, 1, 2, 3, \ldots\}$

The ZFC axioms

Definition 1.2 A set is a well-defined collection of (mathematical) objects, called the elements of the set. To say that a is an element of the set A, we write $a \in A$. Otherwise, we say that $a \notin A$.

Given two sets A & B, we say that

- $A \subseteq B$, A is a subset of B, every element in A is an element in B.
- $A \nsubseteq B$, A is not a subset of B,
- $A \subset B$ · A is a proper subset of B, $A \subseteq B$, but $\exists\, y \in B$ s.t. $y \notin A$.

✓ $A \subsetneq B$

also

how to denote not a proper subset?

not $(A \subsetneq B)$

$\Updownarrow$ or

$A \nsubseteq B$ ~~&~~ $A \neq B$ $A = B$

A: Peano set

S: Successor fn

$B \subseteq A$

P5: If $0 \in B$ and whenever $a \in B$, $S(a) \in B$, then $B = A$.

e.g. $\mathbb{N} = \{0, 1, 2, \ldots\}$

$x \mapsto x+1$

Claim $P(n)$ is true.

Proof: $P(0)$ " $\Rightarrow 0 \in B$ $B = \{k \in \mathbb{N} : P(k) \text{ is true}\}$

Suppose $P(k)$ is true. if $k \in B$ then

Show: $P(k+1)$ " $k+1 \in B$

PMI, $P(n)$ is true $\forall n \in \mathbb{N}$. $\Rightarrow B = \mathbb{N}$.

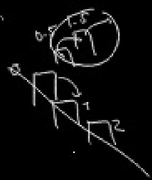
P1–P4

$A = \{0, 0.5, 1, 1.5, 2, \ldots$

$S(x) = x+1$

P5 $B \in$ dist

$a, \in$ $S(a)$



The ZFC axioms

Reason: not any collection can be called set
(read: Russell's Paradox)

A. <u>Basic</u>. Every object is a set. $\{\ \}$ = 1 is a set

B. <u>Extension</u>. Two sets A, B are equal if they have exactly the same elements. In other words, ~~A=~~ $A = B$ iff $A \subseteq B$ & $B \subseteq A$.

C. <u>Existence</u>. There is a set with no elements, called the empty set, denoted by $\emptyset$.

D. <u>Specification</u> Let A be a set. Let $P(a)$ denote a property that applies to every element in A, i.e., for each $a \in A$. Either $P(a)$ is true or it is false. Then there exists a subset

RP: set of all sets?

$$B = \{a \in A : P(a) \text{ is true}\}. \quad \text{set-builder notation.}$$

$a \in B \Leftrightarrow a \in A$ & $P(a)$.

E. Pairing. Given two sets A, B, there exists a set which contains precisely A & B as elements, which we denote by $\{A, B\}$. (In particular, this also gives $\{A\}$, a set with one element, A.) Cons.

In particular, $\{\emptyset\} = A$, $B = \emptyset$
& $\{\emptyset, \{\emptyset\}\} = C$.

$\overset{A}{\{1\}} = \overset{B}{\{1,1,1\}}$

A.o.E. $A = B$ if when $a \in A$, $a \in B$ & when $b \in B$, $b \in A$.

$1 \in A$ observe that $1 \in B$.

$1 \in B$ & $1 \in A$, but B has no other elements.

F. Union. Given a set $\mathcal{F}$ <u>of sets</u>, there exists a set, called the <u>union</u> of the sets in $\mathcal{F}$, denoted by $\bigcup_{A \in \mathcal{F}} A$, ~~such that~~ whose elements are precisely the elements of the elements of $\mathcal{F}$.

i.e. $a \in \bigcup_{A \in \mathcal{F}} A \Leftrightarrow a \in A$ for some $A \in \mathcal{F}$.

Consequence: Intersections of a nonempty set of sets, set differences of two sets $(A \setminus B)$ & ~~direct~~ Cartesian product of 2 sets...

G. Power set. Given a set A, there is a set called the <u>power set</u> of A, denoted by $\mathcal{P}(A)$, whose elements are precisely all the subsets of A.

$\mathcal{P}(A)$

Eg. $A = \{1, \{3\}\}$

$$\mathcal{P}(A) = \{\underset{\{x \in A : x \neq x\}}{\emptyset}, \{1\}, \{\{3\}\}, \{1, \{3\}\}\}$$

<u>Consequence</u>: G. allows us to define ordered pairs as sets, i.e. given $a \in A$ & $b \in B$, (a,b) has meaning in ZFC.
& we can define $A \times B = \{(a,b) : a \in A \text{ \& } b \in B\}$.
↑ Cartesian prod. of A & B.

Def^n: A relation from A to B, is a subset R of $A \times B$. We say that ~~aRb~~ given $a \in A$ & $b \in B$, $aRb \Leftrightarrow (a,b) \in R$.

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Given sets A, B,

Definition 1.3 a) A *relation* R from A to B is a subset of $A \times B$.
We say that $aRb \Leftrightarrow (a,b) \in R$ for any $a \in A$, $b \in B$.

b) The *domain* of R is the set
$$\mathrm{dom}(R) = \{ a \in A : (a,b) \in R \text{ for some } b \in B \}$$

c) The *range* of R is the set
$$\mathrm{ran}(R) = \{ b \in B : (a,b) \in R \text{ for some } a \in A \}.$$

d) R is called a *function* from A to B, denoted as $R \,\text{“}f\text{''}: A \to B$ iff
- $\rightarrow$ $\mathrm{dom}(R) = A$
- $\rightarrow$ for each $a \in A$, there is (at most) one $b \in B$ s.t. $(a,b) \in R$. $(a, f(a))$

Ex. write definitions for 1-1 (injective) & onto (surjective) functions.

Remark: *bijective* fn from A to B is a 1-1 fn from A onto B. fn = function

ZFC
- H. Regularity
- I. Replacement
- C ← J. Choice ✓
- K. Infinity

} skip

Definition 1.4 a) Given a set A, its *successor* is the set
$$A^{+} = A \cup \{A\}$$

pairing, pairing, union: $\mathcal{F} = \{A, \{A\}\}$

$A = \{x \in A : x \in A\}$
$A = \{0,1,2\} =: 3$
$B = \{3\} = \{A\} = \{\{0,1,2\}\}$

b) A set A is called *inductive* if $\phi \in A$ & for each $a \in A$, we have that $a^{+} \in A$.

Infinity: There is an inductive set.

Lemma 1.5. Let $\mathcal{F}$ be a nonempty set of inductive sets. Then,
$$\bigcap_{A \in \mathcal{F}} A \text{ is inductive.}$$

(ω is "omega")

Theorem 1.6. There exists a unique, minimal inductive set ω, i.e., ⊛ (for every inductive set S, $\omega \subseteq S$) and if ω' is another inductive set satisfying ⊛, then $\omega = \omega'$.

Theorem 1.7. ω (as above) is a Peano set when endowed with $S : \omega \to \omega$ given by $S(a) = a^{+}$.

Theorem 1.8. (The Recursion Principle) Let A be a set, $f : A \to A$ be a function & $a \in A$. Then, $\exists$ *unique* function $F : \omega \to A$ s.t.
a) $F(\phi) = a$,
b) $F(b^{+}) = f(F(b))$ $\forall\, b \in \omega$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle F}\uparrow & & \\ \omega & & \end{array}$$

BACK TO NATURAL NUMBERS: We identify $\mathbb{N} = \{0,1,2,\dots\}$ with ω as follows:

Define
$$0 := \phi$$
$$1 := 0^{+} = \{\phi\} = \{0\}$$
$$2 := 1^{+} = \{\phi, \{\phi\}\} = \{0,1\}$$
$$3 := 2^{+} = \{0,1,2\}$$
and so on.