LECTURE 09 07.11.2022

Claim: If $b_n \to b \neq 0$, then $\frac{1}{b_n} \to \frac{1}{b}$.

Want: Given $\epsilon > 0$, find $N \in \mathbb{N}$ s.t. $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \epsilon \ \forall n \geq N$

OR $\frac{|b_n - b|}{|b_n||b|} < \epsilon$

If I could make

(*) $\frac{\cancel{|b_n - b|}}{|b_n||b|} < M_1 \cancel{|b_n - b|}$ for some $M_1 > 0$

Then, I could choose $N \in \mathbb{N}$ s.t. $|b_n - b| < \epsilon / M_1 \ \forall n \geq N$.

Assum. (*) $\implies \forall n \geq N$

$$\frac{|b_n - b|}{|b_n||b|} < \cancel{M_1} \cdot \epsilon / \cancel{M_1} = \epsilon$$

(*) is equivalent to showing that $\exists M_2 > 0$ s.t. $|b_n| > M_2$ (for large n)

$b_n \to b \Downarrow |b_n| \to |b|$

$\implies |b_n| > \frac{|b|}{2}$ for n large

2.2. Infinite series.

(History: Apostol's)

Def^n 2.9. An infinite series (in $\mathbb{R}$) is a formal expression of the form $a_0 + a_1 + a_2 + \cdots$, or $\sum_{n=0}^{\infty} a_n$.

Given $\sum_{n=0}^{\infty} a_n$, its sequence of partial sums (SoPS) is $\{S_n\}_{n=0}^{\infty}$ where

$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$\vdots$$
$$S_n = a_0 + \cdots + a_n.$$

We say that $\sum a_n$ is convergent with sum S if $\lim_{n \to \infty} S_n = S$. Otherwise, we say that $\sum a_n$ is divergent.

$\sum_{n=0}^{\infty} a_n = S$

Examples. (a) (Harmonic series)

Claim. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof. ($\{S_n\}$ is a monotonically increasing sequence.)

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2}$$
$$S_4 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}$$
$$\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8}$$
$$\vdots$$
$$S_{2^k} \geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + \cdots + 2^{k-1} \cdot \frac{1}{2^k}$$
$$= 1 + \frac{k}{2} \to \infty \text{ as } k \to \infty$$

Thus, given any $R \in \mathbb{R}$, $\exists k \in \mathbb{N}$ s.t. $S_{2^k} > R$. (unbdd $\Rightarrow$ divg)

$\implies \{S_n\}$ is divergent. ∎

(b) Claim: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is cvgt. $\left(= \frac{\pi^2}{6}\right)$

$\left(\frac{1}{k} < \frac{1}{k-1}\right)$

Proof. $0 < s_1 = 1$

$0 < s_n = 1 + \sum_{k=2}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{k} \cdot \frac{1}{k-1}$

$= 1 + \sum_{k=2}^{n} \frac{k-(k-1)}{k(k-1)}$

$= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)$

$\sum a_n - a_{n-1}$: $\frac{1}{1} - \cancel{\frac{1}{2}}$ $+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$ $+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$ $\vdots$ $+ \cancel{\frac{1}{n-1}} - \frac{1}{n}$ (Telescoping sum) $\rightsquigarrow 1 - \frac{1}{n}$

$s_n < 1 + 1 - \frac{1}{n} < 2 \;\; \forall n \in \mathbb{N}$.

So, $\{s_n\}$ is an inc. bdd above seq. Thus, it cvgs. ∎

$\sum_{n=1}^{\infty} \left(2 + \frac{1}{n^2}\right)$

$s_n \geq n \cdot 2 \to \infty$.

$S = \sum_{n=1}^{\infty} (-1)^n = \underbrace{-1+1}\; -1 + 1 - 1 + \dots$

$s_1 = -1$

$s_2 = 0$

$s_3 = -1$

$\{s_n\}$ is divg^n.

$\boxed{S = -1 - S} \to S = -\frac{1}{2}$

Theorem 2.10. (Necessary cond^n for convergence). Suppose $\sum a_n$ is convergent. Then, $\lim_{n\to\infty} a_n = 0$.

Proof. Let $\epsilon > 0$. Since $\sum a_n$ is cvgt, say to l, $\exists N \in \mathbb{N}$ s.t.

$$|s_n - l| < \epsilon/2 \;\; \forall n \geq N,$$

where $\{s_n\}$ is the s.o.p.s. of $\sum a_n$.

Now observe that $\boxed{a_n = s_n - s_{n-1}}$.

Thus, for $n > N$, we have

$$|a_n| = |s_n - s_{n-1}| \leq |s_n - l| + |s_{n-1} - l| < \epsilon \;\; \forall n > N.$$

Since $\epsilon > 0$ was arb, $\lim_{n\to\infty} a_n = 0$. ∎

LECTURE 10

09.11.2022

$\sum a_n$

$\{a_n\}$ v/s $\{s_n\}$ s.o.p.s.

$s_n = a_0 + \dots + a_n$

Example. (Geometric series) Let $x \in \mathbb{R}$. Then

$$\sum_{n=0}^{\infty} x^n \begin{cases} = \frac{1}{1-x}, & |x|<1, \\ \text{diverges}, & |x| \geq 1 \end{cases}$$

So, for $|x|<1$, by limit laws,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{1-x} - \frac{x}{1-x}(x^n) = \frac{1}{1-x}.$$

Next, for $|x| \geq 1$, $x \neq 1$,

Once again, we must understand the behavior of $\{x^n\}$ for $|x|\geq 1$, $x \neq 1$.

Kla

Please see typed-up notes for this missing piece!

$$s_n = 1 + x + \dots + x^n \frac{(1-x)}{1-x} \quad x \neq 1$$

$$= \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{1}{1-x}(x^{n+1})$$

Claim. If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$.

If $|x| \geq 1$, $x \neq 1$, $\{x^n\}$ diverges.

Proof. Case 1. $|x|<1$. Suffices to prove for $x>0$.

Note that $(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k$, $y>0$

$> ny$.

If $x<1$, then $\frac{1}{x} > 1$. So, $\frac{1}{x} = 1+y$ for some $y>0$.

Then

$$\left(\frac{1}{x}\right)^n = (1+y)^n > ny = n\left(\frac{1}{x}-1\right).$$

so, $0 \leq x^n < \frac{1}{n} c$, $c = \frac{1}{\frac{1}{x}-1}$

$\searrow 0 \leftarrow \searrow 0$ (HW(p))

Theorem 2.11 (Comparison Test).

Suppose $\exists$ constants $M \in \mathbb{N}$ and $C>0$ s.t.

$$0 \leq a_n \leq C b_n$$

$\forall n \geq M$.

If $\sum b_n$ converges, then $\sum a_n$ converges. (In other words, if $\sum a_n$ diverges, then $\sum b_n$ diverges)

Proof. Let $\{s_n\}$ & $\{t_n\}$ be the s.o.p.s. of $\sum_{n=M}^{\infty} a_n$ & $\sum_{n=M}^{\infty} b_n$, respectively. Note that $\{s_n\}$ & $\{t_n\}$ ~~is an~~ are increasing sequences.

By convergence of $\{t_n\}$, $\exists N \in \mathbb{N}$ & $L>0$ s.t. $t_n < L$ $\forall n \geq N$.

Thus, $0 \leq \underbrace{s_n}_{a_M+\dots+a_{M+n}} \leq C t_n < CL$ $\forall n \geq \max\{M,N\}$ Needs just n

By MCT, $\{s_n\}$ converges.

Thus, $\sum a_n$ converges. ∎

$$\sum_{n=M}^{\infty} a_n \quad \sum_{n=M}^{\infty} b_n$$

$$s_n = a_M + \dots + a_{M+n} \leq C b_M + \dots + C b_{M+n} = C t_n.$$

E.g. $1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{5} + \frac{1}{4} + \dots$

$1, 1+\frac{1}{3}, 1+\frac{1}{3}+\frac{1}{2}, \dots$

(p-series)

Example: Let $p \in \mathbb{R}$

(p-series)

$s_n = a_M + \cdots + a_{M+n}$
$\le Cb_M + \cdots + Cb_{M+n} = Ct_n.$

Example: Let $p \in \mathbb{R}$

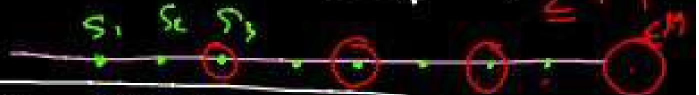
$$p \mapsto \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges, when } p>1 \\ \text{diverges, when } p \le 1 \end{cases}$$

Riemann Zeta function.

Proof. If $p \le 0$, check that $\left(\frac{1}{n}\right)^p \not\to 0$. So $\sum \frac{1}{n^p}$ diverges.

If $0 < p \le 1$, then $n^p \le n \ \forall n \ge 1$.
Thus $\frac{1}{n^p} \ge \frac{1}{n}$. By the C.T., since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^p}$ div.

If $p \ge 2$, then $n^p \ge n^2 \ \forall n$.
So, by C.T. & $\sum \frac{1}{n^2} < \infty$, we have that $\sum \frac{1}{n^p}$ cvgs. $\le M$.



$1 < p < 2$. Note that

$s_1 = 1$

$s_3 = 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{} \le 1 + 2 \cdot \frac{1}{2^p}$

$n \in \mathbb{N}$ $\exists k$ s.t. $n < 2^k - 1$

$\vdots$

$s_7 = 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{} + \underbrace{\frac{1}{4^p} + \cdots + \frac{1}{7^p}}_{} \le 1 + \frac{2}{2^p} + \frac{4}{4^p}$

$\vdots$

$s_{2^k-1} \le 1 + \frac{2}{2^p} + \cdots + \frac{2^{k-1}}{2^{p(k-1)}}$

The RHS are sops of

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n = \frac{1}{1 - \left(\frac{1}{2}\right)^{p-1}}.$$

So, $s_{2^k-1} < \frac{1}{1-\left(\frac{1}{2}\right)^{p-1}}$ $\forall k \in \mathbb{N}, k > 0$. (*)

& S_n is increasing.

(*) By "MCT", $\{S_n\}$ cvgs.

Theorem 2.12 (Ratio Test)
Let $\sum a_n$ be a series of positive terms.
Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$

Then,
a) If $L < 1$, $\sum a_n$ cvgs.
b) If $L > 1$, $\sum a_n$ dvg. $a_n \not\to 0$
c) If $L = 1$, test is inconclusive

Proof c) $\sum 1$ diverges.
$\sum \frac{1}{n^2}$ converges.

Q. Does $\frac{a_{n+1}}{a_n} < 1$ for a cvgt. series. $\forall n$

No.

Thm. 2.13 (~~Ratio~~ Root Test) HW.