Assignment 8

Naman Mishra

23 Dec 2022

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, c) \cup (c, b)$. Show that if $\lim_{x \to c} f'(x) = L$, then $f'(c)$ exists and equals L.

Proof. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that $x \in N_\delta(c) \setminus \{c\} \implies |f'(x) - L| < \varepsilon$.

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L$$

□

Problem 7

We define a function $f : \mathbb{R} \to \mathbb{R}$ to be 'close' at a point $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta > 0$ such that for every $x \in N_\delta(c) \setminus \{c\}$, we have that

$$|f(x) - L| < \varepsilon.$$

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is close at $c \in \mathbb{R}$. Then for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta_\varepsilon > 0$ such that for every $x \in N_{\delta_\varepsilon}(c) \setminus \{c\}$, we have

$$|f(x) - L| < \varepsilon/2.$$

Consider a sequence $\{a_n\} \subset \mathbb{R} \setminus \{c\}$ converging to c. Then there exists an n_0 such that for every $n \geq n_0$, we have

$$|a_n - c| < \delta_\varepsilon \implies |f(a_n) - L| < \varepsilon/2$$

By the triangle inequality,

$$|f(a_n) - f(a_m)| \leq |f(a_n) - L| + |L - f(a_m)| < \varepsilon \; \forall \, m, n \geq n_0$$

Thus $\{f(a_n)\}$ is a Cauchy sequence, and therefore convergent.

Suppose sequences $\{a_n\}$ and $\{b_n\}$ both converge to (but never equal) c, with $\{f(a_n)\}$ and $\{f(b_n)\}$ having different limits L_1 and L_2. Consider the sequence

$$c_n = \begin{cases} a_n & \text{if } n \text{ is even} \\ b_n & \text{if } n \text{ is odd} \end{cases}$$

Clearly $\{c_n\}$ converges to c, but $\{f(c_n)\}$ diverges. This is a contradiction.

Thus there exists a unique $L_0 \in \mathbb{R}$ such that given any sequence $\{a_n\}$ converging to c, we have that

$$\lim_{n\to\infty} f(a_n) = L_0.$$

By the sequential characterization of limits, the limit of f at c exists and equals L_0.

On the other hand, if it is known that f has a limit L_0 at c, setting $L = L_0 \; \forall \, \varepsilon$ proves that f is close at c.

Thus the two definitions are equivalent.