Assignment 06

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Problem 1.

(a)

$$f(x) = \begin{cases} x & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$$

(b)

$$f(x) = x^2$$

Problem 2.

Problem 3. Let the polynomial be $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where n is odd. Define

$$g(x) = \frac{f(x)}{x^n} = \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + a_n.$$

Define the sequence $\{g(m)\}_{m \in \mathbb{N}}$. We have

$$\lim_{m \to \infty} g(m) = a_n.$$

Thus we have $N \in \mathbb{N}$ such that $|g(m) - a_n| < a_n \ \forall\ m \geq N \Rightarrow g(N) > 0 \Rightarrow f(N) = N^n g(N) > 0$.

Define the sequence $\{g(-m)\}_{m \in \mathbb{N}}$. We have

$$g(-m) = -\frac{a_0}{m^n} + \frac{a_1}{m^{n-1}} - \cdots + a_n$$
$$\Rightarrow \lim_{m \to \infty} g(-m) = a_n$$

Thus we have $N' \in \mathbb{N}$ such that $|g(-m) - a_n| < a_n \ \forall\ m \geq N' \Rightarrow g(-N') > 0 \Rightarrow f(-N') = (-N')^n g(-N) = -(N')^n g(-N) < 0$. Also $-N' < 0 < N$. By IVT we have

$$\exists\, c \in [-N', N] : f(c) = 0.$$

Problem 4. Consider $g(x) = \cos x - x^2$. Then $g'(x) = -\sin x - 2x < 0$. So g is decreasing. $g(0) = 1 > 0$. $g(\frac{\pi}{2}) = -\frac{\pi^2}{4} < 0$. Therefore $g(x) = 0$ at exactly one $x = c$ in $[0, \frac{\pi}{2}]$. Thus $g(x) > 0$ for $0 \leq x < c$ and $g(x) < 0$ for $c < x \leq \frac{\pi}{2}$.

For $0 \leq x < c$, $f(x) = \cos x$. For $c < x \leq \frac{\pi}{2}$, $f(x) = x^2$.

Since $\cos x$ is decreasing in $[0, c] \subseteq [0, \frac{\pi}{2}]$, we have $f(x) > f(c)\ \forall\ x \in [0, c)$. Since x^2 is increasing in $[c, \frac{\pi}{2}] \subseteq [0, \frac{\pi}{2}]$, we have $f(x) > f(c)\ \forall\ x \in (c, \frac{\pi}{2}]$

Thus f attains a global minimum at c, where $\cos c = c^2$.

Problem 5.

(a)

$$h \circ g(x) = |x|^3 = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}$$

Since all polynomials are continuous and differentiable at every point, we only need to check at 0.

$$\begin{aligned} \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} \frac{|h|^3}{h} \\ &= \lim_{h \to 0} \frac{h^2|h|}{h} \\ &= \lim_{h \to 0} h|h| \\ &= 0 \end{aligned}$$

Thus $h \circ g$ is differentiable everywhere, so it is continuous everywhere.

(b) We first show that $\cos(\frac{1}{x})$ is differentiable at $x \neq 0$.

$$\begin{aligned} \lim_{x \to p} \frac{\cos(\frac{1}{x}) - \cos(\frac{1}{p})}{x - p} &= \lim_{x \to p} \frac{2\sin(\frac{1}{2x} + \frac{1}{2p})\sin(\frac{1}{2p} - \frac{1}{2x})}{x - p} \\ &= 2\sin(\frac{1}{p}) \lim_{x \to p} \frac{\sin(\frac{x-p}{2xp})}{x - p} \\ &= 2\sin(\frac{1}{p}) \lim_{x \to p} \frac{\sin(\frac{x-p}{2xp})}{\frac{x-p}{2xp}} \frac{1}{2xp} \\ &= \frac{1}{p^2} \sin(\frac{1}{p}) \end{aligned}$$

The limit exists, so $\cos(\frac{1}{x})$ is differentiable everywhere in its domain.

By algebra laws we have $x^2 \cos(\frac{1}{x})$ also differentiable, as is $|x| = -x\ \forall\ x < 0$. So we only need to worry about 0.

In the neighbourhood $(-1, 1)$ about 0, we have $-x^2 \leq f(x) \leq |x|$. By the squeeze theorem, $f(x) + x^2$ tends to 0. By the limit laws, $\lim_{x \to 0} f(x) = 0$. Thus f is continuous everywhere.

$$f(0+h)-f(0)=\begin{cases}-h & h<0\\ h^2\cos(\frac{1}{h}) & h>0\end{cases}$$

$$\frac{f(0+h)-f(0)}{h}=\begin{cases}-1 & h<0\\ h\cos(\frac{1}{h}) & h>0\end{cases}$$

Let $\varepsilon=\frac{1}{4}$. For any $\delta>0$, choose $k=\min\{\frac{1}{2},\frac{\delta}{2}\}$. $-k\le f(k)\le k\Rightarrow f(k)>-k>-\frac{1}{2}$ and $|k-0|<\delta$. Also $f(-k)=-1$. For any L, $|f(k)-L|+|f(-k)-L|\ge|f(k)-f(-k)|=|f(k)+1|\ge\frac{1}{2}=2\varepsilon$. Thus the limit does not exist and so the function is not differentiable at 0.

(c) $|\sin x|=|\sin|x||$. So

$$f(x)=\begin{cases}\frac{|\sin|x||}{\sin|x|} & x\neq n\pi\\ 0 & x=n\pi\end{cases}$$

or

$$f(x)=\begin{cases}1 & \sin|x|>0\\ 0 & \sin|x|=0\\ -1 & \sin|x|<0\end{cases}$$

Since constant functions are continuous and differentiable, $f(x)$ is differentiable in any region where $\sin|x|$ is constant in sign. Thus $f(x)$ is continuous and differentiable in all intervals $(n\pi,(n+1)\pi), n\in\mathbb{Z}$. This leaves only the points $n\pi$, where the function is neither continuous nor differentiable, as

$$\lim_{x\to n\pi} f(x) \text{ does not exist.}$$

Suppose limit exists and is equal to L. Then