Assignment 05

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You may freely use (without proof):

(i) $0 < \cos(x) < \left|\frac{\sin x}{x}\right| < 1$ for all $0 < |x| < \frac{\pi}{2}$.

(ii) Any trigonometric identities that you have seen in school.

(iii) Any limits computed in Lecture 12-14.

Problem 0.1. Prove the squeeze theorem.

Theorem 0.1 (squeeze theorem)**.** Let f, g, h be functions defined on some neighborhood N of p, except perhaps at p. Suppose $f \leq g \leq h$ on N, and

$$\lim_{x \to p} f(x) = \lim_{x \to p} h(x) = L \in \mathbb{R}.$$

Then

$$\lim_{x \to p} g(x) = L.$$

Proof. For any $\varepsilon > 0$, $\exists\, \delta_1, \delta_2$ such that

$$\begin{aligned} |f(x) - a| &< \varepsilon \quad \forall\, x \in N \cap N_{\delta_1}(p) \setminus \{p\} \\ |h(x) - a| &< \varepsilon \quad \forall\, x \in N \cap N_{\delta_2}(p) \setminus \{p\} \end{aligned}$$

Thus for all $x \in N \cap N_\delta(p) \setminus \{p\}$, where $\delta = \min\{\delta_1, \delta_2\}$ so that $N_\delta = N_{\delta_1} \cap N_{\delta_2}$,

$$\begin{aligned} a - \varepsilon < f(x) \leq g(x) \leq h(x) < a + \varepsilon \\ \Rightarrow \lim_{x \to p} g(x) = a \end{aligned}$$

□

Problem 0.2. In each of the following cases, determine whether the limit exists or not, and compute the limit whenever it exists. You may use any of the theorems stated in class, but state what you are using.

(a) $\lim_{x \to 2} \frac{(3x+1)^2 - 49}{x-2}$

(b) $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$

(c) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

(d) $\lim_{x \to p} x^n$ (for fixed $n \in \mathbb{N}$ and $p \in \mathbb{R}$)

Proof. (a) For any $x \neq 2$,

$$\frac{(3x+1)^2 - 49}{x-2} = \frac{(3x+8)(3x-6)}{x-2} = 3(3x+8)$$

By limit laws,

$$\lim_{x \to 2} 3(3x+8) = (\lim_{x \to 2} 3)((\lim_{x \to 2} 3) \lim_{x \to 2} x + \lim_{x \to 2} 8) = 3(3 \cdot 2 + 8) = 42$$

(b) Suppose the function has a limit L. Take $\varepsilon = 1$. Then $\exists\ \delta > 0$ such that $\left|\cos(\frac{1}{x}) - L\right| < 1$ for all $0 < |x - 0| < \delta$. By the Archimedean property, there exists N such that $N2\pi > \frac{1}{\delta} \Rightarrow 0 < \frac{1}{(2N+1)\pi} < \frac{1}{2N\pi} < \delta$.

$$|1 - L| < 1$$
$$|-1 - L| < 1$$
$$2 = |1 - L + L + 1| \leq |1 - L| + |-1 - L| < 1 + 1 = 2$$

Contradiction.

(c)

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

Since $\lim_{x \to 1} \sqrt{x} = 1$, we have $\lim_{x \to 0} \sqrt{1+x} = \lim_{x \to 0} \sqrt{1-x} = 1$. By limit laws,

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{2}{2} = 1$$

(d) We have $\lim_{x \to p} x^0 = \lim_{x \to p} 1 = 1 = p^0$. Suppose $\lim_{x \to p} x^n = p^n\ \forall\ p \in \mathbb{R}$ for some n. Then $\lim_{x \to p} x^{n+1} = \lim_{x \to p} x \cdot \lim_{x \to p} x^n = p \cdot p^n = p^{n+1}\ \forall\ p \in \mathbb{R}$. Thus $\lim_{x \to p} x^n = p^n\ \forall\ p \in \mathbb{R}\ \forall\ n \in \mathbb{N}$. $\square$

Problem 0.3. Let f and g be functions on $\mathbb{R}$ such that

$$\lim_{x\to 0} f(x) = L \quad \text{and} \quad \lim_{y\to 0} g(y) = M,$$

for some $L, M \in \mathbb{R}$. Is it true that

$$\lim_{x\to 0}(g \circ f)(x) = M?$$

If your answer is "yes", prove the above statement. If your answer is "no", provide a counterexample, and give a sufficient condition on g that will make the above statement true.

Proof. NO. Let $f(x) = 0$, $g(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} g \circ f(x) = 1$.

This arises from the fact that $|f(x) - L| < \varepsilon$ does not imply $0 < |f(x) - L|$, so it doesn't guarantee $0 < |y - L| < \delta$.

If we have g continuous at L, then we know g is defined at L as well.

For any $\varepsilon > 0$, $\exists\, \delta_1 > 0$ such that $|y - L| < \delta_1 \Rightarrow g(y)$is defined and$|g(y) - M| < \varepsilon$. Choose $\varepsilon_2 = \delta_1$. Then there exists $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow f(x)$ is defined and $|f(x) - L| < \varepsilon_2 = \delta_1 \Rightarrow |g(f(x)) - M| < \varepsilon$. □

Remarks. We can also enforce a condition on f: if $f(x) \neq L \ \forall\ x \neq 0$, then $\lim_{x\to 0} g(f(x)) = \lim_{x\to L} g(x)$.

This yields a theorem:

Theorem 0.2 (limit of composition)**.** Suppose f and g are functions on $\mathbb{R}$ such that

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to L} g(x) = M.$$

Then we have

$$\lim_{x\to a} g(f(x)) = M$$

if

(a) g is continuous, or

(b) $f(x) \neq L \ \forall\ x \in N_\delta(a) \setminus \{a\}$ for some $\delta > 0$.

Problem 0.4. Prove the sequential characterization of continuity.

Theorem 0.3 (sequential characterization of continuity)**.** Let $f : A \to \mathbb{R}$ be a function and let $p \in A$. Let $P = \left\{\{a_n\} \subseteq A : \lim_{n\to\infty} a_n = p\right\}$. Then f is continuous at p iff $\lim_{n\to\infty} f(a_n) = f(p) \ \forall \ \{a_n\} \in P$.

Proof.

(a) Let f be continuous at p and let $\{a_n\} \in P$.

For every $\varepsilon > 0$ there exists δ such that $f(x) \in N_\varepsilon(f(p)) \ \forall \ x \in N_\delta(p) \cap A$. Morever, $\exists \ N \in \mathbb{N}$ such that $a_n \in N_\delta(p) \cap A \ \forall \ n \geq N$.

Thus for all $n \geq N$, $f(a_n) \in N_\varepsilon(f(p))$. Thus $\lim_{n\to\infty} f(a_n) = f(p)$.

(b) Let $\lim_{n\to\infty} f(a_n) = f(p) \ \forall \ \{a_n\} \in P$. Suppose f is not continuous at p. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists $a \in N_\delta(p) \cap A \setminus \{p\}$ such that $|f(a) - f(p)| \geq \varepsilon$.

Let $\{\delta_n\}_{n\in\mathbb{P}} = \left\{\frac{1}{n}\right\}_{n\in\mathbb{P}}$. Then corresponding to every $\delta_n \ \exists \ a_n \in A \cap N_{\delta_n}(p) \setminus \{p\}$ such that $|f(a_n) - f(p)| \geq \varepsilon$.

Thus $\{a_n\} \to p$ but $\lim_{n\to\infty} f(a_n) \neq f(p)$. Contradiction. □

(This proof uses the axiom of choice. Let me know if you have a proof without it.)

Problem 0.5. Complete the following steps to establish the continuity of the sine and cosine functions on $\mathbb{R}$. Recall (i) and (ii) given at the beginning of this assignment.

(a) Show that $\lim_{x\to 0} \sin(x) = 0$.

(b) Using (a) and a trigonometric identity relating sin and cos, show that $\lim_{x\to 0} \cos(x) = 1$.

(c) Using (a) and (b), show that sin and cos are continuous at any $x \in \mathbb{R}$.

(d) Show that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Proof.

(a)

$$0 < \left|\frac{\sin x}{x}\right| < 1 \Rightarrow -|x| < \sin x < |x| \qquad \forall \ 0 < |x| < \frac{\pi}{2}$$

Since $\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0$, we have $\lim_{x\to 0} \sin x = 0$ by squeeze theorem.

(b) Since $\cos^2(x) + \sin^2(x) = 1$, we have

$$\lim_{x\to 0} \cos^2(x) = \lim_{x\to 0}(1 - \sin^2(x)) = 1 - \lim_{x\to 0} \sin^2 x = 1 - \left(\lim_{x\to 0} \sin x\right)^2 = 1$$

Since $\cos x > 0$ for $|x - 0| < \frac{\pi}{2}$, we have $\sqrt{\cos^2(x)} - \cos x = 0$ for $|x - 0| < \frac{\pi}{2}$. So $\lim_{x\to 0}(\sqrt{\cos^2 x} - \cos x) = 0$. Thus

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \sqrt{\cos^2 x} = \sqrt{\lim_{x\to 0} \cos^2 x} = 1$$

by limit of composition ($\sqrt{\cdot}$ is continuous).

(c) Now

$$\begin{aligned} \sin(x+h) &= \sin x \cos h \\ &\quad + \cos x \sin h \\ \lim_{h\to 0} \sin(x+h) &= \sin x \\ \lim_{y\to x} \sin(y) &= \sin x \end{aligned} \qquad \begin{aligned} \cos(x+h) &= \cos x \cos h \\ &\quad - \sin x \sin h \\ \lim_{h\to 0} \cos(x+h) &= \cos x \\ \lim_{y\to x} \cos(y) &= \cos x \end{aligned}$$

for all $x \in \mathbb{R}$. Thus sin and cos are continuous at any $x \in \mathbb{R}$

(d) Finally, we have

$$\left|\frac{\sin x}{x}\right| = \frac{\sin x}{x} \qquad \forall\, 0 < |x| < \frac{\pi}{2}$$

so

$$\cos x < \frac{\sin x}{x} < 1 \quad \forall\, 0 < |x| < \frac{\pi}{2}.$$

By the squeeze theorem, we get

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \qquad \square$$