UM101: Analysis & Linear Algebra

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Contents

Lecture 1: Introduction; Peano Sets

mon 17 oct '22

Part I

Analysis

1 Set theory & the real number system

Definition 1.1. The set A along with a successor function S is called a Peano set if it obeys the Peano axioms.

(P1) There is an element called 0 in A.

(P2) For every $a \in A$, its successor $S(a)$ is also in A.

(P3) $\forall\, a \in A, S(a) \neq 0$.

(P4) For any $m, n \in A$, $S(m) = S(n)$ only if $m = n$.

(P5) (principle of mathematical induction) For any set $B \subseteq A$, if $0 \in B$ and $a \in B \Rightarrow S(a) \in B$, then $B = A$.

Lecture 2: The ZFC Axioms

wed 19 oct 2022

1.1 The ZFC Axioms

Definition 1.2. A **set** is a well-defined collection of (mathematical) objects, called the *elements* of that set. To say that a is an element of set A, we write $a \in A$. Otherwise, we write $a \notin A$.

Given two sets A and B, we say that:

($A \subseteq B$) A is a subset of B, *i.e.*, every element of A is an element of B.

($A \nsubseteq B$) A is not a subset of B, *i.e.*, there is some element in A which is not an element of B.

($A \subsetneq B$) A is a proper subset of B, *i.e.*, $A \subseteq B$ but $\exists\, b \in B$ such that $b \notin A$.

Remarks. We need ZFC axioms because not any collection can be called a set. Read up on Russell's paradox.

Axiom 1.1 (the basic axiom)**.** Every object is a set.

Axiom 1.2 (axiom of extension)**.** Two sets A, B are equal if they have exactly the same elements. In other words, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

Remarks. As a consequnce, it doesn't matter whether a set contains multiple copies of an element.

$$A = \{1\}$$
$$B = \{1, 1, 1\}$$

Clearly $A \subseteq B$ and $B \subseteq A$, implying $A = B$.

Axiom 1.3 (axiom of existence)**.** There is a set with no elements called the empty set, denoted by the symbol $\varnothing$.

Axiom 1.4 (axiom of specification)**.** Let A be a set. Let $P(a)$ denote a property that applies to every element in A, i.e., for each $a \in A$, either $P(a)$ is true or it is false. Then there exists a subset

$$B = \{a \in A : P(a) \text{ is true}\}$$

Remarks. We are forced to create sets only as subsets of other sets because of Russell's paradox. *From MathGarden:* A somewhat surprising result is that the axiom of specification implies for each set A the existence of an element (a set) x such that $x \notin A$. In other words, there is no set containing all sets of our mathematical universe.

Axiom 1.5 (axiom of pairing)**.** Given two sets A, B, there exists a set which contains precisely A, B as its elements, which we denote by $\{A, B\}$.

Remarks. In particular, by letting $A = B$, we get a set containing only A, i.e., $\{A\}$. For example, we can have $\{\varnothing\}$, and $\{\varnothing, \{\varnothing\}\}$, etc.

Axiom 1.6 (axiom of unions)**.** Given a set $\mathscr{F}$ of sets, there exists a set called the union of the sets in $\mathscr{F}$, denoted by $\bigcup_{A \in \mathscr{F}} A$, whose elements are precisely the elements of the elements of $\mathscr{F}$.

$$a \in \bigcup_{A \in \mathscr{F}} A \Leftrightarrow a \in A \text{ for some } A \in \mathscr{F}$$

Remarks. Intersection of a nonempty set of two or more sets and difference between two sets need not be defined as they follow from the previous axioms. (Exercise)

Proof. By the axiom of specification,

$$A - B = \{a \in A : a \notin B\}$$
$$A \cap B = \{a \in A : a \in B\}$$

□

Axiom 1.7 (axiom of powers)**.** Given a set A, there exists a set called power set of A denoted $\mathscr{P}(A)$, whose elements are precisely all the subsets of A.

Remarks. This axiom allows us to define ordered pairs as sets (assignment) (Isn't pairing sufficient?) and thus direct products, relations and functions.

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

How does this set exist?

How are we able to define $A \times B$, more specifically, ordered pairs?
This is problem 3 of assignment 1.

Lecture 3: Relations and Infinity

fri 21 oct 2022

Definition 1.3. A relation from set A to set B is a subset R of $A \times B$. For any $a \in A, b \in B$ we say $a\, R\, b$ iff $(a, b) \in R$.

- The *domain* of R is the set

$$\operatorname{dom}(R) = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

- The *range* of R is the set

$$\operatorname{ran}(R) = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$$

- R is called a *function* from A to B, denoted as $R : A \to B$ iff
  - $\operatorname{dom}(R) = A$
  - for each $a \in A$ there is (at most) one $b \in B$ such that $(a, b) \in R$.

Exercise: Write definitions for *injective* and *surjective* functions.

Remarks. A *bijective* function from A to B is an injective as well as surjective fuction from A to B.

Axiom 1.8 (axiom of regularity)**.** Read up

Axiom 1.9 (axiom of replacement)**.** Read up

Axiom 1.10 (axiom of choice)**.** Read up

Definition 1.4. Given a set A, its *successor* is the set

$$A^+ = A \cup \{A\}.$$

A set A is said to be *inductive* if $\varnothing \in A$ and for every $a \in A$, we have $a^+ \in A$.

Remarks. The successor of A is guaranteed to exist by axiom of pairing and axiom of unions.
$\{A\}$ exists by axiom of pairing by letting $B = A$.
$A \cup \{A\}$ exists by applying axiom of unions on the set $\{A, \{A\}\}$ formed using axiom of pairing again.

$\{A\}$ can also be created as a subset (axiom of specification) of the power set (axiom of powers) of A.

Remarks. The definition of an inductive set is very similar to the principle of mathematical induction in the Peano axioms.

Axiom 1.11 (axiom of infinity)**.** There exists an inductive set.

Lemma 1.5. Let $\mathscr{F}$ be a nonempty set of inductive sets. (This exists by axiom of infinity and axiom of pairing). Then

$$\bigcap_{A \in \mathscr{F}} A \text{ is inductive.}$$

Proof. Assignment 1, Problem 5 □

Theorem 1.6. There exists a *unique*, *minimal* inductive set ω, *i.e.*, for any inductive set S,

$$\omega \subseteq S$$

and if ω' is any other inductive set satisfying this property,

$$\omega = \omega'$$

Proof under construction. Suppose there is no minimal inductive set. Then for any inductive set ω, we have $\omega \not\subseteq S$ for some inductive S.
Thus there exists a set $\mathscr{F}$ of inductive sets, which contains S, such that $\bigcup_{A \in \mathscr{F}} A$ does not have ω as a subset. □

Theorem 1.7. The ω in theorem 1.6 is a Peano set with successor function $a \mapsto a^+$.

Theorem 1.8 (principle of recursion)**.** Let A be a set, and $f : A \to A$ be a function. Let $a \in A$. Then, there exists a function $F : \omega \to A$ such that

(a) $F(\varnothing) = a$

(b) For some $b \in \omega$, we have $F(b^+) = f(F(b))$

Back to natural numbers!

Lecture 4: Natural Numbers

wed 26 oct 2022

1.2 Natural Numbers

The ZFC axioms give us the existence of $\mathbb{N} = \{0, 1, 2, \ldots\}$ with definitions as follows:

$$\begin{aligned} 0 &:= \varnothing \\ 1 &:= 0^+ = \varnothing \cup \{\varnothing\} = \{\varnothing\} = \{0\} \\ 2 &:= 1^+ = 1 \cup \{1\} = \{0, 1\} \\ 3 &:= 2^+ = 2 \cup \{2\} = \{0, 1, 2\} \end{aligned}$$

This $\mathbb{N}$ is the minimal inductive set ω.

Definition 1.9 (Peano addition)**.** Given a fixed $m \in \mathbb{N}$, the principle of recursion gives a unique function

$$\mathrm{sum}_m : \mathbb{N} \to \mathbb{N}$$

(A1) $\mathrm{sum}_m(0) = m$

(A2) $\mathrm{sum}_m(n^+) = (\mathrm{sum}_m(n))^+$.

Define

$$m + n := \mathrm{sum}_m(n)$$

Proposition 1.10. $2 + 3 = 5$

Proof.

$$
\begin{aligned}
2+3 &= \mathrm{sum}_2(3) \\
&= \mathrm{sum}_2(2^+) \\
&= (\mathrm{sum}_2(2))^+ && \text{((A2))} \\
&= (\mathrm{sum}_2(1^+))^+ \\
&= ((\mathrm{sum}_2(1))^+)^+ && \text{((A2))} \\
&= ((\mathrm{sum}_2(0^+))^+)^+ \\
&= (((\mathrm{sum}_2(0))^+)^+)^+ && \text{((A2))} \\
&= ((2^+)^+)^+ && \text{((A1))} \\
&= (3^+)^+ \\
&= 4^+ \\
&= 5
\end{aligned}
$$

□

Remarks. Note that $m^+ = \mathrm{sum}_m(0)^+ = \mathrm{sum}_m(0^+) = \mathrm{sum}_m(1) = m+1$.
So we will now denote m^+ as $m+1$.

Definition 1.11 (Peano multiplication)**.** Let $m \in \mathbb{N}$. By the recursion principle, $\exists$ a unique function

$$\mathrm{prod}_m : \mathbb{N} \to \mathbb{N}$$

such that

(a) $\mathrm{prod}_m(0) = 0$

(b) $\mathrm{prod}_m(n^+) = m + \mathrm{prod}_m(n)$.

Theorem 1.12. The following hold:

(a) (Commutativity)

$$m + n = n + m$$
$$m \cdot n = n \cdot m$$

for all natural numbers m and n.

(b) (Associativity)

$$m + (n + k) = (m + n) + k$$
$$m \cdot (n \cdot k) = (m \cdot n) \cdot k$$

for all natural numbers m, n, k.

(c) (Distributivity)

$$m \cdot (n + k) = (m \cdot n) + (m \cdot k)$$

for all natural numbers m, n, k.

(d) $m + n = 0 \Leftrightarrow m = n = 0$ for any $m, n \in \mathbb{N}$

(e) $m \cdot n = 0 \Leftrightarrow m = 0$ or $n = 0$ for any $m, n \in \mathbb{N}$

(f) (Cancellation) $m + k = n + k \Leftrightarrow m = n$ for any $m, n, k \in \mathbb{N}$ and if $m \cdot k = n \cdot k$ and $k \neq 0$, then $m = n$.

Proof. (f) We will prove $P(k)$ is true $\forall\, k \in \mathbb{N}$, where $P(k)$ is the property that for some fixed m, n, we have $m + k = n + k \Leftrightarrow m = n$ for all $m, n \in \mathbb{N}$
$P(0)$ is true as $m + 0 = m$, $n + 0 = n$, so $m + 0 = n + 0 \Leftrightarrow m + n$.
Suppose $P(k)$ holds. Then $m + k^+ = n + k^+ \Leftrightarrow (m + k)^+ = (n + k)^+ \Leftrightarrow m + k = n + k$ (P3) which implies $m = n$ by $P(k)$.
Since m, n were arbitrary, $P(k)$ holds for any value of m, n. □

1.2.1 Tao

Lemma 1.13. For any natural number n, $n + 0 = n$.

Proof. Let $P(n)$ be the property that $n + 0 = n$. From (A1), we have $0 + 0 = 0$, *i.e.*, $P(0)$ is true. Suppose $P(n)$ is true for some natural number n. Then $n_{++} + 0 = (n + 0)_{++} = n_{++}$, *i.e.*, $P(n_{++})$ is true. By P5, we have $P(n)$ true for every natural number n. □

Lemma 1.14. For any natural numbers n and m, $n + m_{++} = (n + m)_{++}$

Proof. Let P(n, m) be the property

$$n + m_{++} = (n + m)_{++}$$

$P(0, m)$ is clearly true.
Suppose P(n, m) is true for some n. Then

$$\begin{aligned} n_{++} + m_{++} &= (n + m_{++})_{++} && \text{((A2))} \\ &= ((n + m)_{++})_{++} && \text{(induction hypothesis)} \\ &= (n_{++} + m)_{++} && \text{((A2))} \end{aligned}$$

Therefore $P(n_{++}, m)$ holds. Since we made no assumptions on m, $P(n, m)$ holds for all n, m by (P5) □

Corollary 1.15. $n_{++} = n + 1$.

Proof. From lemmas 1.13 and 1.14, $n + 1 = n + 0_{++} = (n + 0)_{++} = n_{++}$. □

Proposition 1.16. *(Addition is commutative)* For any natural numbers n and m, $n + m = m + n$

Proof. Let $P(n, m) \Leftrightarrow n + m = m + n$

$$\begin{aligned} n_{++} + m &= (n + m)_{++} && \text{((A2))} \\ &= (m + n)_{++} && \text{(induction hypothesis)} \\ &= m + n_{++} && \text{(lemma 1.14)} \end{aligned}$$

Thus $P(n, m) \Rightarrow P(n_{++}, m)$ and we know $P(0, m)$ to be true by (A1).

Since there were no assumptions on m, $P(n, m)$ is true for all n, m by (P5). □

due Thu 27 Oct 2022

Assignment 1

Problem 1.1. Let A be a Peano set and S be the successor function on A (as defined in the first lecture). Show, using only the axioms of Peano, that the range of S is $A \setminus \{0\}$. For this question, please interpret the words "function" and "range" in the way you did in school, and not in the set-theoretic way introduced in class.

Proof. Let $P(n)$ be the property that n is in the range of S, or $n = 0$. $P(0)$ is trivially true. If $P(n)$ is true, then $P(S(n))$ is trivially true, as $S(n)$ is the successor of n.

Thus by the principle of mathematical induction, $P(n)$ is true for all natural n.

This means that for all $n \neq 0 \in A$, n is in the range of S. Since 0 is not the successor of any natural number, it is not in the range of S.

Thus, the range of S is precisely $A \setminus \{0\}$. $\square$

Problem 1.2. We mentioned in class that when listing the ZFC axioms, we do not need to addadditional axioms for the existence of the intersection or the set-difference of two sets. Using the ZFC axioms, prove the following statements.

(a) Given two sets A and B, show that $A \cap B$ exists as a set.

(b) Given two sets A and B, show that $A \setminus B$ exists as a set.

Proof. Using the axiom of specification, we have:

(a)
$$A \cap B = \{a \in A : a \in B\}$$

(b)
$$A \setminus B = \{a \in A : a \notin B\}$$
$\square$

Problem 1.3. Given two objects a, b, let (a, b) denote the set $\{\{a\}, \{a, b\}\}$. First argue why the ZFC axioms guarantee the existence of this set. Then show that $(a, b) = (c, d)$ (as sets) if and only if $a = c$ and $b = d$.

Proof. By the basic axiom, a and b are sets.
By the axiom of pairing, set $\{a, a\} = \{a\}$ (by axiom of equality) exists.
By the axiom of pairing, set $\{a, b\}$ exists.
Applying the axiom of pairing on these two sets, the set $(a, b) = \{\{a\}, \{a, b\}\}$ exists.

If $a = c$ and $b = d$, $(a, b) = (c, d)$. Do I even need to compare the two sets manually? Invoke any of ZFC? Doesn't this follow directly from the notion of "equality"?

If $(a, b) = (c, d)$, then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. By the axiom of equality, $\{a\} \in (c, d) \Rightarrow \{a\} = \{c\}$ or $\{a\} = \{c, d\}$. In either case, the axiom of equality again implies that $c \in \{a\} \Rightarrow c = a$.

By the axiom of equality again, $\{c, d\} = \{a\}$ implying $d = a$, or $\{c, d\} = \{a, b\}$ implying $d \in \{a, b\}$. Thus d is either equal to a or to b. Similarly b is equal to either c or d.

If $b \neq d$, we must have $d = a$ and $b = c = a = d$, a contradiction.
Thus d is necessarily equal to b. $\square$

Problem 1.4. Prove lemma 1.5, *i.e.*, show that if $\mathscr{F}$ is a non-empty set of inductive sets, then
$$\bigcap_{A \in \mathscr{F}} A$$
is an inductive set.

Proof. I shall assume $I = \bigcap_{A \in \mathscr{F}} A$ to be defined as

$$\bigcap_{A \in \mathscr{F}} A = \left\{ a \in \bigcup_{A \in \mathscr{F}} A : a \in A \ \forall\ A \in \mathscr{F} \right\}$$

Since $\varnothing \in A$ for every $A \in \mathscr{F}$, and $\varnothing$ is also in the union of the sets contained in $\mathscr{F}$, $\varnothing \in I$

If an element $a \in I$, then a is in every $A \in \mathscr{F}$. Since $A \in \mathscr{F}$, A is an inductive set, and so $a \in A \Rightarrow a^+ \in A$. Thus $a^+ \in A$ for every $A \in \mathscr{F}$, which implies $a^+ \in I$.

These conditions together imply that I is an inductive set. $\square$

Problem 1.5. Let A, B, C, D be sets. Some of the following statements are always true, and the others are sometimes wrong. Decide which is which. For the ones you declare "always true", provide a proof. For the others, provide one counterexample each.

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$

(c) $C \cap (A \setminus B) = A \cap (C \setminus B)$

(d) $C \cup (A \setminus B) = A \cup (C \setminus B)$

Proof. (a) *True.* Call the LHS set P and the RHS set Q. If $(a, x) \in P$, then $a \in A$ and $x \in B \cup C \Leftrightarrow x \in B$ or $x \in C$. Thus $a \in A$ and $x \in B$, or $a \in A$ and $x \in C$. Thus $(a, x) \in A \times B$ or $(a, x) \in A \times C$. Thus $(a, x) \in (A \times B) \cup (A \times C) = Q$.

If $(a, x) \in Q$, then $(a, x) \in A \times B$ or $(a, x) \in A \times C$. Thus $a \in A$ and $x \in B$, or $a \in A$ and $x \in C$. Thus $a \in A$, and $x \in B$ or $x \in C$. Thus $a \in A$ and $x \in B \cup C$. Thus $(a, x) \in A \times (B \cup C) = P$.

Therefore by the axiom of equality, $P = Q$.

(b) *False.* Consider the sets $A = B = C = \{\varnothing\}, D = \varnothing$. Then $A \times B = \{(\varnothing, \varnothing)\}$ and $C \times D = \varnothing$. Also, $A \setminus C = \varnothing$ and $B \setminus D = \{\varnothing\}$.

LHS $= \{(\varnothing, \varnothing)\}$
RHS $= \varnothing \neq$ LHS

(c) *True.* Call the LHS set P and the RHS set Q.

$$\begin{aligned} P &= \{c \in C : c \in (A \setminus B)\} \\ &= \{c \in C : c \in \{a \in A : a \notin B\}\} \\ &= \{c \in C : c \in A, c \notin B\} \end{aligned}$$

Similarly $Q = \{a \in A : a \in C, a \notin B\}$.

Clearly, every element of P is in Q and vice versa. Thus the two sets are equal.

(d) *False.* Consider the sets $A = \varnothing, B = C = \{\varnothing\}$. Then LHS $= C \cup \varnothing = C = \{\varnothing\}$, but RHS $= \varnothing \cup \varnothing = \varnothing \neq$ LHS. □

Problem 1.6. Let A be a set. Define a relation $\mathbf{R}$ such that for any subsets B and C of A,

$$B\mathbf{R}C \Leftrightarrow B \subseteq C$$

Remember that a relation $\mathbf{R}$ is a subset of a Cartesian product of sets. Is the relation that you've defined a function?

Proof. Since B and C are subsets of A, they can be precisely the elements of the power set of A. So $\mathbf{R}$ is a subset of the $P(A) \times P(A)$.

If $A = \varnothing$, the only subset of A is $\varnothing$. So B and C can only take one value, $\varnothing$, and indeed $\varnothing \subseteq \varnothing \Leftrightarrow \varnothing\mathbf{R}\varnothing$.

Thus, in this case, $\mathbf{R}$ is a function if it is taken to be a relation from $\{\varnothing\}$ to $\{\varnothing\}$. However, $\mathbf{R}$ could also be taken to be a relation from $\{\varnothing, \{\varnothing\}\}$ to $\{\varnothing\}$, in which case it is no longer a function as $\{\varnothing\}\mathbf{R}x$ is false for all $x \in \{\varnothing\}$.

If $A \neq \varnothing$, we have $\varnothing \in P(A)$ and $A \in P(A)$. Thus $\varnothing\mathbf{R}\varnothing$ and $\varnothing\mathbf{R}A$, with $\varnothing \neq A$. Thus for $\varnothing \in P(A)$ there are at least two x such that $(\varnothing, x) \in \mathbf{R}$. Therefore $\mathbf{R}$ is not a function. □

Lecture 5: Fields, Ordered Sets and Ordered Fields

fri 28 oct 2022

1.3 Fields, Ordered Sets and Ordered Fields

We cannot solve

$$3 + x = 2$$
$$3 \cdot x = 2$$

in $\mathbb{N}$.

Definition 1.17. A field is a set F with 2 operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ such that

(F1) $+$ & $\cdot$ are commutative on F.

(F2) $+$ & $\cdot$ are associative on F.

(F3) $+$ & $\cdot$ satisfy distributivity on F, *i.e.*, $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$.

(F4) There exist 2 *distinct* elements, called 0 (additive identity) and 1 (multiplicative identity) such that

$$x + 0 = x$$
$$x \cdot 1 = x$$

for all $x \in F$

(F5) For every $x \in F$, $\exists\, y \in F$ such that

$$x + y = 0$$

(F6) For every $x \in F \setminus \{0\}$, $\exists\, z \in F$ such that

$$x \cdot z = 1$$

Remarks. We are tempted to call y in (F5) "-x" and z in (F6) "$\frac{1}{x}$" but y, z haven't been proven to be unique yet. Prove this. Proved as lemma 1.21.

Once we have proven this, we can also define $a - b := a + (-b)$ and $a/b = a \cdot \frac{1}{b}$.

Theorem 1.18. $(F, +, \cdot)$ is a field. Then for all x,

$$0 \cdot x = x \cdot 0 = 0$$

Proof. By (F1), the first equality holds.
Now by (F4), $1 + 0 = 1$.
By (F3), $x \cdot (1 + 0) = x \cdot 1 + x \cdot 0$
So $x \cdot 1 = x \cdot 1 + x \cdot 0$ or $x = x + x \cdot 0$ (F4)
Adding y to both sides where $x + y = 0$ (F5) and using associativity and commutativity,

$$\begin{aligned} x + y &= x + x \cdot 0 + y \\ 0 &= x + y + x \cdot 0 \\ &= 0 + x \cdot 0 \\ &= x \cdot 0 \end{aligned}$$

By commutativity, $0 \cdot x = 0$. $\square$

Definition 1.19. A set A with a relation $<$ is called an *ordered set* if

(O1) (Trichotomy) For every $x, y \in A$, exactly one of the following holds.

$$x < y, \quad x = y, \quad y < x$$

(O2) (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Notation: $x < y$ is read as "x is less than y"
$x \leq y$ means $x < y$ or $x = y$, read as "x is less that or equal to y".
$x > y$ is read as "x is greater that y" and equivalent to $y < x$.

Example. $\mathbb{N}$ with $a < b$ iff $b = a + k$ for some non-zero k.

Definition 1.20. An *ordered field* is a set that admits two operations $+$ and $\cdot$ and relation $<$ so that $(F, +, \cdot)$ is a field and $(F, <)$ is an ordered set and:

(O3) For $x, y, z \in F$, if $x < y$ then $x + z < y + z$.

(O4) For $x, y \in F$, if $0 < x$ and $0 < y$ then $0 < x \cdot y$.

Lemma 1.21. Given a field $(F, +, \cdot)$: For any element a in a field F, there exists ony one b such that $a + b = 0$. We will denote this b as $-a$. Similarly for any a in $F \setminus \{0\}$ there exists only one $b \in F$ such that $ab = 1$. We will denote this b as $\frac{1}{a}$ or a^{-1}.

Proof. Suppose $a+b_1 = 0$ and $a+b_2 = 0$. Adding b_1 to both sides, we get $a+b_2+b_1 = 0 + b_1 \Rightarrow (a + b_1) + b_2 = b_1 \Rightarrow b_2 = b_1$.
The second part of the proof is analogous. □

Lemma 1.22. $-(-a) = a = (a^{-1})^{-1}$

Proof. $a + (-a) = 0$, so the additive inverse of $-a$ is a.

$aa^{-1} = 1$, so the multiplicative inverse of a^{-1} is a. □

Lemma 1.23. For any field $(F, +, \cdot)$, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.

Proof. By the distributive law, we have

$$(a + (-a))b = ab + (-a)b \Rightarrow 0 = ab + (-a)b \Rightarrow (-a)b = -(ab).$$

It follows that $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$. □

Theorem 1.24. For any field $(F, +, \cdot)$, $0 < 1$.

Proof. By item (O1) and (F4), either $0 < 1$ or $0 > 1$.

If $0 < 1$, we are done.

If $0 > 1$, then on adding -1 on both sides, $0 < -1$ by item (O3). So $0 < (-1)(-1) \Leftrightarrow 0 < 1$, a contradiction. □

Remarks. "a contradiction" is not necessary to state for the proof to be complete. See this discussion at MS Teams.

Assignment 2

due Thu 3 Nov 2022

Problem 2.1.

(a) Prove that for any $m, n \in \mathbb{N}$, exactly one of the following statements holds.

  (*i*) $m = n$;

  (*ii*) there is a $k \in \mathbb{N} \setminus \{0\}$ such that $m + k = n$;

  (*iii*) there is a $k \in \mathbb{N} \setminus \{0\}$ such that $n + k = m$.

  You may use: induction, the definition of sum_m any of its six properties stated in class (as Theorem 1.12), and the fact that the range of the function $f(x) = x + 1$ on $\mathbb{N}$ is $\mathbb{N} \setminus \{0\}$ (Problem 1 in HW1).

(b) Show that $\mathbb{N}$ is an ordered set if we define $<$ as follows: $m < n$ if there is a $k \in \mathbb{N} \setminus \{0\}$ such that $m + k = n$.

Proof. Unless otherwise stated, any lowercase variable denotes a natural number.

(a) Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be a relation such that $a \mathrel{R} b \Leftrightarrow \exists\, k \neq 0$ such that $a + k = b$. Let

$$B = \{m \in \mathbb{N} : m = n, m \mathrel{R} n, \text{ or } n \mathrel{R} m\}$$

Note: If $\exists\, k$ such that $m + k = n$, then $m \in B$ as $k = 0$ gives $m = n$ and $k \neq 0$ gives $m \mathrel{R} n$. Similarly $n + k = m$ also implies $m \in B$.
$0 \in B$ as $0 + n = n$.
If $b \in B$, then:

($b = n$) $S(b) = S(n) = n + 1 \Rightarrow S(b) \in B$.

($b \mathrel{R} n$) $\exists\, k \neq 0$ such that $b + k = n$. Since $k \in \text{ran}(S)$ (HW 1.1), $\exists\, k'$ such that $S(k') = k$. Thus $b + S(k') = n \Rightarrow S(b) + k' = n \Rightarrow S(b) \in B$.

($n \mathrel{R} b$) $\exists\, k \neq 0$ such that $n + k = b$. Then $S(n + k) = S(b) \Rightarrow n + S(k) = S(b) \Rightarrow S(b) \in B$.

Thus $b \in B \Rightarrow S(b) \in B \Rightarrow B = \mathbb{N}$. Since n was arbitrary, one of the three statements holds for each m, n.

Suppose $m = n$. Then if $m + k = n$, then $m + k = m + 0 \Rightarrow k = 0$ by the cancellation law. Similarly $n + k = m$ also implies $k = 0$. Thus $m = n$ cannot hold simultaneously with $m\ R\ n$ or $n\ R\ m$. Now if $m + k = n$ and $n + k' = m$, then $(n + k') + k = n \Rightarrow n + (k + k') = n + 0 \Rightarrow k + k' = 0 \Rightarrow k = k' = 0$. Thus $m\ R\ n$ and $n\ R\ n$ cannot hold simultaneously.

Therefore exactly one of the three statements holds for all $m, n \in \mathbb{N}$.

(b) If we define $m < n$ as $m\ R\ n$ above, from part (a) it is clear that exactly one of $m = n, m < n, n < m$ holds for all $m, n \in \mathbb{N}$. Moreover, if $a < b$ and $b < c$, then there exist natural numbers $k, k' \neq 0$ such that $a + k = b$ and $b + k' = c$. This implies $(a + k) + k' = a + (k + k') = c$. Since $x + y = 0 \Rightarrow x = y = 0$, $x \neq 0$ or $y \neq 0 \Rightarrow x + y \neq 0$. Thus $k + k' \neq 0 \Rightarrow a < c$.

We have shown that $<$ obeys trichotomy and is transitive. Thus $(\mathbb{N}, <)$ is an ordered set. □

Problem 2.2. Let $(F, +, \cdot)$ be a field. According to axiom (F5), given $x \in F$, there is a $y \in F$ such that $x + y = 0$. Show that y is unique, i.e., if there is a $z \in F$ such that if $x + y = x + z = 0$, then $y = z$. Use only the field axioms to justify your answer.

Proof.

$$
\begin{aligned}
x + y &= x + z \\
(y + x) + y &= (y + x) + z \\
y &= z
\end{aligned}
$$
□

Problem 2.3. Let $+$ and $\cdot$ be the usual addition and multiplication on $\mathbb{N}$. You are free to use their well-known properties.

(a) Let $F = \{0, 1, 2, 3\}$. We endow F with addition and multiplication as follows:

$$a \oplus b = c, \quad \text{where } c \text{ is the remainder that } a + b \text{ leaves when divided by } 4$$
$$a \odot b = c, \quad \text{where } c \text{ is the remainder that } a \cdot b \text{ leaves when divided by } 4$$

Is $(F, \oplus, \odot)$ a field? Please justify your answer.

(b) Let $F = \{0, 1\}$. We endow F with addition and multiplication as follows:

$$a \oplus b = c, \quad \text{where } c \text{ is the remainder that } a + b \text{ leaves when divided by } 2$$
$$a \odot b = c, \quad \text{where } c \text{ is the remainder that } a \cdot b \text{ leaves when divided by } 2$$

You may assume that $(F, \oplus, \odot)$ is a field. Is it possible to give F a relation $<$ so that $(F, \oplus, \odot, <)$ is an ordered field? Please justify your answer.

Proof. (a) Clearly 1 is the multiplicative identity.

$$\begin{array}{cccc} 2 \cdot 0 = 0 & 2 \cdot 1 = 2 & 2 \cdot 2 = 4 & 2 \cdot 3 = 6 \\ 2 \odot 0 = 0 & 2 \odot 1 = 2 & 2 \odot 2 = 0 & 2 \odot 3 = 2 \end{array}$$

Thus there is no multiplicative inverse of 2 in F. So $(F, \oplus, \odot)$ is not a field.

(b) If $(F, \oplus, \odot, <)$ is an ordered field and $0 < 1$, then by the field axioms, $0 \oplus 1 < 1 \oplus 1 \Leftrightarrow 1 < 0$ which is a contradiction as it disobeys trichotomy of order. If $1 < 0$ then $1 \oplus 1 < 0 \oplus 1 \Leftrightarrow 0 < 1$, which cannot be true. □

Problem 2.4. Let $(F, +, \cdot, <)$ be an ordered field.

(i) Using only the field axioms, and the uniqueness of the additive inverse, show that for all $a, b, c \in F, a(b - c) = ab - ac$.

(ii) Using the field axioms, the order axioms, and Part (i), show that for all $a, b, c \in F$, if $a < b$ and $c < 0$, then $bc < ac$.

Proof. (i) $a(b + (-c)) = ab + a(-c)$

$$\begin{aligned} a(c + (-c)) &= ac + a(-c) \\ 0 &= ac + a(-c) \\ a(-c) &= -(ac) \end{aligned}$$

Thus $a(b + (-c)) = ab - ac$.

(ii) $c < 0 \Rightarrow c + (-c) < -c \Rightarrow 0 < -c$.
$a < b \Rightarrow a + (-a) < b + (-a) \Rightarrow 0 < b - a$.

Thus

$$\begin{aligned} 0 &< (b + (-a))(-c) && \text{(O4)} \\ 0 &< b(-c) + (-a)(-c) \\ 0 &< -bc + ac \\ bc &< ac \end{aligned}$$

□

Problem 2.5. Apostol defines an ordered field as a field $(F, +, \cdot)$ together with a set $P \subseteq F$ satisfying the following axioms.

(O'1) If $x, y \in P$, then $x + y \in P$ and $x \cdot y \in P$.

(O'2) For every $x \in F$ such that $x \neq 0$, $x \in P$ or $-x \in P$, but not both.

(O'3) $0 \notin P$

Show that our definition of an ordered field is equivalent to that of Apostol's. That is, show that for a field$(F, +, \cdot)$:

(*i*) If there is a relation $<$ satisfying (O1)-(O4), then there is a $P \subseteq F$ satisfying (O'1)-(O'3), and

(*ii*) if there is a $P \subseteq F$ satisfying (O'1)-(O'3), then there is a relation $<$ satisfying (O1)-(O4).

Proof. Suppose there is a relation $<$ on $(F, +, \cdot)$ satisfying (O1)-(O4). Define

$$P = \{x \in F : 0 < x\}.$$

Suppose $x, y \in P \Leftrightarrow 0 < x, y$. Then $-x < x + (-x) \Rightarrow -x < 0 < y \Rightarrow -x < y \Rightarrow 0 < x + y \Rightarrow x + y \in P$ by (O2) and (O3).

If $x, y \in P$, then by (O4), $x \cdot y \in P$. Thus (O'1) holds.

If $0 < x$, $x \in P$. If $x < 0$, then by (O3) $x + (-x) < -x \Rightarrow 0 < -x$, *i.e.*, $-x \in P$. Thus (O'2) is holds.

$0 \not< 0$, so (O'3) holds.

Now suppose there is a subset $P \subseteq F$ which satisfies (O'1)-(O'3). Define relation $<$ on F as $a < b \Leftrightarrow b - a \in P$.

Note that $-(b - a) = a - b$.

(O1) For any $a, b \in F$, exactly one of $b - a = 0$, $b - a \in P$, and $-(b - a) \in P$ holds (by (O'2) and (O'3), as $-0 = 0$). $b - a = 0 \Leftrightarrow a = b, b - a \in P \Leftrightarrow a < b$, and $-(b - a) \in P \Leftrightarrow a - b \in P \Leftrightarrow b < a$. Thus exactly one of $a = b, a < b$, and $b < a$ holds.

(O2) If $a < b$ and $b < c$, then $b - a \in P$ and $c - b \in P$. So by (O'1), $c - b + b - a \in P \Leftrightarrow c - a \in P \Leftrightarrow a < c$.

(O3) If $a < b$ and $c \in F$, then $(b+c) - (a+c) = b + c + (-a) + (-c) = b - a \in P \Rightarrow a + c < b + c$.

(O4) $0 < a \Leftrightarrow a - 0 \in P \Leftrightarrow a \in P$. So $0 < a$ and $0 < b$ implies $0 < a \cdot b$ by (O'1). $\square$

Lecture 6: $\mathbb{R}$, Bounds, Supremums and the LUB Property

mon 31 oct 2022

1.4 Upper bounds & least upper bounds

Throughout this subsection, $(F, +, \cdot, <)$ is an ordered field, and we assume all "basic" properties.
Key example: $(\mathbb{R}, +, \cdot, <)$

Definition 1.25. A non-empty subset $S \subseteq F$ is said to be *bounded above* in F if there exists a $b \in F$ such that

$$a \leq b \, \forall \, a \in S$$

Here, b is called an *upper bound* of S. If $b \in S$, then b is a *maximum* of S.

Example.

$$S = \{x \in F : 0 \leq x \leq 1\}$$
$$T = \{x \in F : 0 \leq x < 1\}$$

Both S and T are bounded above as 1 is an upper bound for both.
1 is in fact, a maximum of S.

Remarks. If a maximum exists, it must be unique (why?).

Proof. Suppose $b_1, b_2 \in S$ are two upper bounds of S. Then $b_1 \leq b_2$ and $b_2 \leq b_1 \Rightarrow b_1 = b_2$. $\square$

Remarks. Upper bounds may not be unique.

Definition 1.26. Let $S \subseteq F$ be bounded above. An element $b \in F$ is said to be a *least upper bound* of S or a *supremum* of S if:

(a) b is an upper bound of S.

(b) If for $c \in F$, $c < b$, then c is not an upper bound of S. In other words, for any $c < b$, $\exists \, s_c \in S$ such that $c < s_c$.
Contrapositive: If c is an upper bound of S, then c is not less than b, *i.e.*, $b \leq c$.

Remarks. There is only one supremum of S.

Proof. Suppose $b_1, b_2 \in F$ are two supremums of S. Then since b_1 is an upper bound, b_1 is not less than b_2. Similarly b_2 is not less than b_1. By (O1), $b_1 = b_2$. $\square$

Example.

$$\sup\{x \in F : 0 \leq x < 1\} = 1$$

Proof. Call the given set T. T is non-empty as $0 \in T$. It is clear that 1 is an upper bound (because $x < 1 \Rightarrow x < 1 \; \forall \, x \in T$).
Let $a \in F$ s.t. $a < 1$. Now if $a < 0$, a is not an upper bound as $0 \in T$.

If $0 \leq a < 1$, first note that

$$\begin{aligned} 0 < 1 \leq a + 1 < 2 \\ \Rightarrow 0 < \frac{1}{2} \leq \frac{a+1}{2} < 1 \end{aligned} \qquad \text{(why?)}$$

Thus, $\frac{a+1}{2} \in T$. Since $a = \frac{a+a}{2} < \frac{a+1}{2}$ (why?), a is not an upper bound. Therefore, 1 is the least upper bound of T. $\square$

1.5 The Real Numbers

We assume the existence of a set $\mathbb{R}$ with operations $+, \cdot$ and relation $<$ such that:

(a) $(\mathbb{R}, +, \cdot, <)$ is an ordered field.

(b) (LUB property) every non-empty bounded above subset in $\mathbb{R}$ has a supremum in $\mathbb{R}$.

Some special subsets of $\mathbb{R}$:

- $x > 0$ is called a positive real number.
- $x < 0$ is called a negative real number.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is a subset of $\mathbb{R}$ and inherits $+, \cdot, <$.
- $\mathbb{P} = \{n \in \mathbb{N} : n \neq 0\}$ is the set of positive natural numbers.
- $\mathbb{Z} = \mathbb{N} \cup \{-n : n \in \mathbb{P}\}$ is the set of integers.
- $\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{P}\right\}$ is the set of rational numbers.
- $\mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.

Theorem 1.27 (Archimedean property of $\mathbb{R}$)**.** Let $x, y \in \mathbb{R}$ and $x > 0$, then $\exists \, n \in \mathbb{N}$ such that

$$n \cdot x > y.$$

Proof. Assert $x > 0$. Let
$$S = \{nx : n \in \mathbb{N}\}$$
Suppose S is bounded above. Since $x \in S$, S is non-empty. Let b be the supremum of S (exists by LUB). Since x is positive, $b - x < b$. Thus $b - x$ is not an upper bound of S, *i.e.*, $\exists\, n \in \mathbb{N}$ such that $nx > b - x$. Thus $(n+1)x > b$, and so b is not an upper bound of S. By contradiction, S cannot be bounded above.

Hence for all $y \in \mathbb{R}$, $\exists\, s \in S$ such that $s > y$, and so the Archimedean property holds. $\square$

Lecture 7: Sequences and Convergence

wed 2 nov 2022

2 Sequences & Series

We will now assume *everything* about real numbers: multiplication, division, exponentiation, etc.

2.1 Sequences

Definition 2.1. A sequence in $\mathbb{R}$ is a function $f : \mathbb{N} \to \mathbb{R}$. We denote this sequence by $\{a_n\}_{n\in\mathbb{N}}$, where
$$a_n = f(n) \quad \forall\, n \in \mathbb{N}$$
and a_n is called the n^{th} term of $\{a_n\}_{n\in\mathbb{N}}$.

Remarks. $\{a_n\} \subseteq \mathbb{R}$ will denote a sequence of real numbers.
The numbering starts at 0 technically, but doesn't really matter. We will often omit the subscript $n \in \mathbb{N}$ and start indexing from some other point.

Definition 2.2. We say that a sequence $\{a_n\} \subseteq \mathbb{R}$ is *convergent* (in $\mathbb{R}$) if $\exists\, L \in \mathbb{R}$ such that for each $\varepsilon > 0$, $\exists\, N_{\varepsilon,L} \in \mathbb{N}$ such that
$$|a_n - L| < \varepsilon \quad \forall\, n \geq N_{\varepsilon,L}$$
We will call L *a* limit of $\{a_n\}$ and we write:
$$a_n \to L \text{ as } n \to \infty$$
A sequence $\{a_n\}$ is said to be *divergent* if it is not convergent, *i.e.*, $\forall\, L \in \mathbb{R}$ and $N_L \in \mathbb{N}$, $\exists\, \varepsilon > 0$ and $N \geq N_L$ such that
$$|a_N - L| > \varepsilon$$

Theorem 2.3 (Uniqueness of limits)**.** Suppose L_1 and L_2 are limits of a (convergent) sequence $\{a_n\} \in \mathbb{R}$. Then $L_1 = L_2$.

Proof (self). Suppose $L_1 \neq L_2$. Then define $\varepsilon = \frac{|L_1 - L_2|}{2}$. There exists N_1 such that $|a_n - L_1| < \varepsilon \;\forall\, n \geq N_1$. So for all $n \geq N_1$,

$$\begin{aligned} |L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\ &\leq |a_n - L_1| + |a_n - L_2| \\ &\leq \varepsilon + |a_n - L_2| \\ \Rightarrow 2\varepsilon &\leq \varepsilon + |a_n - L_2| \\ \varepsilon &\leq |a_n - L_2| \end{aligned}$$

□

Example.

(a) Let $\{a_n\} = \frac{1}{n^p} \;\forall\, n \in \mathbb{P}$, where $p > 0$.

$$\lim_{n \to \infty} a_n = 0$$

Proof. Let $\varepsilon > 0$.
By the Archimedean property of $\mathbb{R}$ applied to $x = \varepsilon^{\frac{1}{p}}$ and $y = 1$, $\exists\, N \in \mathbb{P}$ such that:

$$N\varepsilon^{\frac{1}{p}} > 1 \Rightarrow \varepsilon^{\frac{1}{p}} > \frac{1}{N} \Rightarrow \varepsilon > \frac{1}{N^p}$$

Let $n \geq N$. Then

$$\begin{aligned} \left| \frac{1}{n^p} - 0 \right| &= \frac{1}{n^p} \\ &\leq \frac{1}{N^p} \\ &< \varepsilon \end{aligned}$$

□

Lecture 8: Sequences Continued

fri 4 nov 2022

(b) $\{(-1)^n\}_{n \in \mathbb{P}}$ is divergent.

Proof. Suppose there exists a limit L.

Let $\varepsilon = 1$. Then $\exists\, N \in \mathbb{P}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

$$\begin{aligned} |a_{2N} - L| < 1 \quad &\Rightarrow \quad |L - 1| < 1 \\ |a_{2N+1} - L| < 1 \quad &\Rightarrow \quad |L + 1| < 1 \end{aligned}$$

Thus,

$$\begin{aligned} |1 - L + L + 1| &\leq |L - 1| + |L + 1| \\ 2 &< 2 \end{aligned}$$

Contradiction. □

Definition 2.4. A sequence $\{a_n\}_{n \in \mathbb{N}}$ is said to be *bounded* if $\exists\, M > 0$ such that $|a_n| < M \;\forall\, n \in \mathbb{N}$.

Theorem 2.5. Every convergent sequence is bounded.

Proof. We know that every finite set of real numbers has a minimum and a maximum. □

Definition 2.6. A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotonically increasing* if $a_n \leq a_{n+1} \,\forall\, n \in \mathbb{N}$.

A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotonically decreasing* if $a_n \geq a_{n+1} \,\forall\, n \in \mathbb{N}$.

A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotone* if it is either monotonically increasing or monotonically decreasing.

Theorem 2.7 (Monotone convergence theorem)**.** A monotone sequence is convergent iff it is bounded.

Proof. Assume $\{a_n\}$ is increasing and $\exists\, M > 0$ such that $|a_n| < M \,\forall\, n \in \mathbb{N}$.

Let $S = \{a_n : n \in \mathbb{N}\}$. S is nonempty. S is bounded above. Thus, by LUB, there exists

$$b = \sup S$$

Let $\varepsilon > 0$. Since $a_n \leq b \,\forall\, n \in \mathbb{N}$, we have $a_n < b + \varepsilon$.

Since b is the lowest upper bound, $\exists\, N \in \mathbb{N} : a_N > b - \varepsilon$. Since $\{a_n\}$ is monotonically increasing, $b + \varepsilon > a_n \geq a_N > b - \varepsilon \,\forall\, n \geq N$. $|a_n - b| < \varepsilon \,\forall\, n \geq N$. Thus the given sequence is convergent, with limit b.

A monotone sequence which is unbouded is divergent, as all convergent sequences are bounded (theorem 2.5). □

Remarks (Warning!)*.* Divergent sequences may diverge for different reasons!

- $\{(-1)^n\}$ is bounded but divergent.
- $\{n\}$ is unbounded and divergent, to $+\infty$
- $\{(-1)^n n\}$ is unbounded and divergent, but not to ∞ or $-\infty$.

Definition 2.8. We say that a sequence diverges to $+\infty$ if $\forall\, R \in \mathbb{R}$, $\exists\, N_R \in \mathbb{N}$ such that $a_n > R \,\forall\, n \geq N_R$.
We say that a sequence diverges to $-\infty$ if $\forall\, R \in \mathbb{R}$, $\exists\, N_R \in \mathbb{N}$ such that $a_n < R \,\forall\, n \geq N_R$.
We write $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, but this is purely notational and does not mean "$\{a_n\}$ has a limit".

Assignment 3

due Thu 10 Nov 2022

Problem 3.1. Let $x \in \mathbb{R}$ such that $0 \leq x < \delta$ for every $\delta > 0$. Show that x must be 0. Explicitly state the field and order axioms that you are using.

Proof. By (O1), done. ☐

Problem 3.2. Formulate definitions of the terms "bounded below set", "lower bound" and "greatest lower bound" for subsets of $\mathbb{R}$. Show that $\mathbb{Z}$ is neither bounded above nor bounded below.

Definition 2.9. A subset $S \subseteq \mathbb{R}$ is said to be *bounded below* if there exists an element $b \in \mathbb{R}$ such that $\forall\, s \in S, b \leq s$.

Here, b is called a *lower bound* of S.

b is said to be a (the) *greatest lower bound* if $\forall\, b' > b$, b' is not a lower bound of S, *i.e.*, $\exists\, s \in S$ such that $s < b'$.

Proof. ($\mathbb{Z}$ is unbounded) Suppose $b \in \mathbb{R}$ is an upper bound of $\mathbb{Z}$. By the Archimedean property, there exists $n \in \mathbb{P} \Rightarrow n \in \mathbb{Z}$ such that $n \cdot 1 = n > b$. Hence b is not an upper bound.
Next suppose $b \in \mathbb{R}$ is a lower bound of $\mathbb{Z}$. By the Archimedean property, there exists $n \in \mathbb{P} \Rightarrow n \in \mathbb{Z}$ such that $n \cdot 1 = n > -b \Rightarrow -n < b$. Since the additive inverse of an integer is also an integer, b is not a lower bound.

Thus $\mathbb{Z}$ cannot have an upper bound, nor a lower bound. ☐

Problem 3.3. If x is an arbitrary real number, prove that there is exactly one integer n which satisfies

$$n \leq x < n + 1$$

You may use Theorem 1.28 from Apostol (without proof), which says $\mathbb{P}$ is not bounded above. Other than the least upper bound property of $\mathbb{R}$, you need not specify which axioms you are using in your proof.

Proof. By the Archimedean property, there exists an m such that $m \cdot 1 = m > x$. Thus the set

$$S = \{n \in \mathbb{Z} : n \leq x\}$$

is bounded above. Since $\mathbb{Z}$ is unbounded below, $\exists\, n \in \mathbb{Z} : n < y\ \forall\, y \in \mathbb{R}$. Thus the set S is non-empty.
Therefore the set S has a least upper bound in $\mathbb{R}$. Call this z. Since $z - 1$ is not an upper bound, there exists an $m \in S$ such that

$$z - 1 < m \leq z < m + 1.$$

Since $z < m+1$, $m+1 \notin S$. Thus $m \leq x < m+1$.

Now suppose m_1 and m_2 both satisfy this property. $m_1 \leq x < m_2+1 \Rightarrow m_1 < m_2+1$. Similarly $m_2 < m_1 + 1$. Thus $|m_1 - m_2| < 1$. Since m_1 and m_2 are integers, they cannot be distinct. □

> **Problem 3.4.** Let $\{a_n\} \subset R$ be an arbitrary sequence. Among the statements listed below,exactly one implies that $\{a_n\}$ is convergent, exactly one implies that $\{a_n\}$ is divergent, and the remaining one does not say anything conclusive about the convergence of $\{a_n\}$. Determine which is which. For the conclusive statements, you must give proofs. For the inconclusive statement, you must provide two sequences which satisfy the given statement, but one converges and the other diverges.
>
> (1) There exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < n\varepsilon$ for all $n \geq N$.
>
> (2) There exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \frac{\varepsilon}{n+1}$ for all $n \geq N$.
>
> (3) For every $R > 0$, there exists an $N \in \mathbb{N}$ such that $|a_N| > R$.

Proof. (1) (Inconclusive) Suppose $a_n = 1 \ \forall\ n \in \mathbb{N}$. This converges to 1, and passes the condition using the Archimedean property, and converges to 1. Now suppose $a_n = \sqrt{n}$. Then $|a_n - 0| = |\sqrt{n}|$. For any $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon^2}$. Thus $n \geq N \Rightarrow \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon$. Also $\frac{1}{\sqrt{n}} > 0 > -\varepsilon$. Thus the sequence satisfies the given condition, but diverges.

(2) (Convergent) Since $n+1 > 1$, $\frac{\varepsilon}{n+1} < \varepsilon$. So for any $\varepsilon > 0$, $|a_n - L| < \frac{\varepsilon}{n+1} < \varepsilon \Rightarrow |a_n - L| < \varepsilon \Rightarrow \{a_n\}$ is convergent.

(3) (Divergent) Suppose the sequence converges to a limit L. Then there exists for all $\varepsilon > 0$, an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Let $R = \max(\{|a_n|\}_{n \in \mathbb{N}, n<N} \cup \{|L| + \varepsilon\})$. $\exists\, m \in \mathbb{N}$ such that $|a_m| > R \Rightarrow |a_m| > |L| + \varepsilon$. $|a_m - L| \geq ||a_m| - |L|| > |a_m| - |L| > \varepsilon$.

$m \not< N$ since $R \geq |a_n| \ \forall\ n \in \mathbb{N}, n < N$. Thus $\exists\, m > N$ such that $|a_m - L| > \varepsilon$, which contradicts the assumption that the sequence was convergent.

FALSE alternative: We know that $|a_n|$ diverges to $+\infty$ (not necessarily). Suppose a_n converges to L. Then $||a_n| - |L|| \leq |a_n - L|$. $\exists\, n \in \mathbb{N}$ such that for all $\varepsilon > 0, |a_n - L| < \varepsilon$ for all $n \geq N$.This implies $||a_n| - |L|| < \varepsilon \ \forall\ n \geq N$, *i.e.*, $|a_n|$ converges to $|L|$.

Since $|a_n|$ diverges, a_n must also diverge. □

Problem 3.5. Compute the limit of the following sequences.

(1)
$$\frac{2-3n^2}{n^2+2n+1}$$

Lecture 9: Series and Convergence

mon 7 nov 2022

Theorem 2.10 (Tao Theorem 6.1.19)**.** Suppose $\{b_n\}$ converges to $b \neq 0$ (and $\exists\, M \in \mathbb{N}$ such that $b_n \neq 0\ \forall\, n \geq M$.)
Then $\left\{\frac{1}{b}\right\}_{n\geq M} \to \frac{1}{b}$ as $n \to \infty$.

"Proof". Given $\varepsilon > 0$, find $N \in \mathbb{N}$ such that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon\ \forall\, n \geq N$$

This is equivalent to

$$\frac{|b-b_n|}{|b \cdot b_n|} < \varepsilon$$

Since $b_n \to b$, there exists M such that $|b_n - b| < b/2$ for all $n \geq M$. Thus for all $n \geq M$,

$$\begin{aligned} \frac{b}{2} <{}& \ \ b_n < \frac{3b}{2} \\ \frac{1}{2}|b| <{}& |b_n| < \frac{3}{2}|b| \\ \frac{2}{3|b|} <{}& \frac{1}{|b_n|} < \frac{2}{|b|} \end{aligned}$$

Now, for any $\varepsilon > 0$, there exists M' such that $|b_n - b| < \dfrac{2}{|b|^2}\varepsilon$ for all $n \geq M'$. Thus,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \frac{|b_n - b|}{|b \cdot b_n|} < \frac{2|b_n - b|}{|b|^2} < \varepsilon$$

for all $n \geq M$. $\square$

2.2 Infinite series

Definition 2.11. An infinite series is a *formal expression* of the form

$$a_0 + a_1 + a_2 + \dots, \text{ or, } \sum_{n=0}^{\infty} a_n$$

Given $\sum_{n=0}^{\infty} a_n$, its sequence of partial sums (sops) is $\{s_n\}_{n=0}^{\infty}$ where

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ &\vdots \\ s_n &= a_0 + a_1 + \dots a_n \end{aligned}$$

We say that $\sum a_n$ is *convergent* with sum s if $\lim_{n\to\infty} s_n = s$. Otherwise, we say that $\sum a_n$ is divergent.

Example.

(a) (Harmonic series) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof. $\{s_n\}$ is a monotonically increasing sequence.

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\ s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{8} \\ &> 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} \\ s_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} \\ &> 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + \dots + 2^{k-1} \cdot \frac{1}{2^k} \\ &= 1 + \frac{k}{2} \end{aligned}$$

Thus, given any $R \in \mathbb{R}$, $\exists\, k \in \mathbb{N}$ such that $s_{2^k} > R$.
$\Rightarrow \{s_n\}$ is divergent as it is unbounded (theorem 2.5). □

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof.

$$\begin{aligned}
s_1 &= 1 \\
s_n &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\
&< 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \\
&= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= 1 + 1 - \frac{1}{n} \\
&< 2 \; \forall \; n \in \mathbb{N}
\end{aligned}$$

So $\{s_n\}$ is a monotonically increasing sequence that is bounded above.
$\Rightarrow \{s_n\}$ is convergent. □

Remarks. (Telescoping sum)

Theorem 2.12. Suppose $\sum a_n$ is convergent. Then

$$\lim_{n \to \infty} a_n = 0$$

Proof. Suppose $\sum a_n$ converges to limit L.

Then $\forall \, \varepsilon > 0 \; \exists \, N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \frac{\varepsilon}{2}$. Now

$$\begin{aligned}
|a_n| &= |s_{n+1} - s_n| \\
&= |s_{n+1} - L + L - s_n| \\
&\leq |s_{n+1} - L| + |s_n - L| \\
&< \varepsilon
\end{aligned}$$

for all $n \geq N$, which implies $a_n \to 0$. □

Lecture 10: Geometric Series; Comparison Test

wed 9 nov 2022

Example (Geometric Series). Let $x \in \mathbb{R}$. Then

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & |x| < 1 \\ \text{diverges} & |x| \geq 1 \end{cases}$$

Proof.

$$\begin{aligned} s_n &= \left(1 + x + x^2 + \cdot x^n\right) \cdot \frac{1-x}{1-x} \\ &= \frac{1-x^{n+1}}{1-x} \\ &= \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \end{aligned}$$

Claim: x^n tends to 0 if $|x| < 1$ and diverges if $|x| > 1$

(Case 1) $|x| < 1$. Suffices to prove for $x > 0$.
Note that $(1+y)^n = 1 + ny + \cdots > ny$.
If $x < 1$, then $\frac{1}{x} > 1$. Say $\frac{1}{x} = 1 + y$ for some $y > 0$.
Then

$$\left(\frac{1}{x}\right)^n = (1+y)^n > ny = n\left(\frac{1}{x} - 1\right)$$

So

$$x^n < \frac{1}{n}c, \quad c = \frac{1}{\frac{1}{x} - 1}$$

Since $\frac{1}{n}$ tends to 0, x^n tends to 0 by the squeeze theorem (HW). By limit laws,

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{1}{1-x} - \frac{x}{1-x} \cdot x^n\right) \\ &= \frac{1}{1-x} \end{aligned}$$

(Case 2) $|x| \geq 1, x \neq 1$. $x = -1$ diverges as seen before.
For $|x| > 1$, $|x|^n = (1 + (|x| - 1))^n > n(|x| - 1)$.
Once again, using the "limit laws",

$$s_n = \frac{1}{1-x} - \frac{x}{1-x} \cdot x^n \text{ diverges}$$

(Case 3) x = 1. $\sum(1)^n$ diverges since $a_n = 1^n \to 1 \neq 0$ as $n \to \infty$ □

Theorem 2.13 (Comparison test)**.** Suppose there exist constants $M \in \mathbb{N}$ and $0 < C$ such that

$$0 \leq a_n \leq Cb_n \quad \forall\, n \geq M$$

If $\sum b_n$ converges, then $\sum a_n$ converges. In other words, If $\sum a_n$ diverges, $\sum b_n$ diverges.

Proof. Let $\{s_n\}$ and $\{t_n\}$ be the suquence of partial sums of $\sum_{n=M}^{\infty} a_n$ and $\sum_{n=M}^{\infty} b_n$, respectively. Note that $\{s_n\}$ and $\{t_n\}$ are increasing sequences. By convergent of $\{t_n\}$, there exists $N \in \mathbb{N}$ and $L > 0$ such that

$$t_n < L \quad \forall\, n \geq N$$

Thus, $0 \leq s_n \leq Ct_n < CL \; \forall \; n \geq \max\{M, N\}$.
Thus, $\{s_n\}$ is bounded and by monotone convergence theorem, s_n converges.
$\Rightarrow \sum a_n$ converges. $\square$

Example. Let $p \in \mathbb{R}$. Claim:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

Proof. If $p \leq 0$, check

$$\left(\frac{1}{n}\right) \not\to 0$$

So $\sum \frac{1}{n^p}$ diverges.

If $0 < p \leq 1$, then $n^p \leq n \; \forall \; n \geq 1$. Thus $\frac{1}{n^p} \geq \frac{1}{n}$. By the comparison test theorem, since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{n^p}$

If $p \geq 2$, we have $\frac{1}{n^p} < \frac{1}{n^2}$. By the comparison test, this sum converges.

Finally, if $1 < p < 2$. Note that

$$\begin{aligned} s_1 &= 1 \\ s_3 &= 1 + \frac{1}{2^p} + \frac{1}{3^p} \leq 1 + 2 \cdot \frac{1}{2^p} \\ s_7 &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots \frac{1}{7^p} \leq 1 + 2 \cdot \frac{1}{2^p} + 4 \cdot \frac{1}{4^p} \\ s_{2^k-1} &\leq 1 + \frac{2}{2^p} + \dots + \frac{2^{k-1}}{2^{p(k-1)}} \\ &= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \dots + \frac{1}{2^{(k-1)(p-1)}} \end{aligned}$$

The RHS are sops of

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n = \frac{1}{1 - \left(\frac{1}{2}\right)^{p-1}}$$

and so $s_{2^k-1} < \dfrac{1}{1 - \left(\frac{1}{2}\right)^{p-1}} \; \forall \; k \in \mathbb{P}$. We also know that $\{s_n\}$ is increasing.

Thus by Monotone convergence theorem $\{s_n\}$ converges (prove $\{s_n\}$ is bounded). $\square$

Lecture 11: Series Convergence Tests

fri 11 nov 2022

Theorem 2.14 (Ratio test)**.** Let $\sum a_n$ be a series of positive terms. Suppose

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$$

Then,

(a) If $L < 1$, the series converges.

(b) If $L > 1$, the series diverges.

(c) If $L = 1$, the test is inconclusive.

Proof (self).

(a) There exists an $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} < (L+\varepsilon)\ \forall\ n \geq N$. Thus $a_{N+k} < (L+\varepsilon)^k a_N\ \forall\ k > 0$. Since $L < 1$, choose $\varepsilon = \frac{1-L}{2} \Rightarrow (L+\varepsilon) < 1$. Since the geometric series converges, so does $\sum a_n$ by the comparison test.

(b) There exists an $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} > (L-\varepsilon)\ \forall\ n \geq N$. Thus $a_{N+k} > (L+\varepsilon)^k a_N\ \forall\ k > 0$. Since $L > 1$, choose $\varepsilon = \frac{L-1}{2} \Rightarrow (L-\varepsilon) > 1$. Since the geometric series diverges, so does $\sum a_n$ by the comparison test (contrapositive).

(c) $\sum a_n$ diverges for $a_n = \frac{1}{n}$, converges for $a_n = \frac{1}{n^2}$. Both these sequences satisfy $L = 1$, thus the test is inconclusive. □

Theorem 2.15. Suppose $\sum a_n$ and $\sum b_n$ converge with sums a and b respectively. Then, for constants l and m, $\sum la_n + mb_n$ converges to $la + mb$. Suppose $\sum |a_n|$ and $\sum |b_n|$ converge. Then, so does $\sum |la_n + mb_n|$ for any choice of l and m in $\mathbb{R}$.

Proof (self). Let s_n and t_n be the sops of a_n and b_n. Let S_n and T_n be the sops of la_n and mb_n. We have $S_n = ls_n$ and $T_n = mt_n$, so by the limit laws for sequences we have $\sum la_n = l\sum a_n$ and $\sum mb_n = m\sum b_n$.

By the limit laws for sequences and defining $Q_n =$ sops of $la_n + mb_n = S_n + T_n$, we have $\sum la_n + mb_n = l\sum a_n + m\sum b_n$.

Now suppose σ_n, τ_n and μ_n are the sops of $|a_n|, |b_n|$ and $|a_n + b_n|$. Since $|a_n + b_n| \leq |a_n| + |b_n|$, we have $\mu_n \leq \sigma_n + \tau_n$. Since σ and τ are monotone and convergent, they are bounded. So $\mu_n \leq \sup\{\sigma_n\} + \sup\{\tau_n\}$.

Thus μ is bounded and increasing. By Monotone convergence theorem, it is convergent. □

Corollary 2.16. Suppose $\sum a_n$ converges and $\sum b_n$ diverges. Let $m \in \mathbb{R} \setminus \{0\}$. Then, $\sum(a_n + b_n)$ diverges, and $\sum mb_n$ diverges.

Definition 2.17. A series $\sum a_n$ of real numbers is said to *converge absolutely* if $\sum |a_n|$ converges. A series $\sum a_n$ of real numbers is said to *converge conditionally* if $\sum |a_n|$ diverges but $\sum a_n$ converges.

Theorem 2.18. If $\sum a_n$ converges absolutely, it must converge. Moreover, $|\sum a_n| \le \sum |a_n|$.

Proof. We construct a new series

$$b_n = a_n + |a_n|$$

Observe that $0 \le b_n \le 2|a_n|$. Thus, by the comparison test, $\sum b_n$ converges. Now, by the limit laws for convergent series, $\sum a_n = \sum (b_n - |a_n|)$ converges. $\square$

Example. $\sum \frac{(-1)^n}{n}$ is convergent.

Proof.

$$\begin{aligned} s_1 &= -1 \\ s_3 &= -1 + \left(\frac{1}{2} - \frac{1}{3}\right) > s_1 \\ s_5 &= s_3 + \left(\frac{1}{4} - \frac{1}{5}\right) > s_3 \\ &\vdots \\ s_{2k+1} &= \left(-1 + \frac{1}{2}\right) + (-\frac{1}{3} + \frac{1}{4}) + \dots \left(-\frac{1}{2k+1}\right) < 0 \end{aligned}$$

Thus, $\{s_{2k+1}\}$ being a bounded increasing sequence, converges to some limit l by Monotone convergence theorem

$$\begin{aligned} s_2 &= -1 + \frac{1}{2} \\ s_4 &= -1 + \frac{1}{2} - \left(\frac{1}{3} - \frac{1}{4}\right) < s_2 \\ &\vdots \\ s_{2k} &= -1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{2k}\right) > -1 \end{aligned}$$

Thus, $\{s_{2k}\}$ being a bounded decreasing sequence, converges to some limit m by Monotone convergence theorem Moreover, $s_{2k+1} = s_{2k} - \frac{1}{2k+1}$. So by limit laws for sequences, $l = m$. Why does this suffice to claim that $\{s_n\}$ converges?.

For any $\varepsilon > 0$, there exist n_1, n_2 such that $|s_{2k} - l| < \varepsilon \ \forall\ 2k \ge n_1$ and $|s_{2k+1} - l| < \varepsilon \ \forall\ 2k+1 \ge n_2$. Choose $N = \max\{n_1, n_2\}$. Then $|s_n - l| < \varepsilon$ for all $n \ge N$. $\square$

Theorem 2.19 (Alternating series test)**.** Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive numbers going to 0. Then, $\sum(-1)^n a_n$ converges. Denoting the sum by S, we have that

$$0 < (-1)^n(S - s_n) < a_{n+1}.$$

Also called the Leibniz test.

Proof. Same principle as the example of $\sum \dfrac{(-1)^n}{n}$.

Since a is decreasing,

$$\begin{aligned} a_{2k} &> a_{2k+1} \\ a_{2k} - a_{2k+1} &> 0 \\ s_{2k+1} &> s_{2k-1}. \end{aligned}$$

So $\{s_{2k+1}\}$ is increasing. Thus

$$s_{2k+1} > a_0 - a_1 > 0.$$

Since a is decreasing,

$$\begin{aligned} a_{2k+1} &> a_{2k+2} \\ -a_{2k+1} + a_{2k+2} &< 0 \\ s_{2k+2} &< s_{2k}. \end{aligned}$$

So $\{s_{2k}\}$ is decreasing. Thus

$$s_{2k} < a_0.$$

Thus $\{s_{2k}\}$ and $\{s_{2k+1}\}$ are bounded and monotone, and convergent by Monotone convergence theorem.

Since $\{s_{2k+1}\}$ converges, we have by theorem 2.12 that $a_{2k+1} \to 0$. Since $s_{2k+1} = s_{2k} - a_{2k+1}$, by limit laws we have $\lim s_{2k} = \lim s_{2k+1}$.

Thus the sequence converges. □

Remarks. The estimate in Alternating series test allows us to estimate sums of alternating series within any prescribed error. For instance, to know $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ up to an error of 0.1, it suffices to find n so that

$$|S - s_n| < |a_{n+1}| \leq \frac{1}{100}.$$

$n = 99$ works.

Assignment 4

due Thu 17 Nov 2022

Problem 4.1. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ such that for some $N \in \mathbb{N}, 0 \leq a_n \leq b_n$. Convince yourself that if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$. Using this fact, prove the following statements (you are not allowed to use logarithms for these proofs).

(a) For any $r > 0$, $\displaystyle\lim_{n\to\infty} \sqrt[n]{r} = 1$.

(b) $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. (a) For $r > 1$: By the Archimedean property, there exists $n_0 \in \mathbb{P}$ such that $n_0\varepsilon > r - 1$ for all $\varepsilon > 0$. Thus for all $n \geq n_0$, $n\varepsilon > r - 1 \Rightarrow r < 1 + n\varepsilon \leq (1+\varepsilon)^n \Rightarrow r^{1/n} < 1+\varepsilon$. $r > 1 \Rightarrow r^{1/n} > 1 > 1-\varepsilon$. Thus $\left|r^{1/n} - 1\right| < \varepsilon \; \forall \; n \geq n_0$.

For $r = 1$, $\left|r^{1/n} - 1\right| = 0 < \varepsilon$ for all $n \geq 1, \varepsilon > 0$.

For $r < 1$,

$$r^{1/n} = \frac{1}{\left(\frac{1}{r}\right)^{1/n}}$$

Since $\frac{1}{r} > 1$, by limit laws for sequnces, $\lim_{n\to\infty} r^{1/n} = \frac{1}{1} = 1$.

(b) By the Archimedean property, there exists an $N \in \mathbb{P} > \frac{2}{\varepsilon^2} + 1$. For $n \geq N$, we have

$$\begin{aligned} \frac{n-1}{2} \cdot \varepsilon^2 &> 1 \\ \frac{n(n-1)}{2} \cdot \varepsilon^2 &> n \\ (1+\varepsilon)^n &> n \\ \sqrt[n]{n} &< 1 + \varepsilon \end{aligned}$$

Also since $n \geq 1$, $n^{1/n} \geq 1 \Rightarrow \sqrt[n]{n} > 1 - \varepsilon$. □

Problem 4.2. Show that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. The mathematical constant e is defined as the sum of this series.

Proof. Ratio test. □

Problem 4.3. Let $\{a_n : n \in \mathbb{P}\}$ be an arbitrary collection of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Determine which of the following series will necessarily converge (proof required), and which may either converge or diverge depending on the choice of the a_n's (examples required).

(a) $\sum_{n=1}^{\infty} a_n^2$

(b) $\sum_{n=1}^{\infty} \sqrt{a_n}$

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$

Proof. (a) Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Thus there exists $N \in \mathbb{P}$ such that $a_n < 1 \; \forall \; n \geq N \Rightarrow a_n^2 < a_n \; \forall \; n \geq N$. By the comparison test, $\sum_{n=1}^{\infty} a_n^2$

converges.

(b) $\sum \frac{1}{n^2}$ converges but $\sum \frac{1}{n}$ diverges. $\sum 0$ converges and so does $\sum 0$. Inconclusive.

(c) Let $b_n = a_n + \frac{1}{n^2}$. By the limit laws for series, $\sum b_n$ converges.

$$\frac{\sqrt{a_n}}{n} \leq \frac{\sqrt{b_n}}{n} = \frac{b_n}{n\sqrt{b_n}} \leq \frac{b_n}{n\sqrt{\frac{1}{n^2}}} = b_n.$$

Thus by the comparison test, $\sum\limits_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. □

Problem 4.4. Show that each of the following series converges, and determine its sum.

(a) $\sum\limits_{n=1}^{\infty} \frac{4n^2-1+3^{n-1}}{3^n(2n+1)(2n-1)}$

(b) $\sum\limits_{n=6}^{\infty} \frac{6}{n^2-1}$

(c) $\sum\limits_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$

Proof.

(a)

$$\begin{aligned}\frac{4n^2-1+3^{n-1}}{3^n(2n+1)(2n-1)} &= \frac{1}{3^n} + \frac{1}{3} \cdot \frac{1}{(2n+1)(2n-1)} \\ &= \frac{1}{3^n} + \frac{1}{6} \cdot \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1/3}{1-\frac{1}{3}} = \frac{1}{2}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1} &= \frac{1}{2n-1} - \frac{1}{2(n+1)-1} \\ &= \frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \dots \\ &= 1\end{aligned}$$

Thus by the limit laws,

$$\sum_{n=1}^{\infty} \frac{4n^2 - 1 + 3^{n-1}}{3^n(2n+1)(2n-1)} = \frac{1}{2} + \frac{1}{6} \cdot 1$$
$$= \frac{2}{3}$$

(b)

$$\frac{6}{n^2-1} = \frac{3 \cdot 2}{(n-1)(n+1)}$$
$$= 3\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$
$$= 3\left(\frac{1}{n-1} - \frac{1}{(n+2)-1}\right)$$

Let $\{s_n\}$ be the sops of $\frac{1}{n-1} - \frac{1}{n+1}$. For $n > 5$,

$$s_n = \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \frac{1}{7} - \frac{1}{9} + \cdots + \frac{1}{n-1} + \frac{1}{n+1}$$
$$= \frac{1}{5} + \frac{1}{6} - \frac{1}{n} - \frac{1}{n+1}$$
$$\lim_{n\to\infty} s_n = \frac{1}{5} + \frac{1}{6}$$
$$= \frac{11}{30}$$

$$\Rightarrow \sum_{n=6}^{\infty} \frac{6}{n^2-1} = \frac{11}{10}$$

(c)

$$\frac{n}{(n+1)(n+2)(n+3)} = \frac{1}{2}\frac{3(n+1)-(n+3)}{(n+1)(n+2)(n+3)}$$
$$= \frac{3}{2}\frac{1}{(n+2)(n+3)} - \frac{1}{2}\frac{1}{(n+1)(n+2)}$$
$$= \frac{3}{2}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

Let s_n and t_n be the sops of $\left\{\frac{1}{n+2}-\frac{1}{n+3}\right\}$ and $\left\{\frac{1}{n+1}-\frac{1}{n+2}\right\}$.

$$\begin{aligned}
s_n &= \frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\cdots+\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2} \\
s_n &= \frac{1}{3}-\frac{1}{n+2} \\
\lim_{n\to\infty} s_n &= \frac{1}{3} \\
t_n &= \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+2}-\frac{1}{n+3} \\
t_n &= \frac{1}{2}-\frac{1}{n+3} \\
\lim_{n\to\infty} t_n &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)(n+3)} &= \frac{3}{2}\cdot\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

□

Problem 4.5. For each of the series given below, determine whether it converges or diverges. You need not compute the sum in the case of convergence.

(1) $\sum\limits_{n=1}^{\infty}\frac{n\sin^2(n\pi/3)}{2^n}$

(2) $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{1}{n}}$

(3) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n n^{25}}{(n+2)!}$

(4) $\sum\limits_{n=5}^{\infty}\frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)}$

Proof. (1) Let $b_n=\frac{n}{2^n}$.

$$\frac{b_{n+1}}{b_n}=\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}=\frac{1}{2}\cdot\frac{n+1}{n}$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\frac{1}{2}$$

Thus by the ratio test, $\sum b_n$ converges. Since $a_n<b_n$, $\sum a_n$ converges by the comparison test.

(2) $n\ge 1 \Rightarrow \frac{1}{n}\le 1 \wedge n^{\frac{1}{n}}\le n^1 \Rightarrow \left(\frac{1}{n}\right)^{\frac{1}{n}}\ge\frac{1}{n}$. By the comparison test, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{1}{n}}$ diverges.

(3) Let $a_n = \frac{n^{25}}{(n+2)!}$. Clearly $a_n > 0$.

$$0 < \frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)^{25} \cdot \frac{1}{n+3} \le 2^{25} \cdot \frac{1}{n+3}$$

For any $\varepsilon > 0$, there exists (by the Archimedean property), $N > \frac{2^{25}}{\varepsilon} - 3$. For $n \ge N, -\varepsilon < 0 < \frac{a_{n+1}}{a_n} \le \frac{2^{25}}{n+3} \le \frac{2^{25}}{N+3} < \varepsilon$. Thus the ratio converges to 0. By the ratio test, $\sum_{n=1}^{\infty} a_n$ converges.

Since the given series converges absolutely, it converges.

(4) For $n \ge 5$,

$$\begin{aligned}
0 < \frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)} &= \frac{1}{(\sqrt{n}-1)(n+2)(n-4)} \\
&< \frac{1}{(n+2)(n-4)} \\
&= \frac{1}{n^2 - 2n - 8} \\
&< \frac{1}{n^2 - 4n} \\
&= \frac{1}{\frac{1}{5}n^2 + \frac{4}{5}n(n-5)} \\
&\le \frac{5}{n^2}
\end{aligned}$$

So by the comparison test, $\sum\limits_{n=5}^{\infty} \frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)}$ converges. $\square$

Lecture 12: Limit of a Function mon 14 nov 2022

3 Limits & Continuity

3.1 Limit of a function

Definition 3.1 (Neighborhood)**.** Given a real number p and an $\varepsilon > 0$, the *ε-neighborhood of p* is the open interval

$$N_\varepsilon(p) = (p - \varepsilon, p + \varepsilon) = \{x \in \mathbb{R} : |x - p| < \varepsilon\}.$$

Definition 3.2 (Limit of a function)**.** Given a function f that is defined on some $I = (a, p) \cup (p, b)$ with $a < b$, we say that f has a *limit* L as it approaches p iff for every $\varepsilon > 0\ \exists\ \delta > 0$ such that

(a) $0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$ OR

(b) $x \in N_\delta(p) \setminus \{p\} \Rightarrow f(x) \in N_\varepsilon(L)$.

This is denoted as
$$\lim_{x \to p} f(x) = L.$$

Example.

(a) For $f(x) = c$, $c \in \mathbb{R}$,
$$\lim_{x \to p} f(x) = c.$$
Choose $\delta = 1$. $0 < |x - p| < \delta \Rightarrow |f(x) - c| = 0 < \varepsilon\ \forall\ \varepsilon > 0$.

(b) For $f(x) = x$,
$$\lim_{x \to p} f(x) = p.$$
Choose $\delta = \varepsilon$. $0 < |x - p| < \delta \Rightarrow |f(x) - p| < \varepsilon$.

(c) For $f(x) = \sqrt{x}$ and $p > 0$,
$$\lim_{x \to p} f(x) = \sqrt{p}.$$
$$\begin{aligned} &\left|\sqrt{x} - \sqrt{p}\right| < \varepsilon \\ \Leftrightarrow\ &\frac{|x - p|}{\left|\sqrt{x} + \sqrt{p}\right|} < \varepsilon \end{aligned}$$

Take $\delta = \min\left\{p, \sqrt{p}\varepsilon\right\}$ (this is to make sure f is defined for all points in $N_\delta(p) \setminus \{p\}$). Now $|x - p| < \delta \Rightarrow$
$$\begin{aligned} \frac{|x - p|}{\left|\sqrt{x} + \sqrt{p}\right|} &< \frac{\delta}{\left|\sqrt{x} + \sqrt{p}\right|} \\ &= \frac{\sqrt{p}\varepsilon}{\left|\sqrt{x} + \sqrt{p}\right|} \\ &< \varepsilon \end{aligned}$$

Lecture 13: Limit Laws for Functions

wed 16 nov 2022

(d) For $f(x) = \frac{1}{x}, x \neq 0$,
$$\lim_{x \to 0} f(x) \text{ does not exist}$$

Proof. Suppose $\exists\, L \in \mathbb{R}$ such that

$$\lim_{x \to 0} f(x) = L$$

Choose $\varepsilon = \frac{1}{L}$. Then $\exists\, \delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow |f(x) - L| < 1$. By the Archimedean property, $\exists\, N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$.

Now $0 < \frac{1}{N+2} < \frac{1}{N} < \delta$. Thus by our hypothesis, $|N + 2 - L| < 1$ and $|N - L| < 1 \Rightarrow |2| < |N + 2 - L| + |N - L| < 1 + 1 = 2$. Contradiction. $\square$

Theorem 3.3 (Limit laws for functions)**.** Suppose f and g are functions such that

$$\lim_{x \to p} f(x) = a, \qquad \lim_{x \to p} g(x) = b.$$

Then,

$$\lim_{x \to p} (f \pm g)(x) = a \pm b \tag{1}$$

$$\lim_{x \to p} (f \cdot g)(x) = a \cdot b \tag{2}$$

$$\lim_{x \to p} (f/g)(x) = a/b \tag{3}$$

Proof. Scrapwork: $|f(x)g(x) - ab| = |f(x)g(x) - f(x)b + f(x)b - ab| \leq |f(x)||g(x) - b| + |b||f(x) - a|$.

Let $\varepsilon > 0$. Since $\lim_{x \to p} f(x) = a$, corresponding to $\varepsilon_1 = \frac{\varepsilon}{2(|b|+1)}$ $\exists\, \delta_1 > 0$ such that if $0 < |x - p| < \delta_1$, we have $|f(x) - a| < \frac{\varepsilon}{2(|b|+1)}$.
Also,

$$\begin{aligned} |f(x)| - |a| &\leq |f(x) - a| \\ &< \varepsilon_1 \\ \Rightarrow |f(x)| &< |a| + \varepsilon_1 =: M_\varepsilon > 0 \end{aligned}$$

Let $\varepsilon_2 = \frac{\varepsilon}{2M}$. Since $\lim_{x \to p} g(x) = b$, $\exists\, \delta_2 > 0$ such that if $0 < |x - p| < \delta_2$, then $|g(x) - b| < \varepsilon_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$\begin{aligned} |f(x)g(x) - ab| &\leq |f(x)||g(x) - b| + |b||f(x) - a| \\ &< M_\varepsilon \cdot \frac{\varepsilon}{2M_\varepsilon} + |b| \cdot \frac{\varepsilon}{2(|b| + 1)} \\ &< \varepsilon. \end{aligned}$$

$\square$

Mrigank's trick.

$$\begin{aligned} |f(x)g(x) - ab| &= |(fg - ag - bf + ab) + (ag - ab) + (bf - ab)| \\ &\leq |f(x) - a||g(x) - b| + |a||g(x) - b| + |b||f(x) - a| \end{aligned}$$

Choose δ such that $|g(x) - b|$ and $|f(x) - b| < \min\left\{\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3|a|}, \frac{\varepsilon}{3|b|}\right\}$. Then

$$\begin{aligned} |f(x)g(x) - ab| &< \sqrt{\frac{\varepsilon}{3}}\sqrt{\frac{\varepsilon}{3}} + |a| \cdot \frac{\varepsilon}{3|a|} + |b| \cdot \frac{\varepsilon}{3|b|} \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

□

Lecture 14: Continuity: Examples and Algebra

fri 18 nov 2022

3.2 Continuity

Definition 3.4. Let $S \subseteq \mathbb{R}$ be a (nonempty) subset, $f : S \to \mathbb{R}$ and $p \in S$. We say that f is continuous at p iff:
for every $\varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ such that

$$|x - p| < \delta_\varepsilon \ \wedge \ x \in S \Rightarrow |f(x) - f(p)| < \varepsilon$$

We say that f is continuous on S iff f is continuous at each $p \in S$.

Remarks. It is possible that $\exists\, \delta$ such that $N_\delta(p) \cup S = \{p\}$. *E.g.*, $S = \mathbb{N}, p = 0, \delta \leq 1$

Remarks. If f is defined on some interval (a, b) containing p, then this definition is equivalent to

$$\lim_{x \to p} f(x) = f(p)$$

How? For any $\varepsilon > 0$, choose $\delta = \min\{\delta_\varepsilon, b - p, p - a\}$. Then f is defined on all points in $N_\delta(p)$, and for all $x \in N_\delta(p)$, we have $f(x) \in N_\varepsilon(f(p))$. Thus

$$\lim_{x \to p} f(x) = f(p)$$

Theorem 3.5 (Algebraic laws for continuity)**.** Suppose f and g are continuous at $p \in S$. Then so are $f \pm g, fg$ and if $g(p) \neq 0, f/g$.

Proof. For any $\varepsilon > 0$,

$(f \pm g)$ there exist δ_f, δ_g such that for all $|x - p| < \min\{\delta_f, \delta_g\} \ \wedge \ x \in S$, we have

$$|f(x) - f(p)|, |g(x) - g(p)| < \frac{\varepsilon}{2}.$$

Thus we have

$$|(f \pm g)(x) - (f \pm g)(p)| \leq |f(x) - f(p)| + |g(x) - g(p)| < \varepsilon.$$

(fg) there exist δ_f, δ_g such that for all $|x-p| < \min\{\delta_f, \delta_g\} \wedge x \in S$ we have

$$|f(x) - f(p)|, |g(x) - g(p)| < \min\left\{\frac{\varepsilon}{3g(p)}, \frac{\varepsilon}{3f(p)}, \sqrt{\frac{\varepsilon}{3}}\right\}$$

(handle $f(p) = 0$ and $g(p) = 0$ separately). Then we have

$$\begin{aligned}
|fg(x) - fg(p)| &= |(f(p) + f(x) - f(p))(g(p) + g(x) - g(p)) - fg(p)| \\
&= |f(p)(g(x) - g(p)) + g(p)(f(x) - f(p)) + (f(x) - f(p))(g(x) - g(p))| \\
&\le |f(p)||g(x) - g(p)| + |g(p)||f(x) - f(p)| + |f(x) - f(p)||g(x) - g(p)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

(f/g) Let $h = \frac{1}{g}$ wherever g is non-zero. Since g is continuous and $g(p) \neq 0$, it is non-zero in some neighbourhood δ_2 around it.

Since g is continuous, $\exists\, \delta_0$ such that for all $|x - p| < \delta_0 \wedge x \in S$, we have

$$|g(x) - g(p)| < \frac{1}{2}|g(p)| \Rightarrow |g(x)| > \frac{1}{2}|g(p)|$$

For all $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for all $|x - p| < \delta_1 \wedge x \in S$, we have

$$|g(x) - g(p)| < \frac{1}{2}|g(p)|^2\varepsilon.$$

Now, choose $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. For $|x - p| < \delta$,

$$\begin{aligned}
|h(x) - h(p)| &= \left|\frac{1}{g(x)} - \frac{1}{g(p)}\right| \\
&= \frac{|g(x) - g(p)|}{|g(x)||g(p)|} \\
&< \frac{1}{2}|g(p)|^2\varepsilon \cdot \frac{1}{\frac{1}{2}|g(p)|^2} \\
&= \varepsilon.
\end{aligned}$$

Thus $\frac{1}{h}$ is continuous at p and so is $f \cdot h = f/g$.

□

Lecture 15: Compositions; Borsuk-Ulam Theorem

mon 21 nov 2022

Theorem 3.6. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be continuous functions such that $f(A) \coloneqq range(f) \subseteq B$. Then,

$$g \circ f : x \in A \mapsto g(f(x)) \in \mathbb{R}$$

is continuous.

Proof. Let $p \in A$ and $q = f(p)$. Let $\varepsilon > 0$. Since g is continuous at q, $\exists\, \tau > 0$ such that whenever $y \in B$ and $|y - q| < \tau$, then $|g(y) - g(q)| < \varepsilon$.

Let $\varepsilon_1 = \tau$. Then by the continuity of f at p, $\exists\, \delta > 0$ such that whenever $x \in A \wedge |x - p| < \delta$, we have $|f(x) - f(p)| < \varepsilon_1 = \tau$.

Thus, $|g(f(x)) - g(f(p))| < \varepsilon$. Since $\varepsilon > 0$ and $p \in A$ were arbitrary, $g \circ f$ is continuous. □

Theorem 3.7 (intermediate value theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $y \in \mathbb{R}$ is a number between $f(a)$ and $f(b)$, *i.e.*, $y \in [f(a), f(b)]$. Then $\exists\, c \in [a, b]$ such that

$$f(c) = y$$

Corollary 3.8 (Bolzano's theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)$ and $f(b)$ take opposite signs. Then $\exists\, c \in (a, b)$ such that $f(c) = 0$.

Remarks. Bolzano's statement is equivalent to the IVT (let $g = f - y$).

Theorem 3.9 (the Borsuk-Ulam theorem)**.** Let S^n be the unit n-sphere, *i.e.*, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Let $f : S^n \to \mathbb{R}^n$ be a continuous function. Then f maps some pair of antipodal points to the same point.

$$\exists\, x \text{ such that } f(x) = f(-x)$$

"Proof" for $n = 1$. A continuous function f on S^1 is a 2π-periodic function on $\mathbb{R}$.

View f on $[0, \pi]$. Let
$$g(\theta) = f(\theta) - f(\theta + \pi).$$
g is continuous (Note that $f(\theta+\pi)$ is continous as it can be viewed as a composition).

Then either $g(0) = g(\pi) = 0$, or $g(0) = -g(\pi) \neq 0$. Thus by intermediate value theorem there exists $c \in (0, \pi)$ such that $f(c) = 0$. □

Lemma 3.10. Let a_n, b_n be convergent sequences such that $a_n \leq b_n$ for all n (large enough). Then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Proof. Let $c_n = a_n - b_n$. By limit laws, $\lim_{n\to\infty} c_n$ exists. Since $c_n > 0\ \forall\ n \geq N$, $\lim_{n\to\infty} c_n \geq 0$ (if it were negative, choose $\varepsilon = L$ giving c negative). This gives

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$ □

Lecture 16: IVT; Continuous Functions on Compact Intervals

wed 23 nov 2022

Proof of intermediate value theorem. We may assume: y is strictly between $f(a)$ and $f(b)$. Further, we may assume $f(a) < y < f(b)$. The other case is HW. Let

$$S = \{x \in [a, b] : f(x) < y\}.$$

S is nonempty as $a \in S$. Moreover, b is an upper bound of S.
$\Rightarrow c := \sup S$ exists.

Let $n \in \mathbb{P}$. Then $c - \frac{1}{n}$ is not an upper bound of S. Therefore $\exists\, x_n \in S$ such that

$$c - \frac{1}{n} < x_n \leq c \qquad f(x_n) < y.$$

By squeeze theorem, $\{x_n\} \to c$ as $n \to \infty$. By the sequential character of continuity,

$$\lim_{n\to\infty} f(x_n) = f(c)$$

But $f(x_n) < y \;\forall\, n \in \mathbb{N}$. By **??**, $\lim_{n\to\infty} f(x_n) \leq y \Rightarrow f(c) \leq y$.

Similarly, defining $x_n = c + \frac{1}{n}$ yields $f(c) \geq y$. And thus we have

$$f(c) = y.$$

□

Proof (self). $\Rightarrow c := \sup S$ exists.

Suppose $f(c) < y$. $f(b) > y > f(c) \Rightarrow c \neq b$. There exists $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < |f(c) - y| \Rightarrow f(x) < y$. Thus $c + \delta/2 \in S$, a contradiction.

Now suppose $f(c) > y$. $f(a) < y < f(c) \Rightarrow c \neq a$. There exists $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < |f(c) - y| \Rightarrow f(x) > y$. Thus $c - \delta/2 \notin S$, a contradiction.

So $f(c)$ must be equal to y. □

Definition 3.11. A function $f : S \to \mathbb{R}$ is said to be *bounded above* on S if $\exists\, U \in \mathbb{R}$ such that $f(x) \leq U \;\forall\, x \in S$.

f is said to be *bounded* if $\exists\, M > 0$ such that $|f(x)| < M \;\forall\, x \in S$.

Theorem 3.12 (Continuous functions on compact intervals are bounded)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is a bounded function.

Proof by contradiction (morphed). Let $a_0 = a$, $b_0 = b$, and $I_n = [a_n, b_n]$.

If f is not bounded on I_n, then it is not bounded on at least one of $[a_n, \frac{a_n+b_n}{2}]$ and $[\frac{a_n+b_n}{2}, b_n]$.

Let the leftmost of the subintervals on which it is unbounded be I_{n+1}. Since f is unbounded on I_0, it is unbounded on I_k $\forall$ $k \in \mathbb{N}$. Note that $a_{n+1} \geq a_n$ and that $\|I_{n+1}\| = \frac{1}{2}\|I_n\|$. Thus $\|I_n\| = \frac{1}{2^n}\|I_0\| = \frac{1}{2^n}(b-a)$.

Let a_∞ be the supremum of $\{a_n\}$. Note that b is an upper bound of a_n, so $a_\infty \leq b$. Also, since $a \in \{a_n\}$, $a \leq a_\infty$.

Note that $p, q \in N_\delta(a_\infty) \cap [a,b]$ with $p < q$ implies $[p,q] \subseteq N_\delta(a_\infty) \cap [a,b]$.

By continuity $\exists$ $\delta > 0$ such that $|f(x) - f(a_\infty)| < 1$ $\forall$ $x \in N_\delta(a_\infty) \cap [a,b]$. That is, f is bounded on $N_\delta(a_\infty) \cap [a,b]$.

There exists a $k \in \mathbb{N}$ such that $a_\infty - \delta < a_k \leq a_\infty$. Since $\{a_n\}$ is increasing, we have $a_\infty - \delta < a_n \leq a_\infty$ $\forall$ $n \geq k$. Choose N such that $\frac{b-a}{2^n} = \|I_n\| < \delta$ $\forall$ $n \geq N$.

Letting $n_0 = \max\{k, N\}$, we get $a_\infty - \delta < a_{n_0} \leq b_{n_0} < a_\infty + \delta$. Thus $[a_{n_0}, b_{n_0}] \subseteq N_\delta(a_\infty) \cap [a,b] \Rightarrow I_{n_0}$ is bounded, a contradiction. □

Assignment 5

due Thu 24 Nov 2022

You may freely use (without proof):

(i) $0 < \cos(x) < \left|\dfrac{\sin x}{x}\right| < 1$ for all $0 < |x| < \dfrac{\pi}{2}$.

(ii) Any trigonometric identities that you have seen in school.

(iii) Any limits computed in Lecture 12-14.

Problem 5.1. Prove the squeeze theorem.

Theorem 3.13 (squeeze theorem)**.** Let f, g, h be functions defined on some neighborhood N of p, except perhaps at p. Suppose $f \leq g \leq h$ on N, and

$$\lim_{x\to p} f(x) = \lim_{x\to p} h(x) = L \in \mathbb{R}.$$

Then

$$\lim_{x\to p} g(x) = L.$$

Proof. For any $\varepsilon > 0$, $\exists$ δ_1, δ_2 such that

$$\begin{aligned} |f(x) - a| < \varepsilon \quad & \forall\, x \in N \cap N_{\delta_1}(p) \setminus \{p\} \\ |h(x) - a| < \varepsilon \quad & \forall\, x \in N \cap N_{\delta_2}(p) \setminus \{p\} \end{aligned}$$

Thus for all $x \in N \cap N_\delta(p) \setminus \{p\}$, where $\delta = \min\{\delta_1, \delta_2\}$ so that $N_\delta = N_{\delta_1} \cap N_{\delta_2}$,

$$\begin{aligned} a - \varepsilon < f(x) \le g(x) \le h(x) < a + \varepsilon \\ \Rightarrow \lim_{x \to p} g(x) = a \end{aligned}$$

□

> **Problem 5.2.** In each of the following cases, determine whether the limit exists or not, and compute the limit whenever it exists. You may use any of the theorems stated in class, but state what you are using.
>
> (a) $\displaystyle\lim_{x \to 2} \frac{(3x+1)^2 - 49}{x - 2}$
>
> (b) $\displaystyle\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$
>
> (c) $\displaystyle\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$
>
> (d) $\displaystyle\lim_{x \to p} x^n$ (for fixed $n \in \mathbb{N}$ and $p \in \mathbb{R}$)

Proof. (a) For any $x \neq 2$,

$$\frac{(3x+1)^2 - 49}{x - 2} = \frac{(3x+8)(3x-6)}{x-2} = 3(3x+8)$$

By limit laws,

$$\lim_{x \to 2} 3(3x+8) = (\lim_{x \to 2} 3)((\lim_{x \to 2} 3) \lim_{x \to 2} x + \lim_{x \to 2} 8) = 3(3 \cdot 2 + 8) = 42$$

(b) Suppose the function has a limit L. Take $\varepsilon = 1$. Then $\exists\ \delta > 0$ such that $\left|\cos\left(\frac{1}{x}\right) - L\right| < 1$ for all $0 < |x - 0| < \delta$. By the Archimedean property, there exists N such that $N 2\pi > \frac{1}{\delta} \Rightarrow 0 < \frac{1}{(2N+1)\pi} < \frac{1}{2N\pi} < \delta$.

$$\begin{aligned} |1 - L| &< 1 \\ |-1 - L| &< 1 \\ 2 = |1 - L + L + 1| \le |1 - L| + |-1 - L| &< 1 + 1 = 2 \end{aligned}$$

Contradiction.

(c)

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

Since $\lim_{x \to 1} \sqrt{x} = 1$, we have $\lim_{x \to 0} \sqrt{1+x} = \lim_{x \to 0} \sqrt{1-x} = 1$. By limit laws,

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{2}{2} = 1$$

(d) We have $\lim_{x \to p} x^0 = \lim_{x \to p} 1 = 1 = p^0$. Suppose $\lim_{x \to p} x^n = p^n\ \forall\ p \in \mathbb{R}$ for some n. Then $\lim_{x\to p} x^{n+1} = \lim_{x\to p} x \cdot \lim_{x\to p} x^n = p \cdot p^n = p^{n+1}\ \forall\, p \in \mathbb{R}$. Thus $\lim_{x\to p} x^n = p^n\ \forall\, p \in \mathbb{R}\ \forall\, n \in \mathbb{N}$. $\square$

Problem 5.3. Let f and g be functions on $\mathbb{R}$ such that

$$\lim_{x\to 0} f(x) = L \quad \text{and} \quad \lim_{y\to 0} g(y) = M,$$

for some $L, M \in \mathbb{R}$. Is it true that

$$\lim_{x\to 0} (g \circ f)(x) = M?$$

If your answer is "yes", prove the above statement. If your answer is "no", provide a counterexample, and give a sufficient condition on g that will make the above statement true.

Proof. *NO.* Let $f(x) = 0$, $g(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} g \circ f(x) = 1$.

This arises from the fact that $|f(x) - L| < \varepsilon$ does not imply $0 < |f(x) - L|$, so it doesn't guarantee $0 < |y - L| < \delta$.

If we have g continuous at L, then we know g is defined at L as well.

For any $\varepsilon > 0$, $\exists\ \delta_1 > 0$ such that $|y - L| < \delta_1 \Rightarrow g(y)$is defined and$|g(y) - M| < \varepsilon$. Choose $\varepsilon_2 = \delta_1$. Then there exists $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow f(x)$ is defined and $|f(x) - L| < \varepsilon_2 = \delta_1 \Rightarrow |g(f(x)) - M| < \varepsilon$. $\square$

Remarks. We can also enforce a condition on f: if $f(x) \neq L\ \forall\ x \neq 0$, then $\lim_{x\to 0} g(f(x)) = \lim_{x\to L} g(x)$.

This yields a theorem:

Theorem 3.14 (limit of composition)**.** Suppose f and g are functions on $\mathbb{R}$ such that

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to L} g(x) = M.$$

Then we have

$$\lim_{x\to a} g(f(x)) = M$$

if

(a) g is continuous, or

(b) $f(x) \neq L\ \forall\, x \in N_\delta(a) \setminus \{a\}$ for some $\delta > 0$.

Problem 5.4. Prove the sequential characterization of continuity.

Theorem 3.15 (sequential characterization of continuity)**.** Let $f : A \to \mathbb{R}$ be a function and let $p \in A$. Let $P = \left\{ \{a_n\} \subseteq A : \lim_{n\to\infty} a_n = p \right\}$. Then f is continuous at p iff $\lim_{n\to\infty} f(a_n) = f(p) \; \forall \; \{a_n\} \in P$.

Proof.

(a) Let f be continuous at p and let $\{a_n\} \in P$.

For every $\varepsilon > 0$ there exists δ such that $f(x) \in N_\varepsilon(f(p)) \; \forall \; x \in N_\delta(p) \cap A$. Morever, $\exists \; N \in \mathbb{N}$ such that $a_n \in N_\delta(p) \cap A \; \forall \; n \geq N$.

Thus for all $n \geq N$, $f(a_n) \in N_\varepsilon(f(p))$. Thus $\lim_{n\to\infty} f(a_n) = f(p)$.

(b) Let $\lim_{n\to\infty} f(a_n) = f(p) \; \forall \; \{a_n\} \in P$. Suppose f is not continuous at p. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists $a \in N_\delta(p) \cap A \setminus \{p\}$ such that $|f(a) - f(p)| \geq \varepsilon$.

Let $\{\delta_n\}_{n\in\mathbb{P}} = \left\{\frac{1}{n}\right\}_{n\in\mathbb{P}}$. Then corresponding to every δ_n $\exists \; a_n \in A \cap N_{\delta_n}(p) \setminus \{p\}$ such that $|f(a_n) - f(p)| \geq \varepsilon$.

Thus $\{a_n\} \to p$ but $\lim_{n\to\infty} f(a_n) \neq f(p)$. Contradiction. □

(This proof uses the axiom of choice. Let me know if you have a proof without it.)

Problem 5.5. Complete the following steps to establish the continuity of the sine and cosine functions on $\mathbb{R}$. Recall (i) and (ii) given at the beginning of this assignment.

(a) Show that $\lim_{x\to 0} \sin(x) = 0$.

(b) Using (a) and a trigonometric identity relating sin and cos, show that $\lim_{x\to 0} \cos(x) = 1$.

(c) Using (a) and (b), show that sin and cos are continuous at any $x \in \mathbb{R}$.

(d) Show that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Proof.

(a)

$$0 < \left|\frac{\sin x}{x}\right| < 1 \Rightarrow -|x| < \sin x < |x| \qquad \forall \; 0 < |x| < \frac{\pi}{2}$$

Since $\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0$, we have $\lim_{x\to 0} \sin x = 0$ by squeeze theorem.

(b) Since $\cos^2(x) + \sin^2(x) = 1$, we have

$$\lim_{x\to 0}\cos^2(x) = \lim_{x\to 0}(1-\sin^2(x)) = 1 - \lim_{x\to 0}\sin^2 x = 1 - \left(\lim_{x\to 0}\sin x\right)^2 = 1$$

Since $\cos x > 0$ for $|x-0| < \frac{\pi}{2}$, we have $\sqrt{\cos^2(x)} - \cos x = 0$ for $|x-0| < \frac{\pi}{2}$. So $\lim_{x\to 0}(\sqrt{\cos^2 x} - \cos x) = 0$. Thus

$$\lim_{x\to 0}\cos x = \lim_{x\to 0}\sqrt{\cos^2 x} = \sqrt{\lim_{x\to 0}\cos^2 x} = 1$$

by limit of composition ($\sqrt{\cdot}$ is continuous).

(c) Now

$$\begin{aligned} \sin(x+h) &= \sin x\cos h \\ &\quad + \cos x\sin h \\ \lim_{h\to 0}\sin(x+h) &= \sin x \\ \lim_{y\to x}\sin(y) &= \sin x \end{aligned} \qquad \begin{aligned} \cos(x+h) &= \cos x\cos h \\ &\quad - \sin x\sin h \\ \lim_{h\to 0}\cos(x+h) &= \cos x \\ \lim_{y\to x}\cos(y) &= \cos x \end{aligned}$$

for all $x \in \mathbb{R}$. Thus sin and cos are continuous at any $x \in \mathbb{R}$

(d) Finally, we have

$$\left|\frac{\sin x}{x}\right| = \frac{\sin x}{x} \qquad \forall\, 0 < |x| < \frac{\pi}{2}$$

so

$$\cos x < \frac{\sin x}{x} < 1 \quad \forall\, 0 < |x| < \frac{\pi}{2}.$$

By the squeeze theorem, we get

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

□

Lecture 17: Global Extrema and Extreme Value Theorem

fri 25 nov 2022

Definition 3.16. A function $f : S \to \mathbb{R}$ is said to have a *global maximum* on S at a point $p \in S$ if $f(x) \leq f(p)\ \forall\, x \in S$.

A function $f : S \to \mathbb{R}$ is said to have a *global minimum* on S at a point $p \in S$ if $f(x) \geq f(p)\ \forall\, x \in S$.

Theorem 3.17 (Extreme value theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f attains both a global maximum and a global minimum in $[a,b]$.

Proof. Since f is bounded on $[a,b]$,

$$s = \sup f([a,b])$$

exists. Suppose $f(x) < s \ \forall\, x \in [a, b]$. Let

$$g(x) := s - f(x)$$

on $[a, b]$. Note that g is continuous and positive. Thus (by HW 6.2), there exists a $c > 0$ such that $g(x) \geq c \ \forall\, x \in [a, b] \Rightarrow f(x) \leq s - c \ \forall\, x \in [a, b]$. A contradiction. □

Corollary 3.18. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then (using IVT),

$$f([a, b]) = [\min_{[a,b]} f, \max_{[a,b]} f].$$

Lecture 18: Differentiation

mon 28 nov '22

4 Differentiation

Definition 4.1. Let $f : (a, b) \to \mathbb{R}$ be a function and $p \in (a, b)$. We say that f is differentiable in (a, b) if

$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h} = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

exists, and the limit is called the derivative of f at p, denoted $f'(p)$.

If f is differentiable on each p in (a, b), it is said to be differentiable on (a, b) and $f' : (a, b) \to \mathbb{R}$ is called the derivative of f on (a, b).

We define two more functions:

(a) For any function $f : (a, b) \to \mathbb{R}$ and any $p \in (a, b)$, define

$$f_\delta^p : h \in (a - p, b - p) \setminus \{0\} \mapsto \frac{f(p+h) - f(p)}{h} \in \mathbb{R}.$$

(b) For any differentiable function $f : (a, b) \to \mathbb{R}$ and any $p \in (a, b)$, define

$$f_\Delta^p : h \in (a - p, b - p) \mapsto \begin{cases} \dfrac{f(p+h) - f(p)}{h} & h \neq 0 \\ f'(p) & h = 0 \end{cases} \in \mathbb{R}.$$

Theorem 4.2 (Differentiability $\Rightarrow$ continuity)**.** Let $f : (a, b) \to \mathbb{R}$ be differentiable at $p \in (a, b)$. Then f is continuous at p.

Proof.

$$\begin{aligned} f(x) &= (x-p)\cdot\frac{f(x)-f(p)}{x-p}+f(p) \\ \lim_{x\to p} f(x) &= 0\cdot f'(p)+f(p) && \text{(exists)} \\ &= f(p) \end{aligned}$$

$\square$

Theorem 4.3 (Algebra of derivatives)**.** Let $f, g : (a,b) \to \mathbb{R}$ be differentiable at $p \in (a,b)$. Then

(a) $f+g$ is differentiable at p and $(f+g)' = f' + g'$.

(b) $f-g$ is differentiable at p and $(f-g)' = f' - g'$.

(c) $f\cdot g$ is differentiable at p and $(f\cdot g)' = f'\cdot g + f\cdot g'$.

(d) f/g is differentiable at p if $g \neq 0$ and $(f/g)' = \frac{f'\cdot g - f\cdot g'}{g^2}$.

Proof (Quotient rule). For the special case of $f(x) = 1$, we have

$$\begin{aligned} \lim_{h\to 0}\frac{\frac{1}{g(p+h)}-\frac{1}{g(p)}}{h} &= -\lim_{h\to 0}\frac{g(p+h)-g(p)}{h\cdot g(p+h)\cdot g(p)} \\ &= -\frac{1}{g(p)^2}\lim_{h\to 0}\frac{g(p+h)-g(p)}{h} \\ &= -\frac{1}{g(p)^2}g'(p)\text{exists.} \end{aligned}$$

$\square$

Example.

(a) (Constant function) $f(x) = c$ is differentiable at p for any $c \in \mathbb{R}$ and $f'(p) = 0$.

Proof.

$$\lim_{h\to 0}\frac{f(p+h)-f(p)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

$\square$

(b) $f(x) = x^n, x \in \mathbb{R}, n \in \mathbb{Z}\setminus\{0\}$ is differentiable at p and $f'(p) = n\cdot p^{n-1}$.

Proof.

$$\begin{aligned} \lim_{h\to 0}\frac{f(p+h)-f(p)}{h} &= \lim_{h\to 0}\frac{(p+h)^n-p^n}{h} \\ &= \lim_{h\to 0}\frac{p^n+n\cdot p^{n-1}\cdot h+\cdots+n\cdot h^{n-1}\cdot p+h^n-p^n}{h} \\ &= \lim_{h\to 0}\frac{n\cdot p^{n-1}\cdot h+\cdots+n\cdot h^{n-1}\cdot p}{h} \\ &= n\cdot p^{n-1} \end{aligned}$$

$\square$

As a consequence, we get that polynomials and rational functions are differentiable in their domains.

(c) $f(x) = \sin x$ is differentiable at p and $f'(p) = \cos p$.

Proof.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(p+h)-f(p)}{h} &= \lim_{h\to 0} \frac{\sin(p+h)-\sin p}{h} \\
&= \lim_{h\to 0} \frac{\sin p\cos h + \cos p \sin h - \sin p}{h} \\
&= \cos p \lim_{h\to 0} \frac{\sin h}{h} + \sin p \lim_{h\to 0} \frac{\cos h - 1}{h} \\
&= \cos p + \sin p \lim_{h\to 0} \frac{-2\sin^2 \frac{h}{2}}{h} \\
&= \cos p + \sin p \cdot 0 \\
&= \cos p.
\end{aligned}$$

□

(d) $f(x) = \cos x$ is differentiable at p and $f'(p) = -\sin p$.

Proof.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(p+h)-f(p)}{h} &= \lim_{h\to 0} \frac{\cos(p+h)-\cos p}{h} \\
&= \lim_{h\to 0} \frac{-\sin p\sin h + \cos p \cos h - \cos p}{h} \\
&= -\sin p \lim_{h\to 0} \frac{\sin h}{h} + \cos p \lim_{h\to 0} \frac{\cos h - 1}{h} \\
&= -\sin p + \cos p \lim_{h\to 0} \frac{-2\sin^2 \frac{h}{2}}{h} \\
&= -\sin p + \cos p \cdot 0 \\
&= -\sin p.
\end{aligned}$$

□

(e) $f(x) = |x|$ is continuous but not differentiable at $p = 0$.

Proof.

$$\frac{|h|}{h} = \begin{cases} 1 & h > 0 \\ -1 & h < 0 \\ \text{undefined} & h = 0 \end{cases}$$

Let $a_n = \frac{(-1)^n}{n}$, then $\lim_{n\to\infty} a_n = 0$ but $\lim_{n\to\infty} \frac{|a_n|}{a_n} = \lim_{n\to\infty} (-1)^n$ does not exist. □

Assignment 6

due Thu 1 Dec 2022

Problem 6.1. Give an example each of

(a) a bounded function $f : [-1, 1] \to \mathbb{R}$ that does not achieve either its minimum or its maximum on $[-1, 1]$;

(b) a bounded continuous function $f : (-1, 1) \to \mathbb{R}$ that achieves its minimum but not its maximum on $(-1, 1)$.

(a) $f(x) = \begin{cases} x & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$

(b) $f(x) = x^2$

Problem 6.2. Let f be a continuous function on $[a, b]$ such that $f(x) > 0$ for all $x \in [a, b]$. Show that there is a $c > 0$ such that $f(x) \geq c$ for all $x \in [a, b]$.

Proof. Since f is continuous and non-zero on $[a, b]$, $1/f$ is continuous on $[a, b]$. Since $1/f$ is continuous, it is bounded on $[a, b]$ (theorem 3.12). Thus $f(x) \geq c = \frac{1}{M}\ \forall\, x \in [a, b]$ where M is the supremum of $1/f$ on $[a, b]$. □

Problem 3. Let the polynomial be $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where n is odd. Define

$$g(x) = \frac{f(x)}{x^n} = \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + a_n.$$

Define the sequence $\{g(m)\}_{m\in\mathbb{N}}$. We have

$$\lim_{m\to\infty} g(m) = a_n.$$

Thus we have $N \in \mathbb{N}$ such that $|g(m) - a_n| < a_n\ \forall\, m \geq N \Rightarrow g(N) > 0 \Rightarrow f(N) = N^n g(N) > 0$.

Define the sequence $\{g(-m)\}_{m\in\mathbb{N}}$. We have

$$g(-m) = -\frac{a_0}{m^n} + \frac{a_1}{m^{n-1}} - \cdots + a_n$$
$$\Rightarrow \lim_{m\to\infty} g(-m) = a_n$$

Thus we have $N' \in \mathbb{N}$ such that $|g(-m) - a_n| < a_n\ \forall\, m \geq N' \Rightarrow g(-N') > 0 \Rightarrow f(-N') = (-N')^n g(-N) = -(N')^n g(-N) < 0$. Also $-N' < 0 < N$. By IVT we have

$$\exists\, c \in [-N', N] : f(c) = 0.$$

Problem 4. Consider $g(x) = \cos x - x^2$. Then $g'(x) = -\sin x - 2x < 0$. So g is decreasing. $g(0) = 1 > 0$. $g(\frac{\pi}{2}) = -\frac{\pi^2}{4} < 0$. Therefore $g(x) = 0$ at exactly one $x = c$ in $[0, \frac{\pi}{2}]$. Thus $g(x) > 0$ for $0 \leq x < c$ and $g(x) < 0$ for $c < x \leq \frac{\pi}{2}$.

For $0 \leq x < c$, $f(x) = \cos x$. For $c < x \leq \frac{\pi}{2}$, $f(x) = x^2$.

Since $\cos x$ is decreasing in $[0, c] \subseteq [0, \frac{\pi}{2}]$, we have $f(x) > f(c) \ \forall \ x \in [0, c)$. Since x^2 is increasing in $[c, \frac{\pi}{2}] \subseteq [0, \frac{\pi}{2}]$, we have $f(x) > f(c) \ \forall \ x \in (c, \frac{\pi}{2}]$

Thus f attains a global minimum at c, where $\cos c = c^2$.

Problem 5.

(a)

$$h \circ g(x) = |x|^3 = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}$$

Since all polynomials are continuous and differentiable at every point, we only need to check at 0.

$$\begin{aligned} \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} \frac{|h|^3}{h} \\ &= \lim_{h \to 0} \frac{h^2 |h|}{h} \\ &= \lim_{h \to 0} h|h| \\ &= 0 \end{aligned}$$

Thus $h \circ g$ is differentiable everywhere, so it is continuous everywhere.

(b) We first show that $\cos(\frac{1}{x})$ is differentiable at $x \neq 0$.

$$\begin{aligned} \lim_{x \to p} \frac{\cos(\frac{1}{x}) - \cos(\frac{1}{p})}{x - p} &= \lim_{x \to p} \frac{2 \sin(\frac{1}{2x} + \frac{1}{2p}) \sin(\frac{1}{2p} - \frac{1}{2x})}{x - p} \\ &= 2 \sin(\frac{1}{p}) \lim_{x \to p} \frac{\sin(\frac{x-p}{2xp})}{x - p} \\ &= 2 \sin(\frac{1}{p}) \lim_{x \to p} \frac{\sin(\frac{x-p}{2xp})}{\frac{x-p}{2xp}} \frac{1}{2xp} \\ &= \frac{1}{p^2} \sin(\frac{1}{p}) \end{aligned}$$

The limit exists, so $\cos(\frac{1}{x})$ is differentiable everywhere in its domain.

By algebra laws we have $x^2 \cos(\frac{1}{x})$ also differentiable, as is $|x| = -x \ \forall \ x < 0$. So we only need to worry about 0.

In the neighbourhood $(-1, 1)$ about 0, we have $-x^2 \leq f(x) \leq |x|$. By the squeeze theorem, $f(x) + x^2$ tends to 0. By the limit laws, $\lim_{x \to 0} f(x) = 0$. Thus f is continuous everywhere.

$$f(0+h)-f(0)=\begin{cases}-h & h<0\\ h^2\cos(\frac{1}{h}) & h>0\end{cases}$$
$$\frac{f(0+h)-f(0)}{h}=\begin{cases}-1 & h<0\\ h\cos(\frac{1}{h}) & h>0\end{cases}$$

Let $\varepsilon = \frac{1}{4}$. For any $\delta > 0$, choose $k = \min\{\frac{1}{2}, \frac{\delta}{2}\}$. $-k \le f(k) \le k \Rightarrow f(k) > -k > -\frac{1}{2}$ and $|k-0| < \delta$. Also $f(-k) = -1$. For any L, $|f(k)-L| + |f(-k)-L| \ge |f(k)-f(-k)| = |f(k)+1| \ge \frac{1}{2} = 2\varepsilon$. Thus the limit does not exist and so the function is not differentiable at 0.

(c) $|\sin x| = |\sin|x||$. So

$$f(x)=\begin{cases}\frac{|\sin|x||}{\sin|x|} & x\neq n\pi\\ 0 & x=n\pi\end{cases}$$

or

$$f(x)=\begin{cases}1 & \sin|x|>0\\ 0 & \sin|x|=0\\ -1 & \sin|x|<0\end{cases}$$

Since constant functions are continuous and differentiable, $f(x)$ is differentiable in any region where $\sin|x|$ is constant in sign. Thus $f(x)$ is continuous and differentiable in all intervals $(n\pi, (n+1)\pi), n \in \mathbb{Z}$. This leaves only the points $n\pi$, where the function is neither continuous nor differentiable, as

$$\lim_{x\to n\pi} f(x) \text{ does not exist.}$$

Suppose limit exists and is equal to L. Then

Lecture 19: Invertible Functions: Monotonicity and Continuity

tue 30 nov '22

Definition 4.4 (Inverse function)**.** Let $f : A \to B$ be bijective. Then for any $y \in B$, there exists (unique) $x_y \in A$ such that $f(x_y) = y$. We define the inverse function $f^{-1} : B \to A$ as

$$f^{-1}(y) = x_y.$$

and say that f is invertible on A.

Note that $(f \circ f^{-1})$ and $(f^{-1} \circ f)$ are the identity functions on B and A respectively.

For example, the function $f(x) = x^2$ is invertible on $\mathbb{R}^+$ and its inverse is $f^{-1}(x) = \sqrt{x}$.

Theorem 4.5 (inverse function properties)**.** Let $f : [a, b] \to \mathbb{R}$ be an invertible function on $[a, b]$ with range J.

(i) If f is (strictly) increasing, then so is f^{-1}.

(ii) If f is continuous, then $f : [a, b] \to J$ is strictly monotone and $f^{-1} : J \to [a, b]$ is continuous.

(iii) If f is differentiable at $p \in (a, b)$ with $f'(p) \neq 0$ *and continuous in some neighborhood around* p, then f^{-1} is differentiable at $f(p) = q \in J$ and $(f^{-1})'(q) = \dfrac{1}{f'(p)}$.

Proof.

(i) If $x_1, x_2 \in [a, b]$, then $f(x_1) < f(x_2)$ implies $x_1 < x_2$, and hence $f^{-1}(f(x_1)) < f^{-1}(f(x_2))$.

(ii) Given: $f : [a, b] \to J$ is invertible and continuous. Thus, $J = [A, B]$ (corollary 3.18).

(Case 1) $(f(a) < f(b))$ We prove that f is strictly increasing.

Suppose there exists c in (a, b) such that $f(c) < f(a) < f(b)$. Then by intermediate value theorem, f attains the value $f(a)$ somewhere in (c, b), contradicting that it is invertible.

Similarly for any $x_0 \in (a, b)$, $f(x_0)$ can neither be greater that $f(b)$ nor less than $f(a)$ because of IVT. Then for all $x \in (x_0, b)$, $f(x) > f(x_0)$. Hence f is strictly increasing.

(Case 2) $(f(a) > f(b))$ We prove that f is strictly decreasing.

Suppose there exists c in (a, b) such that $f(c) > f(a) > f(b)$. Then by IVT, f attains the value $f(a)$ somewhere in (c, b), contradicting that it is invertible.

Similarly for any $x_0 \in (a, b)$, $f(x_0)$ can neither be greater that $f(a)$ nor less than $f(b)$ because of IVT. Then for all $x \in (x_0, b)$, $f(x) < f(x_0)$. Hence f is strictly decreasing.

Let $p \in [a, b]$. We show that f^{-1} is continuous at $f(p)$.

Suppose WLOG that f is increasing.

Let $\varepsilon > 0$. Let

$$\delta = \min\big\{f(p) - f^{-1}(\max\{f(p) - \varepsilon, f(a)\}), f^{-1}(\min\{f(p) + \varepsilon, f(b)\}) - f(p)\big\}.$$

This is very ugly, so instead we do this:

Let
$$d = \min\{b - p, p - a\}.$$
It suffices to consider $0 < \varepsilon \le d$. Because if $\varepsilon > d$, we choose δ_ε corresponding to $\varepsilon = d$. Then, whenever $|y - q| < \delta_\varepsilon$, we have $|f^{-1}(y) - f^{-1}(q)| < d < \varepsilon$.

Let $0 < \varepsilon \le d$. Since $(p - \varepsilon, p + \varepsilon) \subseteq (a, b)$, we have $A \le f(p - \varepsilon) < f(p) = q < f(p + \varepsilon) \le B$. Let
$$\delta = \min\{f(p + \varepsilon) - q, q - f(p - \varepsilon)\}$$
Then whenever $y \in N_\delta(q) \cap [A, B]$, we have
$$f(p - \varepsilon) \le q - \delta < q < q + \delta \le f(p + \varepsilon)$$
Thus,
$$p - \varepsilon \le f^{-1}(q - \delta) < p < f^{-1}(q + \delta) \le p + \varepsilon$$
Thus,
$$y \in N_\delta(q) \cap [A, B] \Rightarrow f^{-1}(y) \in N_\varepsilon(p) \cap [a, b].$$

(iii) Given: $f : [a, b] \to J$ is invertible and differentiable at $p \in (a, b)$ with $f'(p) \ne 0$. Thus, $J = [A, B]$. We also require f to be continuous in some neighborhood around p.

We want $\lim_{k\to 0} (f^{-1})^q_\delta(k)$.

JEE Solution: Since $f(p) + k \in \operatorname{dom}(f^{-1})$, it is the image of some $p + h$ in f. Then,
$$\begin{aligned}
\lim_{k\to 0} \frac{f^{-1}(f(p) + k) - f^{-1}(f(p))}{k} &= \lim_{k\to 0} \frac{f^{-1}(f(p + h)) - f^{-1}(f(p))}{f(p + h) - f(p)} \\
&= \lim_{k\to 0} \frac{p + h - p}{f(p + h) - f(p)} \\
&= \lim_{k\to 0} \frac{h}{f(p + h) - f(p)} \\
&= \lim_{h\to 0} \frac{h}{f(p + h) - f(p)} \\
&= \frac{1}{f'(p)}.
\end{aligned}$$

Note that h is a function of k. We don't know that
$$\lim_{k\to 0} \frac{h(k)}{f(p + h(k)) - f(p)} = \lim_{h\to 0} \frac{h}{f(p + h) - f(p)}.$$
Let $h(k) = k \cdot (f^{-1})^q_\delta(k) = f^{-1}(q + k) - f^{-1}(q) = f^{-1}(q + k) - p$ for $k \in$

$\mathrm{dom}((f^{-1})^q_\delta)$. Thus

$$\begin{aligned} h(k)+p &= f^{-1}(q+k) \\ f(h(k)+p) &= q+k \\ k &= f(h(k)+p)-q \\ &= f(h(k)+p)-f(p) \end{aligned}$$

If $k \neq 0$, then $h(k) \neq 0$ as f^{-1} is injective. Thus

Note that since f is continuous in some neighborhood around p, f^{-1} is continuous in some neighborhood around q. Thus,

$$\begin{aligned} \lim_{k\to 0} k &= \lim_{k\to 0} f(h(k)+p) - \lim_{k\to 0} f(p) \\ \lim_{k\to 0} f(h(k)+p) &= f(p) \\ f(\lim_{k\to 0} h(k)+p) &= f(p) \\ f^{-1}(f(\lim_{k\to 0} h(k)+p)) &= f^{-1}(f(p)) \\ \lim_{k\to 0} h(k) &= 0 \end{aligned}$$

If $k \neq 0$, then $h(k) \neq 0$. By the composition theorem,

$$\begin{aligned} \lim_{k\to 0} \frac{f^{-1}(q+k)-f^{-1}(q)}{k} &= \lim_{k\to 0} \frac{h}{f(h(k)+p)-f(p)} \\ &= \lim_{k\to 0} F(h(k)) \end{aligned}$$

where $F(x) = \frac{x}{f(p+x)-f(p)}$. Since $k \neq 0 \Rightarrow h(k) \neq 0$,

$$\begin{aligned} \lim_{k\to 0} F(h(k)) &= \lim_{h\to 0} F(h) \\ &= \frac{1}{f'(p)}. \end{aligned}$$

□

Example. A function continuous at just one point in its domain, defined on all of $\mathbb{R}$.

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

A function differentiable at one point and not continuous in any neighborhood about it.

$$f(x) = \begin{cases} x+x^2 & x \in \mathbb{Q} \\ x & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Example. Derivatives of $x^{\frac{1}{n}}$ for $x \in (0,\infty)$, $n \in \mathbb{N}$.

By product rule, we get derivatives of $x^{p/q} = (x^{1/q})^p$.

Lecture 20

mon 12 dec 2022

Lecture 21: Chain Rule and Extrema

wed 14 dec 2022

Theorem 4.6 (chain rule)**.** Let $f : (a,b) \to \mathbb{R}$ and $g : (c,d) \to \mathbb{R}$ with $f((a,b)) \subseteq (c,d)$ and f differentiable in (a,b). Let g be differentiable at $f(p) := q$. Then $g \circ f : (a,b) \to \mathbb{R}$ is differentiable at p and $(g \circ f)' = g' \circ f \cdot f'$ at p.

Proof. We consider the limit

$$\lim_{k \to 0} \frac{g \circ f(p+k) - g \circ f(p)}{k}.$$

$$\frac{g \circ f(p+k) - g \circ f(p)}{k} = \frac{g(f(p+k)) - g(f(p))}{f(p+k) - f(p)} \cdot \frac{f(p+k) - f(p)}{k}$$

except $f(p+k) - f(p)$ might be zero even for $k \neq 0$. So we instead define $f(p+k) - f(p) = h_k$ and

$$G(h) = \begin{cases} \dfrac{g(q+h) - g(q)}{h} & h \neq 0 \\ g'(q) & h = 0 \end{cases}$$

Now $g(f(p+k)) - g(f(p)) = G(h_k)h_k$ for all k, as $G(0) \cdot 0 = 0$. Note that G is continuous at 0. Also, $\lim_{k\to 0} h_k = 0$ and $\lim_{h \to 0} G(h) = g'(q)$. Thus $\lim_{k \to 0} G(h_k) = g'(q)$.

$$\begin{aligned} \lim_{k \to 0} \frac{g \circ f(p+k) - g \circ f(p)}{k} &= \lim_{k \to 0} \frac{G(h_k)h_k}{k} \\ &= g'(q) \cdot \lim_{k \to 0} \frac{f(p+k) - f(p)}{k} \\ &= g'(q) \cdot f'(p) \end{aligned}$$

□

4.1 Local Extrema

Definition 4.7 (Local Extrema)**.** Let $f : A \to \mathbb{R}$. We say that f attains a local maximum at $a \in A$ iff $\exists\, \delta > 0$ such that

$$f(x) \leq f(a)\ \forall\, x \in N_\delta(a) \cap A.$$

We say that f attains a local minimum at $a \in A$ iff $\exists\, \delta > 0$ such that

$$f(x) \geq f(a)\ \forall\, x \in N_\delta(a) \cap A.$$

Theorem 4.8 (Extremum $\Rightarrow$ Stationary)**.** Let $f : (a,b) \to \mathbb{R}$. Let $c \in (a,b)$ such that f is differentiable at c. If f attains a local extremum at c, then $f'(c) = 0$. Points at which the derivative vanishes are called 'stationary points' and sometimes 'critical points'.

Remarks. Say $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Now, $f_{[a,b]}$ may have local or even global extrema at its end points *w.r.t the new domain* $[a, b]$. Then the derivative of f at the endpoint is not necessarily zero. We cannot comment about the differentiability of $f_{[a,b]}$ at the endpoints.

E.g., $f(x) = x$ on $[0, 1]$.

Remarks. f may have local or global extrema at points where it is simply not differentiable. *E.g.*,

$$f(x) = |x|.$$

Remarks. The converse is not necessarily true. *E.g.*, $f(x) = x^3$ at $x = 0$.

Write a formal proof to say that 0 is neither a point of local maximum nor local minimum for f.

Proof. If f has a local maximum at c, $\exists\, \delta > 0$ such that

$$f(x) \le f(c)\ \forall\, x \in N_\delta(c) \subseteq (a, b).$$

If f has a local minimum at c, $\exists\, \delta > 0$ such that

$$f(x) \ge f(c)\ \forall\, x \in N_\delta(c) \subseteq (a, b).$$

In either case, $(f(x_1) - f(c))(f(x_2) - f(c)) > 0\ \forall\, x \in N_\delta(c) \subseteq (a, b)$. Define

$$F(x) = \begin{cases} \dfrac{f(c+x) - f(c)}{x} & x \in (-\delta, 0) \cup (0, \delta) \\ f'(c) & x = 0 \end{cases}$$

F is continous at 0. Consider the following sequences for $n > \frac{1}{\delta}$.

$$a_n = \frac{1}{n} \qquad\qquad b_n = -\frac{1}{n}$$

$$\lim_{n\to\infty} a_n = 0 \qquad\qquad \lim_{n\to\infty} b_n = 0$$

By sequential characterization of continuity,

$$\lim_{n\to\infty} F(a_n) = \lim_{n\to\infty} F(b_n) = F(0) = f'(c)$$

By limit laws,

$$\lim_{n\to\infty} F(a_n)F(b_n) = f'(c)f'(c) = f'(c)^2 \ge 0.$$

but

$$F(a_n)F(b_n) = \frac{(f(c - \frac{1}{n}) - f(c))(f(c + \frac{1}{n}) - f(c))}{-\frac{1}{n^2}} < 0\ \forall\, n > \frac{1}{\delta}.$$

Thus

$$\lim_{n\to\infty} F(a_n)F(b_n) \le 0.$$

So we have $f'(c)^2 = 0 \Rightarrow f'(c) = 0$. $\square$

Lecture 22: Extreme and Mean Value Theorems

fri 18 dec '22

Assume WLOG that f attains a local maximum at c, and is differentiable at c. Thus there exists a $\delta > 0$ such that $|x - c| < \delta \Rightarrow f(x) < f(c)$. Let $\{a_n\}_{n \geq N}$ be a sequence, $a_n = c + \frac{1}{n}$, where $N > \frac{1}{\delta}$.

$$\frac{f(a_n) - f(c)}{a_n - c} = n \cdot (f(a_n) - f(c)) < 0$$

Thus by **??**, $\lim\limits_{n \to \infty} \dfrac{f(a_n) - f(c)}{a_n - c} \leq 0$. By the sequential characterisation of limits, this limit is equal to $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c)$.

Similarly, considering $b_n = c - \frac{1}{n}$, we get $f'(c) \geq 0$.

Combining these results yields $f'(c) = 0$.

Remarks. The theorem is most helpful in the following way: to identify *potential* points of local extrema within open intervals where f is differentiable. Thereafter, one needs to do some local analysis at the potential points. For the local analysis, one inspects the sign of the derivative 'just a bit before' and 'just a bit after' the point.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = (x-1)^2 + |x| + x^3.$$

The potential extrema are at $x = 0, x = -3$ and $x = \sqrt{2} - 1$. We cannot yet say which of these will be a local maximum.

Theorem 4.9 (Mean Value Theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. **Case 1 (Rolle's Theorem).** $f(a) = f(b)$. Since the function is continuous on $[a,b]$, it achieves a global maximum at c_1 and a global minumum at c_2 by the extreme value theorem. If at least one of c_1 and c_2 lies inside (a,b), we have $f'(c_i) = 0$. Otherwise, we have $f(a) = f(b) = f(c_1) = f(c_2) \Rightarrow f$ is constant. Thus $f'(c) = 0 \; \forall \, c \in (a,b)$.

Case 2. $f(a) \neq f(b)$. Define

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a} \cdot x.$$

Since f is continuous on $[a,b]$, g is continuous on $[a,b]$ and $g(a) = g(b)$. Also g is differentiable on (a,b) with

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

By Rolle's theorem $g'(c) = 0$ for some c in $(a, b) \Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some $c \in (a, b)$. $\square$

due Thu 22 Dec 2022

Assignment 7

Problem 1

(a) is just h replaced with $-h$. Since h could take any real values, including negative ones, it doesn't make any difference to negate it. Since we are replacing h with $-h$, what we have is a composition. The usual difference quotient $(\Delta y/\Delta x)$ is composed with $f(x) = -x$. We have two properties which let us compute the limit of $g \circ f$ given the limits of g and f.

If $\lim_{x\to a} f(x) = L$ and $\lim_{y\to L} g(y) = M$, then $\lim_{x\to a} g(f(x)) = M$ if:

(i) g is continuous.

(ii) f does not equal L anywhere 'around' a (in a sufficiently small neighbourhood).

Here, $f(x) = -x$, which clearly does not equal 0 anywhere except at 0. We will use the second condition to conclude that the limit exists (and is equal to the derivative).

In part (b), however, the points that we are checking are limited in two ways: only very few fixed points around c, as well as only points to the right. We can construct a counterexample exploiting either of these limitations.

(a) True. Let $g(x) = -x$ and $G(x) = \dfrac{f(c) - f(c - x)}{x}$ $\forall\, x \in (a - c, 0) \cup (0, b - c)$. We have

$$\lim_{x\to 0} g(x) = 0 \qquad \lim_{x\to 0} G(x) := L$$

$$g(x) \neq 0 \;\forall\, x \neq 0$$

Thus by limit of compositions, we have

$$\begin{aligned} L &= \lim_{x\to 0} G(g(x)) \\ &= \lim_{x\to 0} \frac{f(c) - f(c - (-x))}{-x} \\ &= \lim_{x\to 0} \frac{f(c + x) - f(c)}{x} \\ &= f'(c) \end{aligned}$$

Thus $f'(c)$ exists and is equal to L.

(b) False. Let $f : (-1, 1) \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}$$

Then the given limit exists for $c = 0$ as $n\big(f\big(\frac{1}{n}\big) - f(0)\big) = n - n = 0 \ \forall\, n \in \mathbb{N}$. However, f is not differentiable at 0 as it is not continuous at 0.

Alternatively, consider $f : (-1, 1) \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Problem 2

Since $\sin, x^3$ and f are differentiable, we have by the chain rule and addition rule:

(a)

$$\begin{aligned} g'(x) &= f'(x^3)(x^3)' + \sin'(f(x))f'(x) \\ &= 3x^2 f'(x^3) + \cos(f(x))f'(x) \end{aligned}$$

(b)

$$g'(x) = f'(f(x))f'(x)$$

Problem 3

$$\frac{f(0+h) - f(0)}{h} = h^{-2/3} \ \forall\, h \neq 0$$

We need to prove that this does not have a limit at 0. This is simple Archimedean,OR direct $\varepsilon - \delta$ contradiction similar to $(-1)^n$. I have given the Archimedean proof, you try $\varepsilon - \delta$. (Hint: Let $\varepsilon = 1$. Consider $x = \pm \min\left\{\dfrac{\delta}{2}, 1\right\}$.)

Consider the sequence $\left\{\frac{1}{n}\right\}_{n \in \mathbb{P}}$. For any $R \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N \cdot 1 > R^{3/2} \Rightarrow N^{2/3} > R \Rightarrow \left(\frac{1}{n}\right)^{-2/3} > R \ \forall\, n \geq N$. Thus, $\left\{\frac{1}{n}\right\}$ converges to 0 as $n \to \infty$ but $\left\{\left(\frac{1}{n}\right)^{-2/3}\right\}$ is unbounded $\Rightarrow$ divergent. Thus limit of $h^{-2/3}$ does not exist at 0.

Problem 4

We want two things:

(i) $f'(c) = 0 \Rightarrow g$ is differentiable at c.

(ii) g is differentiable at $c \Rightarrow f'(c) = 0$.

Let us think about modulus. If $\lim_{x\to a} f(x)$ is known, can we say anything about $\lim_{x\to a}|f(x)|$? The answer is yes, we have $\lim_{x\to a}|f(x)| = |\lim_{x\to a} f(x)|$. Here is a short proof.

Proof. $\lim_{x\to a} f(x) = L \Rightarrow \forall\, \varepsilon > 0 \;\exists\, \delta > 0$ such that $|f(x) - L| < \varepsilon$. But by the triangle inequality, $||f(x)| - |L|| < |f(x) - L| < \varepsilon$. Thus the limit of $|f(x)| = |L|$ as $x \to a$. $\square$

However, can we determine $\lim_{x\to a} f(x)$ from $\lim_{x\to a}|f(x)|$? Let's attempt a similar proof.

Attempt. $\lim_{x\to a}|f(x)| = |L| \Rightarrow \forall\, \varepsilon > 0 \;\exists\, \delta > 0$ such that $||f(x)| - |L|| < \varepsilon$. But by the triangle inequality, $|f(x) - L| > ||f(x)| - |L|| < \varepsilon$. The signs of inequality are not helpful to us anymore. $\square$

Your first thought might be that f can have limit either L or $-L$, but it may not have a limit at all. As an example, consider signum, with limit of $|\text{signum}| = 1$ at 0.

The *essence* of why this is happening is that modulus is not an invertible function. We lose *information* on passing a number through modulus. There is only one special case where we do NOT lose information. What is it?

$$\vdots$$

$$\vdots$$

$|x| = 0 \Rightarrow x = 0$ with full certainty! This is why the given statement is true, and this is how we will prove it. We will use the fact that $\lim_{x\to a} f(x) = 0 \Leftrightarrow \lim_{x\to a}|f(x)| = 0$, which we will prove right now.

Proof.

$$\begin{aligned}
&\lim_{x\to a} f(x) = 0 \\
&\qquad \Leftrightarrow \big(\,\forall\, \varepsilon > 0 \;\exists\, \delta > 0 \text{such that} |x - a| < \delta \Rightarrow |f(x) - 0| < \varepsilon\big) \\
&\qquad \Leftrightarrow \big(\,\forall\, \varepsilon > 0 \;\exists\, \delta > 0 \text{such that} |x - a| < \delta \Rightarrow ||f(x)| - 0| < \varepsilon\big) \\
&\qquad\quad \Leftrightarrow \lim_{x\to a}|f(x)| = 0.
\end{aligned}$$ $\square$

Note that the difference quotients of f and g have the same absolute value. Thus if the limit of either is zero, the limit of the other must also be zero. Thus,

(i) $f'(c) = 0 \Rightarrow g'(c) = 0$.

(ii) $g'(c) = 0 \Rightarrow f'(c) = 0$.

Can you argue why $g'(c) = 0 \Leftrightarrow g$ is differentiable at c to complete the proof?

Problem 5

We know $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$.

Let $g = f^{-1} \Rightarrow g(x) = x^{\frac{1}{n}}$.

Then $f(p) = q \Rightarrow g'(q) = \frac{1}{f'(p)}$.

$$g'(q) = \frac{1}{f'(q^{\frac{1}{n}})} = \frac{1}{nq^{1-\frac{1}{n}}} = \frac{1}{n}q^{\frac{1}{n}-1}.$$

Now let $F(x) = x^{\frac{1}{n}} - 1 - (x-1)^{\frac{1}{n}}, x \in [1, a/b]$. Clearly, $F(1) = 0$.

$$F'(x) = \frac{1}{n}x^{\frac{1}{n}-1} - 0 - \frac{1}{n}(x-1)^{\frac{1}{n}-1} = \frac{1}{n}\Big(x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}\Big).$$

$x \geq 1 \Rightarrow x > x-1 > 0$. $n \geq 2 \Rightarrow \frac{1}{n} \leq \frac{1}{2} \Rightarrow \frac{1}{n} - 1 < 0$. Thus $x^{\frac{1}{n}-1} < (x-1)^{\frac{1}{n}-1} \Rightarrow F'(x) < 0$. Now suppose $F(a/b) \geq 0$. By MVT, there exists a $c \in (1, a/b)$ such that $F'(c) = \dfrac{F(x_0) - F(1)}{x_0 - 1} \geq 0$, a contradiction. Thus,

$$\begin{aligned} F(a/b) &< 0 \\ \Rightarrow \left(\frac{a}{b}\right)^{\frac{1}{n}} - 1 &< \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} \\ \Rightarrow a^{\frac{1}{n}} - b^{\frac{1}{n}} &< (a-b)^{\frac{1}{n}} \end{aligned}$$

Problem 6

$$f(x) = \begin{cases} -nx + \sum_{j=1}^{n} a_j & x < a_1 \\ (2k-n)x - \sum_{j=1}^{k} a_j + \sum_{j=k+1}^{n} a_j & a_k \leq x < a_{k+1} \\ nx - \sum_{j=1}^{n} a_j & a_n \leq x \end{cases}$$

We first prove that a global minimum exists.

Consider f restricted to $[a, b]$. By the extreme value theorem, there exists a $c \in [a, b]$ such that $f(c) \leq f(x) \; \forall \; x \in [a, b]$. Moreover,

$$\begin{aligned} x < a_1 &\Rightarrow -nx > -na_1 \Rightarrow f(x) > f(a_1) \\ x > a_n &\Rightarrow nx > na_n \Rightarrow f(x) > f(a_n) \end{aligned}$$

Thus, $f(c) \leq f(x) \; \forall \; x \in \mathbb{R}$.

For $x \neq a_j$ for some $j \in [1..n]$, $f(x)$ is differentiable with

$$f'(x) = \begin{cases} -n & x < a_1 \\ 2k - n & a_k < x < a_{k+1} \\ n & a_n < x \end{cases}$$

Thus the potential points of global minimum are $\{a_1, a_2, \ldots, a_n\} \cup (a_k, a_{k+1})_{2k=n}$.

Note that f is continuous everywhere.

Thus for $2k < n$, we have $f'(x) < 0 \; \forall \; x \in (a_k, a_{k+1}) \Rightarrow f(a_{k+1}) < f(a_k) \Rightarrow a_k$ is not a global minimum.

For $2k > n$, we have $f'(x) > 0 \; \forall \; x \in (a_k, a_{k+1}) \Rightarrow f(a_k) < f(a_{k+1}) \Rightarrow a_{k+1}$ is not a global minimum.

If n is odd, the only potential point of global minimum is thus $x = a_{\frac{n+1}{2}}$.

If n is even the potential points of global minimum are $\left\{a_{\frac{n}{2}}, a_{\frac{n}{2}+1}\right\} \cup (a_{\frac{n}{2}}, a_{\frac{n}{2}+1}) = [a_{\frac{n}{2}}, a_{\frac{n}{2}+1}]$. Note that the value of f is the same at all these points.

Problem 7

We consider:

$$\frac{f^{-1}(q+k) - f^{-1}(q)}{k}.$$

Let $h(k) = f^{-1}(q+k) - f^{-1}(q) = f^{-1}(q+k) - p$ wherever defined. Thus

$$\begin{aligned} h(k) + p &= f^{-1}(q+k) \\ f(h(k)+p) &= q+k \\ k &= f(h(k)+p) - q \\ &= f(h(k)+p) - f(p) \end{aligned}$$

Note that since f is continuous, f^{-1} is continuous. Thus as $k \to 0$, $h(k) \to 0$.

If $k \neq 0$, then $h(k) \neq 0$. By the composition theorem,

$$\begin{aligned} \lim_{k\to 0} \frac{f^{-1}(q+k) - f^{-1}(q)}{k} &= \lim_{k\to 0} \frac{h}{f(h(k)+p) - f(p)} \\ &= \lim_{k\to 0} F(h(k)) \end{aligned}$$

where $F(x) = \frac{x}{f(p+x)-f(p)}$. Since $k \neq 0 \Rightarrow h(k) \neq 0$,

Lecture 24: TAYLOR EXPANSION

Wed 21 Dec '22

Definition 4.10 (Taylor Polynomial)**.** Let $f : (a,b) \to \mathbb{R}$ be k times differentiable at some $x_0 \in (a,b)$. The k^{th} Taylor polynomial at x_0 is defined as

$$P_k^{x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Theorem 4.11 (Taylor's Theorem)**.** Let $f : (a,b) \to \mathbb{R}$ be an $(n+1)$ times differentiable function on (a,b). Note that this implies $f, f', f'', \dots f^{(n)}$ are continuous. Let $x_0 \in (a,b)$. Then $\forall\ x \in (a,b)\ \exists\ c_x$ between x and x_0 such that

$$f(x) = P_n^{x_0}(x) + f^{(n+1)}(c_x)\frac{(x-x_0)^{n+1}}{(n+1)!}.$$

Remarks. Compare this with the MVT:

$$f(x) = f(x_0) + f'(c_x)(x - x_0).$$

This is in fact Taylor's theorem for $n = 0$.

Copilot's Proof. Let $x \in (a, b)$. We will show that there exists a c_x between x and x_0 such that

$$f(x) = P_n^{x_0}(x) + f^{(n+1)}(c_x)\frac{(x - x_0)^{n+1}}{(n + 1)!}.$$

We will show this by induction on n. The base case is $n = 0$. This is the MVT. For the inductive step, let $n \geq 1$. Let $x \in (a, b)$. We will show that there exists a c_x between x and x_0 such that

$$f(x) = P_n^{x_0}(x) + f^{(n+1)}(c_x)\frac{(x - x_0)^{n+1}}{(n + 1)!}.$$

Let c_x be the point where $f^{(n)}$ is maximized on (x_0, x). Then

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ + f^{(n+1)}(c_x)\frac{(x - x_0)^{n+1}}{(n + 1)!}. \end{aligned} \tag{4}$$

By the inductive hypothesis, there exists a d_x between x and x_0 such that

$$f(x) = P_n^{x_0}(x) + f^{(n+1)}(d_x)\frac{(x - x_0)^{n+1}}{(n + 1)!}.$$

Since $f^{(n+1)}$ is continuous, we have

$$f^{(n+1)}(c_x) = f^{(n+1)}(d_x).$$

Therefore,

$$f(x) = P_n^{x_0}(x) + f^{(n+1)}(c_x)\frac{(x - x_0)^{n+1}}{(n + 1)!}.$$

□

We want a G such that

$$G'(c) = 0 \Leftrightarrow f^{(n+1)}(c) - (n + 1)!\frac{f(x) - P_n^{x_0}(x)}{(x - x_0)^{n+1}} = 0.$$

So let

$$G(t) = f^{(n)}(t) - t(n + 1)!\frac{f(x) - P_n^{x_0}(x)}{(x - x_0)^{n+1}}.$$

However, this G does not satisfy the condition of Rolle's theorem on $[x, x_0]$ or $[x_0, x]$.

Example. $f(x) = \cos x$ with $x_0 = 0$.

$$\begin{aligned} P_2^0(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &= 1 + 0 + \frac{-1}{2}x^2 \\ &= 1 - \frac{x^2}{2} \end{aligned}$$

Since cos is infinitely differentiable on $\mathbb{R}$, we have for $n = 2$,

$$\cos x = 1 - \frac{x^2}{2} + \frac{\sin(c)}{3!}x^3$$

for some c between 0 and x.

$$\cos x - (1 - x^2/2) = \frac{\sin(c)}{6}x^3 \begin{cases} > 0 & x \in (0, \pi) \\ > 0 & x \in (-\pi, 0) \end{cases}$$

Remarks. We will define e^x and $\log x$ rigorously later.

Remarks. We will also *talk about* x^r where r is irrational rigorously. We have defined x^n as well as $x^{1/q}$, which lets us define x to any rational power.

One way to define this is using limit of rational sequences converging to some irrational number. This is natural when defining $\mathbb{R}$ using Cauchy sequence a la Tao.

We can also define it as the supremum of $x^{p/q}$ where p/q is less that r. This is natural when defining $\mathbb{R}$ using Dedekind cuts ala Rudin.

Remarks. We have not defined sine and cosine properly, but we know how they work. Section 2.5 in Apostol outlines some 'rules' for sine and cosine. They derive ALL trigonometric identities from these. However, this is not a constructive definition.

Proof. □

Assignment 8

due never

Problem 7

We define a function $f : \mathbb{R} \to \mathbb{R}$ to be 'close' at a point $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta > 0$ such that for every $x \in N_\delta(c) \setminus \{c\}$, we have that

$$|f(x) - L| < \varepsilon.$$

We define a sequence $\{a_n\} \subset \mathbb{R}$ to be 'close' if for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have that

$$|a_n - L| < \varepsilon.$$

We will prove that if a function $f : \mathbb{R} \to \mathbb{R}$ is close at $c \in \mathbb{R}$, then it admits a limit at c. Here is the outline of the proof.

- Prove that a close sequence is a Cauchy sequence, and therefore convergent.
- Show that if $\{a_n\} \subset \mathbb{R} \setminus \{c\}$ converges to c, then $\{f(a_n)\}$ is a close sequence and thus convergent.
- Show that if two sequences $\{a_n\}, \{b_n\} \subset \mathbb{R} \setminus \{c\}$ converge to c, then $\{f(a_n)\}$ and $\{f(b_n)\}$ have the same limit.
- Use sequential characterization of limits to conclude f has a limit at c.

We first prove that a sequence is close iff it is Cauchy.

(a) Suppose that $\{a_n\}$ is a Cauchy sequence. Then for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have that

$$|a_n - a_m| < \varepsilon \; \forall \; m, n \geq n_0.$$

Let $L = a_{n_0}$. Then for every $n \geq n_0$, we have

$$|a_n - L| < \varepsilon$$

as desired.

(b) Suppose that $\{a_n\}$ is a close sequence. Then for every $\varepsilon > 0$ there exists an $L \in \mathbb{R}$ and an $n_0 \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon/2 \; \forall \; n \geq n_0.$$

Thus for any $m, n > n_0$,

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \varepsilon$$

as desired.

For sequences of real numbers, Cauchy $\Leftrightarrow$ convergent.

Now suppose that $f : \mathbb{R} \to \mathbb{R}$ is close at $c \in \mathbb{R}$. Then for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta_\varepsilon > 0$ such that for every $x \in N_{\delta_\varepsilon}(c) \setminus \{c\}$, we have

$$|f(x) - L| < \varepsilon.$$

Consider a sequence $\{a_n\} \subset \mathbb{R} \setminus \{c\}$ converging to c. Then there exists an n_0 such that for every $n \geq n_0$, we have

$$|a_n - c| < \delta_\varepsilon \Rightarrow |f(a_n) - L| < \varepsilon.$$

Thus $\{f(a_n)\}$ is close, and therefore convergent.

Suppose sequences $\{a_n\}$ and $\{b_n\}$ both converge to c (but never equal it), with $\{f(a_n)\}$ and $\{f(b_n)\}$ having different limits L_1 and L_2. Consider the sequence

$$c_n = \begin{cases} a_n & \text{if } n \text{ is even} \\ b_n & \text{if } n \text{ is odd} \end{cases}$$

Clearly $\{c_n\}$ converges to c, but $\{f(c_n)\}$ diverges. This is a contradiction. Thus there exists a unique $L_0 \in \mathbb{R}$ such that given any sequence $\{a_n\}$ converging to c, we have that

$$\lim_{n\to\infty} f(a_n) = L_0.$$

By the sequential characterization of limits, the limit of f at c exists.

On the other hand, if it is known that f has a limit L_0 at c, setting $L = L_0 \; \forall \, \varepsilon$ proves that f is close at c.

Thus the two definitions are equivalent.

Lecture 26: Integration

mon 26 dec '22

5 Integration

We always integrate functions on closed intervals $[a, b]$.

Definition 5.1 (Partition)**.** A *partition* of $[a, b]$ is a finite subset

$$P = \{x_0, x_1, \dots x_n\} \subseteq [a, b]$$

such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We write

$$P = \{x_0 < x_1 < \cdots < x_n\}.$$

Definition 5.2 (Refinement)**.** Given two partitions P and Q of $[a, b]$, Q is said to be a *refinement* of P if $P \subseteq Q$.

Definition 5.3 (Common refinement)**.** Given two partitions P and Q of $[a, b]$, the *common refinement* of P and Q is the smallest refinement of both P and Q simultaneously. Thus,

$$R = P \cup Q$$

is the common refinement of P and Q.

Definition 5.4 (Step function)**.** Given an interval $[a, b]$, a function $S : [a, b] \to \mathbb{R}$ is called a *step function* is there is some partition $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$ such that for each $j \in [1..n]$, $\exists \, s_j \in \mathbb{R}$ such that

$$s(x) = s_j \quad \forall \, x \in (x_{j-1}, x_j).$$

Remarks. The same function may be a step function corresponding to multiple partitions. For example,

$$s(x) = \lfloor x \rfloor \; \forall \, x \in [0, 100]$$

is a step function corresponding to both

$$P = \{0, 1, 2, \ldots, 99, 100\}$$

and

$$Q = \{0, 0.5, 1, \ldots 99.5, 100\}.$$

Example. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 1 & \text{if } x \in \mathbb{Q}' \cap [0,1] \end{cases}$$

Since there is at least one rational and one irrational number in every open interval, f is not a step function.

Remarks. Given any two step functions $s, t : [a,b] \to \mathbb{R}$,

(a) $s \pm t$

(b) $s \cdot t$

(c) s/t (if t is non-zero for all $x \in [a,b]$)

are also step functions on $[a,b]$. Prove this.

Definition 5.5 (Step Integration)**.** Given a step function $s : [a,b] \to \mathbb{R}$ corresponding to $P = \{x_0 < x_1 < \cdots < x_n\}$, define

$$\int_a^b s(x)\,\mathrm{d}x = \sum_{j=1}^{n} s_j(x_j - x_{j-1}).$$

We also define

$$\int_b^a s(x)\,\mathrm{d}x = -\int_a^b s(x)\,\mathrm{d}x.$$

Lecture 27

wed 28 dec '22

Remarks. This definition is meaningful if and only if:

If $s : [a,b] \to \mathbb{R}$ is a step function corresponding to $P = \{x_0 < x_1 < \cdots < x_m\}$ as well as $Q = \{y_0 < y_1 < \cdots < y_n\}$, then

$$\sum_{j=1}^{m} s_j(x_j - x_{j-1}) = \sum_{j=1}^{n} s_j(y_j - y_{j-1}).$$

Theorem 5.6 (Properties)**.**

(a) $\int_a^b (c_1 s(x) + c_2 t(x))\,\mathrm{d}x = c_1 \int_a^b s(x)\,\mathrm{d}x + c_2 \int_a^b t(x)\,\mathrm{d}x$.

(b) If $s \leq t$ on $[a, b]$, then $\int_a^b s(x)\,\mathrm{d}x \leq \int_a^b t(x)\,\mathrm{d}x$.

(c) $\int_{ka}^{kb} s(x/k)\,\mathrm{d}x = k \int_a^b s(x)\,\mathrm{d}x$.

Definition 5.7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let

$$S_f = \{s : [a, b] \to \mathbb{R} : s \text{ is a step function and } s \leq f \text{ on } [a, b]\}$$

and

$$T_f = \{t : [a, b] \to \mathbb{R} : t \text{ is a step function and } t \geq f \text{ on } [a, b]\}.$$

Lemma 5.8. Let $f : [a, b] \to \mathbb{R}$ be bounded, *i.e.*, $\exists\, M > 0$ such that

$$-M \leq f(x) \leq M \;\forall\, x \in [a, b].$$

Then $\sup s_f$ and $\inf t_f$ exist and

$$-M(b - a) \leq \sup s_f \leq \inf t_f \leq M(b - a)$$

where $s_f = \left\{\int_a^b s(x)\,\mathrm{d}x : s \in S_f\right\}$ and $t_f = \left\{\int_a^b t(x)\,\mathrm{d}x : t \in T_f\right\}$.

Proof. Let $g(x) = -M \;\forall\, x \in [a, b]$ and $h(x) = M \;\forall\, x \in [a, b]$.

Note:

(a) $g \in S_f$ and $h \in T_f$

(b) $g \leq t \;\forall\, t \in T_f \Rightarrow -M(b - a) = \int_a^b g(x)\,\mathrm{d}x \leq \int_a^b t(x)\,\mathrm{d}x \;\forall\, t \in T_f$.

(c) $s \leq h \;\forall\, s \in S_f \Rightarrow \int_a^b s(x)\,\mathrm{d}x \leq \int_a^b h(x)\,\mathrm{d}x = M(b - a) \;\forall\, s \in S_f$.

Thus s_f and t_f exist and

$$-M(b - a) \leq s_f \qquad M(b - a) \geq t_f.$$

For all $s \in S_f$ and $t \in T_f$,

$$s \leq f \leq t$$

and thus

$$\int_a^b s(x)\,\mathrm{d}x \leq \int_a^b t(x)\,\mathrm{d}x$$
$$\Rightarrow \int_a^b s(x)\,\mathrm{d}x \leq t_f$$
$$\Rightarrow s_f \leq t_f.$$

□

Definition 5.9. Given a bounded function $f : [a, b] \to \mathbb{R}$, its *lower integral* is

$$\underline{I}(f) = \sup\left\{\int_a^b s(x)\,\mathrm{d}x : s \in S_f\right\}$$

and its *upper integral* is

$$\bar{I}(f) = \inf\left\{\int_a^b s(x)\,\mathrm{d}x : s \in T_f\right\}.$$

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* (not really) if $\underline{I}(f) = \bar{I}(f)$ and we call this quantity the integral of f over $[a, b]$, denoted by

$$\int_a^b f(x)\,\mathrm{d}x.$$

We also define

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x.$$

Assignment 9

due Thu 12 Jan 2023

Problem 1

Let s be a step function on partition $\mathcal{P} = \{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$. Suppose $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Let $\mathcal{W} = \mathcal{P}' \setminus \mathcal{P} = \{y_1 < y_2 < \cdots < y_m\}$. Let $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_{k+1} = \mathcal{P}_k \cup \{y_{k+1}\}$ Then $\mathcal{P}_m = \mathcal{P} \cup \mathcal{W} = \mathcal{P}'$.

We know that s is a step function on $\mathcal{P}$ with some integral I. Suppose inductively that s is a step function on $\mathcal{P}_k = \{z_0 < z_1 < \cdots < z_{n+k}\}$ with integral I. Since $z_0 = a < y_{k+1} < b = z_n$ and $y_{k+1} \notin \mathcal{P}_k$, we have $z_{j-1} < y_{k+1} < z_j$ for some $j \in \mathbb{N} \cap [1, n+k]$.

$$\mathcal{P}_{k+1} = \{z_0 < z_1 < \cdots < z_{j-1} < y_{k+1} < z_j < \cdots < z_{n+k}\}.$$

Since s is a step function on $\mathcal{P}_k$, $\exists\ c_i \in \mathbb{R}$ such that $s(x) = c_i\ \forall\ x \in (z_{i-1}, z_i)$. Thus $s(x) = c_j\ \forall\ x \in (z_{j-1}, y_{k+1})$ and $s(x) = c_j\ \forall\ x \in (y_{k+1}, z_j)$. Thus s is constant on all $(z_{i-1}, z_i), i \neq j$ as well as (z_{j-1}, y_{k+1}) and $(y_{k+1}, z_j) \Rightarrow s$ is a step function on $\mathcal{P}_{k+1}$.

Moreover,

$$
\begin{aligned}
\int_{\mathcal{P}_{k+1}} s(x)\,\mathrm{d}x &= \sum_{i=1}^{j-1} c_i(z_i - z_{i-1}) + c_j(y_{k+1} - z_{j-1}) + c_j(z_j - y_{k+1}) + \sum_{i=j+1}^{n+k} c_i(z_i - z_{i-1}) \\
&= \sum_{i=1}^{j-1} c_i(z_i - z_{i-1}) + c_j(z_j - z_{j-1}) + \sum_{i=j+1}^{n+k} c_i(z_i - z_{i-1}) \\
&= \sum_{i=1}^{n+k} c_i(z_i - z_{i-1}) \\
&= \int_{\mathcal{P}_k} s(x)\,\mathrm{d}x \\
&= I.
\end{aligned}
$$

Thus by induction, s is a step function on $\mathcal{P}'$ with integral I.

Taking the common refinement $\mathcal{R}$ of $\mathcal{P}$ and $\mathcal{Q}$ yields $\int_{\mathcal{P}} s(x)\,\mathrm{d}x = \int_{\mathcal{R}} s(x)\,\mathrm{d}x = \int_{\mathcal{Q}} s(x)\,\mathrm{d}x$.

Problem 2

(a)

$$
\left\lfloor x - \frac{1}{2} \right\rfloor + \lfloor x \rfloor = \begin{cases} -3 & x \in [-1, -\frac{1}{2}) \\ -2 & x \in [-\frac{1}{2}, 0) \\ -1 & x \in [0, \frac{1}{2}) \\ 0 & x \in [\frac{1}{2}, 1) \\ 1 & x \in [1, \frac{3}{2}) \\ 2 & x \in [\frac{3}{2}, 2) \\ 3 & x = 2 \end{cases}
$$

Thus

$$
\int_{-1}^{2} \left(\left\lfloor x - \frac{1}{2} \right\rfloor + \lfloor x \rfloor \right) \mathrm{d}x = -3 \cdot \frac{1}{2} + (-2) \cdot \frac{1}{2} + \cdots + 2 \cdot \frac{1}{2} = -\frac{3}{2}.
$$

(b)

$$
\lfloor \sqrt{x} \rfloor = \begin{cases} 1 & x \in [1, 4) \\ 2 & x \in [4, 9) \\ 3 & x = 9 \end{cases}
$$

Thus

$$
\int_{1}^{9} \lfloor \sqrt{x} \rfloor \,\mathrm{d}x = 1 \cdot 3 + 2 \cdot 5 = 13.
$$

Problem 3

Given a step function f on $[a, b]$, we have

$$S_f = \left\{ \int_a^b s(x)\,\mathrm{d}x : s \text{ is a step function and } s \leq f \text{ on } [a, b] \right\}.$$

For any step function $s \leq f$, we have $\int_a^b s(x)\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x$ (defined as sum of $f_j(x_j - x_{j-1})$.

Thus $\int_a^b f(x)\,\mathrm{d}x$ is an upper bound of S_f. Moreover, since f is a step function and $f \leq f$ on $[a, b]$, $\int_a^b f(x)\,\mathrm{d}x \in S_f$. Therefore, $\sup S_f = \int_a^b f(x)\,\mathrm{d}x$.

Similarly, $\inf T_f = \int_a^b f(x)\,\mathrm{d}x$ and so the two definitions are concurrent.

Alternatively,

$$\int_a^b s(x)\,\mathrm{d}x \leq \sup S_f \leq \inf T_f \leq \int_a^b t(x)\,\mathrm{d}x.$$

Since $f \leq f$ and $f \geq f$, we can let $s = f$ and $t = f$. So

$$\int_a^b f(x)\,\mathrm{d}x \leq \sup S_f \leq \inf T_f \leq \int_a^b f(x)\,\mathrm{d}x.$$

Thus $\sup S_f = \inf T_f = \int_a^b f(x)\,\mathrm{d}x$.

Problem 4

Suppose f is not Riemann integrable on $[c, d]$. Then $\underline{I}_{[c,d]} \neq \bar{I}_{[c,d]} \Rightarrow \exists\, \varepsilon > 0$ such that $\int_c^d t_{[c,d]}(x)\,\mathrm{d}x - \int_c^d s_{[c,d]}(x)\,\mathrm{d}x > \varepsilon$ for all step functions $s_{[c,d]}, t_{[c,d]} : [c, d] \to \mathbb{R}$ such that $s_{[c,d]} \leq f \leq t_{[c,d]}$ on $[c, d]$.

Now suppose $\hat{s}_{[a,b]}, \hat{t}_{[a,b]} : [a, b] \to \mathbb{R}$ are step functions such that $\hat{s}_{[a,b]} \leq f \leq \hat{t}_{[a,b]}$ on $[a, b]$.

Problem 5

By the expansion property,

$$\int_{-0}^{-a} f(-x)\,\mathrm{d}x = -\int_0^a f(x)\,\mathrm{d}x.$$

If f is even, then $f(-x) = f(x)$ and so

$$\int_{-0}^{-a} f(x)\,\mathrm{d}x = -\int_0^a f(x)\,\mathrm{d}x.$$

So $\int_{-a}^{0} f(x)\,\mathrm{d}x = \int_{0}^{a} f(x)\,\mathrm{d}x$ and thus

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$$

Similarly if f is odd, then $f(-x) = -f(x)$ and so

$$\int_{-0}^{-a} f(x)\,\mathrm{d}x = \int_{0}^{a} f(x)\,\mathrm{d}x.$$

So $\int_{-a}^{0} f(x)\,\mathrm{d}x = -\int_{0}^{a} f(x)\,\mathrm{d}x$ and thus

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0.$$

Problem 6

Suppose f attains a value $y > 0$ at some x_0. Since f is continuous, there exists a neighborhood $N_\delta(x_0)$ with $0 < \delta < \min\{b - x_0, x_0 - a\}$ of size 2δ on which $f > y/2$. Thus $\int_{x_0-\delta}^{x_0+\delta} f(x)\,\mathrm{d}x > y\delta > 0$.

Since $f \geq 0$, we have $\int_{a}^{x_0-\delta} f(x)\,\mathrm{d}x \geq 0$ and $\int_{x_0+\delta}^{b} f(x)\,\mathrm{d}x \geq 0$. Thus

$$\int_{a}^{b} f(x)\,\mathrm{d}x > 0.$$

This is a contradiction, and thus $f(x) = 0 \;\forall\; x \in [a, b]$.

Problem 7

Suppose f is Riemann integrable on $[a, b]$ per the lecture definition. Then $\sup S_f = \inf T_f = I$. Since I is the supremum of S_f, $I - \varepsilon$ is not an upper bound of S_f, and thus there exists an $s_\varepsilon \leq f$ such that $\int_a^b s_\varepsilon(x)\,\mathrm{d}x > I - \varepsilon$.

Similarly there exists a $t_\varepsilon \geq f$ such that $\int_a^b t_\varepsilon(x)\,\mathrm{d}x$. Thus f is Riemann integrable as per the given definition.

Now suppose f is Riemann integrable on $[a, b]$ per the given definition. Defining S_f and T_f as before, we have

$$\int_a^b s(x)\,\mathrm{d}x \leq \sup S_f \leq \inf T_f \leq \int_a^b t(x)\,\mathrm{d}x.$$

Suppose $\int_a^b s(x)\,\mathrm{d}x = I + \varepsilon_0 > I$. Then there exists a t_{ε_0} such that $\int_a^b t_{\varepsilon_0}(x)\,\mathrm{d}x < I + \varepsilon_0 = \int_a^b s(x)\,\mathrm{d}x$. Contradiction. Thus $\int_a^b s(x)\,\mathrm{d}x \leq I$. Also, since $I - \varepsilon$ is not an upper bound of S_f for any $\varepsilon > 0$, I is the least upper bound of S_f.

Similarly, I is the greatest lower bound of T_f. Thus f is Riemann integrable per the lecture definition.

Lecture 28

thu 29 dec '22

Theorem 5.10 (Monotone Integrable)**.** Every bounded monotone function on $[a, b]$ is Riemann integrable on $[a, b]$.

Proof. Assume WLOG that $f : [a, b] \to \mathbb{R}$ is increasing. Let

$$P_n = \{x_0 < x_1 < \cdots < x_n\} \quad n \in \mathbb{P}$$

where $x_j = a + j \cdot \frac{b-a}{n}$. Set

$$s_n(x) = \begin{cases} f(x_{j-1}) & x \in [x_{j-1}, x_j] \\ f(b) & x = b \end{cases} \qquad t_n(x) = \begin{cases} f(x_j) & x \in [x_{j-1}, x_j) \\ f(b) & x = b \end{cases}$$

Note that $s_n \leq f \Rightarrow s \in S_f$ and $t_n \geq f \Rightarrow t \in T_f$.

$$\int_a^b s_n(x)\,\mathrm{d}x = \frac{b-a}{n}\sum_{j=1}^{n} f(x_{j-1}) \qquad \int_a^b t_n(x)\,\mathrm{d} = \frac{b-a}{n}\sum_{j=1}^{n} f(x_j)$$

Note that

$$\int_a^b s_n(x)\,\mathrm{d}x \leq \underline{I}(f) \leq \bar{I}(f) \leq \int_a^b t_n(x)\,\mathrm{d}x$$

$$\Rightarrow 0 \leq \bar{I}(f) - \underline{I}(f) \leq \int_a^b t_n(x)\,\mathrm{d}x - \int_a^b s_n(x)$$

We will show that $\int_a^b t_n(x)\,\mathrm{d}x - \int_a^b s_n(x)\,\mathrm{d}x \to 0$ as $n \to 0$.

$$\int_a^b t_n(x)\,\mathrm{d}x - \int_a^b s_n(x)\,\mathrm{d}x = \frac{(b-a)(f(x_n) - f(x_0))}{n} = \frac{(b-a)(f(b) - f(a))}{n} \to 0$$

Therefore $\bar{I}(f) - \underline{I}(f) = 0$, *i.e.*, f is Riemann integrable. □

Example. Let $b > 0, p \in \mathbb{N}$. Then x^p is Riemann integrable on $[0, b]$ and

$$\int_0^b x^p\,\mathrm{d}x = \frac{b^{p+1}}{p+1}.$$

Proof. Since x^p is increasing on $[0, b]$, we have Riemann integrability on $[0, b]$. Using s_n and t_n as defined in theorem 5.10,

$$\frac{b}{n}\sum_{j=0}^{n-1}\left(\frac{bj}{n}\right)^p \leq \int_0^b x^p\,\mathrm{d}x \leq \frac{b}{n}\sum_{j=1}^{n}\left(\frac{bj}{n}\right)^p$$

$$\frac{b^{p+1}}{n^{p+1}}(1 + 2^p + \cdots + (n-1)^p) \leq \int_0^b x^p\,\mathrm{d}x \leq \frac{b^{p+1}}{n^{p+1}}(1 + 2^p + \cdots + n^p)$$

Prove:

$$1^p + 2^p + \ldots (n-1)^p < \frac{n^p}{p+1} < 1^p + 2^p + \ldots n^p.$$

Let

$$a_n = \frac{1^p + 2^p \cdots + (n-1)^p}{n^p}, \qquad b_n = \frac{1^p + 2^p + \ldots (n-1)^p + n^p}{n^p}$$

Then

$$a_n < \frac{1}{p+1} < b_n = a_n + \frac{1}{n} < \frac{1}{p+1} + \frac{1}{n}.$$

Thus by the squeeze theorem

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \frac{1}{p+1}$$

Therefore

$$\int_0^b x^p \,\mathrm{d}x = \frac{b^{p+1}}{p+1}.$$

□

Lecture 29

mon 2 jan '23

We will take the following properties of Riemann integrals for granted.

Theorem 1.16-1.20 in Apostol (Section 1.27).

- Linearity wrt the integrand.
- Additivity wrt interval of integral.
- Translation invariance.
- Expansion/contraction of the interval.
- Comparison: $f \leq g \Rightarrow \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$.

We won't assume that f, g being Riemann integrable $\Rightarrow fg$ is Riemann integrable.

Definition 5.11 (Uniform Continuity)**.** A function $f : A \to \mathbb{R}$ is said to be *uniformly continuous* if for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that whenever $x, y \in A$ and $|x - y| < \delta_\varepsilon$, then $|f(x) - f(y)| < \varepsilon$.

Remarks.

(a) Obviously this implies continuity.

(b) Uniform continuity is not a local definition. It depends on A.

(c) If f is uniformly continuous on A, then it is uniformly continuous on any subset of A.

Example.

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x$. $\delta = \varepsilon$ works for all $\varepsilon > 0$ irresepective of x.

(b) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. Let $\varepsilon = 1$. Suppose $\delta > 0$. Consider $x = p$, $y = p + \delta/2$. Then $|f(x) - f(y)| = p\delta + \frac{\delta^2}{4} > p\delta$. Choose $p = \varepsilon/\delta = \frac{1}{\delta}$. Thus $\left|p - (p + \frac{\delta}{2})\right| < \delta$ but $|f(x) - f(y)| > \varepsilon$. $\square$

Theorem 5.12 (Closed continuous $\Rightarrow$ uniformly continuous)**.** Every continuous function on a closed, bounded interval is uniformly continuous on $[a, b]$.

One can use the same technique as used for showing boundedness of continuous functions on $[a, b]$. Prove this.

Theorem 5.13 (Continuity $\Rightarrow$ Riemann Integrability)**.** Let f be a continuous function on $[a, b]$. Then f is Riemann integrable on $[a, b]$.

Idea: Squeeze $\underline{I}$ and $\bar{I}$ between $\int_a^b s_n(x)\,\mathrm{d}x$ and $\int_a^b t_n(x)\,\mathrm{d}x$.

Proof. Since f is bounded on $[a, b]$, $\underline{I}$ and $\bar{I}$ exist. By theorem 5.12, there exists a $\delta > 0$ such that whenever $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

By the Archimedean property of $\mathbb{R}$, there exists an $N \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta \;\forall\, n \geq N$. Let $P_n = \{x_0 < x_1 < \cdots < x_n\}$ where $x_j = a + \frac{j(b-a)}{n}$.

Note that $f_{[x_{j-1}, x_j]}$ attains its minimum value m_j at some $l_j \in [x_{j-1}, x_j]$ and its maximum value M_j at some $u_j \in [x_{j-1}, x_j]$.

$$f(l_j) = m_j \leq f(x) \leq M_j = f(u_j) \qquad \forall\, x \in [x_{j-1}, x_j].$$

Let, for the same n,

$$s(n) = \begin{cases} m_j & x \in [x_{j-1}, x_j) \\ f(b) & x = b \end{cases}$$

and

$$t(n) = \begin{cases} M_j & x \in [x_{j-1}, x_j) \\ f(b) & x = b \end{cases}$$

Note that

$$\int_a^b s_n(x)\,\mathrm{d}x \leq \underline{I}(f) \leq \bar{I}(f) \leq \int_a^b t_n(x)\,\mathrm{d}x$$

Thus

$$
\begin{aligned}
0 \le \underline{I}(f) - \bar{I}(f) &\le \int_a^b t_n(x)\,\mathrm{d}x - \int_a^b s_n(x)\,\mathrm{d}x \\
&= \sum_{j=1}^n M_j(x_j - x_{j-1}) - \sum_{j=1}^n m_j(x_j - x_{j-1}) \\
&= \sum_{j=1}^n (f(u_j) - f(l_j))(x_j - x_{j-1}) \\
&< \varepsilon \sum_{j=1}^n (x_j - x_{j-1}) \\
&= \varepsilon(b-a)
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, $\underline{I}(f) = \bar{I}(f)$. □

Lecture 30

wed 4 jan '22

Theorem 5.14 (Mean Value – Integrals)**.** Let f be a continuous function on $[a, b]$. Then there exists a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

Proof. By the extreme value theorem, f attains a minimum m at a and a maximum M at b. Thus

$$
\begin{aligned}
m \le f(x) &\le M \\
\int_a^b m\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x &\le \int_a^b M\,\mathrm{d}x \\
(b-a)m \le \int_a^b f(x)\,\mathrm{d}x &\le (b-a)M \\
m \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x &\le M
\end{aligned}
$$

By the IVT, there exists a number c in $[a, b]$ (or $[b, a]$) such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$ □

Theorem 5.15 (The First Fundamental Theorem of Calculus)**.** Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Let

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \qquad \forall\, x \in [a,b].$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at some $p \in (a,b)$, then F is differentiable at p with $F'(p) = f(p)$.

Remarks.

(a) If f is not continuous at some $p \in (a,b)$, then F may still be differentiable at p. For example

(i)

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Then $F(x) = 0$ is differentiable at $x = 0$.

(ii)

$$f(x) = \begin{cases} 0 & -1 \leq x0 \\ 1 & 0 \leq x \leq 1 \end{cases}$$

Then

$$F(x) = \begin{cases} 0 & -1 \leq x < 0 \\ x & 0 \leq x \leq 1 \end{cases}$$

is not differentiable at $x = 0$.

(b) $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is called an *indefinite integral* of f. (see Apostol Section 2.18)

Proof. We know that f is Riemann integrable on $[a,b]$. We show that F is continuous on $[a,b]$.

Let $x, y \in [a,b]$ be such that $x \leq y$.

$$\begin{aligned} F(y) - F(x) &= \int_a^y f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \\ &= \int_x^y f(t)\,\mathrm{d}t \end{aligned}$$

Since f is bounded on $[a,b]$, there exists $M \in \mathbb{R}$ such that $-M \leq f(x) \leq M \;\forall\, x \in [a,b]$. Thus

$$\int_x^y -M\,\mathrm{d}t \leq \int_x^y f(t)\,\mathrm{d}t \leq \int_x^y M\,\mathrm{d}t$$
$$-M(y-x) \leq F(y) - F(x) \leq M(y-x)$$

Thus

$$|F(y) - F(x)| \leq M(y-x)$$

If $y < x$, we get
$$|F(y) - F(x)| \leq M(x - y)$$
If $x = y$, we have
$$|F(y) - F(x)| = 0 \leq M(y - x)$$
Such a function is called Lipschitz function and this condition implies (*uniform*) continuity and if F is differentiable, it gives boundedness of derivative of F. (Since we have a closed interval, uniform continuity is implied by continuity, but this holds even for open or unbounded intervals).

Now assume that f is continuous at $p \in (a, b)$. Let $h \in \mathbb{R}$ such that $p + h \in (a, b)$.
$$\begin{aligned}
\frac{F(p+h) - F(p)}{h} &= \frac{1}{h}\left(\int_a^{p+h} f(t)\,\mathrm{d}t - \int_a^p f(t)\,\mathrm{d}t\right) \\
&= \frac{1}{h}\int_p^{p+h} f(t)\,\mathrm{d}t \\
&= \frac{1}{h}\int_p^{p+h} (f(t) - f(p) + f(p))\,\mathrm{d}t \\
&= f(p) + \frac{1}{h}\int_p^{p+h} (f(t) - f(p))\,\mathrm{d}t
\end{aligned}$$
Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $0 < |t - p| < \delta \Rightarrow |f(t) - f(p)| < \varepsilon$. Let $|h| < \delta$. Then
$$\begin{aligned}
\left|\frac{F(p+h) - F(p)}{h} - f(p)\right| &= \left|\frac{1}{h}\int_p^{p+h} (f(t) - f(p))\,\mathrm{d}t\right| \\
&\leq \frac{1}{h}\int_p^{p+h} |f(t) - f(p)|\,\mathrm{d}t \\
&\leq \frac{1}{h} \cdot \varepsilon h \\
&= \varepsilon
\end{aligned}$$
□

Lecture 31

fri 6 jan '23

Theorem 5.16 (Integral Triangle Inequality)**.** Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then $|f| : x \in [a, b] \mapsto |f(x)| \in \mathbb{R}$ is Riemann integrable and
$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof. □

Definition 5.17 (Primitive)**.** Given a function $f : (a,b) \to \mathbb{R}$, with $a < b$ and $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, a *primitive* or *antiderivative* of f on (a,b) is a differentiable function $F : (a,b) \to \mathbb{R}$ such that

$$F'(x) = f(x) \; \forall \; x \in (a,b).$$

Remarks (on the First FTOC)*.* We could have also set

$$G(x) = \int_b^x f(t) \, \mathrm{d}t.$$

Note $F - G = \int_a^x f(t) \, \mathrm{d}t - \int_b^x f(t) \, \mathrm{d}t = \int_a^b f(t) \, \mathrm{d}t$ which is a constant. Thus continuity and differentiability of F transfers to G.

Remarks.

(a) Let f, F be as in theorem 5.15. If f is continuous, then F is a primitive of f and

(b) In general, if F and G are primitives of f, then $F' - G' = 0 \Rightarrow F = G + C$, where C is a constant.

Example.

(a) sin and cos are primitives of cos and $-\sin$ on all of $\mathbb{R}$.

(b)
$$\frac{x^{\gamma+1}}{\gamma+1} \qquad \gamma \in \mathbb{R} \setminus \{-1\}, x \in \mathbb{R}^+$$
is a primitive of x^γ on $\mathbb{R}^+$.

(c) Let
$$L(x) = \int_1^x \frac{1}{t} \, \mathrm{d}t \qquad x > 0.$$
By the first fundamental theorem of calculus, L is a primitive of $\dfrac{1}{x}$.

Theorem 5.18 (The Second Fundamental Theorem of Calculus)**.** Let $f : (c,d) \to \mathbb{R}$ be a function such that $f_{[a,b]}$ $([a,b] \subset (c,d))$ is Riemann integrable. Let F be a primitive of f on (c,d). Then

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a).$$

Lecture 33: Logarithms & Exponentiation

11 jan '23

5.1 Logarithms & Exponentiation

Goal: for any $b > 0$, we wish to define "$\log_b(x)$" as *the* real number z which satisfies: $b^z = x$.
There are multiple caveats:

(a) However, we don't even know what b^z means for irrational z.

(b) Why should there exist exactly one z, and for what values of x?

Heuristics: Suppose $z =$ "$\log_b(x)''$ and $w =$ "$\log_b(y)''$, *i.e.*,

$$b^z = x \qquad b^w = y.$$

Then $b^{z+w} = x + y$. Thus we expect

$$\log_b(x) + \log_b(y) = \log_b(xy)\ \forall\ x, y \in \mathrm{dom}(\log) \tag{5}$$

We will call this function l for the moment. Suppose $0 \in \mathrm{dom}(\log)$. Then taking $x = 0$ in the above equation, we get

$$l(0) + l(y) = l(0y) = l(0)\ \forall\ y \in \mathrm{dom}(\log).$$

Thus $l(y) = 0$ for all $y \in \mathrm{dom}(\log)$.

Thus we will exclude 0 from the domain. Let's calculate $l(1)$.

$$l(1) + l(1) = l(1) = 0.$$

Since $(-x)(-x) = (x)(x)$, we get $l(-x) + l(-x) = l(x) + l(x) \Rightarrow$

$$l(-x) = l(x).$$

Now, differentiating the functional equation with respect to x,

$$\begin{aligned} l(xy) &= l(x) + l(y) \\ yl'(xy) &= l'(x) \\ l'(y) &= \frac{l'(1)}{y} \end{aligned}$$

By the second fundamental theorem of calculus,

$$l(y) = \begin{cases} l(y) - l(1) = \int_1^y \frac{l'(1)}{x}\,\mathrm{d}x & y > 0 \\ l(y) - l(-1) = \int_{-1}^y \frac{l'(1)}{x}\,\mathrm{d}x \end{cases}$$

Definition 5.19 (Natural Logarithm)**.** Let $x > 0$. The *natural logarithm* of x is the quantity

$$\ln(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

Theorem 5.20. The function $\ln : \mathbb{R}^+ \to \mathbb{R}$ has the following properties:

(a) $\ln(1) = 0$.

(b) $\ln(x) + \ln(y) = \ln(xy)\ \forall\, x, y \in \mathbb{R}^+$.

(c) ln is continuous and strictly increasing.

(d) ln is differentiable and

$$\ln'(x) = \frac{1}{x}.$$

(e) (Leibniz)

$$\int \frac{1}{t}\,\mathrm{d}t = \ln|t| + C.$$

(f) (Leibniz)

$$\int \ln x\,\mathrm{d}x = x\ln x - x + C.$$

(g) ln is bijective.

Proof.

(a) $\ln(1) = \int_1^1 \frac{1}{t}\,\mathrm{d}t = 0$.

(b) Suppose $x, y > 0$. Then

$$\begin{aligned}
\ln(\frac{1}{x}) &= \int_1^{\frac{1}{x}} \frac{1}{t}\,\mathrm{d}t \\
&= \frac{1}{x}\int_x^1 \frac{x}{t}\,\mathrm{d}t \\
&= -\int_1^x \frac{1}{t}\,\mathrm{d}t \\
&= -\ln(x).
\end{aligned}$$

□

Definition 5.21 (e & Exponentiation)**.** Let e be the unique number that satisfies

$$\ln(e) = 1.$$

Given any $x \in \mathbb{R}$, let $\exp(x)$ be the unique positive y such that

$$\ln(y) = x.$$

That is, exp is the inverse function of ln.

Hard exercise: Prove that this definition of e is identical to the infinite series definiton.

Theorem 5.22. $\exp : \mathbb{R} \to \mathbb{R}^+$ has the following properties:

(a) $\exp(0) = 1$.

(b) $\exp(x + y) = \exp(x)\exp(y)$.

(c) exp is continuous and strictly increasing.

(d) exp is differentiable and

$$\exp'(x) = \exp(x) \;\forall\, x \in \mathbb{R}.$$

(e)

$$\int \exp(x)\,\mathrm{d}x = \exp(x) + C.$$

(f) exp is bijective.

(g) $\exp(r) = e^r \;\forall\, r \in \mathbb{Q}$.

Lecture 34

Fri 13 Jan '23

Part II

Linear Algebra

Linear functions:

$$f(cx) = cf(x)$$
$$f(x + y) = f(x) + f(y)$$

Example (Error Detection/Correction). We wish to transmit a message $m \in \{0, 1\}^2$

over a noisy channel. Say the words have meanings:

$$\begin{aligned}(0,0) &\mapsto \text{up}\\(0,1) &\mapsto \text{down}\\(1,0) &\mapsto \text{left}\\(1,1) &\mapsto \text{right}\end{aligned}$$

If one bit gets flipped, 'left' may become 'right' or 'up'. We can detect whether a bit has been flipped by sending a parity bit.

$$\begin{aligned}(0,0,0) &\mapsto \text{up}\\(0,1,1) &\mapsto \text{down}\\(1,0,1) &\mapsto \text{left}\\(1,1,0) &\mapsto \text{right}\end{aligned}$$

6 Vector Spaces

Recall that a field $(F, \oplus, \odot)$ is a set with two binary operations $\oplus$ and $\odot$ that satisfy 6 axioms Key example: $(\mathbb{R}, +, \cdot)$.

Let us test this definition on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (the real plane), with the operations:

$$\begin{aligned}(a,b) \oplus (c,d) &= (a+c, b+d)\\(a,b) \odot (c,d) &= (a \cdot c, b \cdot d) \ \forall\ (a,b),(c,d) \in \mathbb{R}^2\end{aligned}$$

(a) It is clear that $\oplus$ and $\odot$ are commutative, associative, and distributive.

(b) $(0,0)$ is the unique identity and $(-a,-b)$ is the unique additive inverse of (a,b).

(c) $(1,1)$ is the unique multiplicative identity and $(\frac{1}{a}, \frac{1}{b})$ is the unique multiplicative inverse of (a,b), for $a \neq 0, b \neq 0$. $(1,0)$ does not have a multiplicative inverse, despite being nonzero.

   Proof. Suppose $(a,b) \in \mathbb{R}^2$ is a multiplicative inverse of $(1,0)$. Then

$$\begin{aligned}(a,b) \odot (1,0) &= (a \cdot 1, b \cdot 0)\\&= (a,0)\\&= (1,1)\end{aligned}$$

   Thus $0 = 1$, which is a contradiction. □

Thus these operators don't make $\mathbb{R}^2$ a field. We define (formally)

$$(a,b) = a + ib$$

and define

$$\begin{aligned}(a+ib) \oplus (c+id) &= (a+c) + i(b+d)\\(a+ib) \odot (c+id) &= (ac-bd) + i(ad+bc)\end{aligned}$$

This is a field! The identities are

$$0 := 0 + i0 \quad \text{and} \quad 1 := 1 + i0$$

Note that for real numbers a, b,

$$a \leftrightarrow a + i0$$
$$ib \leftrightarrow 0 + ib$$

Note that

$$(i \cdot 1)^2 = (0 + i) \odot (0 + i) = -1 + i0 = -1$$

We denote $\mathbb{R}^2$ with this structure by $\mathbb{C}$ and call any $a + ib$ a complex number.

Definition 6.1. Let $(F, \oplus, \odot)$ be a field. A *vector space over F* is a set V such that:

(a) Given any two elements $v, w \in V$, there exists a unique element $v + w \in V$ called its sum. (This $+$ may not be the same as $\oplus$).

(b) Given an $a \in F$ and a $v \in V$, there is a unique element $av = a \cdot v \in V$ called the scalar product of a and v.

satisfying the following axioms:

(V1) $v + w = w + v$ for all $v, w \in V$.

(V2) $(v + w) + u = v + (w + u)$ for all $v, w, u \in V$.

(V3) There is an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

(V4) For all $v \in V$, there is a unique element $-v \in V$ called the additive inverse of v such that $v + (-v) = 0$.

(V5) For all $a, b \in F$ and $v \in V$, we have

$$(a \odot b) \cdot v = a \cdot (b \cdot v)$$

Note that this implies $a \cdot (b \cdot v) = b \cdot (a \cdot v)$ by the commutativity of $\odot$.

(V6) Let 1_F be the multiplicative identity of F. Then,

$$1_F \cdot v = v \quad \text{for all} \quad v \in V$$

(V7) For all $a, b \in F$ and $v \in V$, we have

$$(a \oplus b) \cdot v = a \cdot v + b \cdot v$$

(V8) For all $a \in F$ and $v, w \in V$, we have

$$a \cdot (v + w) = a \cdot v + a \cdot w$$

We call the elements of F *scalars* and the elements of V *vectors*.

Example (Key Example). $F = (\mathbb{R}, +, \cdot)$ or *real vector spaces.*

Lecture 35

Mon 16 Jan '23

Remarks. We will notate vector spaces as follows:

$$(V, +, \cdot) \text{ over } (F, \oplus, \odot)$$

where V is the vector space with $+$ vector addition, $(F, \oplus, \odot)$ is the field, and $\cdot$ is scalar multiplication.

The addive identity is unique, as $e_1 = e_1 + e_2 = e_2$.

Example. (a) $(\mathbb{R}, +, \cdot)$ is a vector space over $(\mathbb{R}, +, \cdot)$. In fact, *any* field is a vector space over itself.

(b) $(\mathbb{C}, +, \cdot)$ is a vector space over $(\mathbb{R}, +, \cdot)$.

Proof.

(V1) $(a + ib) + (c + id) = (a + c) + i(b + d)$.

(V2) $((a + ib) + (c + id)) + (e + if) = (a + c + e) + i(b + d + f) = (a + ib) + ((c + id) + (e + if))$.

(V3) $0 + i0 \in \mathbb{C}$ is an additive identity.

(V4) $-a + i(-b)$ is an additive inverse of $a + ib$.

(V5) $(a \cdot b) \cdot (c + id) = (abc) + i(abd) = a \cdot (b \cdot (c + id))$.

(V6) $1 \cdot (a + ib) = a + ib$.

(V7) $(a + b) \cdot (c + id) = (ac + bc) + i(ad + bd) = a \cdot (c + id) + b \cdot (c + id)$.

(V8) $a \cdot ((b + ic) + (d + ie)) = (ab + ad) + i(ac + ae) = a \cdot (b + ic) + a \cdot (d + ie)$. $\square$

(c) $(\mathbb{R}, +, \cdot)$ is a vector space over $(\mathbb{Q}, +, \cdot)$ with

$$r \cdot v = rv \in \mathbb{R} \ \forall \ r \in \mathbb{Q}, v \in \mathbb{R}.$$

(d) In general, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, with addition given by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and scalar multiplication given by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

for $\lambda \in \mathbb{R}$.

(e) The set of all sequences in $\mathbb{R}$ is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined as for $\mathbb{R}^n$.

(f) The set of all polynomials with real coefficients, $\mathcal{P}$, is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined as they usually are for functions.

$$\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1 x + \cdots + a_m x^m, m \in \mathbb{N}, (a_0, a_1, \ldots, a_m) \in \mathbb{R}^n\}$$

(g) Let $\mathcal{P}_d$ be the set of polynomials (over $\mathbb{R}$) with degree *at most* d. Then $\mathcal{P}_d$ is a vector space over $\mathbb{R}$.

(h) The set of all continuous functions on a set $A \subseteq \mathbb{R}$ is a vector space over $\mathbb{R}$, denoted by $\mathscr{C}(A; \mathbb{R}) = \mathscr{C}(A)$.

(i) The set of all continuous functions on $\mathbb{R}$ (or on $[0, 1]$) that are expressible as an infinite series is a vector space over $\mathbb{R}$.

$$f(x) = \sum_{j=0}^{\infty} a_j x^j \quad \forall\, x \in \mathbb{R}.$$

Lecture 36

Wed 25 Jan '23

Proposition 6.2 (Vector properties)**.** Let V be a vector space over F. Then the following hold:

(a) V has a unique additive identity.

(b) $0_F v = 0_V$ for all $v \in V$.

(c) $a0_V = 0_V$ for all $a \in F$.

(d) Each $v \in V$ has a unique additive inverse given by $(-1_F)v$.

(e) If $av = aw$ for some $a \in F \setminus \{0\}$ and $v, w \in V$, then $v = w$.

Proof.

(a) If a and b are additive identities, then $a = a + b = b$.

(b) Let $-0_F v$ be some additive inverse of $0_F v$. Then

$$\begin{aligned} 0_F v &= (0_F + 0_F)v \\ 0_F v &= 0_F v + 0_F v \\ 0_F v + (-0_F v) &= 0_F v + 0_F v + (-0_F v) \\ 0_V &= 0_F v \end{aligned}$$

(c) Let $-a0_V$ be some additive inverse of $a0_V$. Then

$$\begin{aligned} a0_V &= a(0_V + 0_V) \\ a0_V &= a0_V + a0_V \\ a0_V + (-a0_V) &= a0_V + a0_V + (-a0_V) \\ 0_V &= a0_V \end{aligned}$$

(d) Let $-v$ and $-v'$ be two additive inverses of $v \in V$. Then $(-v) = v + (-v') + (-v) = (-v')$.

Also,

$$\begin{aligned}(1_F + -1_F)v &= 1_F v + (-1_F)v \\ 0_F v &= 1_F v + (-1_F)v \\ 0_V &= v + (-1_F)v\end{aligned}$$

Thus $(-1_F)v$ is the additive inverse of v.

(e) Let $av = aw$ for some $a \in F \setminus \{0\}$ and $v, w \in V$. Then

$$\begin{aligned}av &= aw \\ a^{-1}av &= a^{-1}aw \\ 1_F v &= 1_F w \\ v &= w\end{aligned}$$

□

Definition 6.3 (Subspace)**.** Let V be a vector space over some field F. A subset $S \subseteq V$ is a (linear) *subspace* of V if the following hold:

(a) $0_V \in S$.

(b) If $v, w \in S$, then $v + w \in S$.

(c) If $a \in F$ and $v \in S$, then $av \in S$.

These properties together imply that S is also a vector space over F.

S is said to be a *proper* subspace of V if $S \neq V$ but also $S \neq \{0_V\}$.

Example. (a) Any line passing through the origin is a subspace of $\mathbb{R}^2$.

(b) Each P_d is a subspace of P.

(c) $S = \{f \in \mathscr{C}([a, b]) : f(a) = 0\}$.

(d) The space of solutions (as n-tuples) to a homogeneous system of m linear equations in n variables is a subspace of $\mathbb{R}^n$.

Definition 6.4 (Span of finite sequences)**.** Let $v_1, v_2, \ldots, v_m \in V$ be a finite sequence of vectors. A linear combination of $v_1, v_2, \ldots, v_m$ is any vector of the form

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m,$$

where $a_1, a_2, \ldots, a_m \in F$. The *span* of the finite sequence $v_1, v_2, \ldots, v_m$ is the set of all linear combinations of $v_1, v_2, \ldots, v_m$. That is,

$$\operatorname{span}(v_1, v_2, \ldots, v_m) = \left\{ \sum_{j=1}^{m} a_j v_j : a_j \in F \right\}.$$

Remarks. (a) A linear combination of $v_1, v_2, \ldots, v_m$ refers to a vector. The *expression* on the RHS of the definition is called a *representation* of v as a linear combination of $v_1, v_2, \ldots, v_m$.

For example, if $v_1 = (0,0)$ and $v_2 = (1,0)$, the vector $v = (1,0)$ is a linear combination of v_1 and v_2 with infinitely many representations.

$$\begin{aligned} v &= 0v_1 + 1v_2 \\ &= 1v_1 + 1v_2 \\ &= 100v_1 + 1v_2 \\ &= -\pi v_1 + 1v_2 \\ &\vdots \end{aligned}$$

(b) The span of a finite sequence of vectors does not depend on the order or multiplicity of the vectors. Thus we also write $\operatorname{span}(v_1, v_2, \ldots, v_m)$ as $\operatorname{span}\{v_1, v_2, \ldots, v_m\}$

$$\operatorname{span}((1,0),(0,0),(1,0)) = \operatorname{span}\{(1,0),(0,0)\}$$

Lecture 37

Wed 25 Jan '23

Definition 6.5 (Span of sets)**.** Let $S \subseteq V$ be a nonempty set. The *span* of the S is the set

$$\begin{aligned} \operatorname{span} S = \{v \in V :\ & \exists\, a_1, \ldots, a_n \in F \\ & \text{and distinct } v_1, \ldots, v_n \in S \\ & \text{such that } v = a_1 v_1 + \cdots + a_n v_n\}. \end{aligned}$$

Or

$$\operatorname{span} S = \bigcup_{\varnothing \neq \Lambda \subseteq^{\text{finite}} S} \operatorname{span} \Lambda.$$

$\operatorname{span} \varnothing$ is defined to be $\{0\}$.

Example. (a) $\operatorname{span}(0) = \{0\}$.

(b) Let $V \in \mathbb{R}^2$. Let $(a,b) \neq (0,0)$. Then $\operatorname{span}(a,b) = \{(x,y) \in \mathbb{R}^2 : ay - bx = 0\}$.

Proposition 6.6 (Spans are subspaces)**.** Let $S \subseteq V$. Then $\operatorname{span} S$ is a subspace of V. If W is a subspace of V and $S \subseteq W$, then

$$\operatorname{span} S \subseteq W.$$

Proof. □

Definition 6.7 (finite and infinite dimensions)**.** Let V be a vector space over F.

(a) V is *finite-dimensional* if there exists a finite set $S \subseteq V$ such that $V = \operatorname{span} S$.

(b) V is *infinite-dimensional* if it is not finite-dimensional.

Lecture 38

Wed 25 Jan '23

Definition 6.8 (linear independence)**.** (a) A finite sequence of vectors $v_1, \ldots, v_m$ is said to be *linearly dependent* if there exist $c_1, \ldots, c_m \in F$, not all zero, such that

$$c_1 v_1 + \cdots + c_m v_m = 0.$$

We refer to such a relation as a *nontrivial linear relation* between the vectors $v_1, \ldots, v_m$. The vectors are said to be *linearly independent* if they are not linearly dependent. That is,

$$c_1 v_1 + \cdots + c_m v_m = 0 \Leftrightarrow c_1 = \cdots = c_m = 0.$$

(b) A finite set $\{v_1, \ldots, v_m\}$ is said to be linearly independent if the finite sequence $v_1, \ldots, v_m$.

(c) A set $S \subseteq V$ is said to be linearly dependent if there exists a finite subset $L \subseteq S$ such that L is linearly dependent. Similarly, S is said to be linearly independent if L is linearly independent for all finite subsets $L \subseteq S$.

The empty set is declared to be linearly independent.

Proposition 6.9. Let $S = \{v_1, \ldots, v_m\} \subseteq V$ be a linearly dependent set. Then there is some $j \in [1..m]$ such that

$$\operatorname{span} S = \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m),$$

In particular (if $S \neq \{0\}$), v_j can be expressed as a linear combination of the other vectors in S.

Lecture 39

Wed 25 Jan '23

Definition 6.10 (Basis)**.** Given a vector space V over F, a *basis* is a subset $B \subseteq V$ such that

(a) B is a spanning set, *i.e.*, $V = \operatorname{span}(B)$.

(b) B is linearly independent.

Example. Let $V = \mathbb{R}^n$ and $B = \{e_1, e_2, \ldots, e_n\}$. Then B is a basis of V.

Remarks. If B is a basis of V, every $v \in \operatorname{span}(V)$ has a *unique* representation of the form

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for some $c_1, c_2, \ldots, c_m \in F$ and $v_1, v_2, \ldots, v_m \in B$ in the following sense:
If $v = b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$ for some $b_1, b_2, \ldots, b_m \in F$, then $b_1 = c_1, b_2 = c_2, \ldots, b_m = c_m$.

We have span of $B = V$, which gives the above representation of v.

We have that B is linearly independent, which gives uniqueness.

Proof.

$$(c_1 - b_1)v_1 + (c_2 - b_2)v_2 + \cdots + (c_m - b_m)v_m = 0$$

By the linear independence of B, we get

$$c_j = b_j \; \forall \; j \in \{1, 2, \ldots, m\}$$

□

Corollary 6.11. Let V be a finite dimensional vector space over a field F. Let S be a finite spanning set of V. Then S contains as a subset a basis of V.

Corollary 6.12. Every finite dimensional vector space has a basis.

Proof. Let S be a finite spanning set of V. Assume that $\#S = n \in \mathbb{N}$. Let $P(n)$: every spanning set of size n contains a basis.

$P(0)$ is true as the only possibility of S is $\varnothing$. $V = \operatorname{span}(\varnothing) = \{0\}$. But $\varnothing$ is linearly independent. So S itself is a basis.

Suppose $P(j)$ is true for some $j \in \mathbb{N}$. Let S be a spanning set of V of size $j + 1$. If S is linearly independent, then S is a basis. If S is linearly dependent, then there exists $S' \subseteq S$ such that $\#S' = j$ and $\operatorname{span}(S') = \operatorname{span}(S) = V$. By $P(j)$, S' contains a basis.

Thus by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Proposition 6.13. Let $L \subseteq V$ be linearly independent. Then for some $v \in V$, $L \cup \{v\}$ is linearly independent iff $v \notin \operatorname{span}(L)$.

Proof. Assume $v \notin \operatorname{span}(L)$. Suppose $L \cup \{v\}$ is linearly dependent. Thus, there exist $v_1, \dots, v_m \in L$ such that

$$c_1 v_1 + \cdots + c_m v_m + c_{m+1} v = 0$$

for some $c_1, \dots, c_{m+1} \in F$ not all zero. If c_{m+1} is zero, then L is linearly dependent, a contradiction.

If $c_{m+1} \neq 0$, then

$$v = -\frac{c_1 v_1 + \cdots + c_m v_m}{c_{m+1}}$$

is in $\operatorname{span}(L)$, a contradiction. Thus $L \cup \{v\}$ is linearly independent.

The converse is homework. □

Corollary 6.14. Let V be a finite dimensional vector space. Let $L \subseteq V$ be a finite linearly independent set. Then there exist finitely many vectors $w_1, \dots, w_m \in V$ such that $L \cup \{w_1, \dots, w_m\}$ is a basis.

Theorem 6.15. Let V be a finite dimensional vector space. Let $S, L \subseteq V$ be such that S is a *finite* spanning set and L is linearly independent. Then L is finite and

$$\#L \leq \#S.$$

Corollary 6.16. Every (finite) basis of a finite dimensional vector space has the same size.

Proof. Let $B_1, B_2 \subseteq V$ be two finite bases of V. By the above theorem,

$$\#B_1 \leq \#B_2 \qquad \#B_2 \leq \#B_1.$$

Thus $\#B_1 = \#B_2$. □

Question: Is every basis of a finite dimensional vector space finite?

Corollary 6.17. Let S, L, V be as in the previous theorem. If $\#L = \#S$, then both are bases of V.

Proof. If L is not a basis, some finite $L' \supsetneq L$ is a basis. But then

$$\#L' > \#S,$$

a contradiction. □

Lecture 40

Fri 27 Jan '23

> **Corollary 6.18** (Finite Basis of FDVS)**.** Every basis of a finite dimensional vector space is finite.

Proof. A finite dimensional vector space, by definiton, has a finite spanning set S. $\square$

We will now prove the theorem. The idea is:

Assume L is finite.

$$S = \{w_1, \ldots, w_m\}$$
$$L = \{v_1, \ldots, v_n\}$$

We will repeatedly substitute a vector in S with a vector in L such that it remains a spanning set at every iteration. We will run out of vs before we run out of ws, so we will have a $n \leq m$.

Remarks. The list $v_1, \ldots, v_k$ will be written as $(v_1, \ldots, v_k)$ (allowing repetition) v/s the set $\{v_1, \ldots, v_k\}$.

Proof.

(Case 1) L is finite. Let

$$S = \{w_1, \ldots, w_m\}$$
$$L = \{v_1, \ldots, v_n\}$$

Let $P(l)$ be that there exists for all $l < n$, a list $\tilde{S}_l$ such that $\#\tilde{S}_l = m$ containing l distinct elements from L, and the rest from S, and that $l < m$.

$P(0)$ is true: Let $0 < n$. Suppose $m = 0$. This is possible only if $V = \{0\}$ and $S = \varnothing$. In this case, $L = \varnothing$, contradicting that $\#L > 0$. Thus $P(0)$ is vacuously true.

$P(1)$ is true. Let $1 < n = \#L$. Since $\operatorname{span}(S) = V$, the list $(v_1, w_1, \ldots, w_m)$ is linearly dependent. By the linear independence of L, $v_1 \neq 0$. Thus we must have a nontrivial relation between $v_1, w_1, \ldots, w_m$ such that

$$c_0 v_1 + c_1 w_1 + \cdots + c_m w_m = 0$$

where one of the $c_j's$ must be non-zero. Assume WLOG that it is c_1. Then we can write

$$w_1 = -\frac{c_0}{c_1} v_1 - \frac{c_2}{c_1} w_2 - \cdots - \frac{c_m}{c_1} w_m$$

Let

$$\tilde{S}_1 = \{v_1, w_2, \ldots, w_m\}$$

and note that $w_1 \in \operatorname{span}(\tilde{S}_1)$. Thus $\operatorname{span}(\tilde{S}_1) = \operatorname{span}(S) = V$. TODO: Prove this.

Now if $1 < n$, then there exists a v_2 in L independent from v_1. Since $v_2 \in \text{span}(\tilde{S}_1)$, $\tilde{S}_1$ cannot be just (v_1). Thus $\#\tilde{S}_1 = m > l = 1$.

Now suppose inductively that $P(l)$ is true. If $l + 1 \geq n$, then $P(l + 1)$ is vacuously true. Thus we assume $l + 1 < n$. Since $P(l)$ is true, we have a list $\tilde{S}_l = (v_1, \ldots, v_l, w_{l+1}, \ldots, w_m)$ such that $\#\tilde{S}_l = m$ and $l < m$. Consider the list

$$(v_1, \ldots, v_l, v_{l+1}, w_{l+1}, w_{l+2}, \ldots, w_m).$$

Since $v_{l+1} \in \text{span}(\tilde{S}_l)$, we have that the above list is linearly dependent. By the linear independence of L, all v_j's are non-zero. Again by the linear independence of L, there is a non-trivial relation on the above list with at least one of the w_j's coefficients being non-zero. Assume WLOG that it is w_{l+1}. Thus letting

$$\tilde{S}_{l+1} = (v_1, \ldots, v_l, v_{l+1}, w_{l+2}, \ldots, w_m)$$

we get that $\text{span}(\tilde{S}_{l+1}) = \text{span}(S) = V$.

Now, if $l + 1 = m$, then $\tilde{S}_{l+1} = (v_1, \ldots, v_l, v_{l+1})$. But $l + 1 < n \Rightarrow \exists\, v_{l+2} \in L$ which must be in the span of $\tilde{S}_{l+1}$, contradicting the linear independence of L. Thus $l + 1 < m$.

(Case 2) L is infinite. Since V is a finite dimensional vector space, it has a finite spanning set S, say of size m. Since L is infinite, there exists an $L' \subseteq L$ such that L' is finite and $\#L' > m$. How? This contradicts the first case. □

Definition 6.19 (Dimension)**.** Let V be a finite dimensional vector space. We define the length of any of its bases to be the *dimension* of V.

Proposition 6.20. Let V be a finite dimensional vector space. Let W be a subspace of V. Then W is finite dimensional and $\dim(W) \leq \dim(V)$.

Proof. TODO: Formulate induction.

Let $n = \dim(V)$. If $W = \{0\}$, the statement is true. Now assume there exists a non-zero element w_1 in W. Let $S_1 = \{w_1\}$. If $\text{span}(S_1) = W$, we are done (asuuming $n \geq 1$). Otherwise, there exists a non-zero w_2 in $W \setminus \text{span}(S_1)$ and by some earlier proposition,

$$S_2 = S_1 \cup \{w_2\}$$

is linearly independent in V. Repeat to get S_j. This must terminate since a linearly independent set in V can have at most n vectors.

Thus we get some basis of W ($W \setminus \text{span}(S_j)$ is empty) which is finite. Since W is finite dimensional, it has a finite basis B. Since B is linearly independent in V and every basis of V is a spanning set, we have

$$\dim(W) = \#B \leq \dim(V).$$

□

Lecture 41

Fri 03 Feb '23

Lecture 42

Fri 03 Feb '23

Lecture 43

Fri 03 Feb '23

Proposition 6.21. Let $T \in \mathscr{L}(V, W)$. Then $N(T) = \{0\}$ iff T is an injective transformation.

Proof. If T is injective, then for any $v \in N(T)$, we have that

$$\begin{aligned} T(v) &= 0_W = T(0_V) \\ &\Rightarrow v = 0_V \\ &\Rightarrow N(T) \subseteq \{0_V\} \end{aligned}$$

□

Theorem 6.22 (Rank-Nullity Theorem)**.** Let $T \in \mathscr{L}(V, W)$, where V is a finite-dimensional vector space. Then

$$\dim(N(T)) + \dim(R(T)) = \dim(V).$$

Proof. □

Corollary 6.23. If $\dim W < \dim V$, then there is no injective linear transformation from V to W.

Proof.

$$\dim R(T) \leq \dim W < \dim V \Rightarrow \dim N(T) > 0.$$ □

Remarks. Thus a linear transformation from V to W can be bijective iff $\dim W = \dim V$.

Corollary 6.24. Let $T \in \mathscr{L}(V, W)$ where $\dim V = \dim W$. Then the following are equivalent:

(a) T is surjective.

(b) T is injective.

(c) T is invertible *as a linear transformation.* That is, there exists $T^{-1} \in \mathscr{L}(W, V)$ such that $T^{-1}T = I_W$ and $TT^{-1} = I_V$.

Example.

(a) Let $V = \mathscr{S}$. Consider the following maps from $\mathscr{S}$ to $\mathscr{S}$:

$$T_b : \{x_j\}_{j\in\mathbb{N}} \mapsto \{x_{j+1}\}_{j\in\mathbb{N}}$$
$$T_f : \{x_j\}_{j\in\mathbb{N}} \mapsto \{0, x_0, x_1, \ldots\}$$

(i) T_b is surjective, but not injective. Its null space is $\{\{a, 0, 0, \ldots\}, a \in \mathbb{R}\}$.

(ii) T_f is injective, but not surjective. Its null space is $\{0\}$.

(b) Let $T : P_{\leq 2} \to P_{\leq 2}$ such that $T : f \mapsto f'$. Choosing the basis $\{1, x, x^2\}$, we have that T is given by

$$M_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that $N(T) = \{f : f(x) = c, c \in \mathbb{R}\}$. Thus the nullity is 1 and the rank is 2.

Suppose we choose the basis $\{1, x + x^2, x^2\}$. Then T is given by

$$M_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{pmatrix}.$$

Generally, for $T \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$, one uses standard bases to write M_T.