Assignment 8

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Problem 7

We define a function $f : \mathbb{R} \to \mathbb{R}$ to be 'close' at a point $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta > 0$ such that for every $x \in N_\delta(c) \setminus \{c\}$, we have that

$$|f(x) - L| < \varepsilon.$$

We define a sequence $\{a_n\} \subset \mathbb{R}$ to be 'close' if for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have that

$$|a_n - L| < \varepsilon.$$

We will prove that if a function $f : \mathbb{R} \to \mathbb{R}$ is close at $c \in \mathbb{R}$, then it admits a limit at c. Here is the outline of the proof.

- Prove that a close sequence is a Cauchy sequence, and therefore convergent.
- Show that if $\{a_n\} \subset \mathbb{R} \setminus \{c\}$ converges to c, then $\{f(a_n)\}$ is a close sequence and thus convergent.
- Show that if two sequences $\{a_n\}, \{b_n\} \subset \mathbb{R} \setminus \{c\}$ converge to c, then $\{f(a_n)\}$ and $\{f(b_n)\}$ have the same limit.
- Use sequential characterization of limits to conclude f has a limit at c.

We first prove that a sequence is close iff it is Cauchy.

(a) Suppose that $\{a_n\}$ is a Cauchy sequence. Then for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have that

$$|a_n - a_m| < \varepsilon \;\forall\; m, n \geq n_0.$$

Let $L = a_{n_0}$. Then for every $n \geq n_0$, we have

$$|a_n - L| < \varepsilon$$

as desired.

(b) Suppose that $\{a_n\}$ is a close sequence. Then for every $\varepsilon > 0$ there exists an

$L \in \mathbb{R}$ and an $n_0 \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon/2 \; \forall \; n \geq n_0.$$

Thus for any $m, n > n_0$,

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \varepsilon$$

as desired.

For sequences of real numbers, Cauchy $\Leftrightarrow$ convergent.

Now suppose that $f : \mathbb{R} \to \mathbb{R}$ is close at $c \in \mathbb{R}$. Then for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta_\varepsilon > 0$ such that for every $x \in N_{\delta_\varepsilon}(c) \setminus \{c\}$, we have

$$|f(x) - L| < \varepsilon.$$

Consider a sequence $\{a_n\} \subset \mathbb{R} \setminus \{c\}$ converging to c. Then there exists an n_0 such that for every $n \geq n_0$, we have

$$|a_n - c| < \delta_\varepsilon \Rightarrow |f(a_n) - L| < \varepsilon.$$

Thus $\{f(a_n)\}$ is close, and therefore convergent.

Suppose sequences $\{a_n\}$ and $\{b_n\}$ both converge to c (but never equal it), with $\{f(a_n)\}$ and $\{f(b_n)\}$ having different limits L_1 and L_2. Consider the sequence

$$c_n = \begin{cases} a_n & \text{if } n \text{ is even} \\ b_n & \text{if } n \text{ is odd} \end{cases}$$

Clearly $\{c_n\}$ converges to c, but $\{f(c_n)\}$ diverges. This is a contradiction. Thus there exists a unique $L_0 \in \mathbb{R}$ such that given any sequence $\{a_n\}$ converging to c, we have that

$$\lim_{n \to \infty} f(a_n) = L_0.$$

By the sequential characterization of limits, the limit of f at c exists.

On the other hand, if it is known that f has a limit L_0 at c, setting $L = L_0 \; \forall \; \varepsilon$ proves that f is close at c.

Thus the two definitions are equivalent.