LECTURE 04 ":=" defined as $\quad X = \mathbb{N},\ w = \mathbb{N}$ 26.10.2022

1.2 Natural numbers ($\mathbb{N}$)

The ZFC axioms gives us the existence of ω

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

$$\emptyset = 0,\quad 0^+ = 1,\quad 1^+ = 2,\quad 2^+ = 3$$

Definition 1.9 (Peano addition)

Given a fixed $m \in \mathbb{N}$, the recursion principle gives a unique function $sum_m : \mathbb{N} \to \mathbb{N}$ s.t.

a) $sum_m(0) = m$,

b) $sum_m(n^+) = \left(sum_m(n)\right)^+$ $\forall n \in \mathbb{N}$.

Define

$$m+n := sum_m(n) \overset{?}{=} n+m = sum_n(m)$$

$\forall m, n \in \mathbb{N}$.

Proposition 1.10 $2+3=5$.

Proof. We need to show that $sum_2(3) = 5$.

L.H.S. $2+3 = sum_2(3)$

$$= sum_2(2^+) \overset{(b)}{=} \left(sum_2(2)\right)^+ = \left(sum_2(1^+)\right)^+ \overset{(b)}{=} \left(\left(sum_2(1)\right)^+\right)^+$$

$$= \left(\left(sum_2(0^+)\right)^+\right)^+ \overset{(b)}{=} \left(\left(\left(sum_2(0)\right)^+\right)^+\right)^+$$

$$\overset{(a)}{=} \left(\left(2^+\right)^+\right)^+ = \left(3^+\right)^+ = \left(4\right)^+ = 5$$

$= $ R.H.S. ∎

① We could next try $2+8=10$. This would be less tedious if we knew that $2+8 = 8+2$

② Note that $m^+ = \left(sum_m(0)\right)^+ = sum_m(0^+) = sum_m(1) = m+1$

So, we will now always denote m^+ by $m+1$.

$\exists$ = there exists

Definition 1.11 (Peano multiplication).

Let $m \in \mathbb{N}$. By the recursion principle, $\exists$ unique function $prod_m : \mathbb{N} \to \mathbb{N}$ s.t.

a) $prod_m(0) = 0$, $\quad$ ~~$m(n+1) = mn+m$~~

b) $prod_m(n^+) = m + prod_m(n)$.

Define $m \cdot n := prod_m(n)$.

Theorem 1.12 The following hold.

1) (Commutativity)

$m+n = n+m$ & $m\cdot n = n\cdot m$ $\forall\, m,n \in \mathbb{N}$.

2) (Associativity)

$m+(n+k) = (m+n)+k$ & $m\cdot(n\cdot k) = (m\cdot n)\cdot k$ $\forall\, m,n,k \in \mathbb{N}$.

3) (Distributivity)

$m\cdot(n+k) = (m\cdot n)+(m\cdot k)$ $\forall\, m,n,k \in \mathbb{N}$.

4) If $m+n=0$, then $m=n=0$, for any $m,n\in\mathbb{N}$.

5) If $m\cdot n = 0$, then either $m=0$ or $n=0$, for $m,n\in\mathbb{N}$.

6) (Cancellation)

If $m+k=n+k$, then $m=n$ for $m,n,k\in\mathbb{N}$.

If $m\cdot k = n\cdot k$ & $k\neq 0$, then $m=n$ for $m,n,k\in\mathbb{N}$.

Proof of the additive part of (6).

Let $m,n\in\mathbb{N}$ be fixed.

We will prove that $\overset{m,n}{P}(k)$ is true $\forall\, k\in\mathbb{N}$, where for fixed $m,n\in\mathbb{N}$

$P(k)$: if $m+k=n+k$, then $m=n$.

1) $P(0)$. Suppose $m+0=n+0$.

Then, $sum_m(0) = sum_n(0)$.

Then, $m=n$.

2) Suppose $P(k)$ holds.

Suppose $m+(k+1) = n+(k+1)$

Then, $sum_m(k+1) = sum_n(k+1)$

Then, $(sum_m k)+1 = sum_n(K)+1$

$(sum_m(k))^+ = (sum_n(k))^+$

injectivity of $(\ )^+$

$sum_m(k) = sum_n(k)$

$\Rightarrow m=n$. (by $P(k)$)

Thus, $P(k+1)$ holds. So, by PMI $\overset{m,n}{P}(k)$ holds $\forall\, k\in\mathbb{N}$.

Since $m,n\in\mathbb{N}$ were arbitrary, (6) holds for all $m,n,k\in\mathbb{N}$.

LECTURE 05

28.10.2022

fixed m

If $sum_m(n) = sum_m(k)$

$\Downarrow$

then $n=k$.

definition (postulate)

axiom ↔ rules

hypothesis ↔ conclusion

lemma claim

proposition Proof

theorem

Corollary

$\boxed{1.3}$ Fields, ordered set & ordered fields.

Cannot solve

$$3+x=2 \quad \text{in } \mathbb{N}.$$
$$3x=2$$

$$x+0=x$$
$$\& \quad x\cdot 1 = x \qquad \forall x \in F.$$

(F5) For every $x \in F$, $\exists\, y \in F$ s.t. $x+y=0$

(F6) For every $x \in F \setminus \{0\}$, $\exists\, z \in F$ s.t. $x \cdot z = 1$

Definition 1.13. $(F, +, \cdot)$

A field is a set F with 2 operations $+: F\times F \to F$ & $\cdot : F \times F \to F$ such that

(F1) $+$ & $\cdot$ are commutative on F.

(F_2) $+$ & $\cdot$ are associative on F.

(F_3) $+$ & $\cdot$ satisfy ~~the~~ distributivity on F.

(F_4) There exist 2 distinct elements, called 0 (additive identity) & 1 (mult. identity) s.t.

Remark: we are tempted to call y in (F5) "$-x$" & z in (F6) "$1/x$" but why are y & z ~~need not be~~ unique?

Thm. 1.1 & 1.7 in Apostol. ⤳ we will call $y \to -x$
$z \to 1/x$

$$\& \quad a+(-b) =: a-b$$
$$a \cdot 1/b =: a/b \qquad b \neq 0$$

Theorem 1.14 (A. 1.6)

$(F,+,\cdot)$ is a field.

$\forall x \in F \quad 0\cdot x = x\cdot 0 = 0.$

Proof: By (F1), the first equality holds.

Now, by (F4) $(1+0)=1$.

~~By (F3)~~, so,

$$x\cdot(1+0) = x\cdot 1 \overset{(F4)}{=} x$$

$x \overset{\leq}{R} y \Leftrightarrow$ blah blah

Thm 1.1 – 1.7 in the book.

So, by (F3),

$$x\cdot 1 + x\cdot 0 = x$$

So, by (F4)

$$x + (x\cdot 0) = x$$

By (F5), $\exists\, y \in F$ s.t. $x+y=0$

So, $(x + x\cdot 0) + (y) = x+y = 0.$

By (F1) & (F2), $(x+y) + x\cdot 0 = 0$

So, $0 + x\cdot 0 = 0$

So, $x\cdot 0 = 0$ (by (F1) & (F3)).

∎

Definition 1.15. A set A with a relation $<$ is called an ordered set if

(O1) For every $x, y \in A$, exactly one of the 3 foll. statements hold:

$x=y$, $x<y$ or $y<x$.

(O2) If $x<y$ & $y<z$, then $x<z$ $\forall\, x,y,z \in A$. (transitive).

E.g. $\mathbb{N}$ with $<$ defined in HW2.

Not^n & Term. $x<y$ "x less than y"

$x \leq y$ = $x<y$ or $x=y$ ⟶ x "less than or equal to" y.

Definition 1.16. An ordered field is a set that admits 2 operations $+$ & $\cdot$ & a relation $<$ so that $(F,+,\cdot)$ is a field, $(F,<)$ is an ordered set &

(O3) For $x,y,z \in F$, if $x<y$ then $x+z<y+z$.

(O4) For $x,y \in F$, if $0<x$ & $0<y$, then $0<x\cdot y$.

Our (O1) – (O4) are Theorems 1.16 – 1.19 in Apostol.

Theorem 1.17 (Ap. 1.21).

$(F,+,\cdot,<)$ an ord. fld. $\quad 0<1.$

Proof. From (F4), $0 \neq 1$.

So, by (O1), $0<1$ or $1<0$.

If $0<1$, we are done.

$\begin{cases} 0<x \\ 0<y \\ \Downarrow \\ 0<xy \end{cases}$

$0 < 1$ $\quad 0<1$

$0-1 < 1-1$

$-1 < 0$

We will use

$$(-a)\cdot b = -(a\cdot b) \quad \Big| \quad (-a)\cdot(-b) = a\cdot b.$$

Suppose $1<0$.

Then $1+(-1) < 0+(-1)$ (O3)

$0 < -1$ ((F1) & (F4))

So $0 < (-1)(-1)$ (O4)

So, $0<1$.

Contradicts (O1)

So, $0<1$ is the only possibility.

∎