LECTURE 06 31.10.2022

1.4 Upper bounds, least upper bounds.

Throughout this section $(\mathbb{F}, +, \cdot, <)$ is an ordered field, and we assume all "basic" properties.

Key example: $(\mathbb{R}, +, \cdot, <)$

$(\mathbb{Q}, +, \cdot, <)$

Definition 1.18. A non-empty subset $S \subseteq F$ is said to be bounded above in F if $\exists\, b \in F$ s.t. $a \leq b \ \forall\, a \in S$.

Here, b is called an upper bound of S. If $b \in S$, then b is a maximum of S.

Examples. If $\mathbb{R} = F$

$S = \{x \in F : 0 \leq x \leq 1\} = [0,1]$

$T = \{x \in F : 0 \leq x < 1\} = [0,1)$

Both S & T are bounded above, and 1 is an upper bound for both.

1 is, in fact, a max of S.

Remarks: ① If a max exists, it must be unique (why?).

② Upper bounds may not be unique.

"bdd. = bounded"

Definition 1.19 Let $S \subseteq F$ be a bounded above. An element $b \in F$ is said to be a least upper bound of S or a supremum of S if

a) b is an upper bound of S, &

b) If for $c \in F$, $c < b$, then c is not an upper bound of S, i.e., for any $c < b$, $\exists\, s_c \in S$ s.t.

$$c < s_c$$

Aside: $\sqrt{2} \notin \mathbb{Q}$ vs $\mathbb{R}$

$$S = \{x \in \mathbb{Q} : x^2 < 2\}$$

S is bdd. above in both $\mathbb{Q}$ & $\mathbb{R}$, but only admits a l.u.b. in $\mathbb{R}$.

Only assumes a l.u.b. in S, $\mathbb{R}$.

Theorem 1.20. ~~Suppose~~ Let $S \subseteq F$ be a bounded above set. Suppose $b_1, b_2 \in F$ are such that b_1 & b_2 are least upper bounds of S. Then,

$$b_1 = b_2$$

Proof. By (O1), either $b_1 = b_2$, $b_1 < b_2$ or $b_2 < b_1$.

Case 1. $b_1 = b_2$. Nothing to prove.

Case 2. $b_1 < b_2$. Since b_2 is a l.u.b. of S, by Def^n (b), b_1 is not an upper bound of S. But, by Def^n(a) applied to b_1, b_1 is an upper bound of S. This is a contrad^n.

Case 3. $b_2 < b_1$. Exchanging the roles of b_1 & b_2, this is Case 2.

Thus, $b_1 = b_2$ ∎

Remark. The supremum of S is denoted by $\sup S$ or $\text{l.u.b}(S)$

Example: $\sup \overset{T:}{\{x \in F : 0 \le x < 1\}} = 1$.

Proof. (a) ∎ It is clear that 1 is an upper bound (because $x < 1 \Rightarrow x \le 1$) $\forall x \in T$

(b) Let $a \in F$ s.t. $a < 1$.

b: $x \le b \; \forall x \in S$

Suppose $a < 0$. Then, since $0 \in T$, a is not an upper bound of T.

Suppose $0 \le a < 1$.

First note that

$$1 \le a+1 < 2$$

Show $\frac{a+1}{2} \in T$

$\frac{a+1}{2} > a$

So, $0 < \frac{1}{2} \le \frac{a+1}{2} < 1$

$a = a$
$a < 1$

Thus, $\frac{a+1}{2} \in T$.

Next $a = \frac{a+a}{2} < \frac{a+1}{2}$. So, a is not an up. bd of T. ∎

1.5 The set of real numbers.

2 2 T. for an up. bd of

1.5 The set of real numbers.

We assume the existence of a set $\mathbb{R}$ with operations $+$, $\cdot$ & relation $<$ such that

$(\mathbb{R}, +, \cdot, <)$ is an order field. (F1–F6, O1–O4 hold) & the foll. holds.

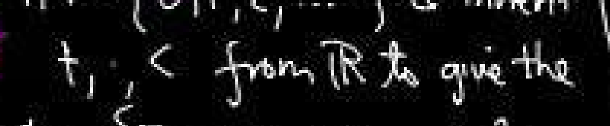
(LUB) every (nonempty) bounded above subset ~~to~~ in $\mathbb{R}$ has a supremum.

Some special subsets of $\mathbb{R}$:

- $x > 0$ is called a positive real no. $x < 0$ " " " negative "
- $\mathbb{N} = \{0, 1, 2, \dots\}$ & inherit $+, \cdot, <$ from $\mathbb{R}$ to give the notions of $+, \cdot, <$ that we know on $\mathbb{N}$.
- $\mathbb{P} = \{n \in \mathbb{N} : n \neq 0\}$
- $\mathbb{Z} = \mathbb{N} \cup \{-n : n \in \mathbb{P}\}$ is the set of integers.
- $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{P}\}$ is the set of rational nos.
- $\mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q}$ is the set of irrational nos

Theorem 1.21 (Archimedean property of $\mathbb{R}$)

Let $x, y \in \mathbb{R}$ & $x > 0$.



Then, $\exists\, n \in \mathbb{P}$ such that

$$n \cdot x > y.$$

Proof. Fix $x > 0$. Let

$$S = \{nx : n \in \mathbb{P}\}.$$

Clearly, $S \neq \emptyset$ since $x \in S$.

Suppose the claim does not hold. Then,

$(*)$ $\quad nx \leq y \quad \forall\, n \in \mathbb{P}$.

Ie, S is bounded above. So, by LUB $\sup S = b$ exists in $\mathbb{R}$.

Thus, $\because b - x < b$, $b - x$ is not an upper bound of S, so $\exists\, m \in \mathbb{P}$ s.t.

$$b - x < mx$$

Then, $b < (m+1)x \in S$.

This is a contrad^n since b is an upper bd of S.

$\blacksquare$

LECTURE 07 2.11.2022

2. Sequences & series.

$\mathbb{Q}$
$\{a_n\} \subseteq \mathbb{R}$

2.1 Sequences

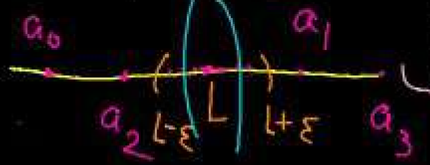
Definition 2.1. A sequence in $\mathbb{R}$ is a function f from $\mathbb{N}$ to $\mathbb{R}$. We denote this sequence by $\{a_n\}_{n\in\mathbb{N}}$, where

$0 \leq x < \epsilon \ \forall\, \varepsilon > 0$

$$a_n = f(n) \quad \forall\, n \in \mathbb{N},$$

and a_n is called the n^{th} term of $\{a_n\}_{n\in\mathbb{N}}$.

Definition 2.2. We say that a sequence $\{a_n\} \subseteq \mathbb{R}$ is convergent (in $\mathbb{R}$) if $\exists\, L \in \mathbb{R}$ such that for each $\varepsilon > 0$, $\exists\, N_{\varepsilon, L} \in \mathbb{N}$ such that



$$\underbrace{|a_n - L| < \epsilon}_{} \quad \forall\, n \geq N_{\varepsilon, L}.$$

$$-\varepsilon < a_n - L < \varepsilon$$
$$-\varepsilon + L < a_n < \varepsilon + L$$

Here, L is called a limit of $\{a_n\}$ & we write: $a_n \to L$ as $n \to \infty$.

A sequence $\{a_n\}$ is said to be divergent if it is not convergent.

Theorem 2.3 (Uniqueness of limits). Suppose l_1 & l_2 are limits of a (convergent) sequence $\{a_n\} \subseteq \mathbb{R}$. Then, $l_1 = l_2$.

Proof. ~~By~~ Let $\delta > 0$. Let $\epsilon = \delta/2 > 0$.

By Def^n 2.2, $\exists\, N_{\epsilon, l_1} \in \mathbb{N}$ s.t. ① $|a_n - l_1| < \epsilon \ \forall\, n \geq N_{\varepsilon, l_1}$

" , $\exists\, N_{\epsilon, l_2} \in \mathbb{N}$ s.t. ② $|a_n - l_2| < \epsilon \ \forall\, n \geq N_{\varepsilon, l_2}$.

Set $N := \max\{N_{\varepsilon, l_1}, N_{\varepsilon, l_2}\}$.

Then,

$$|\ell_1-\ell_2| = |(\ell_1 - a_N)+(a_N - \ell_2)|$$
$$\leq |\ell_1-a_N| + |a_N-\ell_2| < \epsilon + \epsilon = 2\epsilon = \delta$$

Since $N \geq N_{\epsilon,\ell_1}$ & $N \geq N_{\epsilon,\ell_2}$.

Since $\delta > 0$ was arbitrary, we have that

$$0 \leq |\ell_1-\ell_2| < \delta \quad \forall\, \delta > 0.$$

By next HW, $\ell_1 = \ell_2$. ∎

Examples (a) Fix $p>0$. $a_n = \frac{1}{n^p}$, $n \in \mathbb{P} = \{1,2,3,\dots\}$

Claim. $$\lim_{n\to\infty} \frac{1}{n^p} = 0.$$

Let $\epsilon > 0$.
By the Arch. Prop. of $\mathbb{R}$ applied to $x = \epsilon^{1/p}$ & $y=1$, $\exists\, N \in \mathbb{P}$ s.t.

$\overset{\shortparallel}{N_\epsilon}$ $N\epsilon^{1/p} > 1$

Let $\boxed{n \geq N_\epsilon}$ Then

$$\left|\frac{1}{n^p} - \underset{L}{0}\right| = \frac{1}{n^p} \leq \underbrace{\frac{1}{N^p}}_{< (\epsilon^{1/p})^p = \epsilon}$$

Since $\epsilon > 0$ was arbitrary, $\lim_{n\to\infty}\frac{1}{n^p} = 0$

$\epsilon > 0$ find N

$\left|\frac{1}{n^p} - 0\right| < \epsilon \ \forall\, n \geq N$

$\left(\frac{1}{n^p}\right)^{1/p} < (\epsilon)^{1/p} \ \forall\, n \geq N$

or $1 < \epsilon^{1/p} n$

$\lim_{n\to\infty}(a_n - b_n) = 0$
$\Downarrow$
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

$$\left(\begin{gathered} n \geq N \\ n^p \geq N^p \\ \frac{1}{n^p} \leq \frac{1}{N^p} \end{gathered}\right)$$

$$\left(\begin{gathered} N\epsilon^{1/p} > 1 \Rightarrow N > \frac{1}{\epsilon^{1/p}} \\ x \quad y \Rightarrow N^p > \frac{1}{\epsilon} \Rightarrow \frac{1}{N^p} < \epsilon \end{gathered}\right)$$

For a fixed $\epsilon > 0$, we have found N_ϵ s.t.

$$0 \leq \left|\frac{1}{n^p} - 0\right| < \epsilon \quad \forall\, n \geq N_\epsilon.$$

$\epsilon = \frac{1}{100}$ $N_\epsilon = 101$

LECTURE 08 04.11.2022

Recap $\lim_{n\to\infty} \frac{1}{n^p} = 0.$

Given $\epsilon > 0$, $\exists\, N \in \mathbb{N}$

s.t. $\left|\frac{1}{n^p} - 0\right| < \epsilon \;\forall\, n \geq N$

If $n \geq N > \frac{1}{\epsilon^{1/p}}$

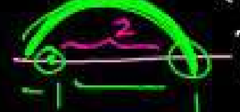
Archimedean property: If $\exists$ Given $x, y \in \mathbb{R}$, $x > 0$, $\exists\, N \in \mathbb{N}$

s.t. $Nx > y$

or $N > \frac{y}{x}$

$\exists$ Natural n $N > x$

(2) Claim. $\{(-1)^n\}_{n\in\mathbb{N}}$ is divergent.



Proof. Suppose $\{a_n\}$ cvgs. to some $L \in \mathbb{R}$.

Let $\epsilon = 1$. Then, by defn, $\exists\, N \in \mathbb{N}$ s.t.

$|a_n - L| < 1 \;\forall\, n \geq N$. — ⊛

Thus, $|a_{2N} - L| < 1$ &

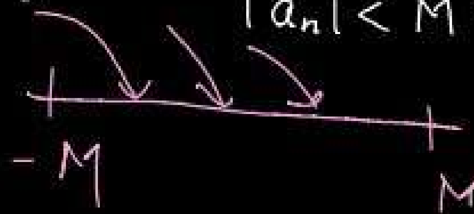
$\Rightarrow$ $|a_{2N+1} - L| < 1$ by Δ ineq

$2 = |a_{2N} - a_{2N+1}| \leq |a_{2N} - L| + |a_{2N+1} - L|$

$< 1 + 1 = 2.$ ✗ ▫

$|a_{2N} - a_{2N+1}|$

Def^n 2.4. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is s.t.b. bounded if $\exists\, M > 0$ s.t. $|a_n| < M \;\forall\, n \in \mathbb{N}$.



Theorem HW3: Every cvgt. sequence is bdd.

But, the converse is not true!

Def^n 2.5. A seq. $\{a_n\} \subseteq \mathbb{R}$ is s.t.b. (monotonically) increasing if $a_n \leq a_{n+1}$ $\forall\, n \in \mathbb{N}$.

decreasing $a_n \geq a_{n+1}$ "

$\{a_n\}$ is monotone if it is one of the above.

$-M < a_n < M$

Theorem 2.6. Bounded monotone sequences are convergent.

Proof. ① Assume $\{a_n\}$ is increasing & $\exists\, M > 0$ s.t. $|a_n| < M \;\forall\, n \in \mathbb{N}$.

Let $S = \{a_n : n \in \mathbb{N}\}$.

→ S is nonempty

→ S is bdd. above (M)

Thus, by LUB, $b = \sup S$ exists in $\mathbb{R}$.

Let $\epsilon > 0$.

Thus, $a_n \leq b < b + \epsilon$. $\forall\, n \in \mathbb{N}$.

and $\exists\, N \in \mathbb{N}$ s.t.

$b - \epsilon < a_N \leq a_n \;\forall\, n \geq N$

So, $\forall\, n \geq N$,

$b - \epsilon < a_n < b + \epsilon \Rightarrow |a_n - b| < \epsilon$.

Since $\epsilon > 0$ was arb., $\lim_{n\to\infty} a_n = b$

② $\{a_n\}$ is decreasing & bdd.

HW ▫

Warning! Divergent sequences may diverge for different reasons!

Eg ① $\{(-1)^n\}$ is bdd & divg.

② $\{n\}$ ~~unbdd~~ not bdd above divg.

③ $\{(-1)^n n\}$ Q: cvg v/s dvg.
bdd v/s not bdd.
$\to +\infty$ v/s $\to -\infty$ v/s neither.

Def^n 2.7. We say that a seq. $\{a_n\}$ diverges to $\pm\infty$ if $\forall R \in \mathbb{R}$, $\exists N_R \in \mathbb{N}$ s.t. $a_n > R \ \forall n \geq N_R$.

$\lim_{n\to\infty} a_n = +\infty$.

Theorem 2.8. (Tao, Theorem 6.1.19)

(4) Suppose $\{b_n\} \subseteq \mathbb{R}$ cvgs. to $b \neq 0$ & ($\exists M \in \mathbb{N}$ s.t. $b_n \neq 0 \ \forall n \geq M$). Then $\left\{\frac{1}{b_n}\right\}_{n\geq M} \longrightarrow \frac{1}{b}$ as $n \to \infty$.

"Proof" Given $\epsilon > 0$. Find $N \in \mathbb{N}$.

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \epsilon \quad \forall n \geq N.$$

OR $$\left|\frac{b_n - b}{b_n b}\right| < \epsilon \quad \forall n \geq N.$$

If I could make $\left|\frac{b_n - b}{b_n b}\right| < M_1 |b_n - b| < \epsilon$, with $M_1 = \frac{2}{|b|^2}$, $|b_n - b| < \epsilon/M$

$$\frac{1}{|b_n||b|} < \frac{2}{|b|^2}$$

$$\left|\frac{b_n - b}{b_n b}\right| < \frac{2}{|b|^2}|b_n - b| \quad \forall n \geq N.$$

We want $\left|\frac{1}{b_n}\right| < M_2$

or $|b_n| > \frac{1}{M_2} > 0$

$\frac{|b|-|b|}{2}$, $\frac{|b|+|b|}{2}$, 0, $|b|$, M, $n \geq N$

$\big||b_n| - |b|\big| \leq |b_n - b| < \frac{|b|}{2}$ reverse Δ

$$\underbrace{\frac{|b|}{2}}_{M} < |b_n| < \frac{3|b|}{2}$$