Assignment 04

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Problem 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ such that for some $N \in \mathbb{N}, 0 \leq a_n \leq b_n$. Convince yourself that if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$. Using this fact, prove the following statements (you are not allowed to use logarithms for these proofs).

(a) For any $r > 0$, $\lim\limits_{n\to\infty} \sqrt[n]{r} = 1$.

(b) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. (a) For $r > 1$: By the Archimedean property, there exists $n_0 \in \mathbb{P}$ such that $n_0\varepsilon > r - 1$ for all $\varepsilon > 0$. Thus for all $n \geq n_0$, $n\varepsilon > r - 1 \Rightarrow r < 1 + n\varepsilon \leq (1+\varepsilon)^n \Rightarrow r^{1/n} < 1+\varepsilon$. $r > 1 \Rightarrow r^{1/n} > 1 > 1-\varepsilon$. Thus $\left|r^{1/n} - 1\right| < \varepsilon\ \forall\ n \geq n_0$.

For $r = 1$, $\left|r^{1/n} - 1\right| = 0 < \varepsilon$ for all $n \geq 1, \varepsilon > 0$.

For $r < 1$,

$$r^{1/n} = \frac{1}{\left(\frac{1}{r}\right)^{1/n}}$$

Since $\frac{1}{r} > 1$, by limit laws for sequnces, $\lim\limits_{n\to\infty} r^{1/n} = \frac{1}{1} = 1$.

(b) By the Archimedean property, there exists an $N \in \mathbb{P} > \frac{2}{\varepsilon^2} + 1$. For $n \geq N$, we have

$$\begin{aligned}
\frac{n-1}{2} \cdot \varepsilon^2 &> 1 \\
\frac{n(n-1)}{2} \cdot \varepsilon^2 &> n \\
(1+\varepsilon)^n &> n \\
\sqrt[n]{n} &< 1 + \varepsilon
\end{aligned}$$

Also since $n \geq 1$, $n^{1/n} \geq 1 \Rightarrow \sqrt[n]{n} > 1 - \varepsilon$.

$\square$

Problem 2. Show that the series $\sum\limits_{n=0}^{\infty} \frac{1}{n!}$ converges. The mathematical constant e is defined as the sum of this series.

Proof. Ratio test. $\square$

Problem 3. Let $\{a_n : n \in \mathbb{P}\}$ be an arbitrary collection of non-negative real numbers such that $\sum\limits_{n=1}^{\infty} a_n$ converges. Determine which of the following series will necessarily converge (proof required), and which may either converge or diverge depending on the choice of the a_n's (examples required).

(a) $\sum\limits_{n=1}^{\infty} a_n^2$

(b) $\sum\limits_{n=1}^{\infty} \sqrt{a_n}$

(c) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$

Proof. (a) Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$. Thus there exists $N \in \mathbb{P}$ such that $a_n < 1 \; \forall \; n \geq N \Rightarrow a_n^2 < a_n \; \forall \; n \geq N$. By the comparison test, $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) $\sum \frac{1}{n^2}$ converges but $\sum \frac{1}{n}$ diverges. $\sum 0$ converges and so does $\sum 0$. Inconclusive.

(c) Let $b_n = a_n + \frac{1}{n^2}$. By the limit laws for series, $\sum b_n$ converges.

$$\frac{\sqrt{a_n}}{n} \leq \frac{\sqrt{b_n}}{n} = \frac{b_n}{n\sqrt{b_n}} \leq \frac{b_n}{n\sqrt{\frac{1}{n^2}}} = b_n.$$

Thus by the comparison test, $\sum\limits_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. $\square$

Problem 4. Show that each of the following series converges, and determine its sum.

(a) $\sum\limits_{n=1}^{\infty} \frac{4n^2-1+3^{n-1}}{3^n(2n+1)(2n-1)}$

(b) $\sum\limits_{n=6}^{\infty} \frac{6}{n^2-1}$

(c) $\sum\limits_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$

Proof. (a)

$$\begin{aligned}\frac{4n^2-1+3^{n-1}}{3^n(2n+1)(2n-1)} &= \frac{1}{3^n}+\frac{1}{3}\cdot\frac{1}{(2n+1)(2n-1)}\\ &= \frac{1}{3^n}+\frac{1}{6}\cdot\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{3^n} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n} &= \frac{1/3}{1-\frac{1}{3}} = \frac{1}{2}\\ \sum_{n=1}^{\infty}\frac{1}{2n-1}-\frac{1}{2n+1} &= \frac{1}{2n-1}-\frac{1}{2(n+1)-1}\\ &= \frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\dots\\ &= 1\end{aligned}$$

Thus by the limit laws,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{4n^2-1+3^{n-1}}{3^n(2n+1)(2n-1)} &= \frac{1}{2}+\frac{1}{6}\cdot 1\\ &= \frac{2}{3}\end{aligned}$$

(b)

$$\begin{aligned}\frac{6}{n^2-1} &= \frac{3\cdot 2}{(n-1)(n+1)}\\ &= 3\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\\ &= 3\left(\frac{1}{n-1}-\frac{1}{(n+2)-1}\right)\end{aligned}$$

Let $\{s_n\}$ be the sops of $\frac{1}{n-1} - \frac{1}{n+1}$. For $n > 5$,

$$\begin{aligned}
s_n &= \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \frac{1}{7} - \frac{1}{9} + \cdots + \frac{1}{n-1} + \frac{1}{n+1} \\
&= \frac{1}{5} + \frac{1}{6} - \frac{1}{n} - \frac{1}{n+1} \\
\lim_{n\to\infty} s_n &= \frac{1}{5} + \frac{1}{6} \\
&= \frac{11}{30}
\end{aligned}$$

$$\Rightarrow \sum_{n=6}^{\infty} \frac{6}{n^2-1} = \frac{11}{10}$$

(c)

$$\begin{aligned}
\frac{n}{(n+1)(n+2)(n+3)} &= \frac{1}{2}\frac{3(n+1)-(n+3)}{(n+1)(n+2)(n+3)} \\
&= \frac{3}{2}\frac{1}{(n+2)(n+3)} - \frac{1}{2}\frac{1}{(n+1)(n+2)} \\
&= \frac{3}{2}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)
\end{aligned}$$

Let s_n and t_n be the sops of $\left\{\frac{1}{n+2} - \frac{1}{n+3}\right\}$ and $\left\{\frac{1}{n+1} - \frac{1}{n+2}\right\}$.

$$\begin{aligned}
s_n &= \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} \\
s_n &= \frac{1}{3} - \frac{1}{n+2} \\
\lim_{n\to\infty} s_n &= \frac{1}{3} \\
t_n &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \\
t_n &= \frac{1}{2} - \frac{1}{n+3} \\
\lim_{n\to\infty} t_n &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} &= \frac{3}{2}\cdot\frac{1}{3} - \frac{1}{2}\cdot\frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

$\square$

Problem 5. For each of the series given below, determine whether it converges or diverges. You need not compute the sum in the case of convergence.

(1) $\sum_{n=1}^{\infty} \frac{n \sin^2(n\pi/3)}{2^n}$

(2) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}$

(3) $\sum_{n=1}^{\infty} \frac{(-1)^n n^{25}}{(n+2)!}$

(4) $\sum_{n=5}^{\infty} \frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)}$

Proof. (1) Let $b_n = \frac{n}{2^n}$.

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} \cdot \frac{n+1}{n}$$
$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{1}{2}$$

Thus by the ratio test, $\sum b_n$ converges. Since $a_n < b_n$, $\sum a_n$ converges by the comparison test.

(2) $n \geq 1 \Rightarrow \frac{1}{n} \leq 1 \wedge n^{\frac{1}{n}} \leq n^1 \Rightarrow \left(\frac{1}{n}\right)^{\frac{1}{n}} \geq \frac{1}{n}$. By the comparison test, $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}$ diverges.

(3) Let $a_n = \frac{n^{25}}{(n+2)!}$. Clearly $a_n > 0$.

$$0 < \frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)^{25} \cdot \frac{1}{n+3} \leq 2^{25} \cdot \frac{1}{n+3}$$

For any $\varepsilon > 0$, there exists (by the Archimedean property), $N > \frac{2^{25}}{\varepsilon} - 3$. For $n \geq N, -\varepsilon < 0 < \frac{a_{n+1}}{a_n} \leq \frac{2^{25}}{n+3} \leq \frac{2^{25}}{N+3} < \varepsilon$. Thus the ratio converges to 0. By the ratio test, $\sum_{n=1}^{\infty} a_n$ converges.

Since the given series converges absolutely, it converges.

(4) For $n \geq 5$,

$$
\begin{aligned}
0 < \frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)} &= \frac{1}{(\sqrt{n}-1)(n+2)(n-4)} \\
&< \frac{1}{(n+2)(n-4)} \\
&= \frac{1}{n^2-2n-8} \\
&< \frac{1}{n^2-4n} \\
&= \frac{1}{\frac{1}{5}n^2+\frac{4}{5}n(n-5)} \\
&\leq \frac{5}{n^2}
\end{aligned}
$$

So by the comparison test, $\sum\limits_{n=5}^{\infty} \frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)}$ converges. $\square$