Assignment 9

Naman Mishra

January 3, 2022

Problem 1

Let s be a step function on partition $\mathcal{P} = \{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$. Suppose $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Let $\mathcal{W} = \mathcal{P}' \setminus \mathcal{P} = \{y_1 < y_2 < \cdots < y_m\}$. Let $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_{k+1} = \mathcal{P}_k \cup \{y_{k+1}\}$ Then $\mathcal{P}_m = \mathcal{P} \cup \mathcal{W} = \mathcal{P}'$.

We know that s is a step function on $\mathcal{P}$ with some integral I. Suppose inductively that s is a step function on $\mathcal{P}_k = \{z_0 < z_1 < \cdots < z_{n+k}\}$ with integral I. Since $z_0 = a < y_{k+1} < b = z_n$ and $y_{k+1} \notin \mathcal{P}_k$, we have $z_{j-1} < y_{k+1} < z_j$ for some $j \in \mathbb{N} \cap [1, n+k]$.

$$\mathcal{P}_{k+1} = \{z_0 < z_1 < \cdots < z_{j-1} < y_{k+1} < z_j < \cdots < z_{n+k}\}.$$

Since s is a step function on $\mathcal{P}_k$, $\exists\ c_i \in \mathbb{R}$ such that $s(x) = c_i\ \forall\ x \in (z_{i-1}, z_i)$. Thus $s(x) = c_j\ \forall\ x \in (z_{j-1}, y_{k+1})$ and $s(x) = c_j\ \forall\ x \in (y_{k+1}, z_j)$. Thus s is constant on all $(z_{i-1}, z_i), i \neq j$ as well as (z_{j-1}, y_{k+1}) and $(y_{k+1}, z_j) \Rightarrow s$ is a step function on $\mathcal{P}_{k+1}$.

Moreover,

$$\begin{aligned}
\int_{\mathcal{P}_{k+1}} s(x)\,\mathrm{d}x &= \sum_{i=1}^{j-1} c_i(z_i - z_{i-1}) + c_j(y_{k+1} - z_{j-1}) + c_j(z_j - y_{k+1}) + \sum_{i=j+1}^{n+k} c_i(z_i - z_{i-1}) \\
&= \sum_{i=1}^{j-1} c_i(z_i - z_{i-1}) + c_j(z_j - z_{j-1}) + \sum_{i=j+1}^{n+k} c_i(z_i - z_{i-1}) \\
&= \sum_{i=1}^{n+k} c_i(z_i - z_{i-1}) \\
&= \int_{\mathcal{P}_k} s(x)\,\mathrm{d}x \\
&= I.
\end{aligned}$$

Thus by induction, s is a step function on $\mathcal{P}'$ with integral I.

Taking the common refinement $\mathcal{R}$ of $\mathcal{P}$ and $\mathcal{Q}$ yields $\int_{\mathcal{P}} s(x)\,\mathrm{d}x = \int_{\mathcal{R}} s(x)\,\mathrm{d}x = \int_{\mathcal{Q}} s(x)\,\mathrm{d}x$.

Problem 2

(a)

$$\left\lfloor x - \frac{1}{2} \right\rfloor + \lfloor x \rfloor = \begin{cases} -3 & x \in [-1, -\frac{1}{2}) \\ -2 & x \in [-\frac{1}{2}, 0) \\ -1 & x \in [0, \frac{1}{2}) \\ 0 & x \in [\frac{1}{2}, 1) \\ 1 & x \in [1, \frac{3}{2}) \\ 2 & x \in [\frac{3}{2}, 2) \\ 3 & x = 2 \end{cases}$$

Thus

$$\int_{-1}^{2} \left(\left\lfloor x - \frac{1}{2} \right\rfloor + \lfloor x \rfloor \right) \mathrm{d}x = -3 \cdot \frac{1}{2} + (-2) \cdot \frac{1}{2} + \cdots + 2 \cdot \frac{1}{2} = -\frac{3}{2}.$$

(b)

$$\lfloor \sqrt{x} \rfloor = \begin{cases} 1 & x \in [1, 4) \\ 2 & x \in [4, 9) \\ 3 & x = 9 \end{cases}$$

Thus

$$\int_{1}^{9} \lfloor \sqrt{x} \rfloor \, \mathrm{d}x = 1 \cdot 3 + 2 \cdot 5 = 13.$$

Problem 3

Given a step function f on $[a, b]$, we have

$$S_f = \left\{ \int_a^b s(x) \, \mathrm{d}x : s \text{ is a step function and } s \leq f \text{ on } [a, b] \right\}.$$

For any step function $s \leq f$, we have $\int_a^b s(x) \, \mathrm{d}x \leq \int_a^b f(x) \, \mathrm{d}x$ (defined as sum of $f_j(x_j - x_{j-1})$.

Thus $\int_a^b f(x) \, \mathrm{d}x$ is an upper bound of S_f. Moreover, since f is a step function and $f \leq f$ on $[a, b]$, $\int_a^b f(x) \, \mathrm{d}x \in S_f$. Therefore, $\sup S_f = \int_a^b f(x) \, \mathrm{d}x$.

Similarly, $\inf T_f = \int_a^b f(x) \, \mathrm{d}x$ and so the two definitions are concurrent.

Alternatively,

$$\int_a^b s(x) \, \mathrm{d}x \leq \sup S_f \leq \inf T_f \leq \int_a^b t(x) \, \mathrm{d}x.$$

Since $f \leq f$ and $f \geq f$, we can let $s = f$ and $t = f$. So

$$\int_a^b f(x) \, \mathrm{d}x \leq \sup S_f \leq \inf T_f \leq \int_a^b f(x) \, \mathrm{d}x.$$

Thus $\sup S_f = \inf T_f = \int_a^b f(x)\,\mathrm{d}x$.

Problem 4

Suppose f is not Riemann integrable on $[c,d]$. Then $\underline{I}_{[c,d]} \neq \bar{I}_{[c,d]} \Rightarrow \exists\, \varepsilon > 0$ such that $\int_c^d t_{[c,d]}(x)\,\mathrm{d}x - \int_c^d s_{[c,d]}(x)\,\mathrm{d}x > \varepsilon$ for all step functions $s_{[c,d]}, t_{[c,d]} : [c,d] \to \mathbb{R}$ such that $s_{[c,d]} \leq f \leq t_{[c,d]}$ on $[c,d]$.

Now suppose $\hat{s}_{[a,b]}, \hat{t}_{[a,b]} : [a,b] \to \mathbb{R}$ are step functions such that $\hat{s}_{[a,b]} \leq f \leq \hat{t}_{[a,b]}$ on $[a,b]$.

Problem 5

By the expansion property,

$$\int_{-0}^{-a} f(-x)\,\mathrm{d}x = -\int_0^a f(x)\,\mathrm{d}x.$$

If f is even, then $f(-x) = f(x)$ and so

$$\int_{-0}^{-a} f(x)\,\mathrm{d}x = -\int_0^a f(x)\,\mathrm{d}x.$$

So $\int_{-a}^0 f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x$ and thus

$$\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x.$$

Similarly if f is odd, then $f(-x) = -f(x)$ and so

$$\int_{-0}^{-a} f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x.$$

So $\int_{-a}^0 f(x)\,\mathrm{d}x = -\int_0^a f(x)\,\mathrm{d}x$ and thus

$$\int_{-a}^a f(x)\,\mathrm{d}x = 0.$$

Problem 6

Suppose f attains a value $y > 0$ at some x_0. Since f is continuous, there exists a neighborhood $N_\delta(x_0)$ with $0 < \delta < \min\{b - x_0, x_0 - a\}$ of size 2δ on which $f > y/2$. Thus $\int_{x_0-\delta}^{x_0+\delta} f(x)\,\mathrm{d}x > y\delta > 0$.

Since $f \geq 0$, we have $\int_a^{x_0-\delta} f(x)\,\mathrm{d}x \geq 0$ and $\int_{x_0+\delta}^b f(x)\,\mathrm{d}x \geq 0$. Thus

$$\int_a^b f(x)\,\mathrm{d}x > 0.$$

This is a contradiction, and thus $f(x) = 0 \; \forall \; x \in [a, b]$.

Problem 7

Suppose f is Riemann integrable on $[a, b]$ per the lecture definition. Then $\sup S_f = \inf T_f = I$. Since I is the supremum of S_f, $I - \varepsilon$ is not an upper bound of S_f, and thus there exists an $s_\varepsilon \leq f$ such that $\int_a^b s_\varepsilon(x)\,\mathrm{d}x > I - \varepsilon$.

Similarly there exists a $t_\varepsilon \geq f$ such that $\int_a^b t_\varepsilon(x)\,\mathrm{d}x$. Thus f is Riemann integrable as per the given definition.

Now suppose f is Riemann integrable on $[a, b]$ per the given definition. Defining S_f and T_f as before, we have

$$\int_a^b s(x)\,\mathrm{d}x \leq \sup S_f \leq \inf T_f \leq \int_a^b t(x)\,\mathrm{d}x.$$

Suppose $\int_a^b s(x)\,\mathrm{d}x = I + \varepsilon_0 > I$. Then there exists a t_{ε_0} such that $\int_a^b t_{\varepsilon_0}(x)\,\mathrm{d}x < I + \varepsilon_0 = \int_a^b s(x)\,\mathrm{d}x$. Contradiction. Thus $\int_a^b s(x)\,\mathrm{d}x \leq I$. Also, since $I - \varepsilon$ is not an upper bound of S_f for any $\varepsilon > 0$, I is the least upper bound of S_f.

Similarly, I is the greatest lower bound of T_f. Thus f is Riemann integrable per the lecture definition.