UM 101 - SAMPLE SOLUTIONS

Problem 1. A set A is said to be *transitive* if every element of A is also a subset of A. Show that the minimal inductive set ω is transitive.

Proof. Let

$$T = \{a \in \omega : a \subseteq \omega\}.$$

In order to show that ω is transitive, we must show that $T = \omega$. We will first show that T is an inductive set.

Note that since ω is an inductive set, $\emptyset \in \omega$. On the other hand, by the axiom of specification, $\emptyset = \{x \in \omega : x \neq x\} \subseteq \omega$. Thus,

$$\phi \in T. \tag{0.1}$$

Now, let $b \in T$. In particular, $b \in \omega$. This gives us two things. First, since ω is inductive, we have that $b^+ \in \omega$. Second, $\{b\} = \{x \in \omega : x = b\} \subseteq \omega$ by specification. Since $b \in T$, we also have that $b \subseteq \omega$. Now, if $A, B \subseteq \omega$, then by specification, $C = \{z \in \omega : z \in A \text{ or } z \in B\} \subseteq \omega$. But by extension, $C = A \cup B$. Thus, $A \cup B \subseteq \omega$ whenever $A, B \subseteq \omega$. We, therefore, have that

$$b^+ \in \omega \text{ and } b^+ = b \cup \{b\} \subseteq \omega. \tag{0.2}$$

Combining (0.1) and (0.2) gives that $T \subseteq \omega$ is inductive. But, ω is the minimal inductive set. Thus, $\omega \subseteq T$. By extension, $T = \omega$. Thus, every element of ω is also a subset of ω. $\square$

Problem 2. Let A be a set. For each subset B of A, let

$$B^c = A \setminus B.$$

Then, for all subsets X and Y of A,

$$(X \cup Y)^c = X^c \cap Y^c.$$

Proof. We will show that $(X \cup Y)^c \subseteq X^c \cap Y^c$ and $X^c \cap Y^c \subseteq (X \cup Y)^c$. We will use the following two facts without proof.

(a) If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.
(b) $(U^c)^c = U$ for any subset $U \subseteq A$.

For the first claim, it suffices to show that

$$(X \cup Y)^c \subseteq X^c \tag{0.3}$$

and

$$(X \cup Y)^c \subseteq Y^c. \tag{0.4}$$

Suppose (0.3) does not hold. Then, there is a $z \in (X \cup Y)^c = A \setminus (X \cup Y)$ such that $z \notin X^c$. Since $z \in A$ and $z \notin X^c$, $z \in A \setminus X^c = (X^c)^c = X$. Thus, $z \in X \cup Y$, which is a contradiction. Thus, (0.3) holds. Exchanging the roles of X and Y in (0.3) yields (0.4).

For the second claim, let $z \in X^c \cap Y^c$. Then, $z \in A$ but z is neither in X nor in Y, i.e., z is not in $X \cup Y$. Thus, $z \in A \setminus (X \cup Y) = (X \cup Y)^c$. Since $z \in X^c \cap Y^c$ was arbitrarily chosen, we have that $X^c \cap Y^c \subseteq (X \cup Y)^c$. $\square$