UM101: Analysis & Linear Algebra

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Proposition 0.1. Let $T \in \mathscr{L}(V, W)$. Then $N(T) = \{0\}$ iff T is an injective transformation.

Proof. If T is injective, then for any $v \in N(T)$, we have that

$$\begin{aligned} T(v) = 0_W = T(0_V) \\ \Rightarrow v = 0_V \\ \Rightarrow N(T) \subseteq \{0_V\} \end{aligned}$$

□

Theorem 0.2 (Rank-Nullity Theorem)**.** Let $T \in \mathscr{L}(V, W)$, where V is a finite-dimensional vector space. Then

$$\dim(N(T)) + \dim(R(T)) = \dim(V).$$

Proof. □

Corollary 0.3. If $\dim W < \dim V$, then there is no injective linear transformation from V to W.

Proof.

$$\dim R(T) \leq \dim W < \dim V \Rightarrow \dim N(T) > 0.$$

□

Remarks. Thus a linear transformation from V to W can be bijective iff $\dim W = \dim V$.

Corollary 0.4. Let $T \in \mathscr{L}(V, W)$ where $\dim V = \dim W$. Then the following are equivalent:

(a) T is surjective.

(b) T is injective.

(c) T is invertible *as a linear transformation.* That is, there exists $T^{-1} \in \mathscr{L}(W, V)$ such that $T^{-1}T = I_W$ and $TT^{-1} = I_V$.

Example.

(a) Let $V = \mathscr{S}$. Consider the following maps from $\mathscr{S}$ to $\mathscr{S}$:

$$T_b : \{x_j\}_{j\in\mathbb{N}} \mapsto \{x_{j+1}\}_{j\in\mathbb{N}}$$
$$T_f : \{x_j\}_{j\in\mathbb{N}} \mapsto \{0, x_0, x_1, \ldots\}$$

(i) T_b is surjective, but not injective. Its null space is $\{\{a, 0, 0, \ldots\}, a \in \mathbb{R}\}$.

(ii) T_f is injective, but not surjective. Its null space is $\{0\}$.

(iii) Let $T : P_{\leq 2} \to P_{\leq 2}$ such that $T : f \mapsto f'$. Choosing the basis $\{1, x, x^2\}$, we have that T is given by

$$M_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that $N(T) = \{f : f(x) = c, c \in \mathbb{R}\}$. Thus the nullity is 1 and the rank is 2.

Suppose we choose the basis $\{1, x + x^2, x^2\}$. Then T is given by

$$M_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{pmatrix}.$$

Generally, for $T \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$, one uses standard bases to write M_T.