UM101: Short Notes

Naman Mishra

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Contents

1 Set theory & the real number system

Definition 1.1. The set A along with a successor function S is called a Peano set if it obeys the Peano axioms.

(P1) There is an element called 0 in A.

(P2) For every $a \in A$, its successor $S(a)$ is also in A.

(P3) $\forall\, a \in A, S(a) \neq 0$.

(P4) For any $m, n \in A$, $S(m) = S(n)$ only if $m = n$.

(P5) (principle of mathematical induction) For any set $B \subseteq A$, if $0 \in B$ and $a \in B \implies S(a) \in B$, then $B = A$.

1.1 The ZFC Axioms

Definition 1.2. A **set** is a well-defined collection of (mathematical) objects, called the *elements* of that set. To say that a is an element of set A, we write $a \in A$. Otherwise, we write $a \notin A$.
Given two sets A and B, we say that:

($A \subseteq B$) A is a subset of B, *i.e.*, every element of A is an element of B.

($A \nsubseteq B$) A is not a subset of B, *i.e.*, there is some element in A which is not an element of B.

($A \subsetneq B$) A is a proper subset of B, *i.e.*, $A \subseteq B$ but $\exists\, b \in B$ such that $b \notin A$.

Remark. We need ZFC axioms because not any collection can be called a set. Read up on Russell's paradox.

Axiom 1.1 (the basic axiom)**.** *Every object is a set.*

Axiom 1.2 (axiom of extension)**.** *Two sets A, B are equal if they have exactly the same elements. In other words, $A = B \iff A \subseteq B$ and $B \subseteq A$*

Remark. As a consequnce, it doesn't matter whether a set contains multiple copies of an element.

$$A = \{1\}$$
$$B = \{1, 1, 1\}$$

Clearly $A \subseteq B$ and $B \subseteq A$, implying $A = B$.

Axiom 1.3 (axiom of existence)**.** *There is a set with no elements called the empty set, denoted by the symbol $\varnothing$.*

Axiom 1.4 (axiom of specification)**.** *Let A be a set. Let $P(a)$ denote a property that applies to every element in A, i.e., for each $a \in A$, either $P(a)$ is true or it is false. Then there exists a subset*

$$B = \{a \in A : P(a) \text{ is true}\}$$

Remark. We are forced to create sets only as subsets of other sets because of Russell's paradox. *From MathGarden:* A somewhat surprising result is that the axiom of specification implies for each set A the existence of an element (a set) x such that $x \notin A$. In other words, there is no set containing all sets of our mathematical universe.

Axiom 1.5 (axiom of pairing)**.** *Given two sets A, B, there exists a set which contains precisely A, B as its elements, which we denote by $\{A,\ B\}$.*

Remark. In particular, by letting $A = B$, we get a set containing only A, i.e., {A}. For example, we can have $\{\varnothing\}$, and $\{\varnothing, \{\varnothing\}\}$, etc.

Axiom 1.6 (axiom of unions)**.** *Given a set $\mathscr{F}$ of sets, there exists a set called the union of the sets in $\mathscr{F}$, denoted by $\bigcup_{A \in \mathscr{F}} A$, whose elements are precisely the elements of the elements of $\mathscr{F}$.*

$$a \in \bigcup_{A \in \mathscr{F}} A \iff a \in A \text{ for some } A \in \mathscr{F}$$

Remark. Intersection of a nonempty set of two or more sets and difference between two sets need not be defined as they follow from the previous axioms. (Exercise)

Axiom 1.7 (axiom of powers)**.** *Given a set A, there exists a set called power set of A denoted $\mathscr{P}(A)$, whose elements are precisely all the subsets of A.*

Remark. This axiom allows us to define ordered pairs as sets (assignment) (Isn't pairing sufficient?) and thus direct products, relations and functions.

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

How does this set exist?

Definition 1.3. A relation from set A to set B is a subset R of $A \times B$. For any $a \in A, b \in B$ we say $a\,R\,b$ iff $(a, b) \in R$.

- The *domain* of R is the set

$$\operatorname{dom}(R) = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

- The *range* of R is the set

$$\operatorname{ran}(R) = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$$

- R is called a *function* from A to B, denoted as $R : A \to B$ iff
  - $\operatorname{dom}(R) = A$
  - for each $a \in A$ there is (at most) one $b \in B$ such that $(a, b) \in R$.

Remark. A *bijective* function from A to B is an injective as well as surjective fuction from A to B.

Axiom 1.8 (axiom of regularity)**.** *Read up*

Axiom 1.9 (axiom of replacement)**.** *Read up*

Axiom 1.10 (axiom of choice)**.** *Read up*

Definition 1.4. Given a set A, its *successor* is the set

$$A^+ = A \cup \{A\}.$$

A set A is said to be *inductive* if $\varnothing \in A$ and for every $a \in A$, we have $a^+ \in A$.

Remark. The successor of A is guaranteed to exist by axiom of pairing and axiom of unions.
$\{A\}$ exists by axiom of pairing by letting $B = A$.
$A \cup \{A\}$ exists by applying axiom of unions on the set $\{A, \{A\}\}$ formed using axiom of pairing again.

$\{A\}$ can also be created as a subset (axiom of specification) of the power set (axiom of powers) of A.

Remark. The definition of an inductive set is very similar to the principle of mathematical induction in the Peano axioms.

Axiom 1.11 (axiom of infinity)**.** *There exists an inductive set.*

Lemma 1.5. *Let $\mathscr{F}$ be a nonempty set of inductive sets. (This exists by axiom of infinity and axiom of pairing). Then*

$$\bigcap_{A \in \mathscr{F}} A \text{ is inductive.}$$

Theorem 1.6. *There exists a* unique, minimal *inductive set* ω, *i.e., for any inductive set* S,

$$\omega \subseteq S$$

and if ω' *is any other inductive set satisfying this property,*

$$\omega = \omega'$$

Theorem 1.7. *The* ω *in theorem 1.6 is a Peano set with successor function* $a \mapsto a^+$.

Theorem 1.8 (principle of recursion)**.** *Let* A *be a set, and* $f : A \to A$ *be a function. Let* $a \in A$. *Then, there exists a function* $F : \omega \to A$ *such that*

(a) $F(\varnothing) = a$

(b) For some $b \in \omega$, *we have* $F(b^+) = f(F(b))$

1.2 Natural Numbers

Definition 1.9 (Peano addition)**.** Given a fixed $m \in \mathbb{N}$, the principle of recursion gives a unique function $\mathrm{sum}_m : \mathbb{N} \to \mathbb{N}$

(a) $\mathrm{sum}_m(0) = m$

(b) $\mathrm{sum}_m(n^+) = (\mathrm{sum}_m(n))^+$

Define

$$m + n := \mathrm{sum}_m(n)$$

Proposition 1.10. $2 + 3 = 5$

Remark. Note that $m^+ = \mathrm{sum}_m(0)^+ = \mathrm{sum}_m(0^+) = \mathrm{sum}_m(1) = m + 1$.
So we will now denote m^+ as $m + 1$.

Definition 1.11 (Peano multiplication)**.** Let $m \in \mathbb{N}$. By the recursion principle, $\exists$ a unique function

$$\mathrm{prod}_m : \mathbb{N} \to \mathbb{N}$$

such that

(a) $\mathrm{prod}_m(0) = 0$

(b) $\mathrm{prod}_m(n^+) = m + \mathrm{prod}_m(n)$

Theorem 1.12. *The following hold:*

(a) (Commutativity)

$$m + n = n + m$$

$$m \cdot n = n \cdot m$$

for all natural numbers m *and* n.

(b) *(Associativity)*

$$m + (n + k) = (m + n) + k$$

$$m \cdot (n \cdot k) = (m \cdot n) \cdot k$$

for all natural numbers m, n, k.

(c) *(Distributivity)*

$$m \cdot (n + k) = (m \cdot n) + (m \cdot k)$$

(d) $m + n = 0 \iff m = n = 0$ *for any* $m, n \in \mathbb{N}$

(e) $m \cdot n = 0 \iff m = 0$ *or* $n = 0$ *for any* $m, n \in \mathbb{N}$

(f) *(Cancellation)* $m+k = n+k \iff m = n$ *for any* $m, n, k \in \mathbb{N}$ *and if* $m \cdot k = n \cdot k$ *and* $k \neq 0$*, then* $m = n$.

1.2.1 Tao

Lemma 1.13. *For any natural number n, $n + 0 = n$*

Lemma 1.14. *For any natural numbers n and m, $n + m_{++} = (n + m)_{++}$*

Corollary 1.15. $n_{++} = n + 1$

Proposition 1.16. (Addition is commutative) *For any natural numbers n and m, $n + m = m + n$*

1.3 Fields, Ordered Sets and Ordered Fields

Definition 1.17. A field is a set F with 2 operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ such that

(F1) $+$ & $\cdot$ are commutative on F.

(F2) $+$ & $\cdot$ are associative on F.

(F3) $+$ & $\cdot$ satisfy distributivity on F, *i.e.*, $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$.

(F4) There exist 2 *distinct* elements, called 0 (additive identity) and 1 (multiplicative identity) such that

$$x + 0 = x$$

$$x \cdot 1 = x$$

for all $x \in F$

(F5) For every $x \in F$, $\exists\, y \in F$ such that

$$x + y = 0$$

(F6) For every $x \in F \setminus \{0\}$, $\exists\, z \in F$ such that

$$x \cdot z = 1$$

Remark. We are tempted to call y in (F5) "-x" and z in (F6) "$\frac{1}{x}$" but y, z haven't been proven to be unique yet. Prove this. Proved as lemma 1.21
Once we have proven this, we can also define $a - b := a + (-b)$ and $a/b = a \cdot \frac{1}{b}$.

Theorem 1.18. *$(F, +, \cdot)$ is a field. Then for all x,*

$$0 \cdot x = x \cdot 0 = 0$$

Definition 1.19. A set A with a relation $<$ is called an *ordered set* if

(O1) (Trichotomy) For every $x, y \in A$, exactly one of the following holds.

$$x < y, \quad x = y, \quad y < x$$

(O2) (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Notation: $x < y$ is read as "x is less than y"
$x \leq y$ means $x < y$ or $x = y$, read as "x is less that or equal to y".
$x > y$ is read as "x is greater that y" and equivalent to $y < x$.

Definition 1.20. An *ordered field* is a set that admits two operations $+$ and $\cdot$ and relation $<$ so that $(F, +, \cdot)$ is a field and $(F, <)$ is an ordered set and:

(O3) For $x, y, z \in F$, if $x < y$ then $x + z < y + z$.

(O4) For $x, y \in F$, if $0 < x$ and $0 < y$ then $0 < x \cdot y$.

Lemma 1.21. *Given a field $(F, +, \cdot)$: For any element a in a field F, there exists ony one b such that $a + b = 0$. We will denote this b as $-a$. Similarly for any a in $F \setminus \{0\}$ there exists only one $b \in F$ such that $ab = 1$. We will denote this b as $\frac{1}{a}$ or a^{-1}.*

Lemma 1.22. $-(-a) = a = (a^{-1})^{-1}$

Lemma 1.23. *For any field $(F, +, \cdot)$, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.*

Theorem 1.24. *For any field $(F, +, \cdot)$, $0 < 1$.*

Remark. "a contradiction" is not necessary to state for the proof to be complete. See this discussion at MS Teams.

1.4 Upper bounds & least upper bounds

Definition 1.25. A non-empty subset $S \subseteq F$ is said to be *bounded above* in F if there exists a $b \in F$ such that

$$a \leq b \;\forall\, a \in S$$

Here, b is called an *upper bound* of S. If $b \in S$, then b is a *maximum* of S.

Example.

$$S = \{x \in F : 0 \leq x \leq 1\}$$
$$T = \{x \in F : 0 \leq x < 1\}$$

Both S and T are bounded above as 1 is an upper bound for both.
1 is in fact, a maximum of S.

Remark. If a maximum exists, it must be unique (why?).

Remark. Upper bounds may not be unique.

Definition 1.26. Let $S \subseteq F$ be bounded above. An element $b \in F$ is said to be a *least upper bound* of S or a *supremum* of S if:

(a) b is an upper bound of S.

(b) If for $c \in F$, $c < b$, then c is not an upper bound of S. In other words, for any $c < b$, $\exists\, s_c \in S$ such that $c < s_c$.
Contrapositive: If c is an upper bound of S, then c is not less than $b \iff c \geq b$.

Remark. There is only one supremum of S.

Example.

$$\sup\{x \in F : 0 \leq x < 1\} = 1$$

1.5 The Real Numbers

Theorem 1.27 (Archimedean property of $\mathbb{R}$)**.** *Let $x, y \in \mathbb{R}$ and $x > 0$, then $\exists\, n \in \mathbb{N}$ such that*

$$n \cdot x > y$$

2 Sequences & Series

2.1 Sequences

Definition 2.1. A sequence in $\mathbb{R}$ is a function $f : \mathbb{N} \to \mathbb{R}$. We denote this sequence by $\{a_n\}_{n\in\mathbb{N}}$, where

$$a_n = f(n) \quad \forall\, n \in \mathbb{N}$$

and a_n is called the n^{th} term of $\{a_n\}_{n\in\mathbb{N}}$.

Remark. $\{a_n\} \subseteq \mathbb{R}$ will denote a sequence of real numbers.
The numbering starts at 0 technically, but doesn't really matter. We will often omit the subscript $n \in \mathbb{N}$ and start indexing from some other point.

Definition 2.2. We say that a sequence $\{a_n\} \subseteq \mathbb{R}$ is *convergent* (in $\mathbb{R}$) if $\exists\, L \in \mathbb{R}$ such that for each $\varepsilon > 0$, $\exists\, N_{\varepsilon,L} \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \forall\, n \geq N_{\varepsilon,L}$$

We will call L *a* limit of $\{a_n\}$ and we write:

$$a_n \to L \text{ as } n \to \infty$$

A sequence $\{a_n\}$ is said to be *divergent* if it is not convergent, *i.e.*, $\forall\, L \in \mathbb{R}$ and $N_L \in \mathbb{N}$, $\exists\, \varepsilon > 0$ and $N \geq N_L$ such that

$$|a_N - L| > \varepsilon$$

Theorem 2.3 (Uniqueness of limits)**.** *Suppose L_1 and L_2 are limits of a (convergent) sequence $\{a_n\} \in \mathbb{R}$. Then $L_1 = L_2$.*

Example. (a) Let $\{a_n\} = \frac{1}{n^p}\ \forall\, n \in \mathbb{P}$, where $p > 0$.

$$\lim_{n\to\infty} a_n = 0$$

Proof. Let $\varepsilon > 0$.
By the Archimedean property of $\mathbb{R}$ applied to $x = \varepsilon^{\frac{1}{p}}$ and $y = 1$, $\exists\, N \in \mathbb{P}$ such that:

$$N\varepsilon^{\frac{1}{p}} > 1 \implies \varepsilon^{\frac{1}{p}} > \frac{1}{N} \implies \varepsilon > \frac{1}{N^p}$$

Let $n \geq N$. Then

$$\begin{aligned}\left|\frac{1}{n^p} - 0\right| &= \frac{1}{n^p}\\ &\leq \frac{1}{N^p}\\ &< \varepsilon\end{aligned}$$

□

(b) $\{(-1)^n\}_{n\in\mathbb{P}}$ is divergent.

Proof. Suppose there exists a limit L.
Let $\varepsilon = 1$.
Then $\exists\, N \in \mathbb{P}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$.
$|a_{2N} - L| < 1 \implies |L - 1| < 1$.
$|a_{2N+1} - L| < 1 \implies |L + 1| < 1$.
$|1 - L + L + 1| \leq |1 - L| + |L + 1| < 2$
$\implies 2 < 2$. Contradiction. □

Definition 2.4. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be *bounded* if $\exists\, M > 0$ such that $|a_n| < M\,\forall\, n \in \mathbb{N}$.

Theorem 2.5. *Every convergent sequence is bounded.*

Definition 2.6. A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotonically increasing* if $a_n \leq a_{n+1}\,\forall\, n \in \mathbb{N}$.
A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotonically decreasing* if $a_n \geq a_{n+1}\,\forall\, n \in \mathbb{N}$.
A sequence $\{a_n\} \subseteq \mathbb{R}$ is said to be *monotone* if it is either monotonically increasing or monotonically decreasing.

Theorem 2.7. *A monotone sequence is convergent iff it is bounded.*

Remark (Warning!)*.* Divergent sequences may diverge for different reasons!

- $\{(-1)^n\}$ is bounded but divergent.
- $\{n\}$ is unbounded and divergent, to $+\infty$
- $\{(-1)^n n\}$ is unbounded and divergent, but not to $\pm\infty$.

Definition 2.8. We say that a sequence diverges to $+\infty$ if $\forall\, R \in \mathbb{R}$, $\exists\, N_R \in \mathbb{N}$ such that $a_n > R \;\forall\, n \geq N_R$.
We say that a sequence diverges to $-\infty$ if $\forall\, R \in \mathbb{R}$, $\exists\, N_R \in \mathbb{N}$ such that $a_n < R \;\forall\, n \geq N_R$.
We write $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, but this is purely notational and does not mean "$\{a_n\}$ has a limit".

Theorem 2.9 (Tao Theorem 6.1.19)**.** *Suppose $\{b_n\}$ converges to $b \neq 0$ (and $\exists\, M \in \mathbb{N}$ such that $b_n \neq 0 \;\forall\, n \geq M$.)*
Then $\left\{\frac{1}{b}\right\}_{n\geq M} \to \frac{1}{b}$ as $n \to \infty$.

2.2 Infinite series

Definition 2.10. An infinite series is a *formal expression* of the form

$$a_0 + a_1 + a_2 + \dots, \text{ or } , \sum_{n=0}^{\infty} a_n$$

Given $\sum_{n=0}^{\infty} a_n$, its sequence of partial sums (sops) is $\{s_n\}_{n=0}^{\infty}$ where

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ &. \\ &. \\ s_n &= a_0 + a_1 + \dots a_n \end{aligned}$$

We say that $\sum a_n$ is *convergent* with sum s if $\lim_{n\to\infty} s_n = s$. Otherwise, we say that $\sum a_n$ is divergent.

Example. (a) (Harmonic series) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof. $\{s_n\}$ is a monotonically increasing sequence.

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} \\
&> 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} \\
s_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} \\
&> 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + \cdots + 2^{k-1} \cdot \frac{1}{2^k} \\
&= 1 + \frac{k}{2}
\end{aligned}$$

Thus, given any $R \in \mathbb{R}$, $\exists\, k \in \mathbb{N}$ such that $s_{2^k} > R$.
$\implies \{s_n\}$ is divergent by MCT. □

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof.

$$\begin{aligned}
s_1 &= 1 \\
s_n &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\
&< 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \\
&= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= 1 + 1 - \frac{1}{n} \\
&< 2 \ \forall\, n \in \mathbb{N}
\end{aligned}$$

So $\{s_n\}$ is a monotonically increasing sequence that is bounded above.
$\implies \{s_n\}$ is convergent. □

Remark. (Telescoping sum)

Theorem 2.11. *Suppose* $\sum a_n$ *is convergent. Then*

$$\lim_{n \to \infty} a_n = 0$$

Example (Geometric Series). Let $x \in \mathbb{R}$. Then

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & |x| < 1 \\ \text{diverges} & |x| \geq 1 \end{cases}$$

Theorem 2.12 (Comparison test)**.** *Suppose there exist constants $M \in \mathbb{N}$ and $0 < C$ such that*

$$0 \leq a_n \leq Cb_n \quad \forall\, n \geq M$$

If $\sum b_n$ converges, then $\sum a_n$ converges. In other words, If $\sum a_n$ diverges, $\sum b_n$ diverges.

Example. Let $p \in \mathbb{R}$. Claim:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

Theorem 2.13 (Ratio test)**.** *Let $\sum a_n$ be a series of positive terms. Suppose*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$$

Then,

(a) If $L < 1$, the series converges.

(b) If $L > 1$, the series diverges.

(c) If $L = 1$, the test is inconclusive.

Theorem 2.14. *Suppose $\sum a_n$ and $\sum b_n$ converge with sums a and b respectively. Then, for constants l and m, $\sum la_n + mb_n$ converges to $la + mb$. Suppose $\sum |a_n|$ and $\sum |b_n|$ converge. Then, so does $\sum |la_n + mb_n|$ for any choice of l and m in $\mathbb{R}$.*

Corollary 2.15. *Suppose $\sum a_n$ converges and $\sum b_n$ diverges. Let $m \in \mathbb{R}\backslash\{0\}$. Then, $\sum(a_n + b_n)$ diverges, and $\sum mb_n$ diverges.*

Definition 2.16. A series $\sum a_n$ of real numbers is said to *converge absolutely* if $\sum |a_n|$ converges. A series $\sum a_n$ of real numbers is said to *converge conditionally* if $\sum |a_n|$ diverges but $\sum a_n$ converges.

Theorem 2.17. *If $\sum a_n$ converges absolutely, it must converge. Moreover, $|\sum a_n| \leq \sum |a_n|$.*

Example. $\sum \frac{(-1)^n}{n}$ is convergent.

Theorem 2.18 (Alternating Series Test/Leibniz Test)**.** *Suppose $\{a_n\}$ is a decreasing sequence of positive numbers going to 0. Then, $\sum(-1)^n a_n$ converges. Denoting the sum by S, we have that*

$$0 < (-1)^n(S - s_n) < a_{n+1}$$

Remark. The estimate is AST allows us to estimate sums of alternating series within any prescribed error. For instance, to know $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ up to an error of 0.01. I need to find n so that

$$|S - s_n| < \frac{1}{100}.$$

Take $n = 99$, or the sum of the first 99 terms.

3 Limits & Continuity

3.1 Limit of a function

Example. (a) For $f(x) = c$, $c \in \mathbb{R}$,

$$\lim_{x \to p} f(x) = c$$

Choose $\delta = 1$. $0 < |x - p| < \delta \implies |f(x) - c| = 0 < \varepsilon \; \forall \, \varepsilon > 0$.

(b) For $f(x) = x$,

$$\lim_{x \to p} f(x) = p$$

Choose $\delta = \varepsilon$. $0 < |x - p| < \delta \implies |f(x) - p| < \varepsilon$.

(c) For $f(x) = \sqrt{x}$ and $p > 0$,

$$\lim_{x \to p} f(x) = \sqrt{p}$$

$$\begin{aligned} &|\sqrt{x} - \sqrt{p}| < \varepsilon \\ \iff &\frac{|x - p|}{|\sqrt{x} + \sqrt{p}|} < \varepsilon \end{aligned}$$

Take $\delta = \min\{p, \sqrt{p}\varepsilon\}$ (this is to make sure f is defined for all points in $N_\delta(p) \setminus \{p\}$). Now $|x - p| < \delta \implies$

$$\begin{aligned} \frac{|x - p|}{|\sqrt{x} + \sqrt{p}|} &< \frac{\delta}{|\sqrt{x} + \sqrt{p}|} \\ &= \frac{\sqrt{p}\varepsilon}{|\sqrt{x} + \sqrt{p}|} \\ &< \varepsilon \end{aligned}$$

(d) For $f(x) = \frac{1}{x}, x \neq 0$,

$$\lim_{x \to 0} f(x) \text{ does not exist}$$

Proof. Suppose $\exists\, L \in \mathbb{R}$ such that

$$\lim_{x \to 0} f(x) = L$$

Choose $\varepsilon = \frac{1}{L}$. Then $\exists\, \delta > 0$ such that $0 < |x - 0| < \delta \implies |f(x) - L| < 1$. By the Archimedean property, $\exists\, N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$.

Now $0 < \frac{1}{N+2} < \frac{1}{N} < \delta$. Thus by our hypothesis, $|N + 2 - L| < 1$ and $|N - L| < 1 \implies |2| < |N + 2 - L| + |N - L| < 1 + 1 = 2$. Contradiction. $\square$

Theorem 3.1 (Limit laws for functions)**.** *Suppose f and g are functions such that*

$$\lim_{x \to p} f(x) = a, \qquad \lim_{x \to p} g(x) = b.$$

Then,

$$\lim_{x \to p} (f \pm g)(x) = a \pm b \tag{1}$$

$$\lim_{x \to p} (f \cdot g)(x) = a \cdot b \tag{2}$$

$$\lim_{x \to p} (f/g)(x) = a/b \tag{3}$$

3.2 Continuity

Definition 3.2. Let $S \subseteq \mathbb{R}$ be a (nonempty) subset, $f : S \to \mathbb{R}$ and $p \in S$. We say that f is continuous at p iff:
for every $\varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that

$$|x - p| < \delta_\varepsilon \;\wedge\; x \in S \implies |f(x) - f(p)| < \varepsilon$$

We say that f is continuous on S iff f is continuous at each $p \in S$.

Remark. It is possible that $\exists\, \delta$ such that $N_\delta(p) \cup S = \{p\}$. *E.g.*, $S = \mathbb{N}, p = 0, \delta \leq 1$

Remark. If f is defined on some interval (a, b) containing p, then this definition is equivalent to

$$\lim_{x \to p} f(x) = f(p)$$

How? For any $\varepsilon > 0$, choose $\delta = \min\{\delta_\varepsilon, b - p, p - a\}$. Then f is defined on all points in $N_\delta(p)$, and for all $x \in N_\delta(p)$, we have $f(x) \in N_\varepsilon(f(p))$. Thus

$$\lim_{x \to p} f(x) = f(p)$$

Theorem 3.3 (Algebraic laws for continuity)**.** *Suppose f and g are continuous at $p \in S$. Then so are $f \pm g, fg$ and if $g(p) \neq 0, f/g$.*

Theorem 3.4. *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be continuous functions such that $f(A) = range(f) \subseteq B$. Then,*

$$g \circ f(x) = g(f(x)) : A \to \mathbb{R}$$

is continuous.

Theorem 3.5 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $y \in \mathbb{R}$ is a number between $f(a)$ and $f(b)$, i.e., $y \in [f(a), f(b)]$. Then $\exists\, c \in [a, b]$ such that*

$$f(c) = y$$

Corollary 3.6 (Bolzano)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)$ and $f(b)$ take opposite signs. Then $\exists\, c \in (a, b)$ such that $f(c) = 0$.*

Remark. Bolzano's statement is equivalent to the IVT (let $g = f - y$).

Theorem 3.7 (The Borsuk-Ulam Theorem : baby version)**.** *Let S be the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$. Let $f : S \to \mathbb{R}$ be a continuous function. Then there exists a pair of antipodal points on the circle which have the same value of f.*

Lemma 3.8. *Let a_n, b_n be convergent sequences such that $a_n \leq b_n$ for all n (large enough). Then*

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Definition 3.9. A function $f : S \to \mathbb{R}$ is said to be *bounded above* on S if $\exists\, U \in \mathbb{R}$ such that $f(x) \leq U \;\forall\, x \in S$.

f is said to be *bounded* if $\exists\, M > 0$ such that $|f(x)| < M \;\forall\, x \in S$.

Theorem 3.10 (Continuous functions on closed, bounded intervals are bounded.)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is a bounded function.*

Definition 3.11. A function $f : S \to \mathbb{R}$ is said to have a *global maximum* on S at a point $p \in S$ if $f(x) \leq f(p) \;\forall\, x \in S$.

Theorem 3.12 (Extreme value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains both a global maximum and a global minimum in $[a, b]$.*

Corollary 3.13. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then (using IVT),*

$$f([a, b]) = [\min_{[a,b]} f, \max_{[a,b]} f]$$

4 Differentiation

Definition 4.1. Let $f : (a, b) \to \mathbb{R}$ be a function and $p \in (a, b)$. We say that f is differentiable in (a, b) if

$$\lim_{h\to 0} \frac{f(p + h) - f(p)}{h} = \lim_{x\to p} \frac{f(x) - f(p)}{x - p}$$

exists, and the limit is called the derivative of f at p, denoted $f'(p)$.

If f is differentiable on each p in (a, b), it is said to be differentiable on (a, b) and $f' : (a, b) \to \mathbb{R}$ is called the derivative of f on (a, b).

Theorem 4.2 (Differentiability $\implies$ continuity)**.** *Let $f : (a, b) \to \mathbb{R}$ be differentiable at $p \in (a, b)$. Then f is continuous at p.*

Theorem 4.3.