Gödel's Incompleteness Theorems

The way Gödel did it

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Formal Proof Systems

- A formal proof system provides a notion of well-formed formulas (WFFs), and a set of axioms along with rules of inference.
- A *proof* is a sequence of WFFs, each of which is either an axiom or follows from previous statements by a rule of inference.
- A *sentence* is a WFF with no free variables.
- A sentence is *provable* if there is a proof of it. For a sentence ϕ, we write

  $$\vdash \phi$$

  to denote that ϕ is provable. We say that ϕ is a *theorem*.

Theories and Theorems

- For a sentence ϕ, we write

$$\models \phi$$

  to denote that ϕ is *true*.

Consistency and Completeness

- A formal proof system (taken with a theory) is *consistent* if every theorem is true.

$$\vdash \phi \implies \vDash \phi$$

- A formal proof system is *complete* if every true sentence is a theorem.

$$\vDash \phi \implies \vdash \phi$$

The Language of Arithmetic

The language of arithmetic consists of:

- the constants 0 and 1,
- the binary operations $+$ and $\cdot$,
- the relation $=$,
- symbols for variables, and
- symbols from first-order logic.

First Incompleteness Theorem

Theorem (1931)
There cannot exist a sound and complete proof system for arithmetic.

Efforts to prove that mathematics is consistent

There existed a significant number of people who thought that mathematics is consistent. That is there exists a set of axioms which can be used to prove every true statement.

Their main idea was to start with a set of axioms and keep on including the statements which are true but unprovable.

The Gigachad Big-Brain Metamove

Let ϕ be the statement “This statement is unprovable.”

An Encoding of Symbols

- We can assign a unique positive integer to each symbol in the language. For example,

$$\begin{array}{rr} \top \mapsto 1 & \bot \mapsto 2 \\ \wedge \mapsto 3 & \vee \mapsto 4 \\ \rightarrow \mapsto 5 & \leftrightarrow \mapsto 6 \\ \neg \mapsto 7 & = \mapsto 8 \\ \forall \mapsto 9 & \exists \mapsto 10 \\ 0 \mapsto 11 & 1 \mapsto 12 \\ + \mapsto 13 & \cdot \mapsto 14 \\ x \mapsto 15 & y \mapsto 16 \\ z \mapsto 17 & \ldots \mapsto \ldots \end{array}$$

An Encoding of WFFs

- Let p_n denote the n-th prime number.
- We define the encoding of a WFF ϕ as

$$\phi = p_1^{\phi_1} \cdot p_2^{\phi_2} \cdot \ldots \cdot p_k^{\phi_k}$$

where $\phi_1, \phi_2, \ldots, \phi_k$ are the symbols in ϕ.