

Homework 3
ECE 6790 - Information Processing Models in Neural Systems

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Problem 1

(a). The estimated marginal distribution is:

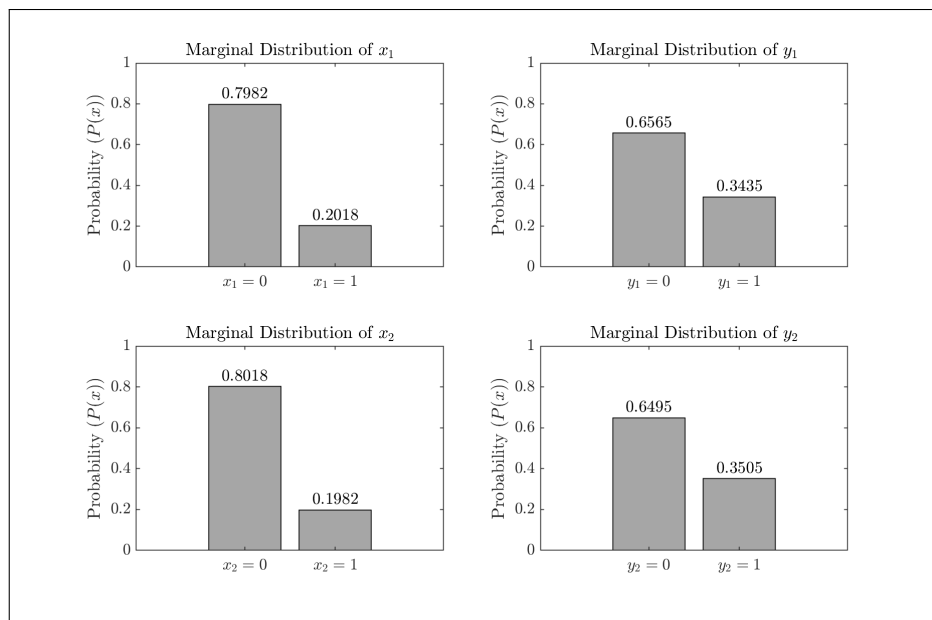
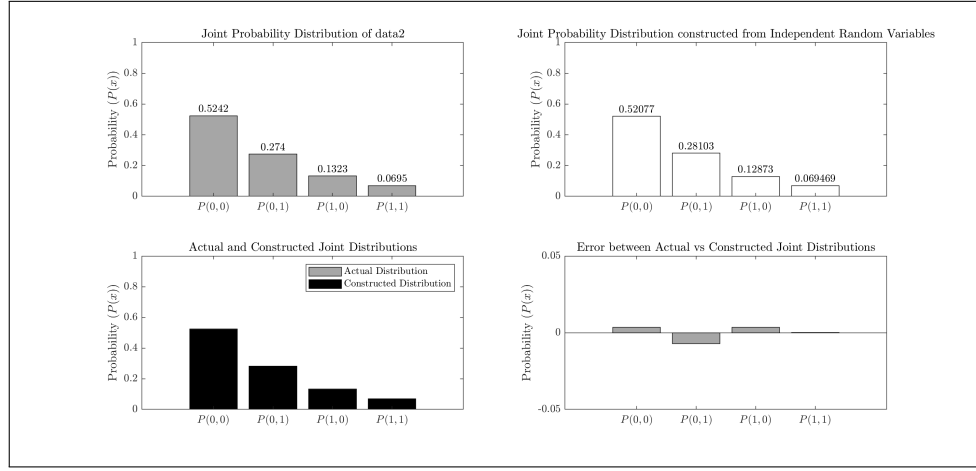


Figure 1: Marginal Probability Distribution of x_1 , y_1 , x_2 and y_2

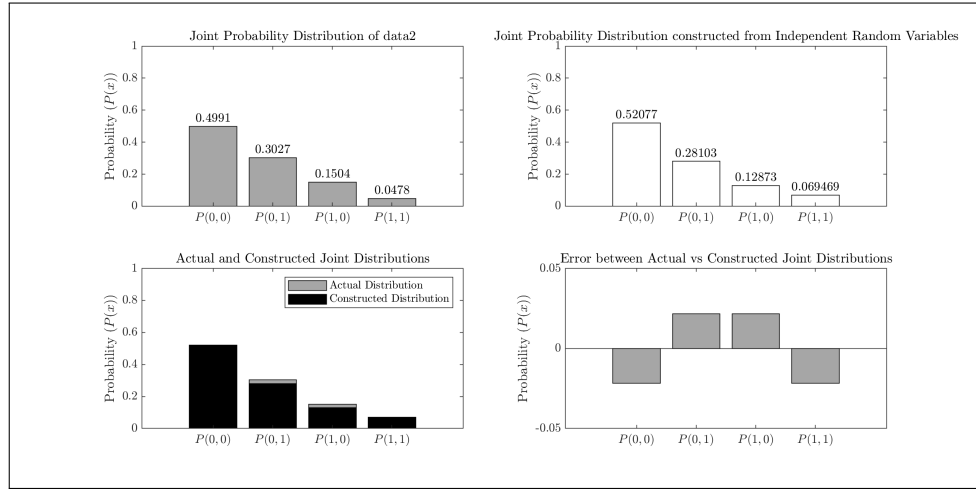
(b). The Probability distribution is said to be independent when,

$$P(X, Y) = P(X) \cdot P(Y)$$

Assuming both datasets are independent, we compare the reconstructed distribution assuming independence and actual distribution. The results of the comparison are as follows:



(a) Actual vs Constructed Distribution for Data 1



(b) Actual vs Constructed Distribution for Data 2

Figure 2: Actual vs Constructed Distribution for Data 1 and Data 2

From observing the error difference between the actual and reconstructed distributions in Data 2, we can safely conclude that $P(X, Y) \neq P(X) \cdot P(Y)$ for data 2 and thus, data 2 is not independent.

Problem 2

(a). The Generated Data is:

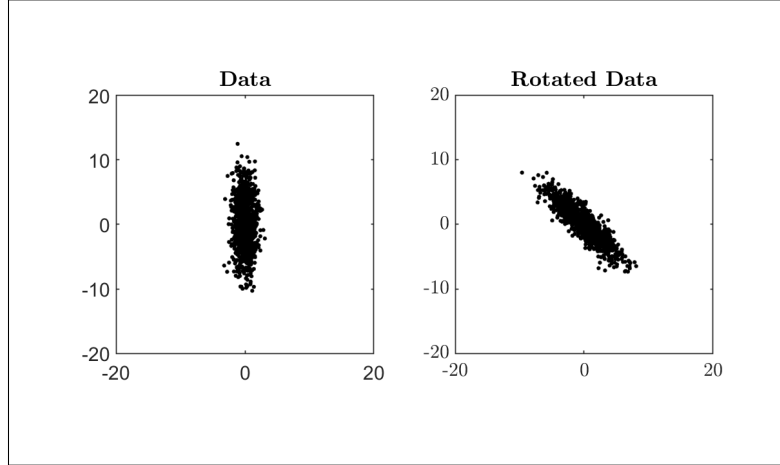


Figure 3: Original and Rotated Distribution

The marginal Distribution of the generated data is:

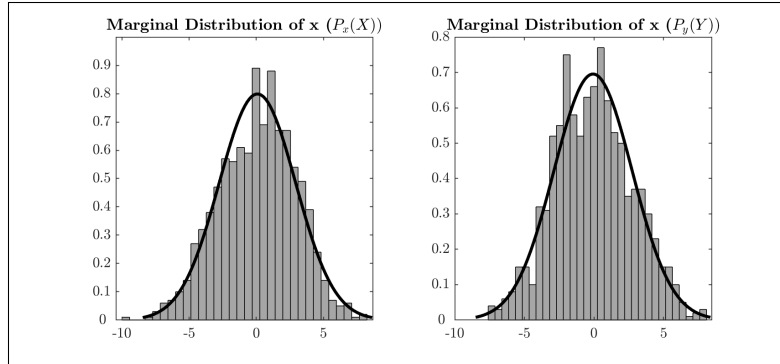


Figure 4: Marginal Distributions

(b). The covariance of a rotated data set with given covariance (Σ_{org}) is:

$$\begin{aligned}\Sigma &= A^{-1}\Sigma_{org}A \\ &= \begin{bmatrix} 8.5 & -7.5 \\ -7.5 & 8.5 \end{bmatrix}\end{aligned}$$

The MATLAB estimated Covariance is:

$$\Sigma = \begin{bmatrix} 8.5136 & -7.4786 \\ -7.4786 & 8.4395 \end{bmatrix}$$

(c). The Eigenvalues of the covariance Matrix are:

$$\lambda_1 = 0.9978, \lambda_2 = 15.9552]$$

The Corresponding Eigenvectors are:

$$\mathbf{V} = \begin{bmatrix} -0.7054 & -0.7089 \\ -0.7089 & 0.7054 \end{bmatrix}$$

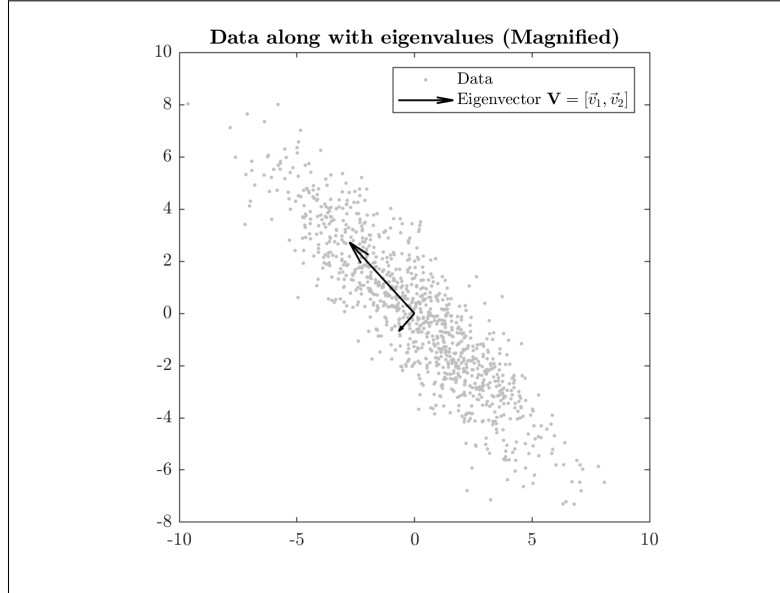


Figure 5: Scaled Eigenvectors of Covariance Matrix overlapped on Data

(d). The new Dataset is:

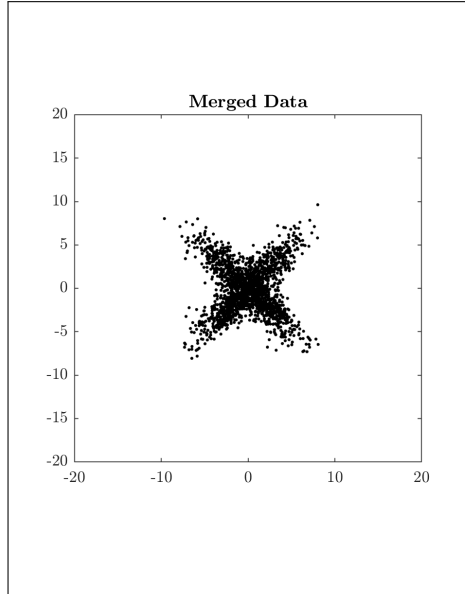


Figure 6: Merged Distribution

The Corresponding Marginal distribution is:

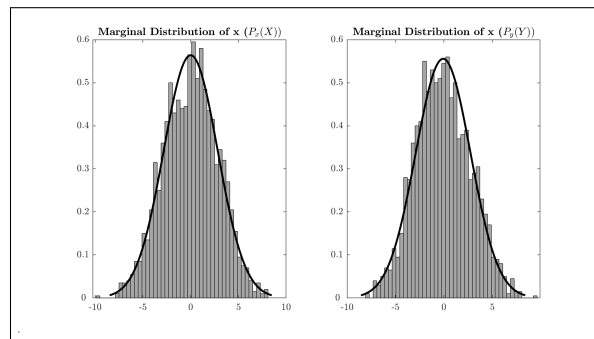


Figure 7: Marginal distribution of the Merged Data

The Data is not Gaussian Distributed anymore.

(d). The Eigenvalues for the new dataset are:

$$\lambda_1 = 8; \lambda_2 = 8$$

The corresponding Eigenvectors are:

$$\mathbf{V} = \begin{bmatrix} 0.0649 & -0.9979 \\ -0.9979 & -0.0649 \end{bmatrix}$$

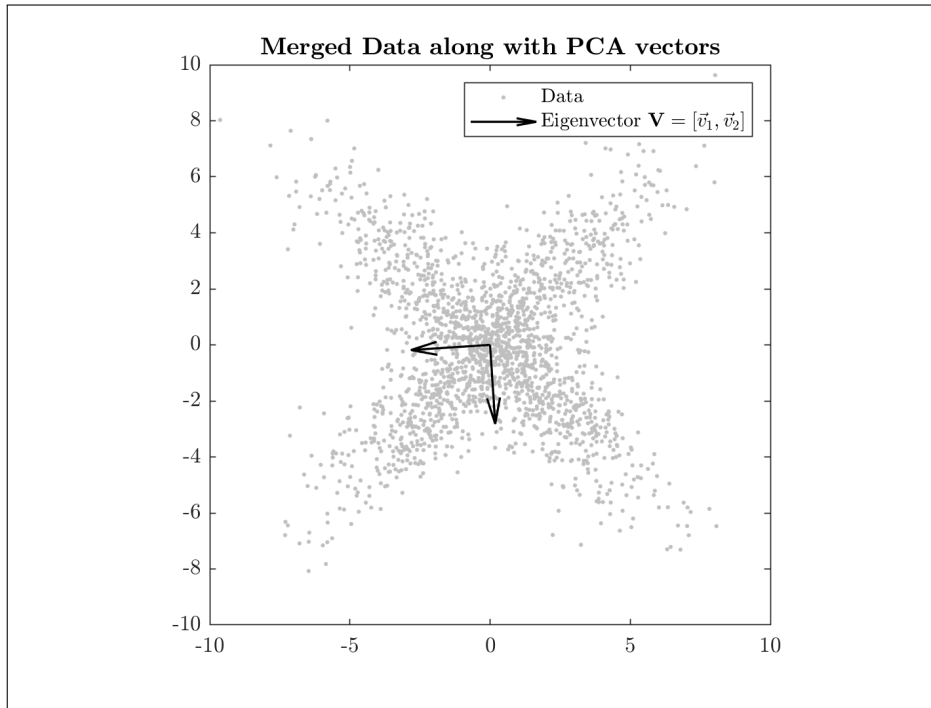


Figure 8: PCA Vectors along with merged data

Problem 3

(a).

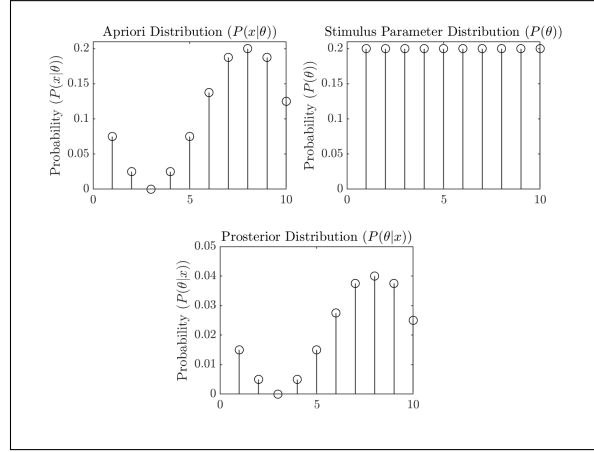


Figure 9: Prosterior Distribution

(b).

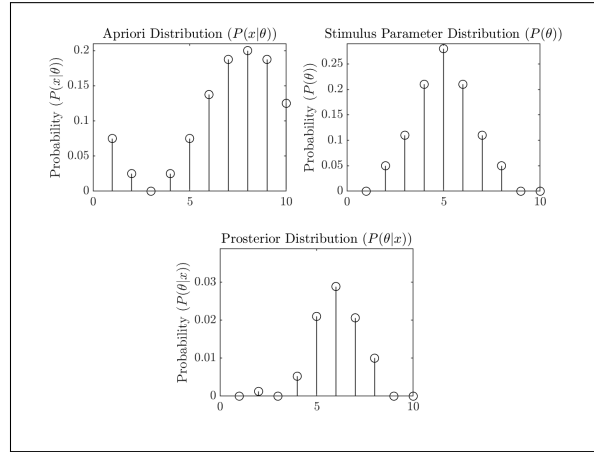


Figure 10: Prosterior Distribution

(c).

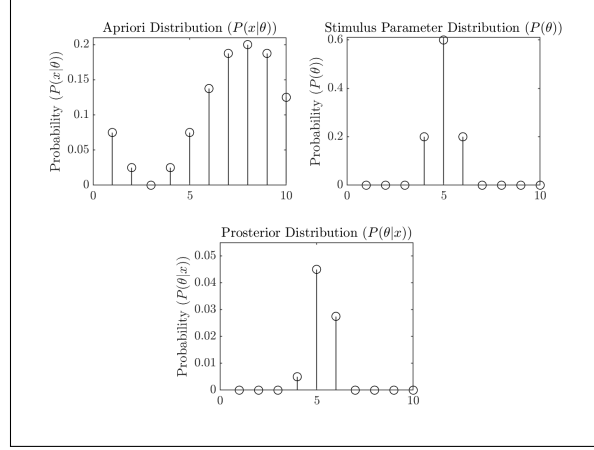


Figure 11: Prosterior Distribution

(d). The MLE and MAP of the distribution is defined as:

$$MLE = \underset{\theta}{argmax} P(x|\theta)$$

$$MAP = \underset{\theta}{argmax} P(\theta|x)$$

In case (a), $P(\theta)$ is equally distributed. Therefore, all the values of θ are equally likely. Thus, the MLE and MAP are the equal ($MLE = MAP = 8$).

In case (b), $P(\theta)$ is more concentrated towards the centre between values 4,5 and 6. However, $P(x|\theta)$ is more concentrated towards values 7, 8 and 9. Thus, the MAP is not equal to MLE as the latter considers A priori distribution which is different from the Prosterior distribution. Here, $MLE = 8$, $MAP = 6$.

In case (c), $P(\theta)$ is centrally concentrated at 5. Thus, considering the prosterior distribution ($P(\theta|x)$), the MAP would be skewed towards 5. Here, $MLE = 8$, $MAP = 5$.

In all the cases, since the A Priori distribution is the same, the MLE remains the same. ($MLE = 8$).

Appendix

Code for Problem 1

```
1 %% Homework - 3
2 % Question: 1
3 % Author: Harsh Bhate
4 %% Pre-loading and Pre-processing.
5 % Loading Data to MATLAB
6 clear all;
7 clc;
8 load ('HW_corr_data.mat');
9 %% Part 1(a). Estimating Marginal Distribution from Joint
    Distribution
10 [px1,py1,px2,py2] = marginal (data1, data2);
11 marginal_dist = [px1,py1,px2,py2]
12 %% Part 1(b). Checking for Independence
13 [p00,p01,p10,p11] = conditional (data1);
14 conditional_dist = [p00,p01,p10,p11]
15 actualVsMarginal (px2,py2, conditional_dist)
16 %% Marginal PDF
17 function [fx1,fy1,fx2,fy2] = marginal(dist1, dist2)
18     % Function to find the marginal PDF and plot it for
19     % two distributions dist1 and dist2.
20
21     % Extracting the x and y points for each
        distribution
22     x1 = dist1(:,1);
23     y1 = dist1(:,2);
24     x2 = dist2(:,1);
25     y2 = dist2(:,2);
26     % Computing the PDF
27     fx1 = [(length(x1) - sum(x1))/length(x1), sum(x1)/
        length(x1)];
28     fy1 = [(length(y1) - sum(y1))/length(y1), sum(y1)/
        length(y1)];
29     fx2 = [(length(x2) - sum(x2))/length(x2), sum(x2)/
        length(x2)];
30     fy2 = [(length(y2) - sum(y2))/length(y2), sum(y2)/
        length(y2)];
31     % Setting the labels for plotting
32     x1Labels ={'$x_{1} = 0$'; '$x_{1} = 1$'};
33     y1Labels ={'$y_{1} = 0$'; '$y_{1} = 1$'};
34     x2Labels ={'$x_{2} = 0$'; '$x_{2} = 1$'};
```

```

35 y2Labels ={'$y_{2} = 0$'; '$y_{2} = 1$'};
36 % Plotting the graph
37 figure();
38 subplot (2,2,1);
39 bar(fx1, 'stacked', 'FaceColor',[0.65 0.65 0.65]);
40 set(gca, 'xticklabel', x1Labels)
41 title ('Marginal Distribution of  $x_{1}$ ', '
    Interpreter','latex')
42 ylabel ('Probability ( $P(x)$ )', 'Interpreter','latex')
43 ylim([0.0 1.0])
44 text(1:length(fx1),fx1,num2str(fx1'),...
45      'vert','bottom','horiz','center', 'Interpreter','
    latex');
46 set(gca, 'TickLabelInterpreter','latex')
47 subplot (2,2,2);
48 bar(fy1, 'stacked', 'FaceColor',[0.65 0.65 0.65]);
49 set(gca, 'xticklabel', y1Labels)
50 title ('Marginal Distribution of  $y_{1}$ ', '
    Interpreter','latex')
51 ylabel ('Probability ( $P(x)$ )', 'Interpreter','latex'
    )
52 ylim([0.0 1.0])
53 text(1:length(fy1),fy1,num2str(fy1'),...
54      'vert','bottom','horiz','center', 'Interpreter','
    latex');
55 set(gca, 'TickLabelInterpreter','latex')
56 subplot (2,2,3);
57 bar(fx2, 'stacked', 'FaceColor',[0.65 0.65 0.65]);
58 set(gca, 'xticklabel', x2Labels)
59 title ('Marginal Distribution of  $x_{2}$ ', '
    Interpreter','latex')
60 ylabel ('Probability ( $P(x)$ )', 'Interpreter','latex'
    )
61 ylim([0.0 1.0])
62 text(1:length(fx2),fx2,num2str(fx2'),...
63      'vert','bottom','horiz','center', 'Interpreter','
    latex');
64 set(gca, 'TickLabelInterpreter','latex')
65 subplot (2,2,4);
66 bar(fy2, 'stacked', 'FaceColor',[0.65 0.65 0.65]);
67 set(gca, 'xticklabel', y2Labels)
68 title ('Marginal Distribution of  $y_{2}$ ', '
    Interpreter','latex')
69 ylabel ('Probability ( $P(x)$ )', 'Interpreter','latex'
    )
70 ylim([0.0 1.0])

```

```

71     text(1:length(fy2),fy2,num2str(fy2'),...
72         'vert','bottom','horiz','center','Interpreter','
           latex');
73     set(gca,'TickLabelInterpreter','latex')
74 end
75 %% Function to plot continuous PDF
76 function [p00, p01, p10, p11] = conditional(dist)
77     % Function to plot conditional prob of binomial
       distribution dist
78
79     % Calculating the conditional PDF
80     p00 = sum(dist(:, 1) == 0 & dist(:, 2) == 0)/length(
       dist);
81     p01 = sum(dist(:, 1) == 0 & dist(:, 2) == 1)/length(
       dist);
82     p10 = sum(dist(:, 1) == 1 & dist(:, 2) == 0)/length(
       dist);
83     p11 = sum(dist(:, 1) == 1 & dist(:, 2) == 1)/length(
       dist);
84     % Labels
85     xLabels ={'$P(0,0)$','$P(0,1)$','$P(1,0)$','$P(1,1)$
       '};
86     % Plotting
87     figure();
88     F = [p00, p01, p10, p11];
89     bar(F, 'stacked','FaceColor',[0.65 0.65 0.65]);
90     set(gca,'xticklabel',xLabels)
91     title('Joint Probability Distribution', ...
92         'Interpreter','latex')
93     ylabel('Probability ($P(x)$)', 'Interpreter','latex'
       )
94     ylim([0.0 1.0])
95     text(1:length(F),F,num2str(F'),...
96         'vert','bottom','horiz','center','Interpreter','
           latex');
97     set(gca,'TickLabelInterpreter','latex')
98 end
99 %% Function to plot Conditional PDF
100 function actualVsMarginal (Fx, Fy, conditional)
101     % Function to plot the marginal and conditional
       prob
102
103     % Extracting Distributions
104     px0 = Fx(1)
105     px1 = Fx(2)
106     py0 = Fy(1)

```

```

107     py1 = Fy(2)
108     % Plotting independence assumed distributions
109     p_00 = px0*py0;
110     p_01 = px0*py1;
111     p_10 = px1*py0;
112     p_11 = px1*py1;
113     P_ind = [p_00, p_01, p_10, p_11]
114     % Labels
115     xLabels ={'$P(0,0)$'; '$P(0,1)$'; '$P(1,0)$'; '$P(1,1)$'
116             '};
117     % Plotting
118     figure();
119     subplot(2,2,1)
120     bar(conditional, 'stacked', 'FaceColor', [0.65 0.65
121         0.65]);
122     set(gca, 'xticklabel', xLabels)
123     title('Joint Probability Distribution of data2', ...
124         'Interpreter', 'latex')
125     ylabel('Probability ($P(x)$)', 'Interpreter', 'latex')
126     ylim([0.0 1.0])
127     text(1:length(conditional), conditional, num2str(
128         conditional'), ...
129         'vert', 'bottom', 'horiz', 'center', 'Interpreter', '
130         latex');
131     set(gca, 'TickLabelInterpreter', 'latex')
132
133     subplot(2,2,2)
134     bar(P_ind, 'stacked', 'FaceColor', [1.0 1.0 1.0]);
135     set(gca, 'xticklabel', xLabels)
136     title('Joint Probability Distribution constructed
137         from Independent Random Variables', ...
138         'Interpreter', 'latex')
139     ylabel('Probability ($P(x)$)', 'Interpreter', 'latex')
140     ylim([0.0 1.0])
141     text(1:length(P_ind), P_ind, num2str(P_ind'), ...
142         'vert', 'bottom', 'horiz', 'center', 'Interpreter', '
143         latex');
144     set(gca, 'TickLabelInterpreter', 'latex')
145
146     subplot(2,2,3)
147     bar(conditional, 'stacked', 'FaceColor', [0.65 0.65
148         0.65]);
149     hold on
150     bar(P_ind, 'stacked', 'FaceColor', [0.0 0.0 0.0]);

```

```

144     set(gca,'xticklabel',xLabels)
145     title ('Actual and Constructed Joint Distributions'
146           '...',
147           'Interpreter', 'Latex')
148     ylabel ('Probability ($P(x)$)', 'Interpreter','latex'
149            )
150     ylim([0.0 1.0])
151     set(gca,'TickLabelInterpreter','latex')
152     leg = legend('Actual Distribution','Constructed
153                Distribution');
154     set(leg,'Interpreter','latex');
155
156     subplot(2,2,4)
157     diff = conditional - P_ind
158     bar(diff, 'stacked','FaceColor',[0.65 0.65 0.65]);
159     set(gca,'xticklabel',xLabels)
160     title ('Error between Actual vs Constructed Joint
161           Distributions', ...
162           'Interpreter','latex')
163     ylim([-0.05 0.05])
164     ylabel ('Probability ($P(x)$)', 'Interpreter','latex'
165            )
166     set(gca,'TickLabelInterpreter','latex')
167 end

```

Code for Problem 2

```

1 %% Homework - 3
2 % Question: 2
3 % Author: Harsh Bhate
4 %% Cleaning Up
5 clear all;
6 clc;
7 %% Part 2(a) Generating Dataset
8 mu = [0 0];
9 sigma = [1 0; 0 16];
10 R = chol(sigma);
11 z = repmat(mu,1000,1) + randn(1000,2)*R;
12 theta = - pi/4;
13 A = [cos(theta) -sin(theta); sin(theta) cos(theta)];
14 data = z*A;
15 figure();
16 subplot(1,2,1);
17 plot (z(:,1),z(:,2),'.k');
18 title ('\textbf{Data}', ...
19         'Interpreter','latex')
20 ylim([-20 20]);
21 xlim([-20 20]);
22 axis square;
23 subplot(1,2,2);
24 plot (data(:,1),data(:,2),'.k');
25 title ('\textbf{Rotated Data}', ...
26         'Interpreter','latex')
27 set(gca,'TickLabelInterpreter','latex')
28 ylim([-20 20]);
29 xlim([-20 20]);
30 set(gca,'TickLabelInterpreter','latex')
31 axis square;
32
33 % Plotting the histograms
34 figure()
35 subplot(1,2,1);
36 h1 = histfit(data(:,1));
37 set(h1(1),'facecolor',[0.65,0.65,0.65]); set(h1(2),'color',
38         'k')
39 yt = get(gca,'YTick');
40 set(gca,'YTick',yt,'YTickLabel',yt/100)
41 title ('\textbf{Marginal Distribution of x ($P_{\{x\}}(X)$)')
42     , ...
43         'Interpreter','latex')
44 set(gca,'TickLabelInterpreter','latex')

```

```

43
44 subplot(1,2,2);
45 h2 = histfit(data(:,2));
46 set(h2(1), 'facecolor', [0.65,0.65,0.65]); set(h2(2), 'color
    ', 'k')
47 yt = get(gca, 'YTick');
48 set(gca, 'YTick', yt, 'YTickLabel', yt/100)
49 title ('\textbf{Marginal Distribution of x ($P_{\{y\}}(Y)$)'}
    , ...
    'Interpreter', 'latex')
50 set(gca, 'TickLabelInterpreter', 'latex')
51 %% Part 2(b) Estimating Covariance
52 sigma_estimated = cov(data);
53 disp('The Estimated Covariance is:');
54 disp(sigma_estimated);
55 %% Part 2(c) Eigenvalue
56 [eigenVectors, D] = eig(sigma_estimated);
57 disp('The Eigenvalues are:');
58 disp(D);
59 disp('The Eigenvectors are:');
60 disp(eigenVectors);
61 v1 = eigenVectors(:,1);
62 v2 = eigenVectors(:,2);
63 eigen = sqrt(diag(D));
64 figure();
65 plot(data(:,1), data(:,2), ...
66     '.', 'color', [0.75 0.75 0.75]);
67 hold on;
68 quiver(0, 0, v1(1), v1(2), eigen(1), ...
69     'LineWidth', 1, 'MaxHeadSize', 1, 'color', [0 0 0]);
70 hold on;
71 quiver(0, 0, v2(1), v2(2), eigen(2), ...
72     'LineWidth', 1, 'MaxHeadSize', 1, 'color', [0 0 0]);
73 hold on;
74 title ('\textbf{Data along with eigenvalues (Magnified)'}
    , ...
    'Interpreter', 'latex')
75 set(gca, 'TickLabelInterpreter', 'latex')
76 axis square
77 set(gca, 'TickLabelInterpreter', 'latex')
78 axis square;
79 leg = legend('Data', 'Eigenvector $\mathbf{V} = [\vec{v}_1
    , \vec{v}_2]$', ...
80     , 'Interpreter', 'latex');
81 set(leg, 'Interpreter', 'latex');
82 %% Part (d). Generating New Data
83 theta = + pi/4;
84

```

```

85 A = [cos(theta) -sin(theta); sin(theta) cos(theta)];
86 data2 = z*A;
87 merged_data = vertcat(data,data2);
88
89 figure();
90 plot (merged_data(:,1),merged_data(:,2),'.k');
91 title ('\textbf{Merged Data}', ...
92         'Interpreter','latex')
93 set(gca,'TickLabelInterpreter','latex')
94 ylim([-20 20]);
95 xlim([-20 20]);
96 set(gca,'TickLabelInterpreter','latex')
97 axis square;
98 % Plotting the histograms
99 figure()
100 subplot(1,2,1);
101 h1 = histfit(merged_data(:,1));
102 set(h1(1),'facecolor',[0.65,0.65,0.65]); set(h1(2),'color',
103         'k')
104 yt = get(gca,'YTick');
105 set(gca,'YTick',yt,'YTickLabel',yt/200)
106 title ('\textbf{Marginal Distribution of x ($P_{\{x\}}(X)$)'}
107         , ...
108         'Interpreter','latex')
109 set(gca,'TickLabelInterpreter','latex')
110
111 subplot(1,2,2);
112 h2 = histfit(merged_data(:,2));
113 set(h2(1),'facecolor',[0.65,0.65,0.65]); set(h2(2),'color',
114         'k')
115 yt = get(gca,'YTick');
116 set(gca,'YTick',yt,'YTickLabel',yt/200)
117 title ('\textbf{Marginal Distribution of y ($P_{\{y\}}(Y)$)'}
118         , ...
119         'Interpreter','latex')
120 set(gca,'TickLabelInterpreter','latex')
121 %% Part (e) PCA
122 sigma_new = cov(merged_data);
123 [Vectors, D] = eig(sigma_new);
124 disp(Vectors)
125 v1 = Vectors(:,1);
126 v2 = Vectors(:,2);
127 eigen = sqrt(diag(D));
128 v1 = v1;
129 v2 = v2;
130 figure();

```



```

127 plot (merged_data(:,1),merged_data(:,2) ,...
128       '.', 'color', [0.75 0.75 0.75]);
129 hold on;
130 quiver(0, 0, v1(1), v1(2), eigen(1), ...
131        'LineWidth', 1, 'MaxHeadSize', 1, 'color', [0 0 0]);
132 hold on;
133 quiver(0, 0, v2(1), v2(2), eigen(2), ...
134        'LineWidth', 1, 'MaxHeadSize', 1, 'color', [0 0 0]);
135 hold off;
136 title ('\textbf{Merged Data along with PCA vectors}', ...
137        'Interpreter','latex')
138 set(gca, 'TickLabelInterpreter','latex')
139 axis square
140 set(gca, 'TickLabelInterpreter','latex')
141 axis square;
142 leg = legend('Data', 'Eigenvector $\mathbf{V} = [\vec{v}_1$
143             , $\vec{v}_2]$', ...
             'Interpreter','latex');

```

Code for Problem 3

```

1 %% Homework - 3
2 % Question: 3
3 % Author: Harsh Bhate
4 %% Cleaning Stuff
5 clc;
6 clear all;
7 %% Common Declarations
8 p_xGivenTheta = [0.075 0.025 0 0.025 0.075 0.1375 0.1875
    0.2 0.1875 0.125];
9 p_theta1 = [0.2 0.2 0.2 0.2 0.2 0.2 0.2 0.2 0.2 0.2];
10 p_theta2 = [0 0.05 0.11 0.21 0.28 0.21 0.11 0.05 0 0];
11 p_theta3 = [0 0 0 0.2 0.6 0.2 0 0 0 0];
12 %% Prob 1(a)
13 p_theta1Givenx = posteriorProb(p_xGivenTheta, p_theta1);
14 %% Prob 1(b)
15 p_theta2Givenx = posteriorProb(p_xGivenTheta, p_theta2);
16 %% Prob 1(c)
17 p_theta3Givenx = posteriorProb(p_xGivenTheta, p_theta3);
18 %% Prob 1(d)
19 [~, MLE] = max(p_xGivenTheta);
20 [~, MAP_theta1] = max(p_theta1Givenx);
21 [~, MAP_theta2] = max(p_theta2Givenx);
22 [~, MAP_theta3] = max(p_theta3Givenx);
23 txt1 = sprintf('MLE = %d', MLE);
24 txt2 = sprintf('MAP(theta1) = %d', MAP_theta1);
25 txt3 = sprintf('MAP(theta2) = %d', MAP_theta2);
26 txt4 = sprintf('MAP(theta3) = %d', MAP_theta3);
27 disp(txt1);
28 disp(txt2);
29 disp(txt3);
30 disp(txt4);
31 %% Function to compute posterior probability
32 function posterior = posteriorProb(apriori, parameter)
33     posterior = apriori.*parameter;
34     figure();
35     X = [1 2 3 4 5 6 7 8 9 10];
36     h(1) = subplot(2,2,1);
37     stem(X, apriori, 'k')
38     title('Apriori Distribution ($P(x|\theta)$)', ...
39         'Interpreter','latex')
40     ylabel('Probability ($P(x|\theta)$)', 'Interpreter',
41         'latex')
42     set(gca, 'TickLabelInterpreter','latex')
43     ylim([0 max(apriori)+0.01])

```

```

43
44 h(2) = subplot(2,2,2);
45 stem(X, parameter, 'k')
46 title ('Stimulus Parameter Distribution ( $P(\theta)$ )', ...
47         'Interpreter','latex')
48 ylabel ('Probability ( $P(\theta)$ )', 'Interpreter','
49         latex')
50 set(gca,'TickLabelInterpreter','latex')
51 ylim([0 max(parameter)+0.01])
52
53 h(3) = subplot(2,2,3);
54 stem(X, posterior, 'k')
55 title ('Posterior Distribution ( $P(\theta|x)$ )', ...
56         'Interpreter','latex')
57 ylabel ('Probability ( $P(\theta|x)$ )', 'Interpreter',
58         'latex')
59 set(gca,'TickLabelInterpreter','latex')
60 ylim([0 max(posterior)+0.01])
61
62 pos = get(h,'Position');
63 new = mean(cellfun(@(v)v(1),pos(1:2)));
64 set(h(3),'Position',[new,pos{end}(2:end)])
65 end

```