$\rightarrow M_j = H\hat{M}_j = H_a H_s H_m \hat{M}_j$ — (1)

$\rightarrow S_i = (H^{-1})^T \hat{S}_i = (H_a^{-1})^T (H_s^{-1})^T (H_m^{-1})^T \hat{S}_i$ — (2)

$\rightarrow H_s = \begin{bmatrix} I & h_s \\ 0 & \end{bmatrix}^{-1}$ & first element of $h_s = 0$.

$$\boxed{\begin{matrix}[h_s^T \hat{S}_i = 1] \\ \text{another condition on } h_s.\end{matrix}} \rightarrow \text{eqn (a)}$$

Let $h_s = \begin{bmatrix} 0 \\ p \\ q \\ r \end{bmatrix}$

$$\Rightarrow H_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & p \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & r \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -p/r & -q/r & 1/r \end{bmatrix}^T$$

$$H_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -p/r \\ 0 & 0 & 1 & -q/r \\ 0 & 0 & 0 & 1/r \end{bmatrix} \quad \text{— (3)}$$

$$\rightarrow H_m = \begin{bmatrix} h_M^T \\ 0 \quad I \end{bmatrix}$$

$$\boxed{\begin{matrix}[h_m^T \hat{M}_j = 1] \\ \text{Condition on } h_M\end{matrix}} \rightarrow \text{eqn (b)}$$

Let $h_M = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

$$\Rightarrow H_M = \begin{bmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{— (4)}$$

$$\rightarrow H_Q = \begin{bmatrix} 1 & 0\ 0 & 0 \\ t & RK & \begin{matrix}0\\0\end{matrix} \\ t^T t - Q_{11} & 2(t^T RK - [Q_{12}\ Q_{13}]) & 1 \end{bmatrix} \quad — (5)$$

$$\rightarrow \text{Let } \hat{M}_j = \begin{bmatrix} \hat{M}_j(1) \\ \hat{M}_j(2) \\ \hat{M}_j(3) \\ \hat{M}_j(4) \end{bmatrix} = \begin{bmatrix} 1 \\ x_j \\ y_j \\ m_j^T m_j \end{bmatrix}, \text{ where } m_j = \begin{bmatrix} x_j \\ y_j \end{bmatrix} \quad — (6)$$

$\rightarrow$ Now consider equation (1)

$$M_j = H_Q H_S H_M \hat{M}_j$$

$$= H_Q \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -p/r \\ 0 & 0 & 1 & -q/r \\ 0 & 0 & 0 & 1/r \end{bmatrix} \begin{bmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \hat{M}_j$$

$$= H_Q \cdot \begin{bmatrix} a & b & c & d \\ 0 & 1 & 0 & -p/r \\ 0 & 0 & 1 & -q/r \\ 0 & 0 & 0 & 1/r \end{bmatrix} \begin{bmatrix} 1 \\ x_j \\ y_j \\ x_j^2 + y_j^2 \end{bmatrix}$$

$$= H_Q \cdot \begin{bmatrix} 1 \\ x_j - p/r(x_j^2 + y_j^2) \\ y_j - q/r(x_j^2 + y_j^2) \\ \frac{1}{r}(x_j^2 + y_j^2) \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{t}1 & 0\ 0 & 0 \\ t & RK & \begin{matrix}0\\0\end{matrix} \\ t^T t - Q_{11} & 2\left(t^T RK - [Q_{12}\ Q_{13}]\right) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_j - p/\gamma\,(x_j^2+y_j^2) \\ y_j - q/\gamma\,(x_j^2+y_j^2) \\ \frac{1}{\gamma}(x_j^2+y_j^2) \end{bmatrix}$$

$$M_j = \begin{bmatrix} 1 \\ t + RK \begin{bmatrix} x_j - p/\gamma\,(x_j^2+y_j^2) \\ y_j - q/\gamma\,(x_j^2+y_j^2) \end{bmatrix} \\ t^T t - Q_{11} + 2\left(t^T RK - [Q_{12}\ Q_{13}]\right) \begin{bmatrix} x_j - p/\gamma\,(x_j^2+y_j^2) \\ y_j - q/\gamma\,(x_j^2+y_j^2) \end{bmatrix} + \frac{1}{\gamma}(x_j^2+y_j^2) \end{bmatrix}$$

Consider $K \begin{bmatrix} x_j - p/\gamma\,(x_j^2+y_j^2) \\ y_j - q/\gamma\,(x_j^2+y_j^2) \end{bmatrix}$

$$= K \left[\begin{bmatrix} x_j \\ y_j \end{bmatrix} - \begin{bmatrix} p/\gamma\,[x_j\ y_j]\begin{bmatrix} x_j \\ y_j \end{bmatrix} \\ q/\gamma\,[x_j\ y_j]\begin{bmatrix} x_j \\ y_j \end{bmatrix} \end{bmatrix} \right]$$

$$= \underbrace{K \left[I - \begin{bmatrix} p/\gamma \\ q/\gamma \end{bmatrix} [x_j\ y_j] \right]}_{\text{If this part becomes } I,} \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

Middle term in $M_j = t + R \begin{bmatrix} x_j \\ y_j \end{bmatrix}$

So, let's assume that

$$K\left[I - \begin{bmatrix} p/\gamma \\ q/\gamma \end{bmatrix} \begin{bmatrix} x_j & y_j \end{bmatrix} \right] = I. \quad — (7)$$

→ consider last element of M_j & apply eqn (7) in it.

$$t^Tt - Q_{11} + 2(t^TRK - [Q_{12} \; Q_{13}]) \begin{bmatrix} x_j - p/\gamma(x_j^2+y_j^2) \\ y_j - q/\gamma(x_j^2+y_j^2) \end{bmatrix} + \frac{1}{\gamma}(x_j^2+y_j^2)$$

$$= t^Tt - Q_{11} + 2t^TR\begin{bmatrix} x_j \\ y_j \end{bmatrix} - 2[Q_{12} \; Q_{13}]K^{-1}\begin{bmatrix} x_j \\ y_j \end{bmatrix} + \frac{1}{\gamma}(x_j^2+y_j^2)$$

— (8).

Now according to M_j structure,

$$[M_j(2) \; M_j(3)]\begin{bmatrix} M_j(2) \\ M_j(3) \end{bmatrix} = M_j(4).$$

LHS ⇒ $\left(t + R\begin{bmatrix} x_j \\ y_j \end{bmatrix}\right)^T \left(t + R\begin{bmatrix} x_j \\ y_j \end{bmatrix}\right)$

$$= \left(t^T + [x_j \; y_j]R^T\right)\left(t + R\begin{bmatrix} x_j \\ y_j \end{bmatrix}\right)$$

$$= t^Tt + t^TR\begin{bmatrix} x_j \\ y_j \end{bmatrix} + [x_j \; y_j]R^Tt + (x_j^2+y_j^2)$$

RHS ⇒ eqn (8).

$$t^Tt - Q_{11} + 2t^TR\begin{bmatrix} x_j \\ y_j \end{bmatrix} - 2[Q_{12} \; Q_{13}]K^{-1}\begin{bmatrix} x_j \\ y_j \end{bmatrix} + \frac{1}{\gamma}(x_j^2+y_j^2).$$

→ Now RHS − LHS

$\Downarrow$

$$-Q_{11} + t^T R \begin{bmatrix} x_j \\ y_j \end{bmatrix} - 2\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix} K^{-1} \begin{bmatrix} x_j \\ y_j \end{bmatrix} - \begin{bmatrix} x_j & y_j \end{bmatrix} R^T t + \left(\frac{1}{\gamma} - 1\right)(x_j^2 + y_j^2) \quad \text{—} (9)$$

Eqn (9) has to be zero, in order to satisfy the structure of M_j, but is not in general.