ALTERNATE WAY TO FIND HQ IN DIRECT COMP. PAPER USING SOURCE CONSTRAINTS (2D)

→ The constraints on S_i can be written as

$$S_i^T B S_i = 0 \quad , \text{where} \quad S_i = \begin{bmatrix} s_i^T s_i & -2x_i & -2y_i & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

→ Justification:-

$$S_i^T B S_i = \begin{bmatrix} s_i^T s_i & -2x_i & -2y_i & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_i^T s_i \\ -2x_i \\ -2y_i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2x_i & -2y_i & -2s_i^T s_i \end{bmatrix} \begin{bmatrix} s_i^T s_i \\ -2x_i \\ -2y_i \\ 1 \end{bmatrix}$$

$$= -2s_i^T s_i + 4x_i^2 + 4y_i^2 - 2s_i^T s_i = 4(x_i^2 + y_i^2) - 4s_i^T s_i$$

$$= 0$$

$$\left[\because s_i^T s_i = \begin{bmatrix} x_i & y_i \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \right]$$

→ Therefore, the sole constraints on s_i are:-

$$\boxed{S_i^T B S_i = 0} \quad \text{— (1)}$$

$\rightarrow$ We have $S_i^T M_j = \hat{S}_i^T H^{-1} H M_j$

$\rightarrow$ So, $S_i^T = \hat{S}_i^T H^{-1}$

$$\Rightarrow \boxed{S_i = (H^{-1})^T \hat{S}_i}$$

$$\Rightarrow S_i = \left[(H_Q H_S H_M)^{-1}\right]^T \hat{S}_i$$

$$= \left[H_M^{-1} H_S^{-1} H_Q^{-1}\right]^T \hat{S}_i$$

$$\boxed{S_i = (H_Q^{-1})^T (H_S^{-1})^T (H_M^{-1})^T \hat{S}_i} \quad \text{—} (2)$$

$\rightarrow$ Now, substitute eqn (2) in (1).

$$S_i^T B S_i = 0$$

$$\hat{S}_i^T H_M^{-1} H_S^{-1} \underbrace{H_Q^{-1} B (H_Q^{-1})^T}_{Q} \underbrace{(H_S^{-1})^T (H_M^{-1})^T \hat{S}_i}_{\hat{S}_i'} = 0.$$

$$\Rightarrow \cancel{\hat{S}_i'^T Q \hat{S}_i' = Q}$$

$$\Rightarrow \boxed{\hat{S}_i'^T Q \hat{S}_i' = 0} \quad \text{—} (3)$$

$\rightarrow$ Now we have $\boxed{Q = H_Q^{-1} B (H_Q^{-1})^T}$ — (4).

→ So, the $\boxed{\text{process}}$ is,

i) Get Q from linear eqns ③. $[\hat{S}_i'^T Q \hat{S}_i' = 0]$

ii) H_Q^{-1} has to be computed, which fits to eqn ④.

→ So, Let's start the process of generalising $\underline{H_Q^{-1}}$.

From eqn ②,

$$S_i = (H_Q^{-1})^T (H_S^{-1})^T (H_m^{-1})^T \hat{S}_i'$$

∴ $(H_Q^{-1})^T$ should ensure that the last row of $\hat{S}_i$ doesn't change.

$$\underbrace{\begin{bmatrix} - & - & - & \\ - & - & - & \\ & - & - & \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(H_Q^{-1})^T} \underbrace{\begin{bmatrix} S_i^T S_i & - & - & \\ -2x_i & - & - & - \\ -2y_i & - & - & - \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{\hat{S}_i}$$

$$\Rightarrow H_Q^{-1} = \begin{bmatrix} \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ \cdot & & & 1 \end{bmatrix}$$

Note:- To get general form for H_Q, $H_Q = (H_Q^{-1})^{-1}$

$$H_Q = \begin{bmatrix} - & - & & 0 \\ - & - & & 0 \\ - & - & & 0 \\ \underbrace{a \quad b \quad c}_{\Downarrow} & & & 1 \end{bmatrix}^{-1}$$

last row of H_Q^{-1}

$$\left[A^{-1} = \frac{[\mathrm{adj}(A)]^T}{|A|}\right]$$

$$\mathrm{Adj}(H_Q) = \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ -\begin{vmatrix} - & - & 0 \\ - & - & 0 \\ - & - & 0 \end{vmatrix} & \begin{vmatrix} - & - & 0 \\ - & - & 0 \\ - & - & 0 \end{vmatrix} & -\begin{vmatrix} - & - & 0 \\ - & - & 0 \\ - & - & 0 \end{vmatrix} & |H_Q^{-1}(1:3,1:3)| \end{bmatrix}$$

$$|H_Q^{-1}| = |H_Q^{-1}(1:3,1:3)|$$

$$\Rightarrow H_Q = (H_Q^{-1})^{-1} = \frac{[\mathrm{adj}(H_Q^{-1})]^T}{|H_Q^{-1}|}$$

$$\Rightarrow H_Q = \begin{bmatrix} - & - & - & 0 \\ - & - & - & 0 \\ - & - & - & 0 \\ - & - & - & 1 \end{bmatrix}$$

→ Also, to ensure that the first row of $\hat{M}_j$ [1 1 1] doesn't change, $H_Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ - & - & - & 0 \\ - & - & - & 0 \\ - & - & - & 1 \end{bmatrix}$

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→ In the same way stated in <u>note</u>, the updated

$$H_Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ - & - & - & 0 \\ - & - & - & 0 \\ - & - & - & 1 \end{bmatrix}$$

→ Now, let's obtain form of Q.

$$Q = (H_Q^{-1})\, B\, (H_Q^{-1})^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ - & - & - & 0 \\ - & - & - & 0 \\ - & - & - & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & - & - & - \\ 0 & - & - & - \\ -2 & - & - & - \end{bmatrix} \begin{bmatrix} 1 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & - & - & - \\ 0 & - & - & - \\ -2 & - & - & - \end{bmatrix}$$

→ Now as Q is symmetric, let's fill Q with few elements.

$$Q = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & Q_{22} & Q_{23} & Q_{24} \\ 0 & Q_{23} & Q_{33} & Q_{34} \\ -2 & Q_{24} & Q_{34} & Q_{44} \end{bmatrix}$$

→ Now, the goal is to find H_Q^{-1} which suits to eqn.

$$\cancel{H_Q^{-1} Q (H_Q^{-1})^T} \quad \boxed{Q = H_Q^{-1} B (H_Q^{-1})^T}$$

→ After many attempts, the following choice of H_Q^{-1} ~~found to~~ found to be the valid one.

$$\boxed{H_Q^{-1} = \begin{bmatrix} 1 & 0\ 0 & 0 \\ \frac{1}{2}\left[KRt - \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix}\right] & KR & \begin{matrix} 0 \\ 0 \end{matrix} \\ \frac{1}{4}\left[t^T t - Q_{44}\right] & t^T & 1 \end{bmatrix}}$$

Justification:

Let's compute $H_Q^{-1} B (H_Q^{-1})^T$.

$$H_Q^{-1} B = \begin{bmatrix} 1 & 0\ 0 & 0 \\ \frac{1}{2}\left[KRt - \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix}\right] & KR & \begin{matrix} 0 \\ 0 \end{matrix} \\ \frac{1}{4}\left[t^T t - Q_{44}\right] & t^T & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0\ 0 & -2 \\ \begin{matrix} 0 \\ 0 \end{matrix} & KR & \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix} - KRt \\ -2 & t^T & -\frac{1}{2}\left[t^T t - Q_{44}\right] \end{bmatrix}$$

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Now,

$$H_Q^{-1} B (H_Q^{-1})^T =$$

$$\begin{bmatrix} 0 & 0 \;\; 0 & -2 \\ \begin{matrix} 0 \\ 0 \end{matrix} & KR & \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix} - KRt \\ -2 & t^T & -\frac{1}{2}\left[t^T t - Q_{44}\right] \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2}\left[t^T R^T K^T - [Q_{24} \;\; Q_{34}]\right] & \frac{1}{4}\left[t^T t - Q_{44}\right] \\ \begin{matrix} 0 \\ 0 \end{matrix} & R^T K^T & t \\ 0 & 0 \;\; 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \quad 0 & -2 \\ \begin{matrix} 0 \\ 0 \end{matrix} & (KR)(R^T K^T) & KRt + \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix} - KRt \\ 0 & \begin{matrix} [Q_{24} \;\; Q_{34}] - t^T R^T K^T \\ + t^T R^T K^T \end{matrix} & \begin{matrix} -\frac{1}{2}\left[t^T t - Q_{44}\right] + t^T t \\ -\frac{1}{2}\left[t^T t - Q_{44}\right] \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & Q_{22} & Q_{23} & Q_{24} \\ 0 & Q_{23} & Q_{33} & Q_{34} \\ -2 & Q_{24} & Q_{34} & Q_{44} \end{bmatrix}$$

[$\because RR^T = I$ & $KK^T = Q_{mid}$ ($K \rightarrow$ cholesky factorization of Q_{mid})]

Lower ogular.

→ Now, we ~~have~~ have $\underbrace{H_Q^{-1}}_{\text{general form}}$, & linear eqns $\hat{S}_i'^T Q \hat{S}_i' = 0$.

→ We compute Q.

→ We obtain K from Q_{mid}, Q_{24}, Q_{34}, Q_{44}.

→ We also have R & t.

→ Thefore, we form H_Q^{-1}.

→ Inverse of H_Q^{-1} gives us H_Q.