ANALYSIS ON FINAL OUTPUT OF DIRECT COMP PAPER BASED ON SOURCE CONSTRAINTS

→ The aim here is to understand the structure of H_Q in Source Constraints case.

→ The structure of H_Q should satisfy

i) $(H_Q^{-1}) \cdot B \cdot (H_Q^{-1})^T = Q$

ii) $\left[-\frac{1}{2} S_i(2)\right]^2 + \left[-\frac{1}{2} S_i(3)\right]^2 = S_i(1)$. $\boxed{2D}$

$\left[-\frac{1}{2}(-2x_i)\right]^2 + \left[-\frac{1}{2}(-2y_i)\right]^2 = x_i^2 + y_i^2 = S_i^T S_i$

after obtaining $S_i = (H_Q^{-1})^T (H_S^{-1})^T (H_M^{-1})^T \hat{S}_i$.

→ We already checked $(H_Q^{-1}) \cdot B \cdot (H_Q^{-1})^T = Q$ condition & it's working. Let's check the (ii) condition.

→ $S_i = (H^{-1})^T \hat{S}_i$

$$S_i = (H_Q^{-1})^T (H_S^{-1})^T (H_M^{-1})^T \hat{S}_i \quad \text{— (1)}$$

→ $H_S = \begin{bmatrix} I & h_s \\ 0 & \end{bmatrix}^{-1}$ $\boxed{h_s^T \hat{S}_i = 1}$ Condition on h_s → eqn (a)

Let $h_s = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$

$$(H_s^{-1})^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & r & s \end{bmatrix} \quad \text{—} (3)$$

$$\rightarrow H_M = \begin{bmatrix} h_m^T \\ 0 \quad I \end{bmatrix}$$

$$\boxed{\begin{array}{c}[h_m^T \hat{M}_j = 1] \\ \text{Condition on } h_m\end{array}}$$

Let $h_m = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$

[Last element of h_m should be '0' to avoid changing the last row of S_i (all ones)]

$$H_M = \begin{bmatrix} a & b & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_i^T B S_i = 0$$

$$\hat{S}_i^T H_m^{-1} H_s^{-1} H_a^{-1} B (H_a^{-1})^T (H_s^{-1})^T (H_m^{-1})^T \hat{S}_i = 0.$$

How constraints are formed?

$$\Rightarrow H_M^{-1} = \begin{bmatrix} 1/a & -b/a & -\frac{c}{a} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(H_m^{-1})^T = \begin{bmatrix} 1/a & 0 & 0 & 0 \\ -b/a & 1 & 0 & 0 \\ -c/a & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{—} (4)$$

$$\rightarrow H_Q^{-1} = \begin{bmatrix} 1 & 0\ 0 & 0 \\ \frac{1}{2}\left[KRt - \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix}\right] & KR & \begin{matrix} 0 \\ 0 \end{matrix} \\ \frac{1}{4}\left[t^T t - Q_{44}\right] & t^T & 1 \end{bmatrix} \quad — (5)$$

$\rightarrow$ Let $\hat{S}_i = \begin{bmatrix} \hat{S}_i(1) \\ \hat{S}_i(2) \\ \hat{S}_i(3) \\ \hat{S}_i(4) \end{bmatrix}$

$\rightarrow$ Now consider equation (1)

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$$S_i = (H_Q^{-1})^T (H_S^{-1})^T (H_M^{-1})^T \hat{S}_i$$

$$= (H_Q^{-1})^T \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & r & s \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ -\frac{b}{a} & 1 & 0 & 0 \\ -\frac{c}{a} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \hat{S}_i$$

$$= (H_Q^{-1})^T \cdot \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ -\frac{b}{a} & 1 & 0 & 0 \\ -\frac{c}{a} & 0 & 1 & 0 \\ \frac{p}{a} - \frac{bq}{a} - \frac{rc}{a} & q & r & s \end{bmatrix} \begin{bmatrix} \hat{S}_i(1) \\ \hat{S}_i(2) \\ \hat{S}_i(3) \\ \hat{S}_i(4) \end{bmatrix} \quad — (6)$$

Here, If $\frac{p}{a} - \frac{bq}{a} - \frac{rc}{a} = p$, the last row of S_i

becomes $[1 \; 1 \; 1 \; 1]$

$$\Rightarrow p\left(\frac{1}{a} - 1\right) + q\left(\frac{-b}{a}\right) + r\left(\frac{-c}{a}\right) = 0.$$

$$\Rightarrow p(1-a) + q(-b) + r(-c) = 0.$$

$$\Rightarrow \boxed{h_s^T \cdot \left[h \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - h_m \right] = 0} \quad \text{— (7)} \text{ is the}$$

additional condition required.

$$S_i = (H_Q^{-1})^T \cdot \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ -\frac{b}{a} & 1 & 0 & 0 \\ -\frac{c}{a} & 0 & 1 & 0 \\ p & q & r & s \end{bmatrix} \begin{bmatrix} \hat{S}_i(1) \\ \hat{S}_i(2) \\ \hat{S}_i(3) \\ \hat{S}_i(4) \end{bmatrix}$$

$$= (H_Q^{-1})^T \cdot \begin{bmatrix} \frac{1}{a}\hat{S}_i(1) \\ \hat{S}_i(2) - \frac{b}{a}\left(\hat{S}_i(1)\right) \\ \hat{S}_i(3) - \frac{c}{a}\left(\hat{S}_i(1)\right) \\ 1 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & \frac{1}{2}\left[t^T R^T K^T - \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix}\right]^T & \frac{1}{4}\left[t^T t - Q_{44}\right] \\ \begin{matrix} 0 \\ 0 \end{matrix} & R^T K^T & t \\ 0 & 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a}\left[\hat{s}_i(1)\right] \\ \hat{s}_i(2) - \frac{b}{a}\left[\hat{s}_i(1)\right] \\ \hat{s}_i(3) - \frac{c}{a}\left[\hat{s}_i(1)\right] \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{1}{a}\left[\hat{s}_i(1)\right] + \frac{1}{2}\left[t^T R^T K^T - \begin{bmatrix} Q_{24} & Q_{34} \end{bmatrix}\right] \begin{bmatrix} \hat{s}_i(2) - \frac{b}{a}\left[\hat{s}_i(1)\right] \\ \hat{s}_i(3) - \frac{c}{a}\left[\hat{s}_i(1)\right] \end{bmatrix} + \frac{1}{4}\left[t^T t - Q_{44}\right] \\ R^T K^T \begin{bmatrix} \hat{s}_i(2) - \frac{b}{a}\left[\hat{s}_i(1)\right] \\ \hat{s}_i(3) - \frac{c}{a}\left[\hat{s}_i(1)\right] \end{bmatrix} + t \\ 1 \end{bmatrix}
$$

From the structure of S_i,

$$\underbrace{\begin{bmatrix} S_i(2) & S_i(3)\end{bmatrix}\begin{bmatrix} S_i(2)\\ S_i(3)\end{bmatrix}}_{LHS} = \underbrace{4\cdot S_i(1)}_{RHS}$$

$$LHS = \left[t + R^TK^T\begin{bmatrix}\hat{S}_i(2) - \frac{b}{a}\left[\hat{S}_i(1)\right]\\ \hat{S}_i(3) - \frac{c}{a}\left[\hat{S}_i(1)\right]\end{bmatrix}\right]^T\left[t + R^TK^T\begin{bmatrix}\hat{S}_i(2) - \frac{b}{a}\left[\hat{S}_i(1)\right]\\ \hat{S}_i(3) - \frac{c}{a}\left[\hat{S}_i(1)\right]\end{bmatrix}\right]$$

$$= t^Tt + t^TRK\begin{bmatrix}\hat{S}_i(2) - \frac{b}{a}\left[\hat{S}_i(1)\right]\\ \hat{S}_i(3) - \frac{c}{a}\left[\hat{S}_i(1)\right]\end{bmatrix}$$

$$+ \begin{bmatrix}\hat{S}_i(2) - \frac{b}{a}\left[\hat{S}_i(1)\right] & \hat{S}_i(3) - \frac{c}{a}\left[\hat{S}_i(1)\right]\end{bmatrix} KRt \; +$$

$$+ \begin{bmatrix}\hat{S}_i(2) - \frac{b}{a}\left[\hat{S}_i(1)\right] & \hat{S}_i(3) - \frac{c}{a}\left[\hat{S}_i(1)\right]\end{bmatrix} \underbrace{KK^T}_{\begin{bmatrix} Q_{22} & Q_{23}\\ Q_{23} & Q_{33}\end{bmatrix}} \begin{bmatrix}\hat{S}_i(2) - \frac{b}{a}\left[\hat{S}_i(1)\right]\\ \hat{S}_i(3) - \frac{c}{a}\left[\hat{S}_i(1)\right]\end{bmatrix}$$

$$RHS = 4\cdot\left[\frac{1}{a}\left[\hat{S}_i(1)\right] + \frac{1}{2}\left[t^TR^TK^T - \begin{bmatrix}Q_{24} & Q_{34}\end{bmatrix}\right]\begin{bmatrix}\hat{S}_i(2) - \frac{b}{a}\left[\hat{S}_i(1)\right]\\ \hat{S}_i(3) - \frac{c}{a}\left[\hat{S}_i(1)\right]\end{bmatrix} \cancel{+\frac{1}{4}t^Tt} + \frac{1}{4}\left[t^Tt - Q_{44}\right]\right]$$

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Now, RHS − LHS =

$$-Q_{44} - 2\begin{bmatrix} \cancel{Q_{12}}\,Q_{24} & \cancel{Q_{13}}\,Q_{34} \end{bmatrix}\begin{bmatrix} \hat{s}_i(2) - \frac{b}{a}[\hat{s}_i(1)] \\ \hat{s}_i(3) - \frac{c}{a}[\hat{s}_i(1)] \end{bmatrix} + \frac{4}{a}[\hat{s}_i(1)]$$

$$- \begin{bmatrix} \hat{s}_i(2) - \frac{b}{a}[\hat{s}_i(1)] & \hat{s}_i(3) - \frac{c}{a}[\hat{s}_i(1)] \end{bmatrix}\begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23} & Q_{33} \end{bmatrix}\begin{bmatrix} \hat{s}_i(2) - \frac{b}{a}[\hat{s}_i(1)] \\ \hat{s}_i(3) - \frac{c}{a}[\hat{s}_i(1)] \end{bmatrix}$$

→ consider the eqn $\hat{s}_i'^{T} Q \hat{s}_i' = 0$ — (8)

where, $\hat{s}_i' = (H_s^{-1})^T (H_m^{-1})^T \hat{s}_i$; $Q = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & Q_{22} & Q_{23} & Q_{24} \\ 0 & Q_{23} & Q_{33} & Q_{34} \\ -2 & Q_{24} & Q_{34} & Q_{44} \end{bmatrix}$

$\Downarrow$ From previous computations

$$\begin{bmatrix} \frac{1}{a}[\hat{s}_i(1)] \\ \hat{s}_i(2) - \frac{b}{a}[\hat{s}_i(1)] \\ \hat{s}_i(3) - \frac{c}{a}[\hat{s}_i(1)] \\ 1 \end{bmatrix}$$

Eqn (8) ⇒ LHS = $\hat{s}_i'^{T} Q \hat{s}_i'$

$$= \begin{bmatrix} \frac{1}{a}[\hat{s}_i(1)] & \hat{s}_i(2) - \frac{b}{a}[\hat{s}_i(1)] & \hat{s}_i(3) - \frac{c}{a}[\hat{s}_i(1)] & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & Q_{22} & Q_{23} & Q_{24} \\ 0 & Q_{23} & Q_{33} & Q_{34} \\ -2 & Q_{24} & Q_{34} & Q_{44} \end{bmatrix}\hat{s}_i'$$

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$$= \left[-2 \left(\begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix} + \left[\left[\hat{S}_i(2) - \frac{b}{a}[\hat{S}_i(1)] \quad \hat{S}_i(3) - \frac{c}{a}[\hat{S}_i(1)]\right] \begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23} & Q_{33} \end{bmatrix}\right]^T \right) \odot \hat{S}_i' \right.$$

dot product $a^T b = a \odot b$

$$\left. -\frac{2}{a}[\hat{S}_i(1)] + \left[\hat{S}_i(2) - \frac{b}{a}[\hat{S}_i(1)] \quad \hat{S}_i(3) - \frac{c}{a}[\hat{S}_i(1)]\right] \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix} + Q_{44} \right]$$

$$= -\frac{2}{a}[\hat{S}_i(1)] + [Q_{24} \quad Q_{34}] \begin{bmatrix} \hat{S}_i(2) - \frac{b}{a}[\hat{S}_i(1)] \\ \hat{S}_i(3) - \frac{b}{a}[\hat{S}_i(1)] \end{bmatrix}$$

$$+ \left[\hat{S}_i(2) - \frac{b}{a}[\hat{S}_i(1)] \quad \hat{S}_i(3) - \frac{c}{a}[\hat{S}_i(1)]\right] \begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} \hat{S}_i(2) - \frac{b}{a}[\hat{S}_i(1)] \\ \hat{S}_i(3) - \frac{c}{a}[\hat{S}_i(1)] \end{bmatrix}$$

$$-\frac{2}{a}[\hat{S}_i(1)] + \left[\hat{S}_i(2) - \frac{b}{a}[\hat{S}_i(1)] \quad \hat{S}_i(3) - \frac{c}{a}[\hat{S}_i(1)]\right] \begin{bmatrix} Q_{24} \\ Q_{34} \end{bmatrix} + Q_{44} = 0.$$

→ Substitute this in RHS − LHS.

⇒ RHS − LHS = 0.

Therefore, Condition (i) & (ii) both are satisfied, with an additional condition.

$$\boxed{h_s^T \cdot \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - h_m\right] = 0}$$