1 Physical Model

- Cart position: $x(t)$
- Pendulum center of gravity x-Position: $x_s(t)$
- Pendulum center of gravity y-Position: $y_s(t)$
- Pendulum angle measured from the top equilibium: $\theta(t)$
- Pendulum length: ℓ
- Pendulum mass: m
- Pendulum moment of inertia: $J = m\ell^2/12$
- Angular friction coefficient: b
- Contact forces between pendulum and cart: $F_x(t), F_y(t)$
- Gravity constant: g

Kinematics:

$$
\begin{aligned}
x_s &= x - \frac{\ell}{2}\sin(\theta) \\
\dot{x}_s &= \dot{x} - \frac{\ell}{2}\dot{\theta}\cos(\theta) \\
\ddot{x}_s &= \ddot{x} - \frac{\ell}{2}\ddot{\theta}\cos(\theta) + \frac{\ell}{2}\dot{\theta}^2\sin(\theta) \\
y_s &= \frac{\ell}{2}\cos(\theta) \\
\dot{y}_s &= -\frac{\ell}{2}\dot{\theta}\sin(\theta) \\
\ddot{y}_s &= -\frac{\ell}{2}\ddot{\theta}\sin(\theta) - \frac{\ell}{2}\dot{\theta}^2\cos\theta
\end{aligned}
$$

Force balance in x-direction of the pendulum:

$$m\ddot{x}_s = F_x \tag{1}$$

Force balance in y-direction of the pendulum:

$$m\ddot{y}_s = F_y - mg \tag{2}$$

Torque balance applied to the pendulum center of gravity:

$$J\ddot{\theta} = F_x\frac{\ell}{2}\cos(\theta) + F_y\frac{\ell}{2}\sin(\theta) - b\dot{\theta} \tag{3}$$

Plug (1) and (2) into (3) to obtain

$$\left(J + m\frac{l^2}{4}\right)\ddot{\theta} = m\ddot{x}\frac{\ell}{2}\cos(\theta) + mg\frac{\ell}{2}\sin(\theta) - b\dot{\theta} \tag{4}$$

Using $(J + ml^2/4) = ml^2/3$ this becomes

$$\ddot{\theta} = \frac{3\cos(\theta)}{2\ell}\ddot{x} + \frac{3\sin(\theta)}{2\ell}g - \frac{3b}{m\ell^2}\dot{\theta} \tag{5}$$

2 Mathematical Model

- States: $x_1(t) \doteq \theta(t)$, $x_2(t) \doteq \dot{\theta}(t)$

- Input: $u(t) \doteq \ddot{x}(t)$, i.e. Input = Cart acceleration

- Output: $y(t) = x_1(t)$
- Model parameters: m, ℓ, b, g

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{3g}{2\ell}\sin(x_1) - \frac{3b}{m\ell^2}x_2 + \frac{3}{2\ell}\cos(x_1)u \end{bmatrix} \tag{6}$$