Module - 2

Simplex Methods

Simplex Method is an iterative procedure for solving lpp in a finite number of steps. It provides an algorithm which consists of moving from the region of one vertex of feasible solution to another in such a manner that the value of objective function at the vertex is less or more. This procedure is repeated since the number of vertices is finite.

Problems:

Maximize $z = 10x_1 + 5x_2$

STC $3x_1 + 3x_2 \le 36$

$2x_1 + 6x_2 \le 60$

$5x_1 + 2x_2 \le 50$

where x_1 & $x_2 \ge 0$. Solve the lpp model by applying simplex methods.

⇒ Step 1: The simplex method is applied only for maximization problem. If the objective function is to minimize, convert it to maximization.

Step 2: Convert each inequality to equality and introduce a slak variable.

$$3x_1 + 3x_2 + S_1 = 36$$

$$2x_1 + 6x_2 + S_2 = 60$$

$$5x_1 + 2x_2 + S_3 = 50$$

Step 3: Represent the equation in matrix format the standard matrix format is $A \cdot X = B$

$$\begin{array}{ccccc} x_1 & x_2 & S_1 & S_2 & S_3 \end{array}$$
$$\begin{bmatrix} 3 & 3 & 1 & 0 & 0 \\ 2 & 6 & 0 & 1 & 0 \\ 5 & 2 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 36 \\ 60 \\ 50 \end{bmatrix}$$

Step 4: Modified objective function is

$$Z = 10x_1 + 5x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3$$

Step 5: $C_B \rightarrow$ introduced variable cost

$Y_B \rightarrow$ introduced variables

$X_B \rightarrow$ RHS value in matrix

Ist Iteration

		C_j	10	5	0	0	0	X_B/x_j
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	36	3	3	1	0	0	$36/3 = 12$
0	s_2	60	2	6	0	1	0	$60/2 = 30$
0	s_3	50	$\boxed{5}$	2	0	0	1	$50/5 = 10$ ← (+min)
$z_j = \Sigma C_B \cdot x_j$			0	0	0	0	0	
$z_j - C_j \geq 0$			−10 ↑ −max	−5	0	0	0	

[5 is the key element]

Calculate $z_j = \Sigma C_B \cdot x_j$

Check if $(z_j - C_j \geq 0)$ carry out iterations until the condition is satisfied

Select the column which has (−max) and calculate X_B/x_j for the corresponding column. Find out (+min) value in X_B/x_j. Select the key element at the point of intersection.

Convert the key element to unity by performing certain row operations

$R_3 : (50 \quad 5 \quad 2 \quad 0 \quad 0 \quad 1) \times 1/5$

$R_3 : \quad 10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5$

Convert the corresponding elements in the column of key elements to zero by applying suitable row operation.

$R3 = R3 \times 2 : (10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5) \times 2$

$\qquad 20 \quad 2 \quad 4/5 \quad 0 \quad 0 \quad 2/5$

$R2 - R3 : 40 \quad 0 \quad 26/5 \quad 0 \quad 1 \quad -2/5$

$R2 : 40 \quad 0 \quad 26/5 \quad 0 \quad 1 \quad -2/5$

$R3 \times 3 : (10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5) \times 3$

$\qquad 30 \quad 3 \quad 6/5 \quad 0 \quad 0 \quad 3/5$

$-$

$R1 : 36 \quad 3 \quad 3 \quad 1 \quad 0 \quad 0$

$R1 = R3 - R1$

$R1 : 6 \quad 0 \quad 9/5 \quad 1 \quad 0 \quad -3/5$

IInd Iteration.

		C_j	10	5	0	0	0	x_B/x_j
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	
0	S_1	6	0	9/5	1	0	-3/5	$6 \times 5/9 = 10/3 \leftarrow$ (min)
0	S_2	40	0	26/5	0	1	-2/5	$40 \times 5/26 = 100/3$
10	x_1	10	1	2/5	0	0	1/5	$10 \times 5/2 = 25$
$Z_j = \sum C_B \cdot X_j$			10	4	0	0	2	
			0	-1	0	0	2	

(−max)

$R1 : (6 \quad 0 \quad 9/5 \quad 1 \quad 0 \quad -3/5) \times 5/9$

$R1 : 10/3 \quad 0 \quad 1 \quad 5/9 \quad 0 \quad -1/3$

$R2 = R2 - (R1 \times 26/5)$

$R1 \times 26/5 : (10/3 \quad 0 \quad 1 \quad 5/9 \quad 0 \quad -1/3) \times 26/5$

$52/3 \quad 0 \quad 26/5 \quad 26/9 \quad 0 \quad -26/27$

$40 \quad 0 \quad 26/5 \quad 0 \quad 1 \quad -2/5$

$R2 : \frac{68}{3} \quad 0 \quad 0 \quad -\frac{26}{9} \quad 1 \quad \frac{76}{135}$

$R3 = R3 - (R1 \times 2/5)$

$(10/3 \quad 0 \quad 1 \quad 5/9 \quad 0 \quad -1/3) \times 2/5$

$\frac{4}{3} \quad 0 \quad 2/5 \quad 2/9 \quad 0 \quad -2/15$

$10 \quad 1 \quad 2/5 \quad 0 \quad 0 \quad 1/5$

$R3 : 26/3 \quad 1 \quad 0 \quad -2/9 \quad 0 \quad 3/5$

IIIrd Iteration

		C_j	10	5	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
5	x_2	10/3	0	1	5/9	0	-1/3
0	s_2	68/3	0	0	-26/9	1	76/135
10	x_1	26/3	1	0	-2/9	0	3/5
$z_j = \sum C_B \cdot x_j$			10	5	5/9	0	13/3
$z_j - C_j \geq 0$			0	0	5/9	0	13/3

The condition $z_j - c_j \geq 0$ is satisfied

$$\max z = \sum C_B \cdot X_B$$

$$= (5 \times 10/3) + (10 \times 26/3)$$

$$= 50/3 + 260/3$$

$$= 310/3$$

To verify, substitute the value of x_1, x_2, x_3 in the objective function

$$z = 10x_1 + 5x_2$$

$$= 10\left(\frac{26}{3}\right) + 5\left(10/3\right)$$

$$= \frac{310}{3}$$

2) Use simplex method to solve LPP

$$\max z = 3x_1 + 2x_2$$

STC $\quad x_1 + x_2 \leq 4$

$\quad x_1 - x_2 \leq 2$

where $x_1 \,\&\, x_2 \geq 0$

$\Rightarrow$ Convert inequality to equality

$$x_1 + x_2 + S_1 = 4$$

$$x_1 - x_2 + S_2 = 2$$

Represent the equation in matrix format

$A \cdot X = B$

$$
\begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array}
$$
$$
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}
$$

The modified objective function is

$z = 3x_1 + 2x_2 + 0 \cdot s_1 + 0 \cdot s_2$

Ist Iteration

		C_j	3	2	0	0	x_B/x_j	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2		
0	s_1	4	1	1	1	0	4	
0	s_2	2	$\boxed{1}$	-1	0	1	2	← +min
$z_j = \sum c_B \cdot x_j$			0	0	0	0		
$z_j - c_j \geq 0$			-3	-2	0	0		
			↑ (-max)					

$R_1 = R_1 - R_2$

$$
\begin{array}{ccccc} 4 & 1 & 1 & 1 & 0 \\ - \; 2 & 1 & -1 & 0 & 1 \end{array}
$$

R_1: 2 0 2 1 -1

IInd Iteration

		C_j	3	2	0	0	x_B/x_i
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	
0	s_1	2	0	2	1	-1	1 ← +min
3	x_1	2	1	-1	0	1	-2
$Z_j = \Sigma C_B \cdot x_j$			3	-3	0	3	
$Z_j - C_j \geq 0$			0	-5	0	3	

↑ (max)

$R_1 : (2 \quad 0 \quad 2 \quad 1 \quad -1) \times 1/2$

$\quad\;\; 1 \quad 0 \quad 1 \quad 1/2 \quad -1/2$

$R_2 : \quad 2 \quad 1 \quad -1 \quad 0 \quad 1$

$+ R_1 : \quad 1 \quad 0 \quad 1 \quad 1/2 \quad -1/2$

$R_2 : \quad 3 \quad 1 \quad 0 \quad 1/2 \quad 1/2$

III iteration

		C_i	3	2	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
2	x_2	1	0	1	1/2	-1/2
3	x_1	3	1	0	1/2	-1/2
$Z_j = \Sigma C_B \cdot x_j$			3	2	5/2	3/4
$Z_j - C_j \geq 0$			0	0	5/2	3/4

The condition $z_j - c_j \geq 0$ is satisfied

max $z = (2 \times 1) + (3 \times 3)$

$= 2 + 9$ $\quad x_1 = 3 \quad x_2 = 1$

$= 11$ //

To verify $z = 3x_1 + 2x_2$

$= 3(3) + 2(1)$

$= 9 + 2$

$= 11$ //

Big-M method

1) Use Big-M method to solve the following
lpp minimize $z = 5x + 3y$

$$2x + 4y \leq 12$$

$$2x + 2y = 10$$

$$5x + 2y \geq 10$$

⇒ Convert the inequality to equality and introduce
slak variable

$$2x + 4y + S1 = 12$$

$$2x + 2y + A1 = 10$$

$$5x + 2y - S2 + A2 = 10$$

[Since the 3rd constraint does not have a boundry introduce a (-S) surplus variable and an artificial variable (A)]

Convert the given problem to maximization

$z = -5x - 3y$

Write the equation in matrix format

$$\begin{array}{cccccc} x & y & S1 & S2 & A1 & A2 \end{array}$$

$$\begin{bmatrix} 2 & 4 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 5 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ S1 \\ S2 \\ A1 \\ A2 \end{bmatrix} \begin{bmatrix} 12 \\ 10 \\ 10 \end{bmatrix}$$

modified objective function:

$z = -5x - 3y + 0 \cdot S1 + 0 \cdot S2 - MA1 - MA2$

Ist Iteration

		Ci	-5	-3	0	0	-M	-M	x_B / x_j
C_B	Y_B	X_B	x	y	S1	S2	A1	A2	
0	S1	12	2	4	1	0	0	0	6
-M	A1	10	2	2	0	0	1	0	5
-M	A2	10	5	2	0	-1	0	1	2 ← (+min)
$z_j = \Sigma C_B \cdot x_j$			-7M	-4M	0	+M	-M	-M	
$z_j - C_j \geq 0$			-7M-5	-4M-3	0	M	0	0	

↑

(-max)

R3 : (10 5 2 0 -1 0 1) 1/5

$\boxed{R3}$: 2 1 2/5 0 -1/5 0 1/5

R3 x 2 : 4 2 4/5 0 -2/5 0 2/5

R2 : 10 2 2 0 0 1 0

$\boxed{R2}$: 6 0 6/5 0 2/5 1 -2/5

R1 : 4 2 4/5 0 -2/5 0 2/5

12 2 4 1 0 0 0

$\boxed{R1}$: 8 0 16/5 1 2/5 0 -2/5

IInd Iteration

		C_j	-5	-3	0	0	-M	-M	x_B/x_1
C_B	Y_B	X_B	x	y	S1	S2	A1	A2	
0	S1	8	0	$\boxed{16/5}$	1	2/5	0	-2/5	5/2 ← (+min)
-M	A1	6	0	6/5	0	2/5	1	-2/5	5
-5	x	2	1	2/5	0	-1/5	0	1/5	5
$Z_j = \Sigma C_B \cdot X_j$			-5	$\frac{-6M}{5} - 2$	0	$\frac{-2M}{5} + 1$	-M		
$Z_j - C_j \geq 0$			0	$\frac{-6M}{5} + 1$	0	$-\frac{2M}{5} + 1$	0		
				↑ (-max)					

R1 : (8 0 16/5 1 2/5 0) x 5/16

$\boxed{R1}$: 5/2 0 1 5/16 1/8 0

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$R1 \times 6/5 \quad \left(\frac{5}{2} \quad 0 \quad 1 \quad \frac{5}{16} \quad 1/8 \quad 0\right) \times 6/5$

	3	0	6/5	3/8	3/20	0
R2 −	3	0	6/5	3/8	2/5	1
	6	0	6/5	0	2/5	1
				−3/8	1/4	1
$\boxed{R2}$:	3	0	0			

R1 × 2/5	1	0	2/5	1/8	1/20	0
− R3	2	1	2/5	0	−1/5	0
R3 :	1	1	0	−1/8	−1/4	0

C_j			−5	−3	0	0	−M
C_B	Y_B	X_B	x	y	S1	S2	A1
−3	y	5/2	0	1	5/16	1/8	0
−M	A1	3	0	0	−3/8	1/4	1
−5	x	1	1	0	−1/8	−1/4	0
$Z_j = \Sigma C_B \cdot x_j$			−5	−3	$+\frac{3M}{8}-\frac{5}{16}$	$\frac{7}{8}-\frac{M}{4}$	−M
$Z_j - C_j \geq 0$			0	0	1/6	5/8	0

$Z = (-5\times 1) + (-3 \times 5/2)$

$= -5 - \frac{15}{2}$

$= -25/2$

To verify: $Z = -5x - 3y$

$= -5(1) - 3(5/2)$

$= -5 - 15/2 = -25/2$ //

Two Phase Simplex Method.

The Two Phase Simplex Method is another method to solve the given LPP involving some artificial variables.

Phase 1: In this phase we construct an auxillary LPP to a final simplex table containing a basic feasible solution to the original problem

Step 1: Assign a cost(-1) to each artificial variable and a cost (0) to all other variables and get a new objective function.

Step 2: Write down the auxillary LPP in which the new objective function is to be maximized subject to the given set of constraints. ~~step 3~~

Step 3: Solve the auxillary LPP by simplex method either of the following cases arise.

1) max $z < 0$ and atleast 1 artificial variable appears at +ve level

2) max $z = 0$ and atleast 1 artificial variable appears at 0 level

3) max $z = 0$ and no artificial variables appears

Note: In case 1, given LPP does not process any feasible solution where as in case 2 and 3 we go to phase ~~1~~ 2.

Phase 2 : Use the optimum basic feasible solution of phase 1 as a starting solution for the original LPP. Assign the actual cost to the variable in the objective function, and a zero cost to every artificial variable at zero level, delete the artificial variable column that is eliminated from the phase 1. Apply simplex method to the modified simplex table obtain at the end of phase 1 till an optimum basic feasible solution is obtained.

1) Use two phase method to solve

$\max z = 3x_1 - x_2$

STC $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 2$

$x_2 \leq 4$

where $x_1, x_2 \geq 0$

⇒ Step 1: Convert the given inequality to equality

$z = 3x_1 - x_2$

$2x_1 + x_2 - S_1 + A_1 = 2$

$x_1 + 3x_2 + S_2 = 2$

$x_2 + S_3 = 4$

Step 2: Rewrite the equation in matrix format

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & A_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \\ A_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

Phase I

			0	0	0	0	0	-1	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	A_1	x_B/x_1
-1	A_1	2	$\boxed{2}$	1	-1	0	0	1	1 ← (+ve min)
0	s_2	2	1	3	0	1	0	0	2
0	s_3	4	0	1	0	0	1	0	∞
$Z_j = \Sigma C_B \cdot x_j$			-2	-1	1	0	0	-1	
$Z_j - C_j \geq 0$			-2	-1	1	0	0	0	
			↑ (-max)						

R1: (2 2 1 -1 0 0 1) / 2

1 1 1/2 -1/2 0 0 1/2

R2 2 1 3 0 1 0 0

R2: 1 0 5/2 1/2 1 0 -1/2

R3: 4 0 1 0 0 1 0

			0	0	0	0	0		
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	A_1	x_B/x_j
0	x_1	1	1	1/2	-1/2	0	0	1/2	
0	s_2	1	0	5/2	1/2	1	0	-1/2	
0	s_3	4	0	1	0	0	1	0	
$Z_j = \Sigma C_B \cdot x_j$			0	0	0	0	0	0	

$z_{max} = 0$ and no artificial variable goto phase II

		C_j	3	-1	0	0	0	X_B/x_j
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	
3	x_1	1	1	1/2	-1/2	0	0	-2
0	S_2	1	0	5/2	$\boxed{1/2}$	1	0	2 ← +min
0	S_3	4	0	1	0	0	1	∞ ← +min
$z_j = \Sigma C_B \cdot X_B$			3	3/2	-3/2	0	0	
$z_j - C_j \geq 0$			0	5/2	-3/2	0	0	
					↑ (-max)			

$R_2 : (1 \quad 0 \quad 5/2 \quad 1/2 \quad 1 \quad 0) \times 2$

$R_2 \quad 2 \quad 0 \quad 5 \quad 1 \quad 2 \quad 0$

$(R_2/2) \quad 1 \quad 0 \quad 5/2 \quad 1/2 \quad 1 \quad 0$

$R_1 + \quad 1 \quad 1 \quad 1/2 \quad -1/2 \quad 0 \quad 0$

$R_1 \quad 2 \quad 1 \quad 3 \quad 0 \quad 1 \quad 0$

		C_j	3	-1	0	0	0	X_B/x_j
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	
3	x_1	2	1	3	0	1	0	
0	S_2	2	0	5	1	2	0	
0	S_3	4	0	1	0	0	1	
$z_j = \Sigma C_B \cdot X_j$			3	9	0	3	0	
$z_j - C_j$			0	10	0	3	0	

$z_{max} = 6$ //

$x_1 = 2 \qquad x_2 = 0$

$Z = 3x_1 - x_2$

$= 3(2) - 0$

$= 6$ //

2) min $z = x_1 - 2x_2 - 3x_3$

STC $-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

where $x_1, x_2, x_3 \geq 0$

Step 1: convert min to max problem

$z = -x_1 + 2x_2 + 3x_3$

Step 2: Convert inequality to equality.

$-2x_1 + x_2 + 3x_3 + A_1 = 2$

$+2x_1 + 3x_2 + 4x_3 + A_2 = 1$

Step 3: Write in matrix form.

$$\begin{bmatrix} x_1 & x_2 & x_3 & A_1 & A_2 \\ -2 & 1 & 3 & 1 & 0 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

	C_j		0	0	0	-1	-1	x_B/x_j
C_B	Y_B	X_B	x_1	x_2	x_3	A_1	A_2	
-1	A_1	2	-2	1	3	1	0	2/3
-1	A_2	1	2	3	$\boxed{4}$	0	1	1/4 ← +min
$z_j = \Sigma C_B \cdot x_j$			-4	-4	-7	-1	-1	
$z_j - C_j$			-4	-4	-7	0	0	

↑ (−max)

R2: (1 2 3 4 0 1) / 4

1/4 1/2 3/4 1 0 1/4

(R2 x 3)	3/4	3/2	9/4	3	0	3/4
- R1	2	-2	1	3	1	0
R1 :	5/4	-7/2	-5/4	0	1	-3/4

		C_j	0	0	0	-1	
C_B	Y_B	X_B	x_1	x_2	x_3	A_1	A_2
-1	A_1	5/4	-7/2	-5/4	0	1	-3/4
0	x_3	1/4	1/2	3/4	1	0	1/4
$z_j = \Sigma C_B \cdot x_j$			7/2	5/4	0	-1	3/4
$z_j - C_j$			7/2	5/4	0	0	

$\max z = (-1 \times 5/4) + 0$

$= -5/4$

$z_{max} \not< 0$ and one artificial variable appears.

No feasible solution.