1 SNR-Weighted Least-Squares (WLS)

We assume n_{BS} anchors at known 2-D positions $\mathbf{a}_j = (x_j, y_j),\ j = 1, \ldots, n_{\mathrm{BS}}$, and for snapshot i we have measured ranges $d_{i,j}$ and SNRs in dB $\mathsf{snr}_{i,j}$. Let S_i be the set of the k anchors with highest SNR, and convert to linear weights

$$w_{i,j} \;=\; 10^{\mathsf{snr}_{i,j}/10}, \quad w_{i,j} \leftarrow \frac{w_{i,j}}{\sum_{m\in S_i} 10^{\mathsf{snr}_{i,m}/10}}, \quad j \in S_i.$$

Choose the first anchor in S_i as reference, denoted (x_1, y_1, d_1). For each $j \in S_i \setminus \{j_1\}$ linearize the range-difference equation:

$$2(x_j - x_1)\,\hat{p}_x \;+\; 2(y_j - y_1)\,\hat{p}_y = (x_j^2 - x_1^2) + (y_j^2 - y_1^2) - (d_j^2 - d_1^2). \tag{1}$$

Stacking these for all $j = 2, \ldots, k$ gives

$$A_i\,p \;=\; b_i, \quad A_i \in R^{(k-1)\times 2},\ b_i \in R^{(k-1)}.$$

Form the diagonal weight matrix

$$W_i = \mathrm{diag}\big(w_{i,2},\, w_{i,3},\, \ldots,\, w_{i,k}\big)$$

and solve the weighted normal equations in closed form:

$$\hat{p}_i^{\mathrm{WLS}} \;=\; \big(A_i^\top W_i\, A_i\big)^{-1} A_i^\top W_i\, b_i. \tag{2}$$

If ground-truth positions $\{p_i^{\mathrm{true}}\}$ are available, the per-snapshot error is $e_i = \big\|\hat{p}_i^{\mathrm{WLS}} - p_i^{\mathrm{true}}\big\|$, and the overall RMSE is

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} e_i^2}.$$

2 Iterative MLE via Levenberg–Marquardt

Under the Gaussian noise model, we assume that a link with 0 dB SNR has 1 m standard deviation (i.e. $\sigma_0 = 1$ m at 0 dB). Hence each observed range d_j^{obs} has standard deviation

$$\sigma_j = \frac{\sigma_0}{\sqrt{10^{\mathsf{snr}_j/10}}} = \frac{1}{\sqrt{10^{\mathsf{snr}_j/10}}}\,.$$

So the negative log-likelihood becomes the weighted least squares

$$J(p) = \tfrac{1}{2}\sum_{j=1}^{k}\Big(\frac{d_j^{\mathrm{obs}} - \|p - a_j\|}{\sigma_j}\Big)^2.$$

Define the residual vector $r \in R^k$ and Jacobian $J \in R^{k\times 2}$ by

$$r_j(p) = \frac{d_j^{\mathrm{obs}} - \|p - a_j\|}{\sigma_j}, \qquad \frac{\partial r_j}{\partial p} = -\,\frac{(p - a_j)^\top}{\|p - a_j\|\,\sigma_j}. \tag{3}$$

At each iteration, stack $r = [r_1, \dots, r_k]^\top$, compute $H = J^\top J, \quad g = J^\top r$, and solve the Levenberg–Marquardt step

$$(H + \lambda I)\, \delta = -\, g.$$

Update $p \leftarrow p + \delta$ and adapt the damping λ via the standard gain-ratio rule. Convergence is declared when any of

$$\|g\|_\infty < 10^{-12}, \quad \|\delta\| \le \mathrm{TolStep}\,(\|p\| + \mathrm{TolStep}), \quad |\Delta J| < \mathrm{TolFun}$$

is met or the maximum iteration count is reached.

3 Range–Angle Fusion via Extended Kalman Filter

To combine our 2-D MLE position estimates $\{\hat{p}_k^{\mathrm{MLE}}\}$ with a per-snapshot bearing from anchor A, we formulate a 4-D constant-velocity state and apply an EKF.

3.1 State Model

Let the state at time k be

$$x_k = \begin{bmatrix} x_k \\ y_k \\ v_{x,k} \\ v_{y,k} \end{bmatrix}.$$

We assume a discrete constant-velocity evolution with sampling interval Δt:

$$x_{k+1} = F\, x_k + w_k, \qquad F = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad w_k \sim \mathcal{N}\big(0, Q\big). \tag{4}$$

Here $Q \in R^{4\times 4}$ is the process-noise covariance.

3.2 Measurement Model

At each time k we collect

$$z_k = \begin{bmatrix} \hat{x}_k^{\mathrm{MLE}} \\ \hat{y}_k^{\mathrm{MLE}} \\ \hat{\theta}_k \end{bmatrix} = h(x_k) + v_k, \quad v_k \sim \mathcal{N}\big(0, R_k\big). \tag{5}$$

The nonlinear measurement function is

$$h(x) = \begin{bmatrix} x \\ y \\ \mathrm{atan2}\big(y - y_A,\ x - x_A\big) \end{bmatrix}, \tag{6}$$

and

$$R_k = \mathrm{diag}\big(\sigma_x^2,\ \sigma_y^2,\ \sigma_\theta^2\big),$$

reflects the MLE and DOA error variances (optionally scaled by SNR).

3.3 EKF Recursion

Denote the predicted (prior) state by $\hat{x}_{k|k-1}, P_{k|k-1}$ and the updated (posterior) by $\hat{x}_{k|k}, P_{k|k}$. Then at each step:

1. **Predict:**

$$\hat{x}_{k|k-1} = F\,\hat{x}_{k-1|k-1}, \quad P_{k|k-1} = F\,P_{k-1|k-1}\,F^\top + Q.$$

2. **Linearize:** compute

$$H_k = \frac{\partial h}{\partial x}\bigg|_{\hat{x}_{k|k-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{y-y_A}{d^2} & \dfrac{x-x_A}{d^2} & 0 & 0 \end{bmatrix}, \quad d^2 = (x-x_A)^2 + (y-y_A)^2.$$

3. **Update:**

$$K_k = P_{k|k-1}\,H_k^\top\big(H_k\,P_{k|k-1}\,H_k^\top + R_k\big)^{-1}, \tag{7}$$

$$r_k = z_k - h(\hat{x}_{k|k-1}), \tag{8}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\,r_k, \tag{9}$$

$$P_{k|k} = \big(I - K_k\,H_k\big)\,P_{k|k-1}. \tag{10}$$

This filter fuses the Cartesian MLE position with the AoA bearing to produce a smoothed, dynamically consistent track.

RSSI Averaging, SNR Computation, and Range Resolution

3.1 RSSI Averaging

At each snapshot i and for each anchor b we obtain a raw RSSI measurement in dBm, denoted $\mathrm{RSSI}_{i,b}$. To smooth out fast fluctuations we

1. convert to linear power:

$$P_{i,b} = 10^{\,\mathrm{RSSI}_{i,b}/10} \quad (\mathrm{mW}),$$

2. apply a causal moving average of length N,

$$\overline{P}_{i,b} = \frac{1}{N}\sum_{n=0}^{N-1} P_{i-n,b}\,,$$

3. convert back to dBm,

$$\overline{\mathrm{RSSI}}_{i,b} = 10\log_{10}\bigl(\overline{P}_{i,b}\bigr).$$

3.2 Noise Floor and SNR

Thermal noise power (in watts) over our bandwidth B is

$$P_{\mathrm{th}} = k_{\mathrm{B}}\,T\,B,$$

where $k_{\mathrm{B}} = 1.38 \times 10^{-23}\,\mathrm{J/K}$ and $T = 290\,\mathrm{K}$. With a receiver noise figure NF (in dB), the noise-floor in dBm is

$$P_{\mathrm{n,dBm}} = 10\log_{10}(1000\,P_{\mathrm{th}}) \;+\; \mathrm{NF}.$$

Thus the per-link SNR in dB becomes

$$\mathrm{SNR}_{i,b} = \overline{\mathrm{RSSI}}_{i,b} \;-\; P_{\mathrm{n,dBm}}.$$

3.3 Theoretical Range Resolution

Given a signal bandwidth B (Hz), the fundamental two-way range resolution (for e.g. pulse or FMCW) is

$$\Delta R = \frac{c}{2\,B},$$

where c is the speed of light. In our case, with $B = 10\,\mathrm{MHz}$, this yields

$$\Delta R = \frac{3 \times 10^{8}}{2 \times 10^{7}} = 15\ \mathrm{m}.$$

This represents the best-case granularity with which one could hope to distinguish two closely-spaced targets, and provides a useful benchmark when examining our inverted-range errors.

4 Log-Distance Inversion of Measured RSSI

We assume the common log-distance model with log-normal shadowing. Let

$$d \quad [\mathrm{m}] \quad \text{be the true transmitter–receiver distance}, \quad d_0 = 1\,\mathrm{m} \quad \text{(reference distance)},$$

$$P_{\mathrm{tx}} \quad [\mathrm{dBm}] = \text{transmit power}, \quad G_{\mathrm{tx}}, G_{\mathrm{rx}} \quad [\mathrm{dBi}] = \text{antenna gains}, \quad L_{\mathrm{sys}} \quad [\mathrm{dB}] = \text{system losses}.$$

We define the *reference* received power at d_0 by

$$P_0 \;=\; P_{\mathrm{tx}} + G_{\mathrm{tx}} + G_{\mathrm{rx}} - L_{\mathrm{sys}} \;-\; \mathrm{PL}(d_0),$$

where $\mathrm{PL}(d_0)$ is the chosen path-loss (e.g. FSPL or 3GPP LOS) at d_0.

At an arbitrary distance d, the measured RSSI (in dBm) is

$$\mathrm{RSSI} = P_0 \;-\; 10\,\beta\,\log_{10}\bigl(\tfrac{d}{d_0}\bigr) \;+\; n, \quad n \sim \mathcal{N}\bigl(0, \sigma^2_{\mathrm{shadow}}\bigr). \tag{11}$$

Since our RSSI already *includes* the random shadowing n (and optionally small-scale fading), we drop n when inverting.

Solving (11) for d gives

$$d = d_0\, 10^{\frac{P_0 - \text{RSSI}}{10\,\beta}}. \tag{12}$$

The path-loss exponent β and shadow-standard deviation σ_{shadow} are taken from 3GPP TR 38.901 for the UMa scenario:

$$\beta = \begin{cases} 2.2, & \text{LOS links,} \\ 3.5, & \text{NLOS links,} \end{cases} \qquad \sigma_{\text{shadow}} = \begin{cases} 4\,\text{dB}, & \text{LOS,} \\ 6\,\text{dB}, & \text{NLOS.} \end{cases}$$

In our implementation, for each measured RSSI we

1. compute P_0 from the known Tx/Rx parameters and reference loss,
2. determine β via the LOS/NLOS flag,
3. invert via (12) to yield an estimated range.

5 MSE

$$\begin{aligned} \text{MSE}(\hat{\theta}) &= E\big[(\hat{\theta} - \theta)^2\big] \\ &= \underbrace{E\big[(\hat{\theta} - E[\hat{\theta}])^2\big]}_{\text{Var}(\hat{\theta})} + \underbrace{\big(E[\hat{\theta}] - \theta\big)^2}_{\text{Bias}(\hat{\theta})^2}. \end{aligned}$$

$$\text{Bias} = \mu_e = \frac{1}{N}\sum_{i=1}^{N} e_i,$$

$$\text{Var}(e) = \frac{1}{N}\sum_{i=1}^{N}\big(e_i - \mu_e\big)^2,$$

$$\text{MSE} = \frac{1}{N}\sum_{i=1}^{N} e_i^2 = \mu_e^2 + \text{Var}(e),$$

$$\text{RMSE} = \sqrt{\text{MSE}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} e_i^2},$$

$$\text{MAE} = \frac{1}{N}\sum_{i=1}^{N}\big| e_i\big|.$$

6 CRLB calculation

1. RSS-to-Range Noise Conversion

Assuming a log-distance path-loss model:

$$P_j(d_j) \approx P_{0,j} \; - \; 10\,\beta_j \, \log_{10}\!\big(d_j/d_0\big) + n_j, \quad n_j \sim \mathcal{N}(0, \sigma_{p,j}^2)\,.$$

Linearizing yields

$$\frac{dP_j}{dd_j} \; = \; -\,\frac{10\,\beta_j}{\ln(10)}\,\frac{1}{d_j} \quad \Longrightarrow \quad \big|\tfrac{dd_j}{dP_j}\big| = \frac{\ln(10)}{10\,\beta_j}\,d_j.$$

Thus the standard deviation of the range error is

$$\sigma_{d,j} = \frac{\ln(10)}{10\,\beta_j}\,d_j\;\sigma_{p,j}\,. \tag{13}$$

2. Range Measurement Jacobian

For the j-th anchor at $\mathbf{a}_j = [x_j,\; y_j]^T$ and target $\mathbf{x} = [x, y]^T$,

$$d_j = \|\mathbf{x} - \mathbf{a}_j\| = \sqrt{(x - x_j)^2 + (y - y_j)^2}\,.$$

The gradient w.r.t. $\mathbf{x}$ is

$$\nabla_{\mathbf{x}}\, d_j = \begin{bmatrix} \dfrac{\partial d_j}{\partial x} \\ \dfrac{\partial d_j}{\partial y} \end{bmatrix} = \frac{1}{d_j}\begin{bmatrix} x - x_j \\ y - y_j \end{bmatrix}. \tag{14}$$

Stacking for $j = 1, \ldots, K$ gives

$$\mathbf{J}_r = \begin{bmatrix} \frac{x-x_1}{d_1} & \frac{y-y_1}{d_1} \\ \frac{x-x_2}{d_2} & \frac{y-y_2}{d_2} \\ \vdots & \vdots \\ \frac{x-x_K}{d_K} & \frac{y-y_K}{d_K} \end{bmatrix}, \quad \mathbf{J}_r \in R^{K\times 2}.$$

3. Range-Only FIM

Under independent Gaussian range noise $w_{d,j} \sim \mathcal{N}(0, \sigma_{d,j}^2)$,

$$\mathbf{F}_r = \sum_{j=1}^{K} \frac{1}{\sigma_{d,j}^2}\,\big(\nabla_{\mathbf{x}}\, d_j\big)\big(\nabla_{\mathbf{x}}\, d_j\big)^T \;=\; \mathbf{J}_r^T\,\big[\mathrm{diag}(1/\sigma_{d,1}^2, \ldots, 1/\sigma_{d,K}^2)\big]\,\mathbf{J}_r\,, \tag{15}$$

where $\mathbf{F}_r \in R^{2\times 2}$.

4. Bearing (DOA) Jacobian and FIM

If a bearing measurement is available from a sensor at $\mathbf{a}_A = [x_A, y_A]^T$, the true bearing is

$$\theta = \operatorname{atan2}(y - y_A,\ x - x_A), \quad \tilde{\theta} = \theta + w_b, \quad w_b \sim \mathcal{N}(0, \sigma_b^2).$$

The gradient w.r.t. $\mathbf{x}$ is

$$\nabla_{\mathbf{x}}\, \theta = \frac{1}{r^2} \begin{bmatrix} -\big(y - y_A\big) \\ \big(x - x_A\big) \end{bmatrix}, \quad r^2 = (x - x_A)^2 + (y - y_A)^2. \tag{16}$$

Denote this row as

$$\mathbf{J}_b = \begin{bmatrix} -\dfrac{y - y_A}{r^2} & \dfrac{x - x_A}{r^2} \end{bmatrix}.$$

The bearing FIM is then

$$\mathbf{F}_b = \frac{1}{\sigma_b^2}\, \mathbf{J}_b^T\, \mathbf{J}_b \ = \ \frac{1}{\sigma_b^2} \begin{bmatrix} \left(\frac{y-y_A}{r^2}\right)^2 & -\frac{(x-x_A)(y-y_A)}{r^4} \\ -\frac{(x-x_A)(y-y_A)}{r^4} & \left(\frac{x-x_A}{r^2}\right)^2 \end{bmatrix}. \tag{17}$$

5. Total FIM and CRLB

Total FIM (with regularization) Combine range and bearing information (regularize by ϵI_2):

$$\mathbf{F} = \mathbf{F}_r \ + \ \mathbf{F}_b \ + \ \epsilon\, \mathbf{I}_2, \quad \epsilon > 0\,. \tag{18}$$

Check positive definiteness of $\mathbf{F}$ (e.g. via Cholesky) and increase ϵ if needed.

CRLB Covariance The CRLB lower bound on the 2D position covariance is

$$\boldsymbol{\Sigma} = \mathbf{F}^{-1}. \tag{19}$$

Its trace is the sum of variances:

$$\operatorname{tr}(\boldsymbol{\Sigma}) = \operatorname{Var}(x) \ + \ \operatorname{Var}(y). \tag{20}$$

CRLB-RMS Radial Error Define the RMS radial error bound as

$$\mathrm{RMSE}_{\mathrm{CRLB}} = \sqrt{\operatorname{tr}(\boldsymbol{\Sigma})} = \sqrt{\operatorname{Var}(x) + \operatorname{Var}(y)}. \tag{21}$$