SUBJECT : Mathematics 2

SUBJECT CODE:BMATS201

MODULE:01 to 05

VERY IMPORTANT QUESTIONS WITH SOLUTIONS

- Can score 70+ marks in VTU sem exam with minimal efforts.
- Specially designed for last moment exam preparation.
- This material includes the questions from previous year question papers, model papers and also expert professors predicted important questions.

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NOTE:

- **Very Important questions with solutions will be explained in the video Course by Expert VTU Engineering Professors.**
- **Video course will include all 5 modules and will be covered within 4-5 hours, which is Specially made for last moment exam preparation .**

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INTEGRAL CALCULAS - MODULE - 01

SUBCODE : BMATS201/C201/M201

(1) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

:- $I = \int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$

$$= \int_{-c}^{c}\int_{-b}^{b}\left[x^2(z)+y^2z+\frac{z^3}{3}\right]_{-a}^{a}dy\,dx$$

$$= \int_{-c}^{c}\int_{-b}^{b}\left(2ax^2+2ay^2+\frac{2a^3}{3}\right)dy\,dx$$

$$= 2a\int_{-c}^{c}\left(x^2y+\frac{y^3}{3}+\frac{a^2}{3}y\right)_{-b}^{b}dx$$

$$= 2a\int_{-c}^{c}\left(2bx^2+\frac{2b^3}{3}+\frac{2a^2b}{3}\right)dx$$

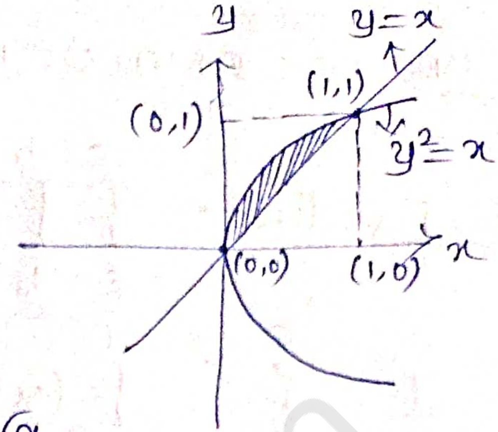
$$= 4ab\int_{-c}^{c}\left(x^2+\frac{b^2}{3}+\frac{a^2}{3}\right)dx$$

$$= 4ab\left[\frac{x^3}{3}+\frac{b^2}{3}x+\frac{a^2}{3}x\right]_{-c}^{c}$$

$$= 4ab\left[\frac{2c^3}{3}+\frac{2b^2c}{3}+\frac{2a^2c}{3}\right]$$

$$I = \frac{8abc}{3}(a^2+b^2+c^2)$$

2) Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by changing the order of integration.



:- $I = \int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$

$x : 0 \to 1$; $y : x \to \sqrt{x}$

$\Rightarrow y = x$, $y = \sqrt{x}$

$\Rightarrow y^2 = x$

Now, $y : 0 \to 1$

$x : y^2 \to y$

$$I = \int_0^1 \left[x \frac{y^2}{2} \right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \frac{x}{2} \left[x - x^2 \right] dx = \frac{1}{2} \int_0^1 (x^2 - x^3) \, dx$$

$$I = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{3} - \frac{1}{4} \right] = \frac{1}{2} \left[\frac{4-3}{12} \right]$$

$$I = \frac{1}{24}$$

(3) S.T $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

:- WKT,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{---} (1)$$

$$\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1} dx \quad \text{---} (2)$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2n-1} dy \quad \text{---} (3)$$

Consider,

$$\Gamma(m+n) = 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr \quad \text{---} (4)$$

consider,

$$\Gamma(m)\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2m-1} dx \times 2\int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dy\, dx \quad \text{---} (*)$$

The above integral is solved by changing to polar co-ordinates,

$x = r\cos\theta$, $y = r\sin\theta$, $r: 0 \to \infty$ $\quad dx\,dy = r\,dr\,d\theta$

$\theta: 0 \to \pi/2$

$\therefore (*) \Rightarrow$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \times 2\int_0^{\infty} e^{-r^2} \frac{r^{2m-1} r^{2n-1} \cdot r\,dr}{r^{2m+2n-1}}$$

$$\Gamma(m)\Gamma(n) = \beta(m,n) \times \Gamma(m+n)$$

By Gayathri V, Assistant Professor,EPCET

(4) Show that $\Gamma(1/2) = \sqrt{\pi}$

:- WKT, $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

put $m = n = 1/2$

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}$$

$$\beta(1/2, 1/2) = [\Gamma(1/2)]^2 \text{ ---- } (1)$$

Let us consider,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$m = n = 1/2$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2}$$

$$= \cancel{2}\left(\frac{\pi}{\cancel{2}}\right) = \pi$$

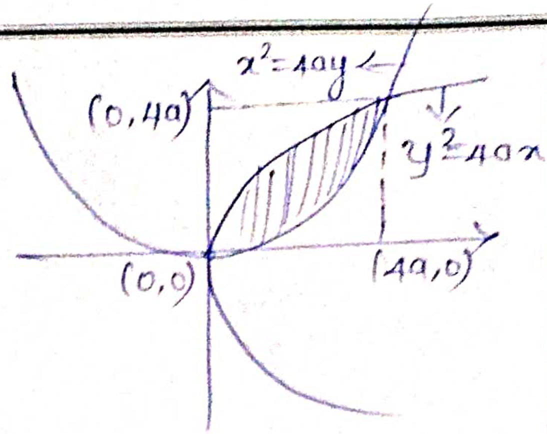
$\therefore (1) \Rightarrow \quad \beta(1/2, 1/2) = \pi = [\Gamma(1/2)]^2$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$$

(5) Find by double integration, the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

:- We know that,

$$A = \iint_R dy\,dx$$



$$A = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

$$= \int_0^{4a} \left[y\right]_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \cancel{\int_{0}} \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\left(\frac{x^3}{3}\right)\right]_0^{4a}$$

$$= \frac{\cancel{4}}{3}\sqrt{a}\left[4a\right]^{3/2} - \frac{1}{12a}(4a)^3$$

$$= \frac{4}{3}a^{1/2}(4a)^{3/2} - \frac{64}{12}a^2 = \frac{4a^2 \cdot 8}{3} - \frac{16}{3}a^2$$

$$= \cancel{4a^2} - \cancel{\frac{64a^2}{12}} = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{}{\cancel{12}} \qquad A = \frac{16a^2}{3} \text{ sq units}$$

SUB CODE : BMATS201 / C201 / E201

IMPORTANT QUESTION PAPER SOLUTIONS :→ MODULE-02

(1) Find $\nabla\phi$, if $\phi = x^3+y^3+z^3-3xyz$ at the point $(1,-1,2)$

:- Given $\phi = x^3+y^3+z^3-3xyz$

To find $\nabla\phi$

WKT $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$

at $(1,-1,2)$

$\nabla\phi = (3+6)\hat{i} + (3-6)\hat{j} + (12+3)\hat{k}$

$= 9\hat{i} + (-3)\hat{j} + 15\hat{k}$

$= 3(3\hat{i} - \hat{j} + 5\hat{k})$ //

(2) If $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$ find div $\vec{F}$ and curl $\vec{F}$

:- Given $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

$\vec{F} = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$

div $\vec{F} = \nabla\cdot\vec{F}$

$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$$\nabla \cdot \vec{F} = 6x + 6y + 6z$$

$$\text{div}\,\vec{F} = 6(x+y+z)$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3yx \end{vmatrix}$$

$$= \hat{i}(-3x+3x) - \hat{j}(-3y+3y) + \hat{k}(-3z+3z)$$

$$\text{curl}\,\vec{F} = \vec{0}$$

(3) If $\vec{F} = \nabla(xy^3z^2)$, find $\text{div}\,\vec{F}$ and $\text{curl}\,\vec{F}$ at $(1,-1,1)$

:- Given $\vec{F} = \nabla(xy^3z^2)$

$$\vec{F} = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$$

$$= \frac{\partial}{\partial x} y^3z^2 + \frac{\partial}{\partial y} 3xy^2z^2 + \frac{\partial}{\partial z} 2xy^3z$$

$$= 0 + 6xyz^2 + 2xy^3$$

$$\text{div}\,\vec{F} = 2xy(y+z^2)$$ at $(1,-1,1)$

$$\text{div}\,\vec{F} = -8$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= \hat{i}\left(6xy^2z - 6xy^2z\right) - \hat{j}\left(2y^3z - 2y^3z\right)$$
$$+ \hat{k}\left(3y^2z^2 - 3y^2z^2\right)$$

$$\text{curl } \vec{F} = \vec{0}$$

At $(1,-1,0)$, curl $\vec{F} = \vec{0}$ //

Repeated

(4) Find the Directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

:- WKT Directional derivative $= \nabla\phi \cdot \hat{n}$

Given, $\phi = x^2yz + 4xz^2$, $(1,-2,-1)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \left(2xy + 4z^2\right)\hat{i} + \left(x^2z\right)\hat{j} + \left(x^2y + 8xz\right)\hat{k}$$

at $(1,-2,-1)$ $\quad \nabla\phi = 0\hat{i} - \hat{j} - 10\hat{k}$

WKT

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|}, \quad \text{Given } \vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$$

$$\because |\vec{a}| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

$$\therefore D.D = \nabla\phi \cdot \hat{n}$$

$$= (0\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

$$= 0 + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{21}{3} = 7$$

(5) Find the angle between the surfaces $x^2+y^2-z^2=4$ and $z=x^2+y^2-13$ at $(2,1,2)$

:- Given,

$\phi_1 = x^2+y^2-z^2-4$ & $\phi_2 = z-x^2-y^2+13$.

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\hat{i} + \frac{\partial\phi_1}{\partial y}\hat{j} + \frac{\partial\phi_1}{\partial z}\hat{k}$$

$$= 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

at $(2,1,2)$

$$\nabla\phi_1 = 4\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} + \frac{\partial\phi_2}{\partial z}\hat{k}$$

$$= -2x\hat{i} - 2y\hat{j} + \hat{k}$$

at $(2,1,2)$

$$\nabla\phi_2 = -4\hat{i} - 2\hat{j} + \hat{k}$$

WKT,

Angle b/w the surfaces.

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4\hat{i}+2\hat{j}-4\hat{k})\cdot(-4\hat{i}-2\hat{j}+\hat{k})}{(\sqrt{36})\,(\sqrt{21})}$$

$$= \frac{-16-4-4}{6\sqrt{21}}$$

$$= \frac{-24}{6\sqrt{21}}$$

$$\cos\theta = \frac{-4}{\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{-4}{\sqrt{21}}\right) //$$

(6) Show that the vector $\vec{F} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$ is both solenoidal and irrotational

:- Given $\vec{F} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + 0$$

$$= \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{-\cancel{x^2}+\cancel{y^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2}$$

$\operatorname{div}\vec{F} = 0 \Rightarrow \vec{F}$ is Solenoidal.

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= \vec{0}$$

$\Rightarrow \vec{F}$ is irrotational.

(7) Using Mathematical tools, write a code to find curl of $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

:- PROGRAM:

```
from sympy import *
```

```
from sympy import symbols
N = CoordSys3D ('N')
x, y, z = symbols ('x y z')
A = N.x* N.y**2 * N.i + 2*N. x**2. N.y *N.z
    * N.j - 3* N.y * N.z**2* N.k
delop = Del()
curl A = delop. cross(A)
print(delop(A))
print (f"\n curl of {A} is \n")
print (curl(A))
```

(8) Using Mathematical tools, write the code to find gradient of $\phi = x^2yz$

:- PROGRAM:

```
from sympy. vector import*
from sympy import symbols
N = CoordSys3D('N')
x, y, z = symbols ('x y z')
A = N.x**2 *N.y * N.z
delop= Del()
print (delop(A))
grad(A) = gradient (A)
print("the gradient is", grad(A))
```

(9) Show that the Spherical coordinate system is orthogonal.

:- For the Spherical system, we have

$$\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

Let $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ be the basic vectors

We have, $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

Consider

$$\hat{e}_r = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta} = \frac{1}{r}\left(r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} + r(-\sin\theta)\hat{k}\right)$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial\phi} = \frac{1}{r\sin\theta}\left(-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\hat{k}\right)$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Consider,

$$\hat{e}_r\cdot\hat{e}_\theta = \sin\theta\cos\theta\left(\cos^2\phi + \sin^2\phi\right) - \sin\theta\cos\theta = 0$$

lllly $\hat{e}_\theta\cdot\hat{e}_\phi = 0$ & $\hat{e}_\phi\cdot\hat{e}_r = 0$

Thus the Spherical co-ordinate system is orthogonal

(10) Show that cylindrical co-ordinate system is orthogonal.

:- Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a position vector.

Cyclindrical co-ordinate system,

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

$\therefore \vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

Let $\hat{e}_\rho, \hat{e}_\phi, \hat{e}_\theta$ be the basic vectors.

$$\therefore \hat{e}_\rho = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0k \qquad \because h_1 = 1$$

$$\hat{e}_\phi = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\phi} = \frac{1}{\rho}\left(-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j} + 0\hat{k}\right) \quad \because h = \rho$$

$$\hat{e}_z = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial z} = 0\hat{i} + 0\hat{j} + 1\hat{k} = \hat{k} \qquad \because h_3 = 1$$

Consider,

$\hat{e}_\rho \cdot \hat{e}_\phi = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0$

$\hat{e}_\phi \cdot \hat{e}_z = 0$

$\hat{e}_z \cdot \hat{e}_\rho = 0$

$\therefore$ Cyclindrical system is orthogonal.

MODULE-03 : Vector Space and Linear Transformation

Important Question paper Solution

SUBCODE - BMATS201

1. prove that the subset $W = \{(x, y, z) / x - 3y + 4z = 0\}$ of the vector space R^3 is a subspace of R^3.

:- Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any two elements of W, $x_1, x_2, y_1, y_2, z_1, z_2 \in R$

Then $ax_1 + by_1 + cz_1 = 0$ & $ax_2 + by_2 + cz_2 = 0$

$$\Rightarrow \underbrace{x_1 - 3y_1 + 4z_1}_{u} = 0 \text{ and } \underbrace{x_2 - 3y_2 + 4z_2}_{v} = 0$$

For $\alpha \in R$, $\alpha u + v = \alpha(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$$= (\alpha x_1, \alpha y_1, \alpha z_1) + (x_2, y_2, z_2)$$

$$= (\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2)$$

Where $\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2 \in R$

Now, We've $x - 3y + 4z = 0$

$$1(\alpha x_1 + x_2) - 3(\alpha y_1 + y_2) + 4(\alpha z_1 + z_2)$$

$$= \alpha(x_1 - 3y_1 + 4z_1) + (x_2 - 3y_2 + 4z_2)$$

$$= \alpha(0) + 0 \quad (\text{or}) \quad = \alpha(u) + v$$

$$= 0$$

Hence W is a Subspace of R^3

(2) Let $V = R^3$ be a vector space and consider the subset W of V consisting of vectors of the form (a, a^2, b), where the second component is the square of the first. Is W a subspace of V?

:- Let $V = \mathbb{R}^3 = \{(a,b,c) / a,b,c \in R\}$

Subspace $W = \{(a, a^2, b) / a, b \in R\}$

Condn, (i) $\forall\ w_1, w_2 \in W \ni: w_1 + w_2 \in W$

(ii) $\forall\ \alpha \in F,\ w \in W \ni: \alpha w \in W$

$\because \alpha \in F$, let $\alpha = -1 \in R$ ($\because$ R is a field)

Let $a = 1$, $b = 0$ $\therefore w = (1,1,0) \in W$

Consider, $\alpha . w = -1(1,1,0)$

$= (-1, -1, 0) \notin W$

Scalar Multiplication fails.

$\Rightarrow$ W is not a subspace of V.

(4) Find the basis and the dimension of the subspace spanned by the vectors $\{(2,4,2),(1,-1,0),(1,2,1),(0,3,1)\}$ in $V_3(R)$

:- Let $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

$R_1 \to \frac{1}{2} R_1$

$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

$R_4 \to R_4 + R_2$

$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of A are

$\because \{(1,2,1),(0,-3,-1)\}$ are linearly independent vectors

$\therefore$ Basis of A is $\{(1,2,1),(0,-3,-1)\}$

$\therefore$ WKT Dimensions is No of Non-zero rows

$\therefore$ Dimension (A) = dim [A] = 2

(5) Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a Linear Combination of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, in the Vector Space M_{22} of 2×2 matrices.

:- Let,

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \beta \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} \alpha+2\beta & -3\beta+\gamma \\ 2\alpha+2\gamma & \alpha+2\beta \end{bmatrix} \text{ ---- (1)}$$

$\therefore$ On equating we get

$\underbrace{\alpha+2\beta=-1}_{(1)}$; $\underbrace{-3\beta+\gamma=7}_{(2)}$; $\underbrace{2\alpha+2\gamma=8}_{(3)}$; $\underbrace{\alpha+2\beta=-1}_{(4)}$

$\alpha+2\beta=-1$ $\quad \times^{ly}(2)$

$\underset{(-)}{2\alpha} \underset{(-)}{+} 2\gamma \underset{(-)}{=} 8$

$4\beta-2\gamma=-10$

$\Rightarrow 2\beta-\cancel{\gamma}=-5$

$-3\beta+\cancel{\gamma}=7$

$-\beta=2 \Rightarrow \boxed{\beta=-2}$

$\therefore (1) \Rightarrow \alpha+2(-2)=1$

$\boxed{\alpha=5}$

$(2) \Rightarrow -3\beta+\gamma=7$

$-3(-2)+\gamma=7 \Rightarrow \boxed{\gamma=1}$

$$(1) \Rightarrow \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 7 \\ 12 & 1 \end{bmatrix}$$

which is not equal

The matrix is not a linear combination of the given vectors.

(6) Show that the set $S = \{(1,2,4), (1,0,0), (0,1,0), (0,0,1)\}$ is linearly dependent.

:- Let

$(1,2,4) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$(1,2,4) = (a, b, c)$

$\Rightarrow a = 1, \; b = 2, \; c = 4$

$\therefore (1,2,4) = 1(1,0,0) + 2(0,1,0) + 4(0,0,1)$

$= (1, 2, 4)$

$\therefore$ The set is linearly dependent because one of the element in set S can be expressed as linear combination of others vectors.

(7) prove that the transformation $T: R^2 \to R^2$ defined by $T(x,y) = (3x, x+y)$ is linear. Find the images of the vectors $(1,3)$ and $(-1,2)$ under this transformation.

:- Let $u = (x_1, y_1) \Rightarrow T(u) = T(x_1, y_1) = (3x_1, x_1+y_1)$

$v = (x_2, y_2) \Rightarrow T(v) = T(x_2, y_2) = (3x_2, x_2+y_2)$

(i) $T(u+v) = T(x_1+x_2, y_1+y_2)$

$= (3(x_1+x_2), x_1+x_2+y_1+y_2)$

$= (3x_1+3x_2, x_1+x_2+y_1+y_2)$

$= (3x_1, x_1+y_1) + (3x_2, x_2+y_2)$

$= T(u) + T(v)$

(ii) $T(\alpha u) = T(\alpha x_1, \alpha y_1)$

$= (\alpha 3x_1, \alpha(x_1+y_1))$

$= \alpha(3x_1, x_1+y_1)$

$= \alpha T(u)$

$\Rightarrow T(x,y)$ is linear.

$\therefore$ Image of $(1,3)$, $T(1,3) = (3,4)$

Image of $(-1,2)$, $T(-1,2) = (3(-1), -1+2) = (-3, 1)$

(8) Find the Kernel and range of the linear operator $T(x,y,z) = (x+y, z)$ of $R^3 \to R^2$

:- Linear operator means Domain and Co-domain must be same and also same dimension

ex: $T: \mathbb{R}^2 \longrightarrow R^2$

$T: V_3(R) \longrightarrow V_3(R)$

here,

$\dim[\text{Domain (or) } \mathbb{R}^3(R)] = 3$

$\dim[\text{Co-domain (or) } \mathbb{R}^2(R)] = 2$

Null space

$Ker(T) = N(T) = \{v \in R^3(R) / T(v) = 0\}$

Let $(x,y,z) \in N(T)$

$T(x,y,z) = 0 = (0,0)$

$\Rightarrow (x+y, z) = (0,0)$

$x = -y, \quad z = 0$

$\therefore N(T) = (x,y,z) = \{(x, -x, 0)\}$

x is non-trivial entry.

hence $\dim(N(T)) = 1$ //

Range Space

Let $\{(1,0,0), (0,1,0), (0,0,1)\}$ be stand Basis. for $R^3(R)$

Their image $T(1,0,0) = (1,0)$, $T(0,0,1) = (0,1)$

$T(0,1,0) = (1,0)$

$R(T) = \{x(1,0) + y(1,0), z(0,1) / x, y, z \in \mathbb{R}\}$

$= \{x+y, z / x, y, z \in \mathbb{R}\}$

Let $(1,0) = 1(1,0) + 0(0,1)$

$= (1,0)$ //

$\Rightarrow$ (1,0) can be written as linear combination of remaining two vectors $\therefore$ It is L.T

$\dim(R(T)) = \text{rank} = \rho(T) = 2$

$\therefore$ Rank + Nulity = dimension of Domain

$2+1 = 3$

$3 = 3$ // verified.

(9) Find the matrix of the Linear transformation $T: V_2(R) \to V_3(R) \ni: T(-1,1) = (-1,0,2)$ and $T(2,1) = (1,2,1)$

:- Let $(x,y) = \alpha(-1,1) + \beta(2,1)$

$= (-\alpha + 2\beta, \alpha + \beta)$

$\Rightarrow -\alpha + 2\beta = x \quad ; \quad \alpha + \beta = y$

$\therefore -\alpha + 2\beta = x$

$\alpha + \beta = y$

$3\beta = x + y \qquad \beta = \frac{x+y}{3}$

$$\therefore \alpha+\beta = y$$

$$\alpha + \left(\frac{x+y}{3}\right) = y$$

$$\alpha = \frac{3y-x-y}{3}$$

$$\alpha = \frac{2y-x}{3}$$

Now,

$$(x,y) = \left(\frac{2y-x}{3}\right)(-1,1) + \left(\frac{x+y}{3}\right)(2,1)$$

$$T(x,y) = \left(\frac{2y-x}{3}\right)T(-1,1) + \left(\frac{x+y}{3}\right)T(2,1)$$

$$= \left(\frac{2y-x}{3}\right)(-1,0,2) + \left(\frac{x+y}{3}\right)(1,2,1)$$

$$= \left(-1\left(\frac{2y-x}{3}\right) + \left(\frac{x+y}{3}\right),\ 0+2\left(\frac{x+y}{3}\right),\ 2\left(\frac{2y-x}{3}\right) + \left(\frac{x+y}{3}\right)\right)$$

$$T(x,y) = \left(\frac{2x-y}{3},\ \frac{2}{3}(x+y),\ \frac{5y-x}{3}\right)$$

$\therefore$ T is linear transformation.

MODULE-04 - NUMERICAL METHOD-01

SUBCODE: BMATS201/C201/E201/M201

IMPORTANT QUESTION PAPER - SOLUTIONS → MODULE-4

(1) Find an approximate soln of the root of the equation $xe^x = 3$, using the Regula-Falsi method, carry out three iterations.

:- Given that

$$y(x) = xe^x - 3. \quad (\text{or}) \quad f(x) = xe^x - 3$$

WKT Regula-Falsi formula,

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

To find (a, b)

Let $x = 0$, $f(0) = -3 < 0 \Rightarrow a = 0$

$x = 1$, $f(1) = e - 3 = -0.28 < 0 \Rightarrow a = 1$

$x = 2$, $f(2) = 2e^2 - 3 = 11.77 > 0 \Rightarrow b = 2$

The root may lie between $(1, 2) = (a, b)$

1st approximation $(a, b) = (1, 2)$

$$x_1 = \frac{1(11.77) - 2(-0.28)}{(11.77) - (-0.28)} = 1.0232$$

$\therefore f(1.0232) = -0.1533 \Rightarrow a = 1.0232$

2nd approximation $(1.0232, 2)$

$$x_2 = \frac{1.0232(11.77) - 2(-0.1533)}{11.77 - (-0.1533)} = 1.0357$$

$\therefore f(1.0357) = -0.0823 < 0 \Rightarrow a = 1.0357$

<u>3rd approximation</u> $(a, b) = (1.0357, 2)$

$$x_3 = \frac{1.0357(11.77) - 2(-0.0823)}{11.77 - (-0.0823)} = 1.0423$$

Thus the required root is $x = 1.0423$

(2) Find an approximate root of the equation $x^3 - 3x + 4 = 0$ using the method of false position, correct to three decimal which lies b/w -3 & -2

:- Given that,

$f(x) = x^3 - 3x + 4$ and $(a, b) = (-2, -3)$

WKT Regula-falsi formula,

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$f(-2) = 2$

$f(-3) = -14$

<u>1st approximation:</u>

$$x_1 = \frac{-2(-14) - (-3)(2)}{-14 - (2)} = -2.125$$

$\therefore f(-2.125) = (-2.125)^3 - 3(-2.125) + 4 = 0.7792$

$\Rightarrow a = -2.125$

2nd approximation: $(a,b) = (-2.185, -3)$

$$x_2 = \frac{-2.125(-14) - (-3)(0.7792)}{-14 - (0.7792)} = -2.1711$$

$\therefore f(-2.1711) = 0.2794$

$\Rightarrow a = -2.1711$

3rd approximation: $(a,b) = (-2.1711, -3)$

$$x_3 = \frac{(-2.1711)(-14) - (-3)(0.2794)}{-14 - 0.2794} = -2.1873$$

Thus the required root is $x = -2.1873$

(3) Find the real root of the equation $3x = \cos x + 1$ correct to three decimal places using Newton's Raphson method.

:- Given that

$f(x) = 3x - \cos x - 1$ & $f'(x) = 3 + \sin x$

To find (a,b)

$x=0, \quad f(0) = 0 - 1 - 1 = -2 < 0$

$x=1, \quad f(1) = 3 - 0.9998 - 1 = 1.0002 > 0$

$\therefore$ The root lies between $(a,b) = (0,1)$

WKT Newton's Raphson's formula,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

<u>1st approximation</u> ; $x_0 = 0$

$$x_1 = 0 - \frac{(-2)}{3} = 0.6666$$

$$\therefore f(0.6666) = -0.0001$$

$$f'(0.6666) = 3.0116$$

<u>2nd approximation</u> ; $x_1 = 0.6666$

$$x_2 = x_2 - \frac{f(x_1)}{f'(x_1)} = 0.6666 - \frac{(-0.0001)}{3.0116}$$

$$x_2 = 0.66663$$

Thus required root is $x = 0.6666$

(4) Find the real root of the equation $\cos x = xe^x$ which is nearer to $x = 0.5$ by the Newton-Raphson method, correct to 3 decimal places.

:- Given that,

$$f(x) = \cos x - xe^x, \quad x_0 = 0.5$$

$$f'(x) = -\sin x - (xe^x + e^x)$$

$$= -\sin x - e^x(x+1)$$

(5) Given, $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$. find $\sin 48^\circ$ using Newton's forward interpolation formula

:- Need to find $\sin 48^\circ$

i.e $x = 48^\circ$ near to x_0

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		$\Delta y_0 = 0.0589$		
			$\Delta^2 y_0 = -5.7\times10^{-3}$	
50°	0.7660			$\Delta^3 y_0 = -7\times10^{-4}$
		$\Delta y_1 = 0.0532$		
			$\Delta^2 y_1 = -6.4\times10^{-3}$	
55°	0.8192			
		$\Delta y_2 = 0.0468$		
60°	0.8660			

$h = 5$, $x = 48$, $n = \dfrac{x - x_0}{h} = \dfrac{48-45}{5} = 0.6$

Newton's forward interpolation formula,

$$y = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0$$

$$y = 0.7071 + (0.6)(0.0589) + \frac{0.6(0.6-1)}{2}(-5.7\times10^{-3})$$

$$+ \frac{(0.6)(0.6-1)(0.6-2)}{6}(-7\times10^{-4})$$

$= 0.74308$ // i.e $\sin 48^\circ = 0.7430$ //

WKT,

Newton - Raphson method,

1st approximation ; $x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(0.5) = 0.1756$

$f'(0.5) = -2.4818$

$$= 0.5 - \left[\frac{0.1756}{-2.4818}\right]$$

$x_1 = 0.5707$

2nd approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(0.5707) = -0.0099$

$f'(0.5707) = -2.7893$

$$= 0.5707 - \left[\frac{-0.0099}{-2.7893}\right]$$

$x_2 = 0.5671$

3rd approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$f(0.5671) = 0.00017$

$f'(0.5671) = -2.7729$

$$= 0.5671 - \frac{(0.00017)}{-2.7729}$$

$x_3 = 0.5671$ // the required root is 0.5671 //

(6) Using Newton's appropriate interpolation formula, find the values of y at $x=8$ and at $x=22$ from the following table,

x	0	5	10	15	20	25
y	7	11	14	18	24	32

:- Here we need to find y at $x=8$ and at $x=22$

i.e $x=8$ near to x_0 & $x=22$ near to x_n

x	y	1st Diff	2nd Diff	3rd Diff	4th Diff	5th Diff
0	y_0 7					
		$\Delta y_0 = 4$	$\Delta^2 y_0$			
5	11		-1	$\Delta^3 y_0$		
		3		2	$\Delta^4 y_0$	
10	14		1		-1	
		4		1		0
15	18		2		-1 $\nabla^4 y_n$	
		6		0 $\nabla^3 y_n$		
20	24		2 $\nabla^2 y_n$			
		8 ∇y_n				
25	32 y_n					

To find $y(8)$;

Newton Forward interpolation formula,

$$n = \frac{x - x_0}{h} = \frac{8-0}{5} = 1.6$$

$$y = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 y_0$$

$$y = 7 + 1.6(4) + \frac{(1.6)(0.6)}{2}(-1) + \frac{(1.6)(0.6)(-0.4)}{6} 2$$

$$+ \frac{(1.6)(0.6)(-0.4)(-1.4)}{24}(-1)$$

$$y(8) = 7 + 6.4 + (-0.48) - 0.128 - 0.0224$$

$$= 12.7696$$

<u>To find</u> $y(22)$, Newton's Backward interpolation formula

$$x = 22, \quad n = \frac{x - x_n}{h} = \frac{22-25}{5} = -0.6$$

$$y = y_n + n \nabla y_n + \frac{n(n+1)}{2!} \nabla^2 y_n + \cdots$$

$$= 32 + (-0.6)8 + \frac{(-0.6)(0.4)}{2} 2 + \frac{(-0.6)(0.4)(1.4)}{6}(0) +$$

$$\frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-1)$$

$$= 32 - 4.8 - 0.24 + 0.0336$$

$$y = 26.9936$$

(7) Using Newton's Divided difference formula, evaluate $f(8)$ from the following

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

:- WKT, Newton's Divided difference formula,

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \cdots$$

x	$f(x)$	1st D.D	2nd D.D	3rd D.D	4th D.D
4	48				
		$\frac{100-48}{5-4} = \underline{\underline{52}}$			
5	100		$\frac{97-52}{7-4} = \underline{\underline{15}}$		
		97		$\underline{\underline{1}}$	
7	294		21		$\underline{\underline{0}}$
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

$$f(8) = 48 + (8-4)52 + (8-4)(8-5)15 + (8-4)(8-5)(8-7) \\ 1 + 0$$

$$= 448 //$$

(8) Find y at $x=5$ if $y(1)=-3$, $y(3)=9$, $y(4)=30$, $y(6)=132$ using Lagrange's interpolation formula.

:- Given that

	x_0	x_1	x_2	x_3
x	1	3	4	6
y	-3	9	30	132

WKT, Lagrange's interpolation formula,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

To find y at $x=5$,

$$y = \frac{(5-3)(5-4)(5-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(5-1)(5-4)(5-6)}{(3-1)(3-4)(3-6)} 9 +$$

$$\frac{(5-1)(5-3)(5-6)}{(4-1)(4-3)(4-6)} 30 + \frac{(5-1)(5-3)(5-4)}{(6-1)(6-3)(6-4)} 132.$$

$$= -0.2 + (-6) + 40 - 35.2$$

$$y = -1.4$$

(9) Evaluate $\int_0^{\pi/2} \sqrt{\cos x}\, dx$ by using the trapezoidal rule by taking 7 ordinates,

:- $f(x) = \sqrt{\cos x}$

$a = 0, \quad b = \pi/2, \quad n+1 = 7$

$\Rightarrow n = 6.$

$$h = \frac{\frac{\pi}{2} - 0}{6} \Rightarrow h = \frac{\pi}{12}.$$

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1	0.9999	0.999	0.9999	0.9999	0.9998	0.9998

Trapezoidal rule,

$$y = \cancel{y}\, \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots) + y_n\right]$$

$$= \frac{\pi}{24}\left[1 + 2(0.9999 + 0.9999 + 0.9999 + 0.9999 + 0.9998) + 0.9998\right]$$

$$y = 1.5706$$

(10) Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using the Trapezoidal rule by taking 6 divisions,

:- $a = 0$, $b = 1$, $n = 6$

$\therefore h = \frac{b-a}{n} = \frac{1}{6} = 0.166$

x	0	1/6	2/6	1/2	4/6	5/6	1
y	1	0.9729	0.9	0.8	0.6923	0.5901	0.5

Trapezoidal rule,

$$I = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n\right]$$

$$= \frac{0.166}{2}\left[1 + 2(0.9729 + 0.9 + 0.8 + 0.6923 + 0.5901) + 0.5\right]$$

$$= 0.083\left[1 + 7.9106 + 0.5\right]$$

$$I = 0.7810$$

(11) Evaluate $\int_0^3 \frac{1}{4x+5}\,dx$ by using Simpson's $1/3^{rd}$ rule by taking 7 ordinates.

:- $f(x) = \frac{1}{4x+5}$, $a = 0$, $b = 3$, $n = 6$.

$h = \frac{3-0}{6} = 0.5$

x	0	0.5	1	1.5	2	2.5	3
y	0.2	0.142	0.111	0.09	0.076	0.06	0.058

Simpson's $1/3^{rd}$ rule,

$$I = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{0.5}{3}\left[(0.2+0.058) + 4(0.142+0.09+0.06) + 2(0.111+0.076)\right]$$

$$= 0.1666\,(0.258 + 1.168 + 0.374)$$

$$I = 0.2998$$

(12) Evaluate $\int_0^1 \frac{1}{1+x}\,dx$ by taking 7 ordinates and by using Simpson's $3/8^{th}$ rule.

:- $f(x) = \frac{1}{1+x}$, $a=0$, $b=1$, $n=6$

$h = \frac{1}{6}$.

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6$
y	1	0.8571	0.75	0.666	0.6	0.5454	0.5

Simpson's $3/8^{th}$ rule

$$I = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{3}{8(6)}\left[(1+0.5) + 3(0.8571+0.75+0.6+0.5454) + 2(0.666)\right]$$

$$= 0.0625\left[1.5 + 8.2575 + 1.332\right]$$

$$I = 0.6930$$

SUB CODE : BMATS201/C201/M201/E201

IMPORTANT QUESTION PAPER SOLUTIONS : MODULE-05

(Numerical Methods - 02)

1. Employ Taylor's Series method to obtain approximate value of y when $x=0.2$ given that $\frac{dy}{dx} = x+y$ and $y=1$ when $x=0$.

:- WKT Taylor's Series method,

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$$

Given that

$y'(x) = x+y \quad , x_0 = 0, y_0 = 1$

$\therefore$ At $x=0$, $\quad y'(0) = 0+1 = 1$

At $x=0$, $\quad y''(x) = 1+y' \Rightarrow y''(0) = 2$

At $x=0$, $\quad y'''(x) = y'' \Rightarrow y'''(0) = 2$

At $x=0$, $\quad y^{iv}(x) = y''' \Rightarrow y^{iv}(0) = 2$

$\therefore$ Taylor's series method,

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{iv}(x_0)$$

$$y(0.2) = 1 + (0.2)1 + \frac{(0.2)^2}{2}2 + \frac{(0.2)^3}{6}2 + \frac{(0.2)^4}{24}2$$

$$= 1.2428 //$$

(Q) By Taylor's series method, find the value of y at $x=0.1$ and $x=0.2$ to 5 places of decimals from $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$

:- Given, $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$

$$y(x_0) = y_0 \Rightarrow y(0) = 1$$

$y' = x^2y - 1$, at $x = 0$, $\boxed{y'(0) = -1}$

$y'' = x^2y' + 2xy$, at $x=0$, $\boxed{y''(0) = 0}$

$$y''' = x^2y'' + 2xy' + 2xy' + 2y$$

$= x^2y'' + 4xy' + 2y$, at $x=0$, $\boxed{y'''(0) = 2}$

$$y^{iv} = x^2y''' + 2xy'' + 4xy'' + 4y' + 2y'$$

$= x^2y''' + 6xy'' + 6y'$, at $x=0$, $\boxed{y^{iv}(0) = -6}$

$$y^{v} = x^2y^{iv} + 2xy''' + 6xy''' + 6y'' + 6y''$$

$= x^2y^{iv} + 8xy''' + 12y''$, at $x=0$, $\boxed{y^{v}(0) = 0}$

Taylor's Series Method,

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$$

At $x = 0.1$

$$y(0.1) = 1 + (0.1)(-1) + \cancel{\frac{(0.1)^2}{2}(0)}^{=0} + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(-6) + \cancel{0}$$

$$y(0.1) = 1 + (-0.1) + 0.00033 - 0.000025$$
$$= 0.900305$$

$$y(0.2) = 1 + (0.2)(-1) + 0 + \frac{(0.2)^3 2}{6} + \frac{(0.2)^4(-6)}{24} + 0$$
$$= 1 - 0.2 + 0.00266 - 0.0004$$
$$= 0.80226$$

---

(3) Solve using Modified Euler's Method

$y'(x) = 3x + \frac{y}{2}$, $y(0) = 1$ then find $y(0.2)$ with

$h = 0.2$

:- Given $y' = 3x + \frac{y}{2}$, $y(0) = 1$

from Euler method,

$$y_1 = y_0 + h f(x_0, y_0)$$
$$y_1 = 1 + (0.2)\left[0 + \frac{1}{2}\right] = 1.1$$

Modified Euler's method

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 1 + \frac{(0.2)}{2}\left[\left(0 + \frac{1}{2}\right) + \left(3(0.2) + \frac{1.1}{2}\right)\right]$$

$$y_1^{(1)} = 1.165$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(2)} = 1 + \frac{0.2}{2}\left[0.5 + \left(3(0.2) + \frac{1.165}{2}\right)\right]$$

$$= 1.1682$$

$$y_1^{(3)} = 1 + \frac{0.2}{2}\left[0.5 + \left(3(0.2) + \frac{1.1682}{2}\right)\right]$$

$$= 1.1684$$

The required root is 1.1684

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(4) Using the Modified Euler's method, find $y(0.1)$ given that $\frac{dy}{dx} = x^2 + y$ and $y(0) = 1$ take step $h = 0.05$ and perform two modification in each stage

:- Given that

$$\frac{dy}{dx} = x^2 + y, \quad y_0 = 1, \quad x_0 = 0, \quad h = 0.05$$

To find $y(0.1) = ?$

I[st] stage

$$x_1 = 0.05, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.05$$

Euler's method,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.05(1) = 1.05$$

Modified Euler's method,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + 0.025[1 + (0.0025 + 1.05)]$$
$$= 1.0513.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$
$$= 1 + 0.025[1 + (0.0025 + 1.0513)]$$
$$= 1.0513 //$$

II^nd stage

$x_2 = 0.1$, $h = 0.05$, $x_1 = 0.05$, $y_1 = 1.0513$.

Euler's formula

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= 1.0513 + 0.05(1.0538)$$
$$= 1.1039$$

Modified Euler's Method

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$
$$= 1.0513 + 0.025[1.0538 + 1.1139]$$
$$= 1.1054$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$
$$= 1.0513 + 0.025[1.0538 + 1.1154]$$
$$= 1.1055 //$$

The required root is $1.1055 //$

(5) Apply Runge-Kutta method of fourth order to find an approximate value of y when $x=0.2$ given that $\frac{dy}{dx}=x+y$ and $y(0)=1$

:- Given that, $y'=x+y$, $x_0=0$, $y_0=1$, $h=0.2$

WKT Runge-Kutta 4th order formula,

$$y_1 = y_0 + \frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$$

Where

$K_1 = hf(x_0, y_0)$

$= 0.2\left[0+1\right]$

$K_1 = 0.2$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$

$= 0.2f(0.1, 1.1)$

$= 0.2\left[0.1+1.1\right] = 0.24$

$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$

$= 0.2f(0.1, 1.12)$

$= 0.244$

$K_4 = hf\left(x_0+h, y_0+K_3\right)$

$= 0.2f(0.2, 1.244)$

$= 0.2888$

$$\therefore y_1 = 1 + \frac{1}{6}\left[0.2+2(0.24)+2(0.244)+0.2888\right]$$

$y_1 = 1.2428$ at $x_1=0.2$

---

(6) Using Runge-Kutta Method of fourth order, find $y(0.1)$ given that $\frac{dy}{dx}=3e^x+2y$, $y(0)=0$ taking $h=0.1$

:- Given, $\frac{dy}{dx}=3e^x+2y$, $x_0=0$, $y_0=0$, $h=0.1$

WKT Runge-Kutta 4^{th} order formula,

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

Where,

$K_1 = hf(x_0, y_0)$

$= 0.1 f(0,0)$

$= 0.1 \times 3$

$= 0.3$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.1 f(0.05, 0.15)$

$= 0.1 (3.4538)$

$= 0.3453$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= 0.1 \left[f(0.05, 0.172)\right]$

$= 0.3497$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1 f(0.1, 0.3497)$

$= 0.4014$

$$y_1 = 0 + \frac{1}{6}\left[0.3 + 2(0.3453) + 2(0.3497) + 0.4014\right]$$

$= 0.3490$ //

---

(7) Using the Runge-Kutta method of fourth order find $y(0.2)$ given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0.1) = 1.0912$ taking $h = 0.1$

:- Given $\frac{dy}{dx} = \frac{y-x}{y+x}$, $x_0 = 0.1$, $y_0 = 1.0912$

$h = 0.1$

WKT, Runge-Kutta 4th order method,

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

Where

$k_1 = hf(x_0, y_0)$

$= 0.1 f(0.1, 1.0912)$

$= 0.0832$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1 f(0.15, 1.1328)$

$= 0.0766$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1 f(0.15, 1.1295)$

$= 0.0765$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1 f(0.2, 1.1675)$

$= 0.0707$

$$y_1 = 1.0912 + \frac{1}{6}\left[0.0832 + 2(0.0766) + 2(0.0765) + 0.0707\right]$$

$= 1.1678$ //

---

(8) using the Runge-Kutta method of fourth order, solve $y' = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$, at $x = 0.2$

:- Given that

$y' = \frac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

WKT Runge-kutta 4^{th} order method,

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

Where

$K_1 = hf(x_0, y_0)$

$= 0.2 f(0, 1)$

$= 0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.2 f(0.1, 1.1)$

$= 0.1967$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= 0.2 f(0.1, 1.0983)$

$= 0.2(0.9835)$

$= 0.1967$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.2 f(0.2, 1.1967)$

$= 0.1891$

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= 1 + \frac{1}{6}\left[0.2 + 2(0.1967) + 2(0.1967) + 0.1891\right]$$

$$= 1.1959$$

---

(9) Given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$ compute $y(0.4)$ using Milne's method

:- Given $y' = xy + y^2$,

$x_0 = 0$ $\quad x_1 = 0.1$ $\quad x_2 = 0.2$ $\quad x_3 = 0.3$

$y_0 = 1$ $\quad y_1 = 1.1169$ $\quad y_2 = 1.2773$ $\quad y_3 = 1.50$

To find $y_4 = ?$ at $x_4 = 0.4$

x	y	$y' = xy + y^2$
$x_0 = 0.0$	$y_0 = 1$	$y_0' = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$y_1' = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$y_2' = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$y_3' = 2.7161$
$x_4 = 0.4$	$y_4 = ?$	$y_4' = ?$

Milne's predictor formula,

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(1.3591) - (1.8869) + 2(2.7161)\right]$$

$$y_4^{(P)} = 1.8330$$

$$\therefore\ y_4' = (0.4)(1.8330) + (1.8330)^2 \quad (\because y' = xy + y^2)$$

$$y_4' = 4.0930$$

corrector formula

$$y_4^{c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'^{(P)}\right]$$

$$= 1.2773 + \frac{0.1}{3}\left[1.8869 + 4(2.7161) + 4.0930\right]$$

$$y_4^{c} = 1.8387$$

Thus reqd soln $y_4 = 1.838$

$y_4' = 4.0930$

(10) Given $\frac{dy}{dx} = x - y^2$, data $y(0) = 0$, $y(0.2) = 0.02$

$y(0.4) = 0.0795$, $y(0.6) = 0.1762$ Compute y at $x = 0.8$

by applying Milne's Method.

:- $y' = x - y^2$,

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = 0.3936$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = 0.5689$
$x_4 = 0.8$	$y_4 =$	$y_4' =$

Milne's predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - (0.3936) + 2(0.5689)\right]$$

$$y_4^{(P)} = 0.266\left[1.1434\right] = 0.3041$$

$$\therefore y_4'^{(P)} = 0.8 - (0.3041)^2 = 0.7075$$

Corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'^{(P)}\right]$$

$$= 0.0795 + \left(\frac{0.2}{3}\right)\left[0.3936 + 4(0.5689) + 0.7075\right] = 0.3046$$

The real value $y_4 = 0.3046$