Optimization Assignment - Advanced

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October 2022

Problem

Find the shortest distance between the following lines.
$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$

Solution

From given question
We can write in terms of vector form of line:

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}$$

We have

$$L_1 : \mathbf{x} = \mathbf{a_1} + \lambda_1 \mathbf{m_1}$$

$$L_2 : \mathbf{x} = \mathbf{a_2} + \lambda_2 \mathbf{m_2}$$

where, $\mathbf{a_i}, \mathbf{m_i}$ are positional and slope vectors of line L_i respectively
Now, let us assume that L_1 and L_2 are intersecting at a point.
Therefore,

$$\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} \tag{1}$$

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \left(7, -6, 1\right) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 1 & 7 \\ -2 & -6 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -8 \end{pmatrix} \tag{3}$$

Row reducing the Augmented matrix

$$\begin{pmatrix} 1 & 7 & -4 \\ -2 & -6 & -6 \\ 1 & 1 & -8 \end{pmatrix} \to R_3 - R_1 \to \begin{pmatrix} 1 & 7 & -4 \\ -2 & -6 & -6 \\ 0 & -6 & -4 \end{pmatrix} \tag{4}$$

$$\to R_2 + 2R_1 \to \begin{pmatrix} 1 & 7 & -4 \\ 0 & 8 & -14 \\ 0 & -6 & -4 \end{pmatrix} \tag{5}$$

$$\to R_3 + \frac{3R_2}{4} \to \begin{pmatrix} 1 & 7 & -4 \\ 0 & 8 & -14 \\ 0 & 0 & -\frac{29}{2} \end{pmatrix} \tag{6}$$

Therefore the above matrix has $rank = 3$,Hence the lines do not intersect.Hence the lines L_1 and L_2 are skew lines

Let d be the shortest distance between L_1 and L_2 and $\mathbf{p_1}$ and $\mathbf{p_2}$ be the positional vectors of its end points.
For d to be the shortest , we know that

$$\mathbf{p_1} = \mathbf{a_1} + \lambda_1 \mathbf{m_1} \tag{7}$$

$$\mathbf{p_2} = \mathbf{a_2} + \lambda_2 \mathbf{m_2} \tag{8}$$

$$d^2 = \|\mathbf{p_1} - \mathbf{p_2}\| = (\mathbf{p_1} - \mathbf{p_2})(\mathbf{p_1} - \mathbf{p_2})^T \tag{9}$$

$$d^2 = (\mathbf{a_1} + \boldsymbol{\lambda_1}\mathbf{m_1} - \mathbf{a_2} - \boldsymbol{\lambda_2}\mathbf{m_2})^2 \tag{10}$$

By substituting the values of $\mathbf{a_1}$, $\mathbf{a_2}$,$\mathbf{m_1}$ and $\mathbf{m_2}$ in above equation we get a Quadratic equation in two variables λ_1 and λ_2
Therefore, the equation is given as follows

$$d^2 = (4 + \lambda_2 - 7\lambda_2)^2 + (6 - 2\lambda_1 + 6\lambda_2)^2 + (8 + \lambda_1 - \lambda_2)^2 \tag{11}$$

$$d^2 = 6\lambda_1^2 + 86\lambda_2^2 - 40\lambda_1\lambda_2 + 116 \tag{12}$$

After solving the above equation we get the shortest distance between two lines

Therefore,

$$ShortestDistanced = 10.77 \tag{13}$$

Execution

Verify the above Solution in the following code.

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https://github.com/bhavani360/FWC_assignments
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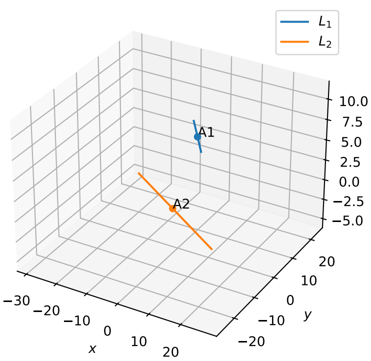
Construction


L_1
L_2
A1
A2
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−30
−20
−10
0
10
20
x
20
10
0
−10
−20
y