Discrete Structures Tutorial-1

(Section-A)

Question-1

Determine whether each of these statements is true or false.

a) $0 \in \emptyset$

b) $\emptyset \in \{0\}$

c) $\{0\} \subset \emptyset$

d) $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\emptyset\} \subseteq \{\emptyset\}$

Solution-1

a) This is false, since the empty set has no elements.

b) This is false. The set on the right has only one element, namely the number 0, not the empty set.

c) This is false. In fact, the empty set has *no* proper subsets.

d) This is true. Every element of the set on the left is, vacuously, an element of the set on the right; and the set on the right contains an element, namely 0, that is not in the set on the left.

e) This is false. The set on the right has only one element, namely the number 0, not the set containing the number 0.

f) This is false. For one set to be a proper subset of another, the two sets cannot be equal.

g) This is true. Every set is a subset of itself.

Question-2

Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$.

b) $(A \cap B) \cup (A \cap \overline{B}) = A$.

Solution-2

a) This is clear, since both of these sets are precisely $\{x \mid x \in A \wedge x \notin B\}$.

b) One approach here is to use the distributive law; see the answer section for that approach. Alternatively, we can argue directly as follows. Suppose $x \in (A \cap B) \cup (A \cap \overline{B})$. Then we know that either $x \in A \cap B$ or $x \in A \cap \overline{B}$ (or both). If either case, this forces $x \in A$. Thus we have shown that the left-hand side is a subset of the right-hand side. For the opposite direction, suppose $x \in A$. There are two cases: $x \in B$ and $x \notin B$. In the former case, x is then an element of $A \cap B$ and therefore also an element of $(A \cap B) \cup (A \cap \overline{B})$. In the latter cases, $x \in \overline{B}$ and therefore x is an element of $A \cap \overline{B}$ and therefore also an element of $(A \cap B) \cup (A \cap \overline{B})$.

Question-3

Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

where, $A \oplus B = (A - B) \cup (B - A)$.

Solution-3

Yes. To show that $A = B$, we need to show that $x \in A$ implies $x \in B$ and conversely. By symmetry, it will be enough to show one direction of this. So assume that $A \oplus C = B \oplus C$, and let $x \in A$ be given. There are two cases to consider, depending on whether $x \in C$. If $x \in C$, then by definition we can conclude that $x \notin A \oplus C$. Therefore $x \notin B \oplus C$. Now if x were *not* in B, then x *would* be in $B \oplus C$ (since $x \in C$ by assumption). Since this is not true, we conclude that $x \in B$, as desired. For the other case, assume that $x \notin C$. Then $x \in A \oplus C$. Therefore $x \in B \oplus C$ as well. Again, if x were *not* in B, then it could not be in $B \oplus C$ (since $x \notin C$ by assumption). Once again we conclude that $x \in B$, and the proof is complete.

Question-4

Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.

Solution-4

Each of the properties is a universally quantified statement. Because the domain is empty, each of them is vacuously true.

In mathematics and logic, a **vacuous truth** is a conditional or universal statement (a universal statement that can be converted to a conditional statement) that is true because the antecedent cannot be satisfied.[1] For example, the statement "all cell phones in the room are turned off" will be true when no cell phones are in the room. In this case, the statement "all cell phones in the room are turned *on*" would also be vacuously true, as would the conjunction of the two: "all cell phones in the room are turned on *and* turned off", which would otherwise be incoherent and false. For that reason, it is sometimes said that a statement is vacuously true because it does not really say anything.[2]

Solution-4 (cont)

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *reflexive*, if $(a, a) \in R, \forall a \in A$
  $\Longrightarrow {}_aR_a$ holds for every $a \in A$.
- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *symmetric*, if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
  In other words, ${}_aR_b \Rightarrow {}_bR_a$ for every $a, b \in A$.
- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *transitive*, if ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c, \forall a, b, c \in A$.

Bonus Question: Is the given relation R anti-symmetric?

Question-5

How many relations are there on a set with n elements that are

a) symmetric?

b) antisymmetric?

c) asymmetric?

d) irreflexive?

e) reflexive and symmetric?

f) neither reflexive nor irreflexive?

Note: Only solve **f)** for now. You can derive the rest later as exercise

Solution-5

These are combinatorics problems, some harder than others. Let A be the set with n elements on which the relations are defined.

a) To specify a symmetric relation, we need to decide, for each unordered pair $\{a, b\}$ of distinct elements of A, whether to include the pairs (a, b) and (b, a) or leave them out; this can be done in 2 ways for each such unordered pair. Also, for each element $a \in A$, we need to decide whether to include (a, a) or not, again 2 possibilities. We can think of these two parts as one by considering an element to be an unordered pair with repetition allowed. Thus we need to make this 2-fold choice $C(n+1, 2)$ times, since there are $C(n+2-1, 2)$ ways to choose an unordered pair with repetition allowed. Therefore the answer is $2^{C(n+1,2)} = 2^{n(n+1)/2}$.

b) This is somewhat similar to part **(a)**. For each unordered pair $\{a,b\}$ of distinct elements of A, we have a 3-way choice—either include (a,b) only, include (b,a) only, or include neither. For each element of A we have a 2-way choice. Therefore the answer is $3^{C(n,2)}2^n = 3^{n(n-1)/2}2^n$.

c) As in part **(b)** we have a 3-way choice for $a \neq b$. There is no choice about including (a,a) in the relation—the definition prohibits it. Therefore the answer is $3^{C(n,2)} = 3^{n(n-1)/2}$.

d) For each ordered pair (a,b), with $a \neq b$ (and there are $P(n,2)$ such pairs), we can choose to include (a,b) or to leave it out. There is no choice for pairs (a,a). Therefore the answer is $2^{P(n,2)} = 2^{n(n-1)}$.

e) This is just like part **(a)**, except that there is no choice about including (a,a). For each unordered pair of distinct elements of A, we can choose to include neither or both of the corresponding ordered pairs. Therefore the answer is $2^{C(n,2)} = 2^{n(n-1)/2}$.

f) We have complete freedom with the ordered pairs (a,b) with $a \neq b$, so that part of the choice gives us $2^{P(n,2)}$ possibilities, just as in part **(d)**. For the decision as to whether to include (a,a), two of the 2^n possibilities are prohibited: we cannot include all such pairs, and we cannot leave them all out. Therefore the answer is $2^{P(n,2)}(2^n - 2) = 2^{n^2-n}(2^n - 2) = 2^{n^2} - 2^{n^2-n+1}$.

Extra Practice Problems

Question-6

Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$

b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$

c) $\{\emptyset\} \in \{\emptyset\}$

d) $\{\emptyset\} \in \{\{\emptyset\}\}$

e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Question-7

If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

Question-8

Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

Question-9

Find the error in the "proof" of the following "theorem."

"*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.

"*Proof*": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

Question-10

Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.