Functions Tutorial

1.

Show that the following functions are neither one-one nor onto ($\mathbb{Z}$ in (a), (b); and $\mathbb{R}$ in (c), (d), and (e)).

a. $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = 4x^2 + 3$ for all $x \in \mathbb{Z}$.

b. $f : \mathbb{Z} \to \mathbb{Z}$, defined by for all $n \in \mathbb{Z}$

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ -1, & \text{if } n \text{ is odd.} \end{cases}$$

c. $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = |x| + 1$ for all $x \in \mathbb{R}$.

d. $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \frac{x}{x^2+1}$ for all $x \in \mathbb{R}$.

e. $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \cos x$ for all $x \in \mathbb{R}$.

2.

Show that the following functions are one-one, but not onto $\mathbb{Z}$.

a. $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 9n + 1$ for all $n \in \mathbb{Z}$.

b. $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 3^n$ for all $n \in \mathbb{Z}$.

3. Determine which of the following functions are one-one, onto or both

a. $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 4n - 3$ for all $n \in \mathbb{Z}$.

b. $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$, defined by $f(n, m) = \frac{n}{m}$ for all $n \in \mathbb{Z}$ and for all $m \in \mathbb{N}$.

c. $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 - x$ for all $x \in \mathbb{R}$.

d. $f : \mathbb{Z} \to \mathbb{Q}$, defined by $f(n) = 2^n$ for all $n \in \mathbb{Z}$.

e. $f : \mathbb{R}^+ \to \mathbb{R}^+$, defined by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^+$.

f. $f : \mathbb{R} \to \mathbb{R}^+$, defined by $f(x) = 3^x$ for all $x \in \mathbb{R}$.

4.

Let $f : X \to Y$ and $g : X \to Y$ be functions. Show that $f = g$ if and only if $f(x) = g(x)$ for all $x \in X$.

5.

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be functions defined by $f(x) = \sqrt{x}$ and $g(x) = 3x + 1$ for all $x \in \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of all positive real numbers. Find $f \circ g$ and $g \circ f$. Is $f \circ g = g \circ f$?

6.

Let $f : \mathbb{Q}^+ \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 + \frac{1}{x}$ for all $x \in \mathbb{Q}^+$ and $g(x) = x + 1$ for all $x \in \mathbb{R}$, where $\mathbb{Q}^+$ is the set of all positive rational numbers. Find $g \circ f$.

7.

Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 5$ for all $n \in \mathbb{N}$.

a. Show that f is one-one but not onto.

b. Let $g : \mathbb{N} \to \mathbb{N}$ be defined by: For all $n \in \mathbb{N}$,

$$g(n) = \begin{cases} 5 & \text{if } n \le 5 \\ n - 5 & \text{if } n > 5. \end{cases}$$

Show that $g \circ f = i_{\mathbb{N}}$.

c. Show that $f \circ g \neq i_{\mathbb{N}}$.

8.

Given $f : X \to Y$ and $A, B \subseteq X$, prove that

a. $f(A \cup B) = f(A) \cup f(B)$,

b. $f(A \cap B) \subseteq f(A) \cap f(B)$,

c. $f(A - B) \subseteq f(A) - f(B)$ if f is one-one.

9.

Given $f: X \to Y$, let $S \subseteq Y$. Define $f^{-1}(S) = \{x \in X \mid f(x) \in S\}$. Let $A, B \subseteq Y$. Prove that

a. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$,

b. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$,

c. $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.

10.

For the following statement, write the proof if the statement is true, otherwise give a counterexample.

A function $f: A \to B$ is one-one if and only if for all subsets C of A, $f(A - C) \supseteq B - f(C)$.

11. Suppose the area of a rectangle is equal to its perimeter, show that the function mapping one side length to the other is an involution. Also find this function when the domain and range is positive integers (Z^+)

12. Prove that if a function is both idempotent and an involution, it must be an identity function.