$S$

$S' = \{(a,a) \mid a \in S\}$

$R'$ is an irreflexive relation

$A \subset S'$ $A = \emptyset$

$R' \cup A$ : Neither ref. nor irref.

$$(2^n - 2)\, 2^{n(n-1)} = 2^{n^2} - 2^{n^2 - n + 1}$$

$R' = \{(a,b) \mid a \neq b,\ a,b \in S\}$

$n^2 - n$ $\quad 2^{n^2 - n}$

$A = \{(a,a) \mid a \in S''\}$ $S'' \subset S$ and $S'' \neq \emptyset$

$R' \cup A$

$(2^{n^2 - n})(2^n - 2) \rightarrow$ Ans.

# Discrete Structures Tutorial-1 (Extra Practice Problems solutions)

(SECTION-A)

# Question-6

**Determine whether these statements are true or false.**

**a)** $\emptyset \in \{\emptyset\}$

**b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$

**c)** $\{\emptyset\} \in \{\emptyset\}$

**d)** $\{\emptyset\} \in \{\{\emptyset\}\}$

**e)** $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

**f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

**g)** $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

# Solution-6

a) True

b) True

c) False

d) True

e) True

f) True

g) True

K.H. Rosen 2.1 Exercise Q10

# Question-7

If $A$, $B$, $C$, and $D$ are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

# Solution-7

| | $(A \oplus B) \oplus (C \oplus D)$ | $(A \oplus D) \oplus (B \oplus C)$ |
|---|---|---|
| $A \cap B \cap C \cap D$ | Excluded | Excluded |
| $A \cap B \cap C \cap D^C$ | Included | Included |
| $A \cap B \cap C^C \cap D$ | Included | Included |
| $A \cap B \cap C^C \cap D^C$ | Excluded | Excluded |
| $A \cap B^C \cap C \cap D$ | Included | Included |
| $A \cap B^C \cap C \cap D^C$ | Excluded | Excluded |
| $A \cap B^C \cap C^C \cap D$ | Excluded | Excluded |
| $A \cap B^C \cap C^C \cap D^C$ | Included | Included |
| $A^C \cap B \cap C \cap D$ | Included | Included |
| $A^C \cap B \cap C \cap D^C$ | Excluded | Excluded |
| $A^C \cap B \cap C^C \cap D$ | Excluded | Excluded |
| $A^C \cap B \cap C^C \cap D^C$ | Included | Included |
| $A^C \cap B^C \cap C \cap D$ | Excluded | Excluded |
| $A^C \cap B^C \cap C \cap D^C$ | Included | Included |
| $A^C \cap B^C \cap C^C \cap D$ | Included | Included |
| $A^C \cap B^C \cap C^C \cap D^C$ | Excluded | Excluded |

With this table of Inlcusion/Exclusion of partitions, we can conclusively prove that
$(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$

K.H. Rosen 2.2 Exercise Q43

# Question-8

Suppose that the relation $R$ is irreflexive. Is $R^2$ necessarily irreflexive? Give a reason for your answer.

# Solution-8

It is not necessarily true that $R^2$ is irreflexive when $R$ is. We might have pairs $(a, b)$ and $(b, a)$ both in $R$, with $a \neq b$; then it would follow that $(a, a) \in R^2$, preventing $R^2$ from being irreflexive. As the simplest example, let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 1)\}$. Then $R$ is clearly irreflexive. In this case $R^2 = \{(1, 1), (2, 2)\}$, which is not irreflexive.

K.H. Rosen 9.1 Exercise Q9

# Question-9

Find the error in the "proof" of the following "theorem."

"*Theorem*": Let $R$ be a relation on a set $A$ that is symmetric and transitive. Then $R$ is reflexive.

"*Proof*": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because $R$ is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

# Solution-9

The second sentence of the proof asks us to "take an element $b \in A$ such that $(a, b) \in R$." There is no guarantee that such an element exists for the taking. This is the only mistake in the proof. If one could be guaranteed that each element in $A$ is related to at least one element, then symmetry and transitivity would indeed imply reflexivity. Without this assumption, however, the proof and the proposition are wrong. As a simple example, take the relation $\emptyset$ on any nonempty set. This relation is vacuously symmetric and transitive, but not reflexive. Here is another counterexample: the relation $\{(1,1),(1,2),(2,1),(2,2)\}$ on the set $\{1,2,3\}$.

K.H. Rosen 9.1 Exercise Q49

# Question-10

Show that the relation $R$ on a set $A$ is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.

# Solution-10

Let $R$ be a relation on set $A$, then $R$ is anti-symmetric if and only if for $a, b \in A$, ${}_aR_b,{}_b\,R_a \Rightarrow a = b$
and $R^{-1} = \{(b, a)|a, b \in A,{}_a\,R_b\}$
Also note that $R, R^{-1} \subseteq A \times A$
Now if possible, let $\exists x, y \in A$ such that $x \neq y,{}_x\,R_y,{}_y\,R_x$
Then, ${}_xR_y \Rightarrow_y\,R_x^{-1}$ and ${}_yR_x \Rightarrow_x\,R_y^{-1}$
$\therefore (x, y), (y, x) \in R \cap R^{-1}$
But it is given that $R \cap R^{-1} \subseteq \Delta$ where $\Delta = \{(a, a)|a \in A\}$
$\therefore (x, y), (y, x) \notin \Delta$ and thus, $(x, y), (y, x) \notin R \cap R^{-1}$
Hence, if ${}_xR_y,{}_y\,R_x$ then $x = y$. And R is anti-symmetric

K.H. Rosen 9.1 Exercise Q52