Discrete Structures (MA5.101)

Solution for Mid Semester Examination (Monsoon 2021)

International Institute of Information Technology, Hyderabad

Time: 90 Minutes Total Marks: 30

Instructions: This is online examination.

Write at the top of your answer book the following:

Discrete Structures (MA5.101)

Mid Semester Examination (Monsoon 2021)

Date: 10-Jan-2022

Name:

Roll Number:

Submit your scanned hand-written answer script in the moodle

with the file name: RollNo_MidSem_SecNo_10Jan2022.pdf

NOTE: No email submissions for the answer scripts are allowed even if you are facing Internet issues or moodle problem from your end. In that case, viva-voce of mid semester examination will be considered later for fair evaluations to all.

10 January 2022

1. (a) We know that the composite gf of any two injections $f : S \to T$ and $g : T \to V$ is an injection. Extend the definition of the composite gf to the case in which the domain of g contains the codomain of f. Prove or disprove:

   (i) the composite gf of any two injections is an injection.

   (ii) the composite gf of any two surjectios is a surjection.

   Solution: Consider the following Figure 1.

   We have $f : S \to T$, $g : U \to V$ and $T \subset U$. The extended composition of f and g is defined by $gf : S \to V$ such that $(gf)(x) = g[f(x)], \forall x \in S$.

   (i). **TRUE**
   RTP: gf is injection (one-one).
   Let $s_1, s_2 \in S$ such that $s_1 \neq s_2$.
   Then,

$$
\begin{aligned}
s_1 &\neq s_2 \\
\Rightarrow f(s_1) &\neq f(s_2), \text{ since } f : S \to T \text{ is one-one} \\
\Rightarrow g[f(s_1)] &\neq g[f(s_2)], \text{ since } g : U \to V \text{ is one-one and } T \subset U \\
\Rightarrow (gf)(s_1) &\neq (gf)(s_2)
\end{aligned}
$$

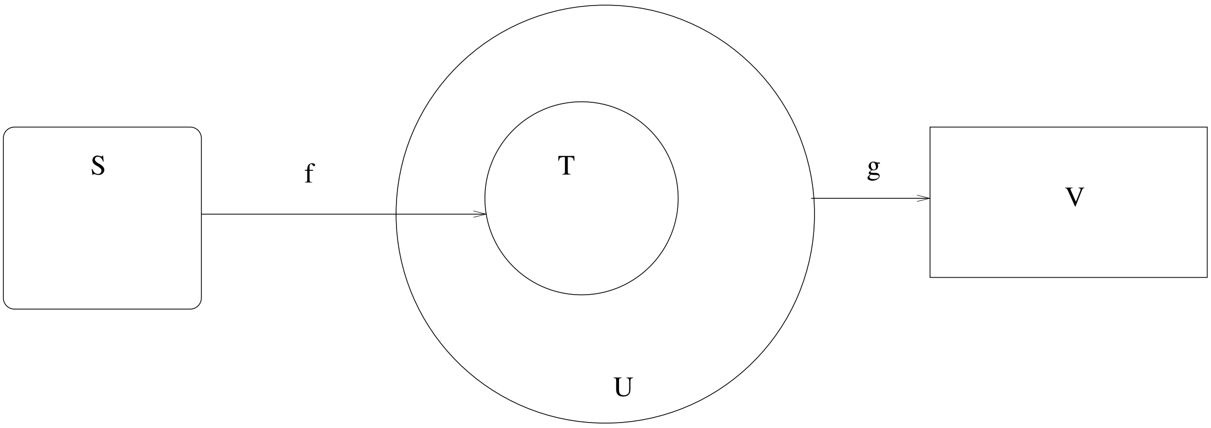


Figure 1: Extended composition gf of f and g.

As a result, we have the following:

$$\begin{aligned} s_1 &\neq s_2 \\ \Rightarrow (gf)(s_1) &\neq (gf)(s_2) \end{aligned}$$

Hence, gf is injection.

$\square$

(ii). **FALSE**
RTP: gf is not sujection (onto).
Let $v \in V$ be such that $\forall u$, $g(u) = v$.
Then $u \in U - T$.
Hence, gf needs not be a sujection.

$\square$

(b) A set S is said to be *infinite* if there is a one-to-one correspondence (bijection) between S and a proper subset of S. Using this definition, prove that the set of real numbers is infinite.

Solution: According to the given definition, the set S is called *infinite* if $\exists$ a bijection $f : S \rightarrow S'$, where $S' \subset S$ (S' is a proper subset of S).
Here S = the set of real numbers, R. Let $S' = (-1, 1) = \{x \in R| -1 < x < 1\}$.
Construct a mapping $f : S \rightarrow S'$ such that

$$f(x) = \begin{cases} \frac{x}{1+x}, \text{if x is positive or 0} \, (x \geq 0) \text{ real number} \\ \frac{x}{1-x}, \text{if x is negative} \, (x < 0) \text{ real number} \end{cases}$$

Note that range of $f = [0, 1)$ when $x \geq 0$, and $(-1, 0)$ when $x < 0$. Thus, the function f is well-defined.
We now claim that f is bijective.
(i) f is one-one: Required to prove that if $f(x_1) = f(x_2)$, then $x_1 = x_2$, for $x_1, x_2 \in R$.

Case I: $0 \leq x < 1$
Let $f(x_1) = f(x_2)$. Then, $\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \Rightarrow x_1 + x_1x_2 = x_2 + x_1x_2 \Rightarrow x_1 = x_2$.
Hence, $f : R \rightarrow [0, 1)$ is one-one.

Case II: $-1 < x < 0$

Let $f(x_1) = f(x_2)$. Then, $\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2 \Rightarrow x_1 = x_2$.
Hence, $f : R \to (-1, 0)$ is one-one.

Combining above two cases, $f : R \to (-1, 1)$ is then one-one.

(ii) f is onto: Required to prove that $\forall y \in (-1, 1)$, $\exists x \in S = R$, such that $y = f(x)$.

Case I: $y \in [0, 1)$

Then, by definition of f, $y = \frac{x}{1+x} \Rightarrow x = \frac{y}{1-y} \in R$. Hence, for every $y \in [0, 1)$, $\exists x \in R$ such that $y = f(x)$ and then $f : R \to [0, 1)$ is onto.

Case II: $y \in (-1, 0)$

Then, by definition of f, $y = \frac{x}{1-x} \Rightarrow x = \frac{y}{1+y} \in R$. Hence, for every $y \in (-1, 0)$, $\exists x \in R$ such that $y = f(x)$ and then $f : R \to (-1, 0)$ is onto.

Combining above two cases, $f : R \to (-1, 1)$ is then onto.
Thus, f is bijective and hence, the set of real numbers is infinite.

$\square$

[5 + 5 = 10]

2. (a) Let A, B, and C be the subsets of a set U with the properties that $B \cap C = \emptyset$ and $A = B \cup C$. Construct a bijection map $b : \mathcal{P}(A) \to \mathcal{P}(B) \times \mathcal{P}(C)$, where $\mathcal{P}(X)$ represents the power set of a given set X.

Solution (b): Consider the mapping $b : \mathcal{P}(A) \to \mathcal{P}(B) \times \mathcal{P}(C)$ such that $S \subseteq A \to (S \cap B, S \cap C)$, that is, $b(S) = (S \cap B, S \cap C)$.
Required to prove (RTP) that b is bijective, that is
(i) b is one-one, and
(ii) b is onto.

(i) RTP: $b(S_1) = b(S_2) \Rightarrow S_1 = S_2$.
Now, $b(S_1) = b(S_2)$
$\Rightarrow (S_1 \cap B, S_1 \cap C) = (S_2 \cap B, S_2 \cap C)$.
Since these are ordered pairs, so we have: $S_1 \cap B = S_2 \cap B$ and $S_1 \cap C = S_2 \cap C$.
We have, $(S_1 \cap B) \cup (S_1 \cap C) = (S_2 \cap B) \cup (S_2 \cap C)$
$\Rightarrow S_1 \cap (B \cup C) = S_2 \cap (B \cup C)$
$\Rightarrow S_1 \cap A = S_2 \cap A$
$\Rightarrow S_1 = S_2$, since $S_1 \subseteq A$ and $S_2 \subseteq A$.
Thus, b is one-one.

(ii) RTP: $\forall S_1 \subseteq B$ and $S_2 \subseteq C$, $\exists S \subseteq A$ such that $b(S) = (S_1, S_2)$, where $S = S_1 \cup S_2$.
We have,

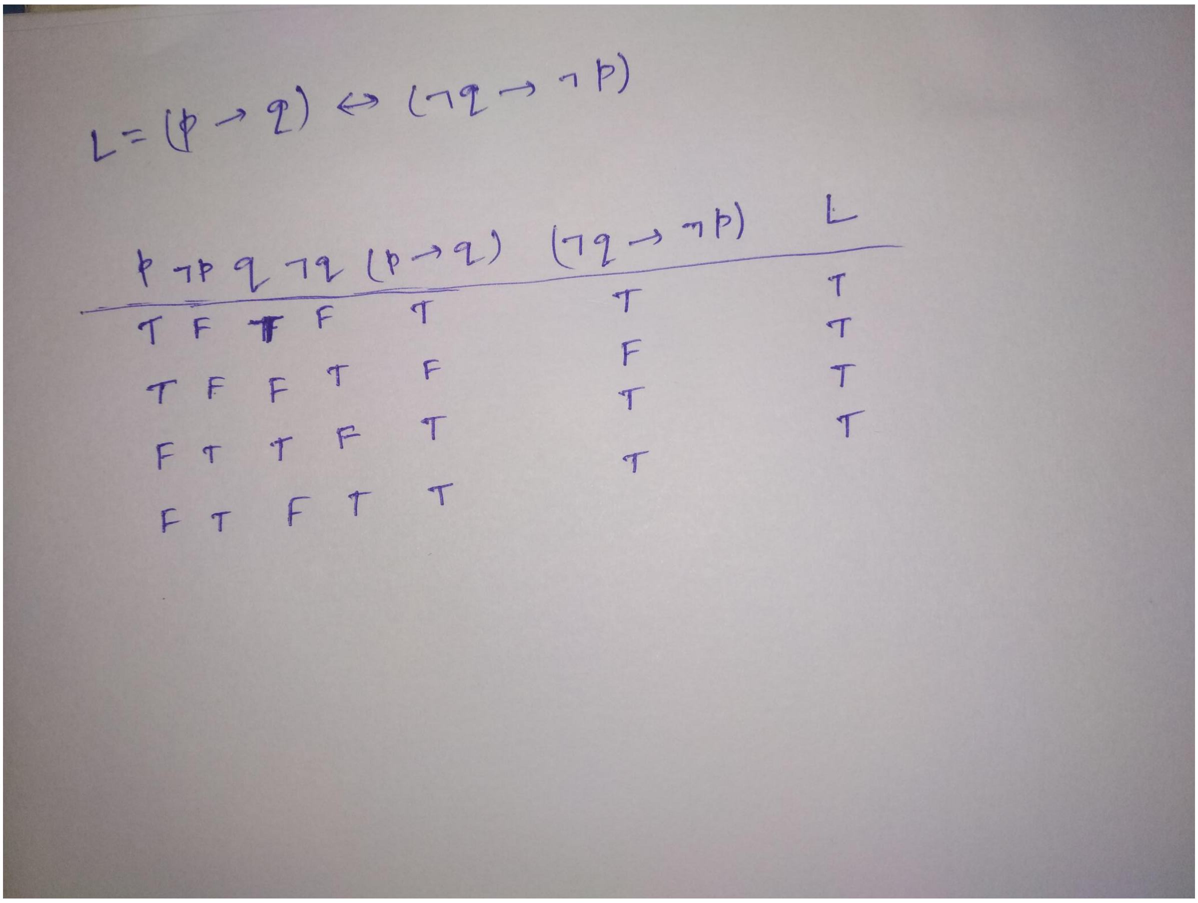
$$\begin{aligned}
b(S) &= (S \cap B, S \cap C) \\
&= ((S_1 \cup S_2) \cap B, (S_1 \cup S_2) \cap C) \\
&= ((S_1 \cap B) \cup (S_2 \cap B), (S_1 \cap C) \cup (S_2 \cap C)), \text{ using the distributive laws} \\
&= ((S_1 \cap B) \cup \emptyset, \emptyset \cup (S_2 \cup C)) \\
&= (S_1, S_2),
\end{aligned}$$

since $S_1 \subseteq B$ and $S_2 \subseteq C$.

□

(b) Construct a truth table for each of these compound propositions:

(i) $(p \to q) \leftrightarrow (\neg q \to \neg p)$
Answer:


$L = (p \to q) \leftrightarrow (\neg q \to \neg p)$

p	$\neg p$	q	$\neg q$	$(p \to q)$	$(\neg q \to \neg p)$	L
T	F	T	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T

(ii) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

[5 + (2.5 + 2.5) = 10]

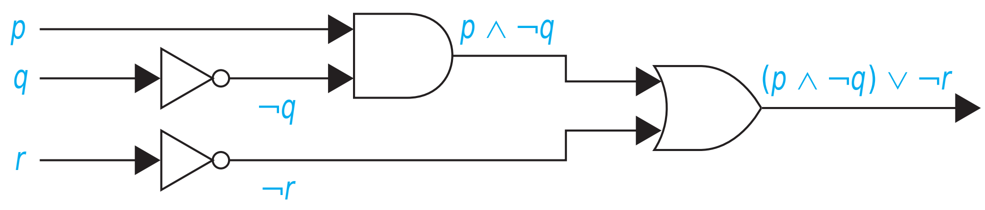
Answer: 3-a

$$\begin{aligned}
&\neg p \to (q \to r) && \\
\equiv\ & p \lor (q \to r) && \text{by implication law} \\
\equiv\ & p \lor (\neg q \lor r) && \text{by implication law} \\
\equiv\ & \neg q \lor (p \lor r) && \text{by commutative and associative laws} \\
\equiv\ & q \to (p \lor r) && \text{by implication law}
\end{aligned}$$

3. (a) Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent by a series of logical equivalences.

(b) (i) Build a digital circuit that produces the output $(a \vee \neg c) \wedge (\neg a \vee (b \vee \neg c))$, where input bits a, b, and c are given.

Answer: (3-b-i) Similar answer ($p = a$, $q = b$, $r = c$)



(ii) Show that $(a \vee b) \wedge (\neg a \vee c) \rightarrow (b \vee c)$ is a tautology.

Answer: (3-b-ii) Similar answer ($p = a$, $q = b$, $r = c$)

$(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$	
$\equiv \neg[(p \vee q) \wedge (\neg p \vee r)] \vee (q \vee r)$	by implication law
$\equiv [\neg(p \vee q) \vee \neg(\neg p \vee r)] \vee (q \vee r)$	by de Morgan's law
$\equiv [(\neg p \wedge \neg q) \vee (p \wedge \neg r)] \vee (q \vee r)$	by de Morgan's law
$\equiv [(\neg p \wedge \neg q) \vee q] \vee [(p \wedge \neg r) \vee r]$	by commutative and associative laws
$\equiv [(\neg p \vee q) \wedge (\neg q \vee q)] \vee [(p \vee r) \wedge (\neg r \vee r)]$	by distributive laws
$\equiv (\neg p \vee q) \vee (p \vee r)$	by negation and identity laws
$\equiv (\neg p \vee p) \vee (q \vee r)$	by communicative and associative laws
$\equiv T$	by negation and domination laws

[5 + (2.5 + 2.5) = 10]

**************** **End of Question Paper** *******************