Date 1 / 9 / 23

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Parameterized Complexity
Assignment - 2.

2. Input → A graph $G = (V, E)$, V is the no. of vertices & E is the no. of edges in a graph.

Parameter → Size of solution, denoted by k.

Question/Output → Does there exist a vertex cover of size atmost k in G.

Step 1. Remove isolated vertices :-

Remove all vertices of degree 0 from the graph. As they don't contribute to the vertex cover, removing them does not affect the existance of a solution.

Step 2. Reduction Rule 1 → Degree 1 Vertex Rule.

If there exists a vertex v of degree 1, remove v & its incident edge. & Decrease k by 1.

$$(G, k) \xrightarrow{\text{Degree 1 Vertex Rule.}} (G', k-1)$$

Proof
let {u, v} be the only edge incident to v. To cover edge (u, v). We must either include u or v in the vertex cover. Since v is removed we included u in the vertex cover. This does not increase the size of the vertex cover.

Saathi

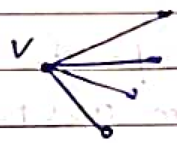
Date __/__/__

Step 3 Reduction Rule 2 → Degree K Vertex Rule.

If there exists a vertex v of degree $k>2$, remove v & its incident edges & Decrease parameter by 1

Proof
let v be a vertex of degree $k>2$. To cover all incident edges we must include atleast one neighbour of v in the vertex cover. By removing v we include all its neighbours in the vertex cover. Since $k>2$, we still $k-1$ available vertices in the vertex cover.

$(G,k) \longrightarrow (G'k')$



Step 4. Kernal Completion.

If $k<0$, set $k=0$, Remove all vertices of degree $>k$ & their incident edges.

Proof
Suppose a vertex u has degree $>k$. Atleast $k+1$ vertices are required to cover its incident edges. Thus we can easily remove its edges & reduce the vertex cover size.

Date __ / __ / ____

Steps. Runtime Analysis.

n is the input size. So $n = |V| + |E|$.

(a) Removing degree-0 & degree-1 vertices takes $O(n)$ time.

(b) Contracting degree-2 vertices takes $O(n)$ time.

(c) Deleting excess vertices takes $O(|V|)$ time.

So overall kernalization process takes polynomial time.

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Date 9 / 2 / 23.

Parameterized Complexity

Assignment - 2.

1.(a) To Prove → G contains a set of l vertices (disjoint) K_{10}'s if & only if F contains l pairwise disjoint sets.

As it is if & only if so we need to prove in forward & backward direction as following :-

(a) If G contains a set of l vertices (disjoint) K_{10}'s then F contains l pairwise disjoint sets

(b) If F contains l pairwise disjoint sets then G contains a set of l vertex disjoint K_{10}'s

Forward Direction.

→ If G contains a set of l vertices (disjoint) K_{10}'s then F contains l pairwise disjoint sets.

Proof

let's consider the l vertices disjoint K_{10}'s present in G. Each K_{10} in G corresponding to a subset of 10 vertices from $V(G)$ that induces a complete graph. These subset are $S_1, S_2 \ldots S_l$

As K_{10}'s are disjoint vertices So. $S_i \cap S_j$ $1 < i \neq j < l$

$$S_1 \cap S_2 = S_1 \cap S_2 = \ldots S_l = \phi$$

So F contains l pairwise disjoint set $\{S_1 \ldots S_l\}$, Date ___ / ___ / _____

Saathi

Backward Direction.

→ If F contains l pairwise disjoint sets. then G contains a set of l vertex disjoint K_{10}'s

Proof.

As each subset F induces a K_{10} in G, we have l vertex disjoint K_{10}' present in G, Corresponding to sub $\{S_1 \dots S_l\}$.

As each subset of F is a 10-sized vertex subset both K_{v_1} & K_{v_2} correspond to subsets in F. As F contains pairwise disjoint sets So $S_{v_i} \cap S_{v_j} = \phi$

This contradicts that the vertex belong to both K_{v_1} & K_{v_2}. So our assumption that there exists a vertex belonging to two different K_{10}'s is false.

Thus l K_{10}'s corresponding to the subsets $\{S_1 \dots S_l\}$ are indeed vertex - disjoint in G.

Date __/__/____

1.2 (b) ~~K10~~:

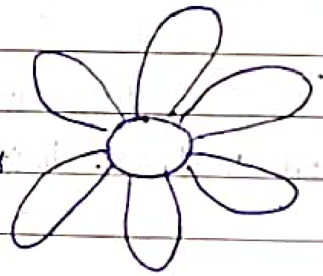
Given →

K10: A complete graph on 10 vertices. The graph where every pair of distinct vertices is connected by an edge.

~~Repre~~ Input → Graph G
Parameter → k integer.
Output → If there exist atleast k vertex disjoint K10 subgraph in G.

Sunflower lemma → It states that for any +ve integer r & k there exists a +ve integer G(r, k) such that any collection of G(r, k) sets of size atleast r can be partitioned into disjoint sunflower of size r, & each sunflower has a common center & its petals are disjoint.

If the graph is sufficiently ~~gra~~ large size & each set having unique elements ~~Here~~ if there are more than $(k-1)^{r}$ sets in F, where F is the family of sunflower lemma then there must be a sunflower with k petals



Now let's proof the above lema

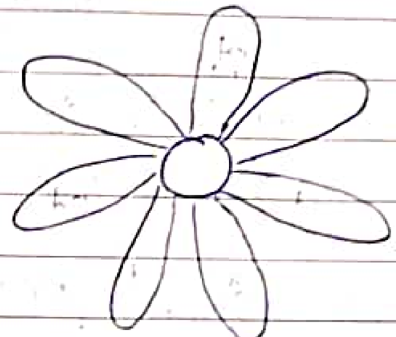
~~Pro~~ Proof

Saathi

Date __/__/__

Lemma → For any integer $k \geq 2$ & $r \geq 1$ if a family F of sets has more then $(k-1)^r$ sets each having unique / disjoint value atleast r then F contains a sunflower lemma with k petals.

Proof →



1. Assume there are $(k-1)^r$ sets in F
2. Select any element x from each set in F
3. As we have $> (k-1)^r$ different choices of x. then there must be atleast two different sets say S_1 & S_2 & they are identical.
4. $A = \{S_1/S_2\}$ & $B = \{S_2/S_1\}$
   ↳ $A = \{S_1 - S_2\}$ ↳ $B = \{S_2 - S_1\}$

   Both A & B are unique & S.T. $A, B > r$
5. S_1 & S_2 are identical excepts for elements in A & B.
6. The sets in F in element A forms a sunflower with center A & ~~~~ with petals B as A & B are disjoint.
7. This sunflower has k petals (one from each element in B) & $\subseteq$ F

Saathi

Date ___ / ___ / _____

ATQ deriving Alogrithm

Given → G graph
Parameter → k (integer)
Find → If there exists atleast k vertex disjoint K_{10}'s in G using ~~an~~ sunflower lemma.

Step 1. Set $r = 10$ (as K_{10}s) & a family F consisting of all the possible K_{10} subgraph of G.

Step 2. Check $|F| > (k-1)^r$.
↳ If not, return No as can't be k vertex disjoint K_{10}'s

Step 3. Apply Sunflower lemma to find a sunflower with k petals in F.

Step 4. If a sunflower with k petals is found.
↳ Return Yes, That there exist a vertex-disjoint in G

Analysis / Time Complexity.

1. Generating all possible K_{10} subgraph of G is done in a polynomial time
2. Checking if $|F| > (k-1)^r$ is done in polynomial time.
3. Sunflower lemma, ~~but~~ finding sunflower with k petals has polynomial time

Thus overall algo runs in a polynomial time.

Saathi

Date __ / __ / ____

we construct a sunflower of F^* family of k petals with the above algorithm. & The center of the sunflower is one of the set in F^* & petals are from rest of the sunflower.

F^* has a family of size $O(k^r)$ & $r = 10$ (given)
$= O(k^{10})$

1(c) To Prove → (i) If (G, k) is a yes instance of K_{10} - collecting, then $(G-v, k)$ is a yes instance of K_{10}.

(ii) If $(G-v, k)$ is a yes instance of K_{10} collecting then (G, k) is a yes instance of K_{10} collecting.

(i) Given → (G, K) is a yes instance of K_{10} - collecting. Here exist k vertex disjoint K_{10} in G.

To Prove → $(G-V, k)$ is also a yes instance.

Proof →

We have to proof this by contradiction

let $(G-V, k)$ is not a yes instance of K_{10} collecting means there are not k vertex disjoin in $G-V$. This implies that after removing v there must be $< k$ vertex-disjoint in G.

This implies that the original graph G will have fewer than $(k+1)$ vertex disjoint. This contradicts the fact that (G, k) is a yes instance which states that there are atleast $(k+1)$ vertex disjoint in G.

Thus our Prove that $(G - v, k)$ is indeed a yes instance of K10-collecting.

~~(ii) If $(G-v, k)$~~

(ii) Given → ~~If~~ $(G-v, k)$ is a yes instance of K_{10}-collecting, there exists k vertex disjoint in $(G-v)$

To Prove → (G, k) is a yes instance of K_{10}-collecting

Proof →

~~If let~~ $(G-v, k)$ is a yes instance means there exist k vertex disjoint K_{10}s in $G-v$. As from to prove vertex v in v is not the part of any set of F^*. then we we can add a set in F^* to create a collection of k vertex disjoint of K_{10} in G. Adding v vertex does not change anything because v is not connected to any vertex in the set of F^*

Thus if we had a valid collection of k vertex disjoint K_{10} in $G-v$ by adding v to any one of set we create a valid / correct collection of G. Thus (G, k) is also a yes instance of K_{10} collecting.

This by (i) & (ii) we prove that If there exist a vertex $v \in V(G)$ S.T. v is not any set in F^* then (G, k) is yes instance of K-10 collecting if & only if $(G-v, k)$ is a yes instance of K_{10}-collecting.

Saathi

Date __ / __ / ____

(d) Using a, b, c we have to show that K-10 collecting admits a kernal with $O(K^{10})$ vertices

Proof

From(b) given an algorithm that find F^* a family of size at most $O(K^{10})$ containing atleast k pairwise disjoint sets iff F contains k pairwise disjoint sets.

From(c) Vertex removal doesnot effect the instance if removed vertex is not in any set of F^*

By above two statements

↳ ~~The vertices~~ We can iterate through the vertices & removal all vertices not in F^*, results a keral with $O(k^{10})$ vertices.

Time Analysis

1. Generating F^* takes $O(\cancel{\log}\ n^{10})$ time.

2. Iterating through vertices & checking that a subset of F^* takes $O(n)$ time.

Overall time complexity $\Rightarrow O(n^{10}+n) \rightarrow O(n^{10})$

Thus admits a kernal with ~~$O(k^{10})$~~ $O(K^{10})$ vertices & the running time analysis is $O(K^{10})$.