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Saathi

Parametrized Complexity
Assignment 1

1 (a) Given a graph (G) & X is the vertex cover of G & X is a subset of V(G) or ($X \subseteq V(G)$) of size k. Dominating set parametrized by vertex cover problem S of min size such that every vertex v in G such that either v is in S or atleast one neighbour of v is in S

We need to design an kernalization algorithm so that we can prove that the dominating set parametrized by vertex cover admits a kernal.

$$(G, X, k) \longrightarrow \boxed{\text{Kernalization}} \longrightarrow (G', X', k')$$

Kernalization Algorithm.

To design an kernalization algo that has a yes instance of (G, X, k) of a dominating set and ~~param~~ produces a yes instance of (G' X', k')

$|X'| \leq f(k) \Rightarrow$ Vertex cover is bounded by a function of k

To design an algorithm we have several reduction rules as →

Reduction Rule 1 ⇒ Isolated vertices.

If the graph G has an isolated vertex (vertex with no neighbour) we can safely remove it from G & reduce k by 1

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As isolated vertex can not contribute in dominating set & don't affect the problem solution.

Reduction Rule 2 $\Longrightarrow$ Degree 1 Vertices

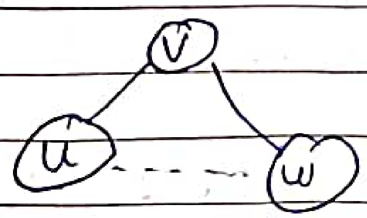
If G contains a vertex V with degree 1. We can add its neighbour to the vertex cover X & remove both V & its neighbour must be satisfying the dominating condition.

Reduction Rule 3 $\Rightarrow$ Vertex Cover Edges.

For each edge {u, v} in G where both u & v in vertex cover (X) then we can safely remove the edge from G since both end points are already in the vertex cover. This does not affect the dominating set property. Count the no. of such edges & say C.

Reduction Rule 4 $\Rightarrow$ Degree 2 Vertices

If G contains a vertex V with degree 2. Say neighbour u & w we can replace it as {u, v} & {v, w} edges with a single {u, w}. Remove V from G & reducing k by 1 for each step. This is correct because either u or w must be in dominating set to satisfy the condition for V



Reduction Rule 5 $\Rightarrow$ Romaing Vertex Cover.

If the current vertex cover X' is larger than f(k) we can replace X' with a smaller set of size f(k)

$$(G, X, k) \xrightarrow[\text{Algorithm}]{\text{Kernalization}} (G' X' k')$$

Kernal Size Analysis / Observations

1 The above Reduction Rule 1 & 2 remove atmost k vertices.

2 Reduction Rule 3 removes atmost c vertices

3. Reduction Rule 4 removes atmost $\frac{k}{2}$ vertices as each removed vertex has atleast degree 2.

Total no. of vertices is $= k + c + \frac{k}{2} = f(k)$

Runtime Analysis

Reduction Rule can be executed in polynomial time checking for isolated, degree 1 vertices as well as Update vertex cover & degree all take polynomial time & Total no. of vertices remains polynomially bounded throughout the reduction process.

By applying these rules iteratively we end up with an equivalent instance (G', X' K') such that $|X'| \leq f(k)$.

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4(b) We are given a graph $G(V, E)$ along with a vertex cover $X \subseteq V$ of size k. We need to find a dominating set $S \subseteq V$ such that every vertex $v \in V$ satisfies either $v \in S$ or atleast one neighbour of v is in S.

Observations

1. Since X is a vertex cover, every edge ~~in E~~ in E is incident on atleast one vertex in X

2. If a vertex $v \in X$ then it can be calculated & will be included in dominating set S to satisfy the dominating set property.

3. If a vertex $v \notin X$ then all of its neighbour must be included in the dominating set S to satisfy the dominating set property.

Algorithm

1. Initialize the dominating set S as an empty set.

2. For each vertex $v \in X$ add v to S

3. For each vertex $v \notin X$ add all neighbour ~~to~~ of v to S

4. Return the set S as the final dominating set.

Proof.

Constructed set S is valid dominating set if that its size is atmost 2^k

1. Every vertex in X is either in the dominating set S or has its neighbour which satisfy the dominating set property.

2. If vertex $\notin X$ then their neighbour are marked as covered, taking care that every vertex is dominated.

$|S| \leq k + 2k$ ~~$|S| \leq k$~~

→ Size of S is bounded by the size of X (at most k) + the no. of vertices whose neighbour are marked as covered. (at most $2k$ because each vertex in X can cover at most 2 neighbours)

Time Analysis

1. Construct the vertex cover X takes polynomial time say $O(n^c)$ for some constant c

2. Iterating through each vertex & its neighbour takes $O(n+m)$ time $m \Rightarrow$ no. of edges bounded by graph $(O(m) \leq O(n^2))$

Overall $\Rightarrow$ $O(n^c) + O(n^2) \Rightarrow O(n^2)$

Since k is used to bound the size of the dominating set. The total Time taken

$$\boxed{2^{n^{O(k^2)}} \cdot O n^{O(1)}}$$

2(a) Given a graph G (V, E) where V => set of vertices
E => set of Edges. & a parameter k

We need to prove that iff (G', k') is a yes instance then (G, k) is also a yes instance
such that

$$|V(G)| \leq O(k^3)$$

i.e., the no. of vertices in the reduced instance is polynomial in k

Observations 1.

1. If there exists a vertex of atleast 3 degree in G then we can remove it without changing solution because after removing it will decrease the degree of 3 vertices.

2. If there exists a vertex of degree 2 in G which is not neighbour / adjacent to any other vertex of degree > 2 then we can remove it without changing solution.

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1. Now after the above 2 observations we have a G' graph with almost 3k vertices

How? (See Ans. below)
→ Each time we remove a vertex we decrease the parameter k by 1. Since we only remove vertices with degree ≥ 2 & vertices of degree 2 not neighbours to higher degree vertices we can remove almost 3 vertices in each step.
So
For k steps we have 3k vertices in G'

2. If there exist a solution set S for graph G (G, k) then there exist a solution for (G', k)

How? (Explained below)
→ We only removed vertices in that way which does not effect our parameter k so (G', k) is also a yes instance.

3. If there exist a solution of (G', k) then there exist a solution for (G, K)

How? (Explained below)
→ We only removed vertices & we can also add vertices back into G' so that we can obtain the optimal solution.

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So by the above 3 case & merging them we can conclude that

(G', k) has atmost $3k$ vertices

And.

If (G, k) is a yes instance. iff (G', k) is also a yes instance.