A simple explicit homogenization solution for the macroscopic response of isotropic porous elastomers

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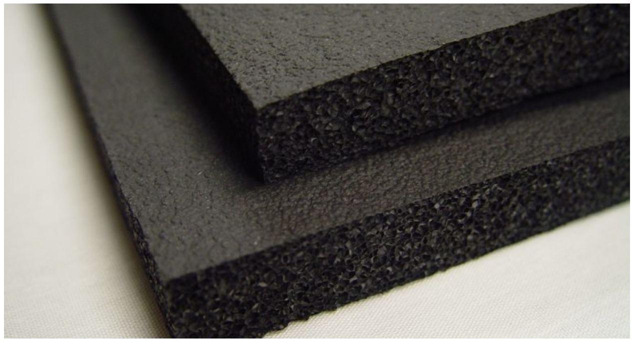
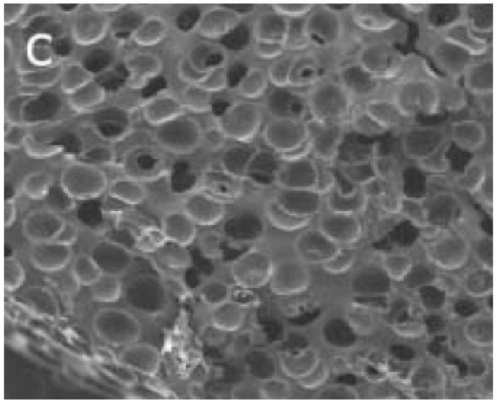
Closed-cell elastomeric foams: Applications

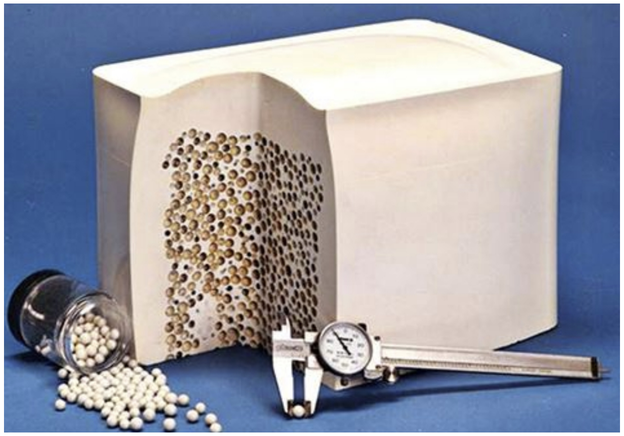
Polyurethane foam


Closed-cell neoprene



SIS block copolymer foam

SBR Foam


Syntactic foam

Microscopic description of Elastomers

$$W_{\mathtt{m}}(\mathbf{QFK}) = W_{\mathtt{m}}(\mathbf{F}) \quad \forall \quad \mathbf{Q}, \mathbf{K} \in Orth^{+}$$

$$W_{\mathtt{m}}(\mathbf{F}) = \begin{cases} \Psi_{\mathtt{m}}(I_1) & \text{if} \quad J = 1 \\ +\infty & \text{otherwise} \end{cases}$$

$$I_1 = \mathbf{F} \cdot \mathbf{F} \qquad J = \det \mathbf{F}$$

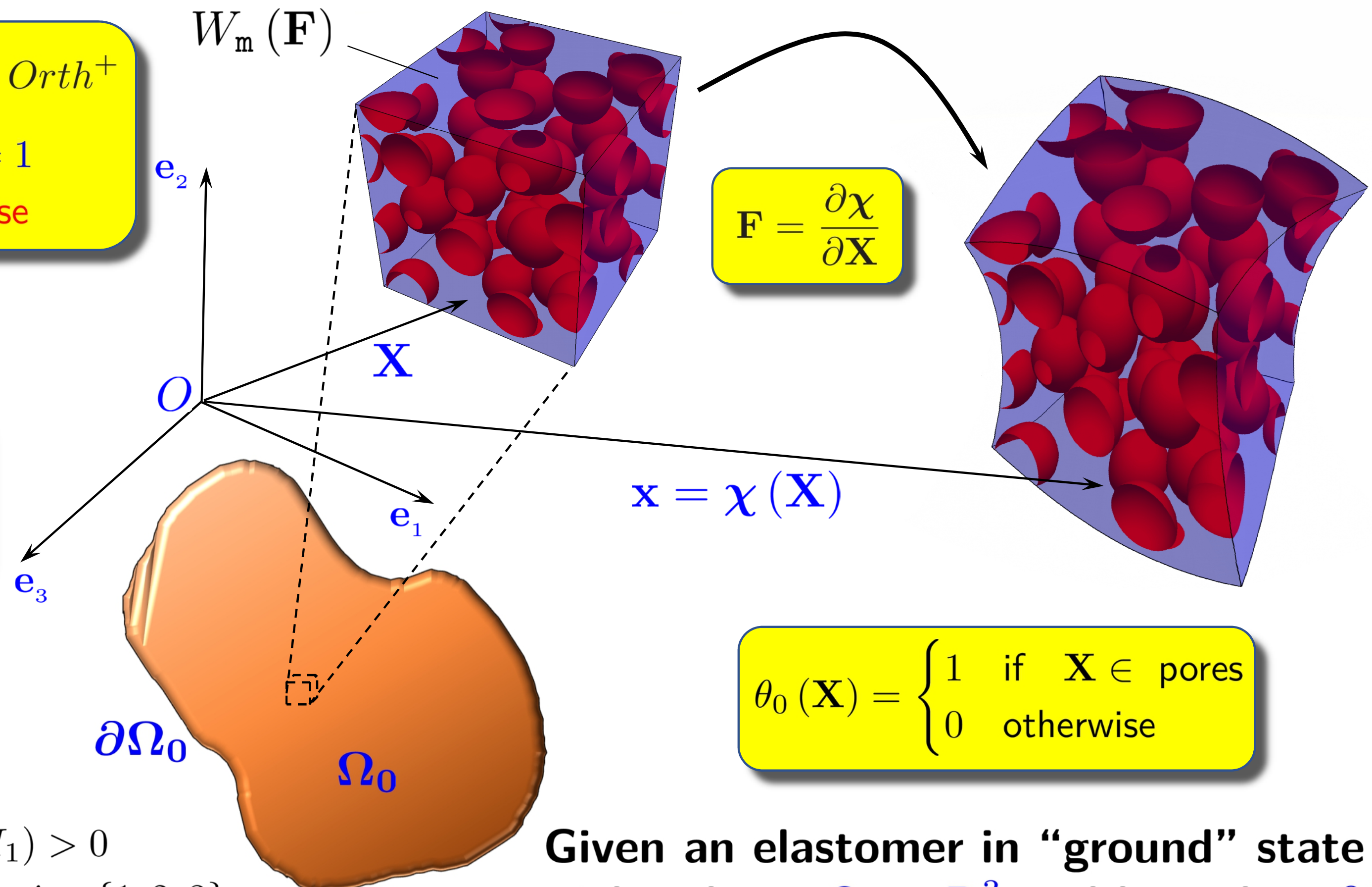
$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{X}, \mathbf{F});$$
$$W(\mathbf{X}, \mathbf{F}) = [1 - \theta_0(\mathbf{X})] W_{\mathtt{m}}(\mathbf{F})$$

$\Psi_{\mathtt{m}}(3) = 0$ and $\Psi'_{\mathtt{m}}(3) = \dfrac{\mu}{2}$

$\Psi'_{\mathtt{m}}(I_1) > 0;$

$$\Psi'_{\mathtt{m}}(I_1) + 2\left(I_1 - \lambda_i^2 - \frac{2}{\lambda_i}\right)\Psi''_{\mathtt{m}}(I_1) > 0$$
$$i = \{1, 2, 3\}$$

Zee and Sternberg, Arch. Rat. Mech. Anal. *(1983)*



$$\mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}}$$

$$\theta_0(\mathbf{X}) = \begin{cases} 1 & \text{if} \quad \mathbf{X} \in \text{pores} \\ 0 & \text{otherwise} \end{cases}$$

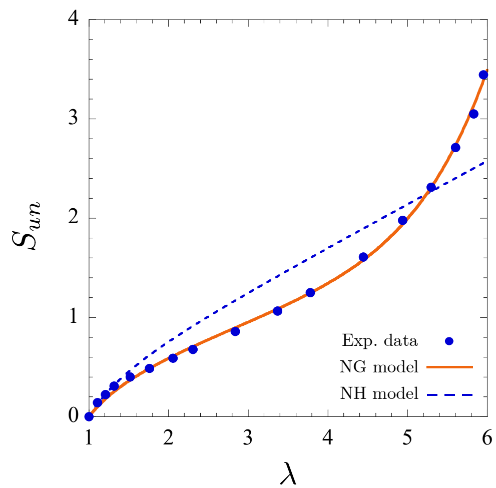
Given an elastomer in “ground” state with volume $\Omega_0 \subset \mathbb{R}^3$ and boundary $\partial\Omega_0$

Microscopic description of Elastomers...

Examples: **Gent Model, Arruda-Boyce Model, LP Model**

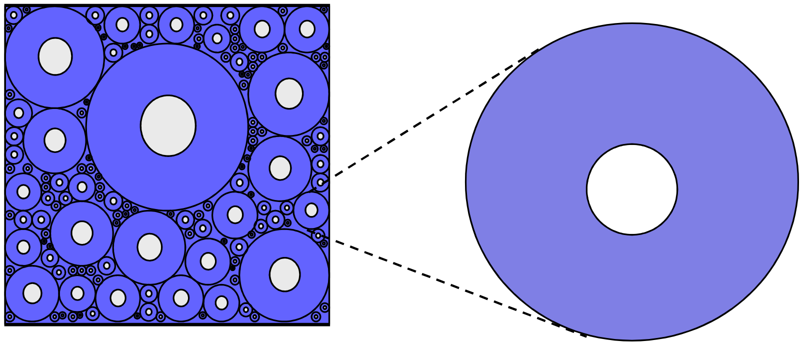
$$\Psi_{\mathtt{m}}(I_1) = \frac{\mu}{2}[I_1 - 3]$$

$$\Psi_{\mathtt{m}}(I_1) = \frac{3^{1-\alpha_1}}{2\alpha_1}\mu_1[I_1^{\alpha_1} - 3^{\alpha_1}] + \frac{3^{1-\alpha_2}}{2\alpha_2}\mu_2[I_1^{\alpha_2} - 3^{\alpha_2}]$$



Existing Homogenization Solutions:

- **Ogden's (1978)** Voight bound for isochoric deformations, i.e. when $\det(\overline{\mathbf{F}}) = 1$
- **Hashin's (1985)** solution for the HSA microstructure under isotropic deformations, i.e. $\overline{\mathbf{F}} = \overline{J}^{1/3}\mathbf{I}$
- **Danielsson et al. (2004)** making use of the trial field by **Hou and Abeyaratne(1992)**



Treolar, Trans. of the Far. Soc. *(1944)* Lopez-Pamies, Com. Rend. Mec. *(2010)*

Microscopic description of Elastomers...

Existing Results (contd...):

- Lopez-Pamies and Ponte Castañeda (2007), making use of a "linear" comparison medium method, constructed a solution that is applicable to arbitrary deformation gradients ($\overline{\mathbf{F}}$)

$$\boxed{f \neq \frac{\overline{J}-1}{\overline{J}} + \frac{f_0}{\overline{J}}}$$

- Numerical solutions by Danielsson et al.(2004), Moraleda et al.(2004) and Wang and Hennan (2016)

Lopez-Pamies, J. Mech. Phys. Sol. *(2011)*
Lopez-Pamies and Ponte-Castañeda , J. Mech. Phys. Sol. *(2007)*

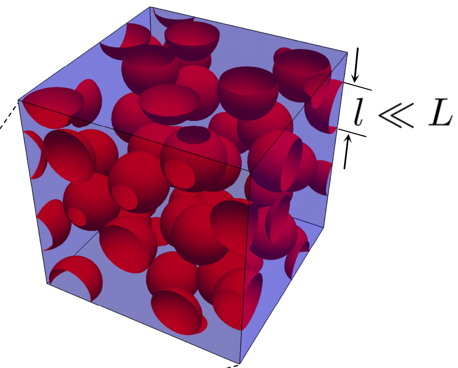
Macroscopic Response

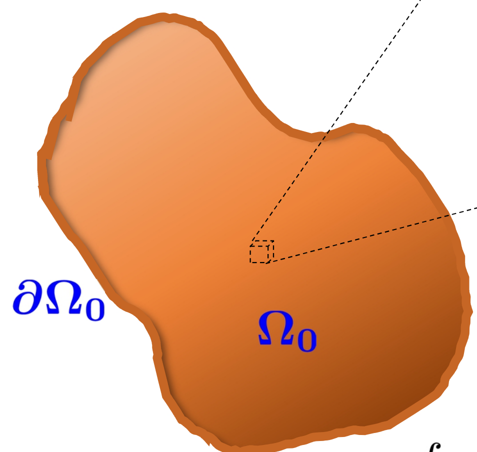
Definition: : Relationship between the volume averages of $\mathbf{S}$ and $\mathbf{F}$

$$\overline{\mathbf{S}} \doteq \frac{1}{|\Omega_0|}\int_{\Omega_0} \mathbf{S}(\mathbf{X})\, d\mathbf{X} \qquad \overline{\mathbf{F}} \doteq \frac{1}{|\Omega_0|}\int_{\Omega_0} \mathbf{F}(\mathbf{X})\, d\mathbf{X}$$

$$\overline{\mathbf{S}} = \frac{\partial \overline{W}}{\partial \overline{\mathbf{F}}}(\overline{\mathbf{F}}, f_0),$$

where $\overline{W}(\overline{\mathbf{F}}, f_0) = \min_{\mathbf{F}\in\mathcal{K}} \int_{\Omega_0} W(\mathbf{X}, \mathbf{F})\mathrm{d}\mathbf{X},$





Affine B.C.'s:

$\mathbf{x} = \overline{\mathbf{F}}\mathbf{X}$ on $\partial\Omega_0$

$$f_0 \doteq \frac{1}{|\Omega_0|}\int_{\Omega_0} \theta_0(\mathbf{X})\, d\mathbf{X}$$

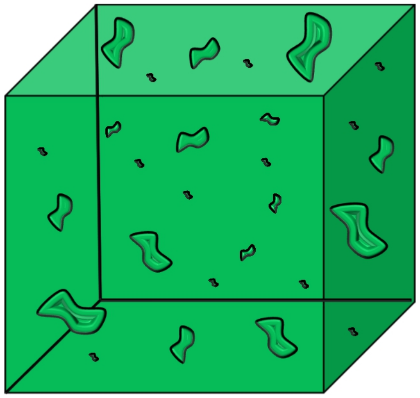
- $\overline{W}\left(\overline{\mathbf{F}}\right)$ **is an isotropic function of** $\overline{\mathbf{F}}$
- $\overline{W}(\overline{\mathbf{F}}, f_0) = \overline{U}(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, f_0) = \overline{\Psi}(\overline{I}_1, \overline{I}_2, \overline{J}, f_0)$

$$\overline{I}_1 = \overline{\mathbf{F}}\cdot\overline{\mathbf{F}}, \quad \overline{I}_2 = \frac{1}{2}\left[\overline{I}_1^2 - (\overline{\mathbf{F}}^T\overline{\mathbf{F}})\cdot(\overline{\mathbf{F}}^T\overline{\mathbf{F}})\right], \quad \overline{J} = \det\overline{\mathbf{F}}.$$

Hill, Proc. Roy. Soc. A. *(1972)*

Main Result

- For any type of non-percolative isotropic distribution of equi-axed closed-cell vacuous pores embedded in a Gaussian or Non-Gaussian elastomeric matrix



$$\overline{W}(\overline{\mathbf{F}}, f_0) = (1 - f_0)\,\Psi_{\mathtt{m}}\left(\frac{\mathcal{I}_1}{1 - f_0} + 3\right)$$

$$\mathcal{I}_1 = \frac{3(1-f_0)}{3+2f_0}\left[\overline{I}_1 - 3\right] + \frac{3}{\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1-f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3} + 2f_0\right)}{3+2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\left(2\overline{J} + f_0 - 2\right)}{\left(\overline{J} - 1 + f_0\right)^{1/3}}\right]$$

- $\Psi_{\mathtt{m}}\left(\overline{\mathbf{F}}\right)$: Stored-energy function (finite branch) for the elastomeric matrix

$$\overline{I}_1 = \overline{\mathbf{F}} \cdot \overline{\mathbf{F}}, \quad \overline{I}_2 = \frac{1}{2}\left[\overline{I}_1^2 - (\overline{\mathbf{F}}^T\overline{\mathbf{F}}) \cdot (\overline{\mathbf{F}}^T\overline{\mathbf{F}})\right], \quad \overline{J} = \det \overline{\mathbf{F}}.$$

Shrimali, Lefèvre, Lopez-Pamies, J. Mech. Phys. Sol. *(2019)*

Outline

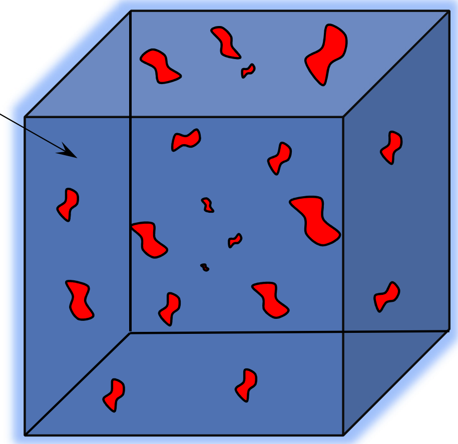
- **Neo-Hookean elastomers:**
  - **Infinitely polydisperse and abstract shape**
  - **Iterated homogenization**
  - **A simple closed form solution**

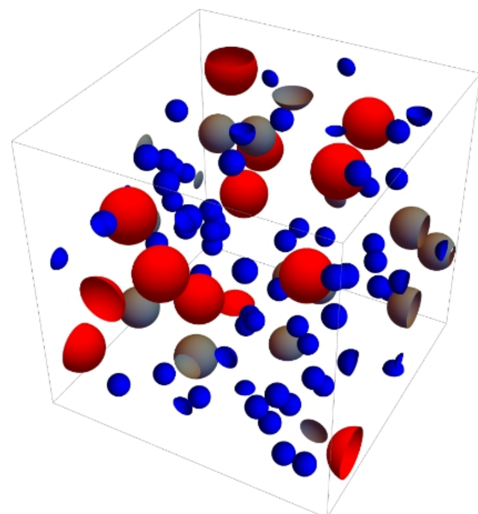
$$\overline{W}^{NH}(\overline{\mathbf{F}}, f_0) = \ldots \qquad \overline{W}^{NG}(\overline{\mathbf{F}}, f_0) = \ldots$$

  - **Numerical homogenization**
    finitely polydisperse/monodisperse
    equi-axed pores

- **Non-Gaussian elastomers:**
  - **Nonlinear comparison medium**
  - **A simple explicit solution**

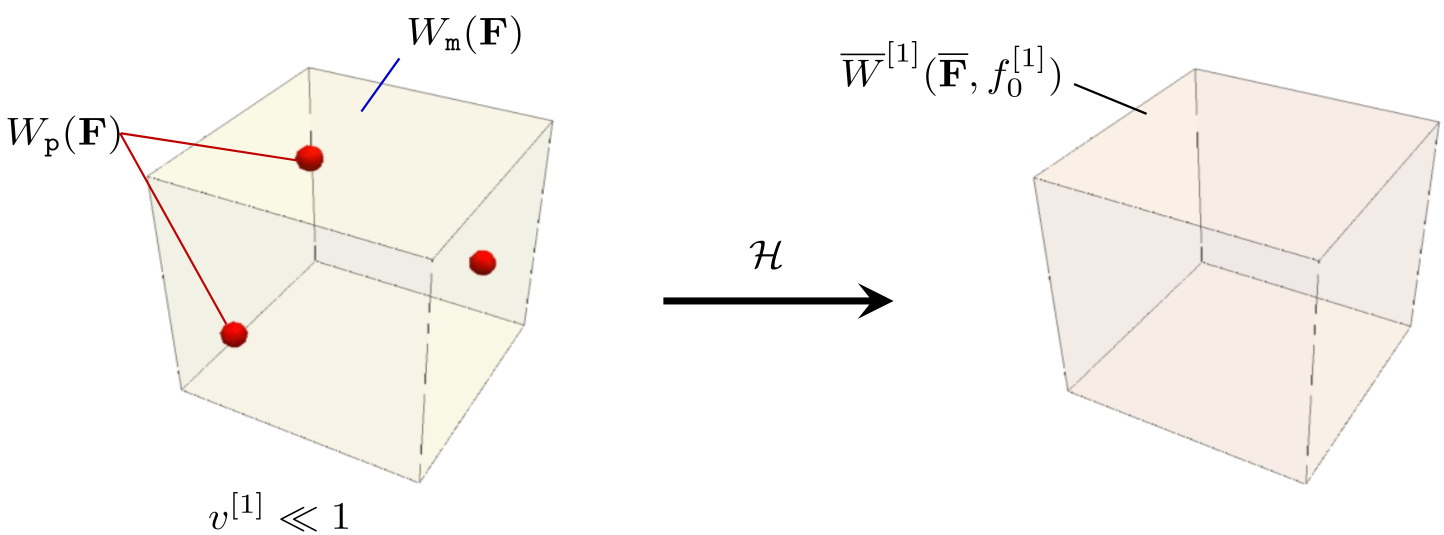
$$\Psi_{\mathtt{m}}(I_1) = \frac{\mu_{\mathtt{m}}}{2}[I_1 - 3]$$





An exact solution via iterated homogenization

Iterated dilute homogenization: iteration 1

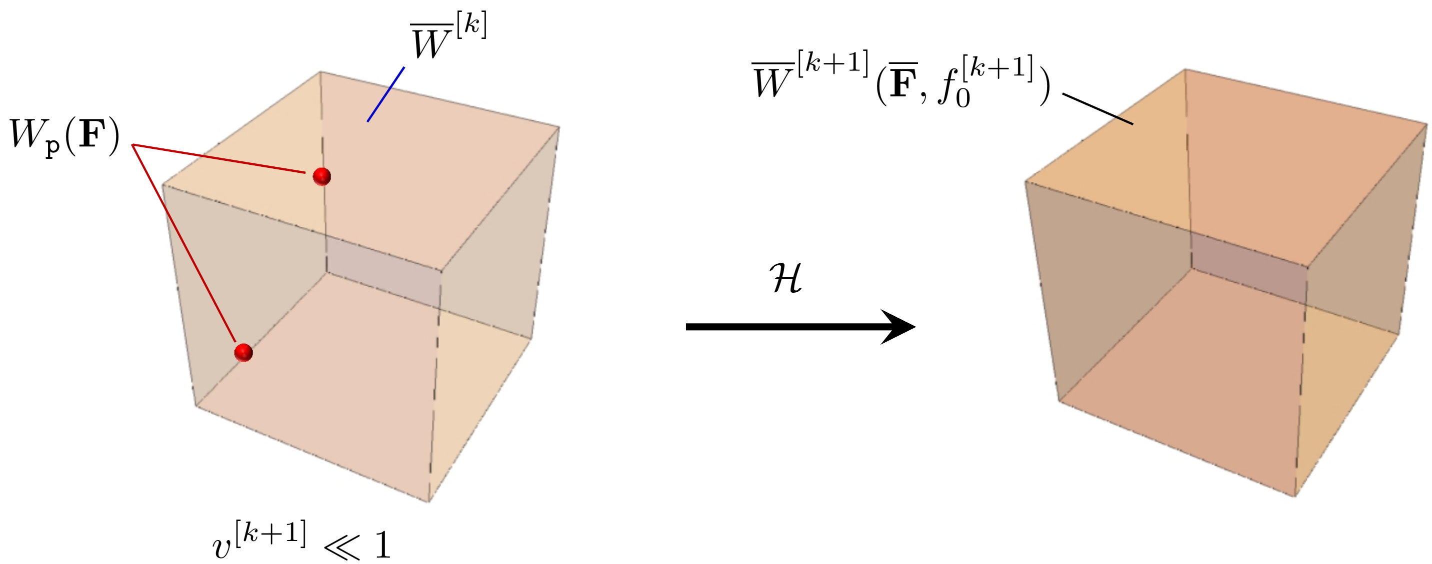


$$\boxed{\overline{W}^{[1]}(\overline{\mathbf{F}}, f_0^{[1]}) = W_{\mathtt{m}}(\overline{\mathbf{F}}) + v^{[1]}\mathcal{H}(W_{\mathtt{m}}, W_{\mathtt{p}}; \overline{\mathbf{F}})}$$

total volume fraction of pores: $f_0^{[1]} = v^{[1]}$

An exact solution via iterated homogenization...

Iterated dilute homogenization: iteration $k+1$



$$\boxed{\overline{W}^{[k+1]}(\overline{\mathbf{F}}, f_0^{[k+1]}) = \overline{W}^{[k]}(\overline{\mathbf{F}}, f_0^{[k]}) + v^{[k+1]}\mathcal{H}(\overline{W}^{[k]}, W_{\mathrm{p}}; \overline{\mathbf{F}})}$$

total volume fraction of the pores: $f_0^{[k+1]} = v^{[k+1]} + f_0^{[k]}(1 - v^{[k+1]})$

An exact solution via iterated homogenization...

Iterated dilute homogenization: iteration $k \longrightarrow +\infty$

$$(1-f)\,\frac{\partial \overline{W}}{\partial f}\left(\overline{\mathbf{F}}, f\right) - \mathcal{H}\left(\overline{W}, W_{\mathtt{p}}; \overline{\mathbf{F}}\right) = 0 \quad \textbf{with} \quad \overline{W}(\overline{\mathbf{F}}, 0) = W_{\mathtt{m}}(\overline{\mathbf{F}})$$

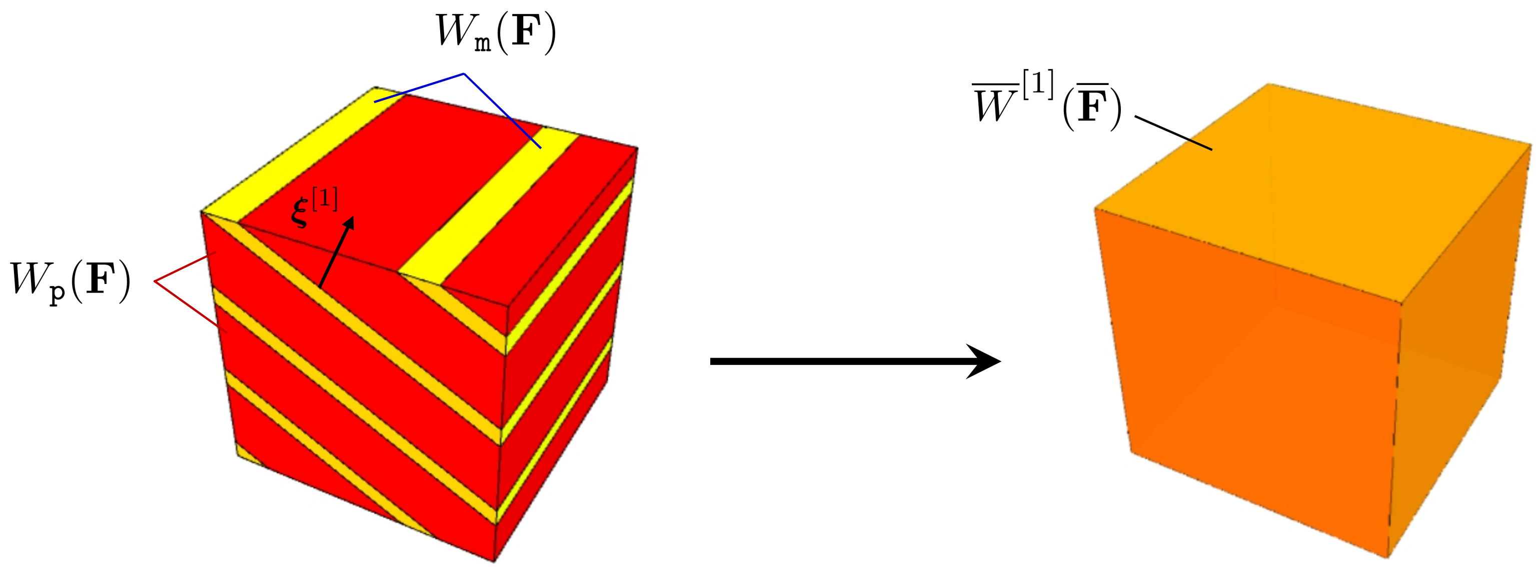
Instead start with $W_{\mathtt{p}}$ (or 100% pores) and add dilute amounts of matrix

$$f\,\frac{\partial \overline{W}}{\partial f}\left(\overline{\mathbf{F}}, f\right) - \mathcal{H}\left(\overline{W}, W_{\mathtt{m}}; \overline{\mathbf{F}}\right) = 0 \quad \textbf{with} \quad \overline{W}\left(\overline{\mathbf{F}}, 1\right) = W_{\mathtt{p}}(\overline{\mathbf{F}})$$

- The pores are infinitely polydisperse
- Valid for any choice of $W_{\mathtt{m}}$ and $W_{\mathtt{p}}$
- Arbitrary large macroscopic deformations $\left(\overline{\mathbf{F}}\right)$

provided that $\mathcal{H}$ is known

An analytically tractable $\mathcal{H}$: Coated laminates



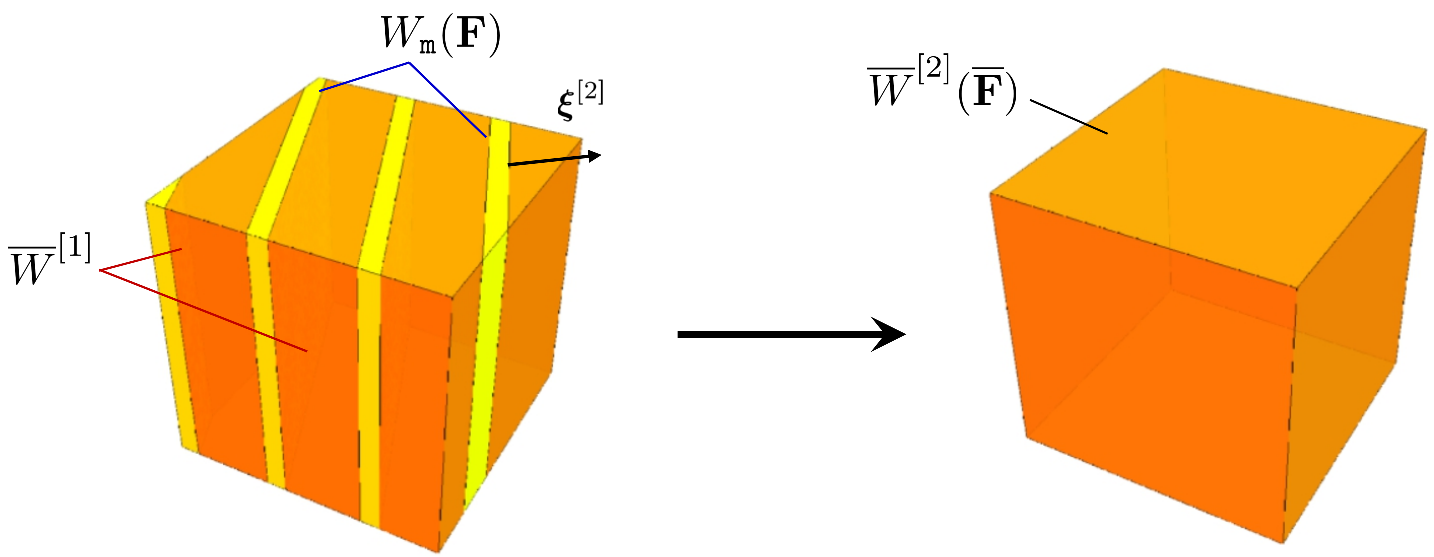
Rank - 1 laminate

$$\mathbf{F}(\mathbf{X}) = \begin{cases} \mathbf{F}^{(1)} = \overline{\mathbf{F}} + f^{[1]}\boldsymbol{\alpha}^{[1]} \otimes \boldsymbol{\xi}^{[1]} & \text{if } \mathbf{X} \in \Omega_{\mathtt{m}} \\ \mathbf{F}^{(2)} = \overline{\mathbf{F}} - (1 - f^{[1]})\boldsymbol{\alpha}^{[1]} \otimes \boldsymbol{\xi}^{[1]} & \text{if } \mathbf{X} \in \Omega_{\mathtt{p}} \end{cases}$$

$$\overline{W}^{[1]}(\overline{\mathbf{F}}) = \min_{\boldsymbol{\alpha}^{[1]}}\{(1 - f^{[1]})W_{\mathtt{m}}(\mathbf{F}^{(1)}) + f^{[1]}W_{\mathtt{p}}(\mathbf{F}^{(2)})\}$$

Tartar, *(1985)* Francfort and Murat, *(1986)*

An analytically tractable $\mathcal{H}$: Coated laminates



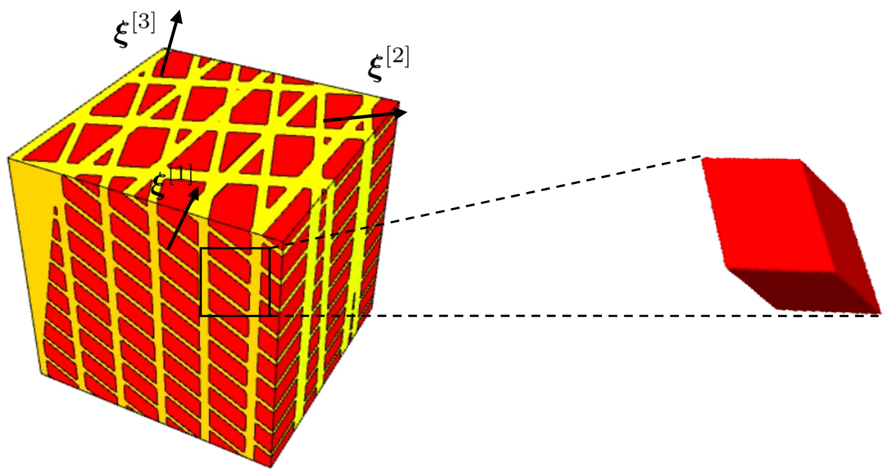
Rank - 2 laminate

$$\mathbf{F}(\mathbf{X}) = \begin{cases} \mathbf{F}^{(1)} = \overline{\mathbf{F}} + f^{[2]}\boldsymbol{\alpha}^{[2]} \otimes \boldsymbol{\xi}^{[2]} & \text{if } \mathbf{X} \in \Omega_{\mathtt{m}} \\ \mathbf{F}^{(2)} = \overline{\mathbf{F}} - (1 - f^{[2]})\boldsymbol{\alpha}^{[2]} \otimes \boldsymbol{\xi}^{[2]} & \text{if } \mathbf{X} \in \Omega^{[1]} \end{cases}$$

$$\overline{W}^{[2]}(\overline{\mathbf{F}}) = \min_{\boldsymbol{\alpha}^{[2]}}\{(1 - f^{[2]})W_{\mathtt{m}}(\mathbf{F}^{(1)}) + f^{[2]}\overline{W}^{[1]}(\mathbf{F}^{(2)})\}$$

An analytically tractable $\mathcal{H}$: Coated laminates

Rank - 3 laminate



Rank - m laminate

$$\overline{W}^{[m]}(\overline{\mathbf{F}}) = \min_{\substack{\alpha^{[i]} \\ i=1,\dots,m}} \{\dots\} \text{ and then take the limit } m \to +\infty$$

- **Dilute limit of pores** $f_0 \to 0^+$ **does depend on the lamination sequence**
- **Dilute limit of matrix** $f_0 \to 1^-$ **does not**

de Botton, *(2005)* Idiart, *(2008)*

An analytically tractable $\mathcal{H}$: Coated laminates

- The solution is given by

$$\boxed{\overline{W}(\overline{\mathbf{F}}) = W_{\mathtt{p}}(\overline{\mathbf{F}}) + (1-f_0)\,\mathcal{H}(W_{\mathtt{m}}, W_{\mathtt{p}}; \overline{\mathbf{F}})}$$

$$\boxed{\mathcal{H}(W_{\mathtt{m}}, W_{\mathtt{p}}; \overline{\mathbf{F}}) = -W_{\mathtt{p}}(\overline{\mathbf{F}}) - \max_{\boldsymbol{\alpha}(\boldsymbol{\xi})} \int_{|\boldsymbol{\xi}|=1} \left[\boldsymbol{\alpha} \cdot \frac{\partial W_{\mathtt{p}}}{\partial \overline{\mathbf{F}}}\boldsymbol{\xi} - W_{\mathtt{m}}\left(\overline{\mathbf{F}} + \boldsymbol{\alpha} \otimes \boldsymbol{\xi}\right)\right] \nu(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}}$$

$$\boxed{\nu(\boldsymbol{\xi}) = \frac{1}{(2\pi)^N} \int_{\Omega_0} \int_0^{\infty} \frac{p_0^{(\mathtt{ii})}(\mathbf{X}) - f_0^2}{f_0(1-f_0)} \rho^{N-1} e^{i\rho\boldsymbol{\xi}\cdot\mathbf{X}}\, \mathrm{d}\rho\, \mathrm{d}\mathbf{X}}$$

- For an isotropic distribution of pores

$$\nu(\boldsymbol{\xi}) = \frac{1}{4\pi}$$

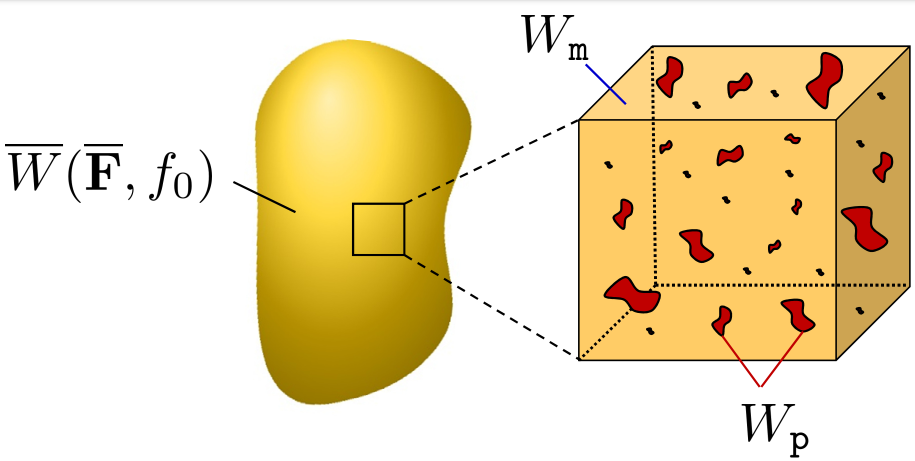
LOPEZ-PAMIES, IDIART, NAKAMURA *(2011)*

Putting things together

$$\begin{cases} \dfrac{\partial \overline{W}}{\partial f} + \mathcal{H}\left(\overline{\mathbf{F}}, f, \overline{W}, \dfrac{\partial \overline{W}}{\partial \overline{\mathbf{F}}}\right) = 0, \quad \overline{\mathbf{F}} \in \mathfrak{S}, \quad f \in \mathfrak{T} \\ \overline{W}(\overline{\mathbf{F}}, 1) = W_{\mathtt{p}}(\overline{\mathbf{F}}), \quad \overline{\mathbf{F}} \in \mathfrak{S} \end{cases}$$

$$\mathfrak{S} = \{\mathbf{A} \in \mathbb{R}^{N\times N} : \det\, \mathbf{A} > 0\},$$
$$\mathfrak{T} = [f_0, 1)$$

$$\mathcal{H}\left(\overline{\mathbf{F}}, f, \overline{W}, \frac{\partial \overline{W}}{\partial \overline{\mathbf{F}}}\right) = -W_{\mathtt{p}}(\overline{\mathbf{F}}) - \max_{\boldsymbol{\alpha}(\boldsymbol{\xi})} \int_{|\boldsymbol{\xi}|=1} \frac{1}{4\pi}\left[\boldsymbol{\alpha}\cdot\frac{\partial W_{\mathtt{p}}}{\partial \overline{\mathbf{F}}}\boldsymbol{\xi} - W_{\mathtt{m}}\left(\overline{\mathbf{F}} + \boldsymbol{\alpha}\otimes\boldsymbol{\xi}\right)\right] \mathrm{d}\boldsymbol{\xi}$$



WENO Solutions for arbitrary finite deformations

- **A fifth order Runge-Kutta scheme that integrates the IVP from $f_0 = 1$ to the desired value of the initial porosity $f_0 \in (0, 1)$**
- **Monotone and consistent Hamiltonian: Crandall and Lions (1983, 1984)**
- **WENO 5^{th}-order: Osher and Sethian (1988), Jiang and Shu (1996)**

Computational Domain:

$$x \doteq \ln(\overline{\lambda}_1/\overline{\lambda}_2) \qquad y \doteq \ln(\overline{\lambda}_2/\overline{\lambda}_3) \qquad z \doteq \ln(\overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3) \qquad t \doteq -\ln(f_0)$$

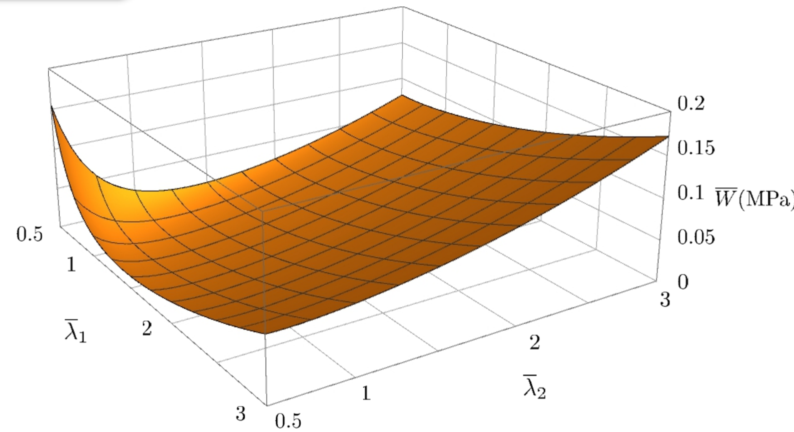
$$f_0 = 0.05 \begin{cases} x & \in (0, 2.1) \\ y & \in (0, 2.1) \\ z & \in (-0.05, 2.1) \\ t & \in (0, 3.0) \end{cases} \qquad f_0 = 0.15 \begin{cases} x & \in (0, 2.1) \\ y & \in (0, 2.1) \\ z & \in (-0.15, 2.1) \\ t & \in (0, 1.9) \end{cases} \qquad f_0 = 0.25 \begin{cases} x & \in (0, 2.1) \\ y & \in (0, 2.1) \\ z & \in (-0.25, 2.1) \\ t & \in (0, 1.4) \end{cases}$$

$$h_x = 0.02, \; h_y = 0.04, \; h_z = 0.01; \qquad \Delta t = \min\{h_x, h_y, h_z\}/10 = 10^{-3}$$

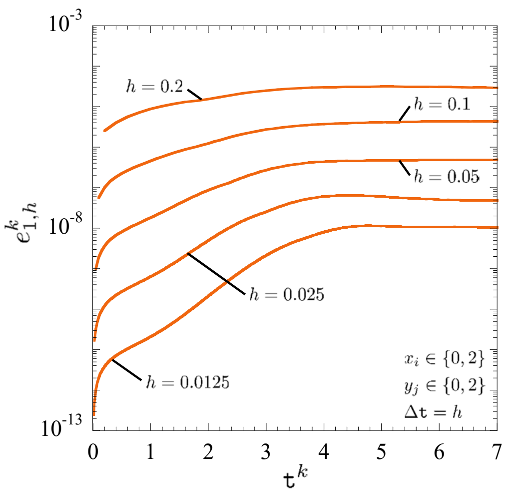
Lefèvre et al., Comp. Meth. Appl. Mech. Engg. *(2019)*

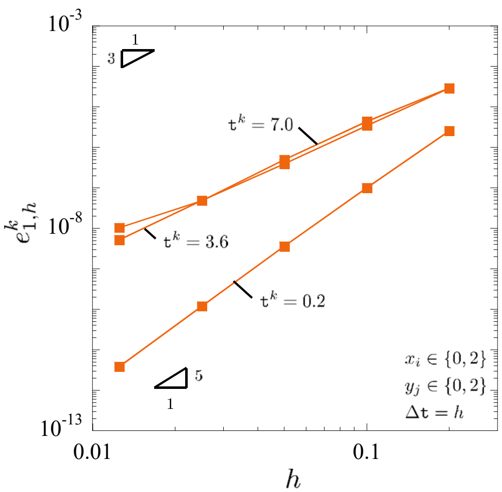
WENO Solutions for arbitrary finite deformations...

$$\overline{W} = \frac{\mu_{\mathtt{m}}}{2}\frac{1-f_0}{1+f_0}\left[\overline{\lambda}_1^2 + \overline{\lambda}_2^2 - 2\right] + \frac{\mu_{\mathtt{m}}}{2}(\overline{\lambda}_1\overline{\lambda}_2 - 1)\left[\ln\left(\frac{\overline{\lambda}_1\overline{\lambda}_2 + f_0 - 1}{f_0\overline{\lambda}_1\overline{\lambda}_2}\right) - 2\frac{1-f_0}{1+f_0}\right]$$



$$e_{1,h}^k = \frac{||W_{i,j}^k - W(x_i, y_j, \mathtt{t}^k)||_1}{||W(x_i, y_j, \mathtt{t}^k)||_1}$$





$$W_{\mathtt{m}}(\mathbf{F}) = \begin{cases} \frac{\mu_{\mathtt{m}}}{2}\left[\mathbf{F}\cdot\mathbf{F} - 2\right], & \det\mathbf{F} = 1 \\ +\infty & \text{otherwise} \end{cases}$$

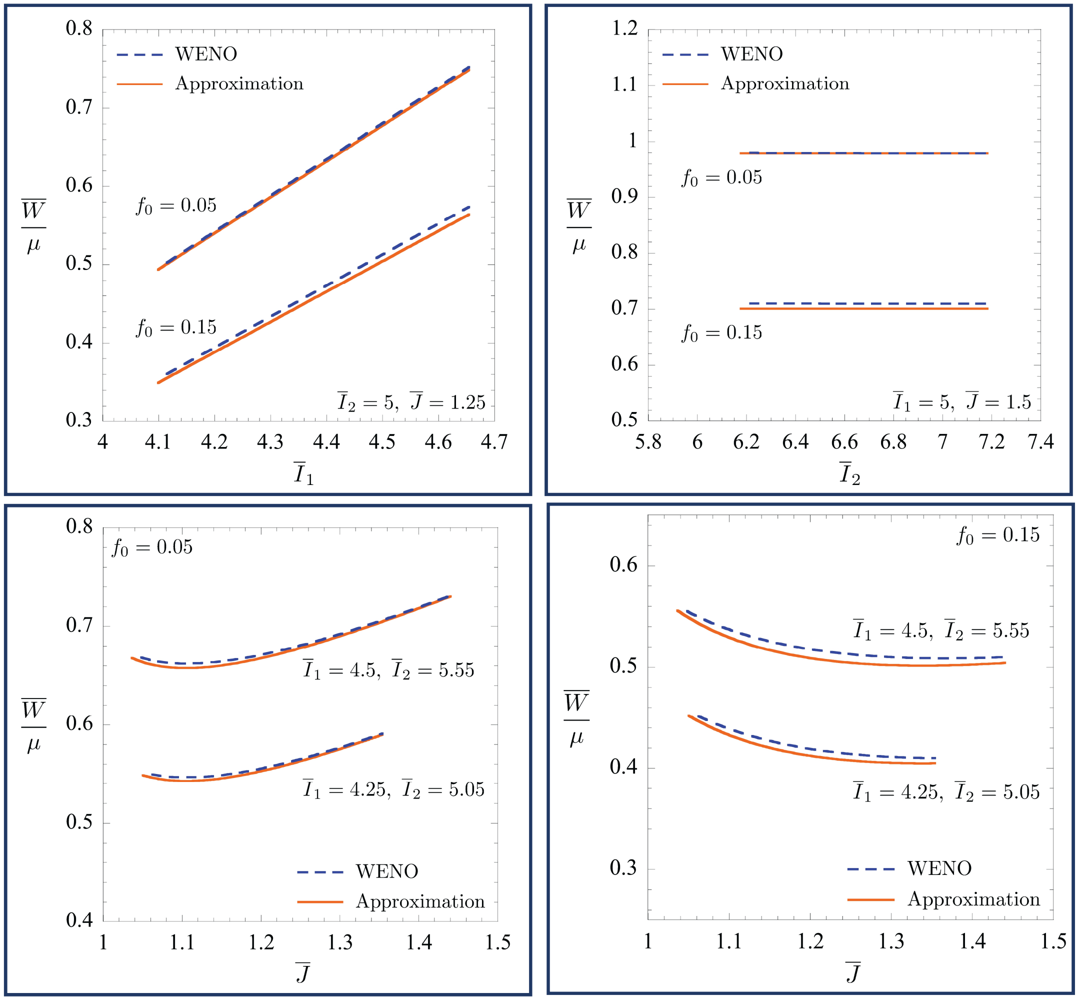
WENO Solutions for arbitrary finite deformations...

- **The solution is roughly linear in** $\overline{I}_1(\overline{\mathbf{F}})$
- **The solution is independent of** $\overline{I}_2(\overline{\mathbf{F}})$
- **The solution is nonlinear in** $\overline{J}(\overline{\mathbf{F}})$
- **Of the form**

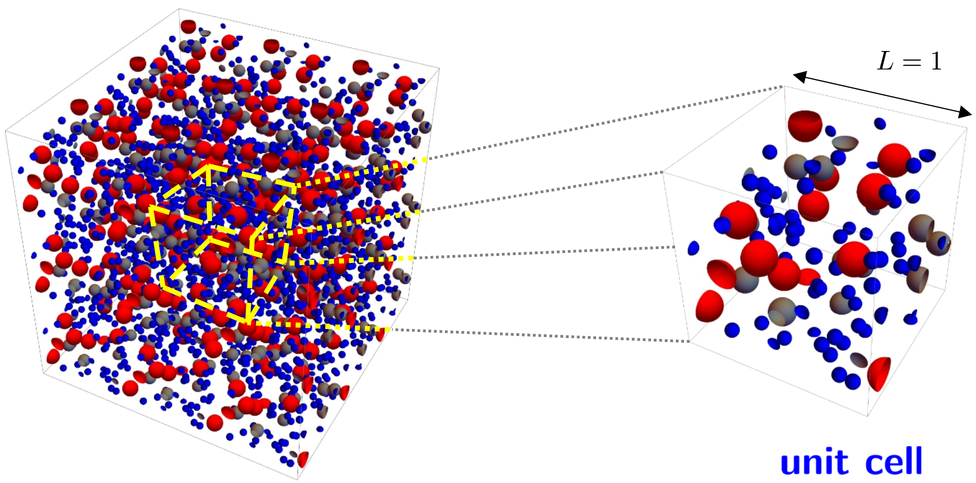
$$\overline{W} = \overline{W}_{\overline{I}_1}\left(\overline{I}_1, f_0\right) + \overline{W}_{\overline{J}}\left(\overline{J}, f_0\right)$$

Recall:

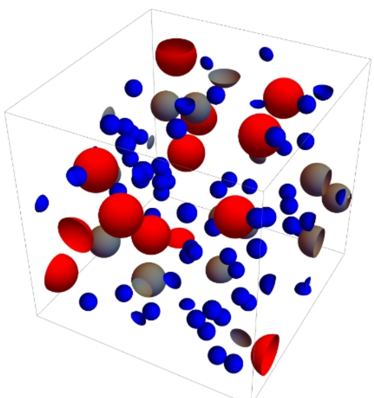
- **Microstructures *Infinitely polydisperse*, hence percolate at** $f_0 = 1$
- **The behavior in the limit of small and infinitely large deformations is exactly as noted above**



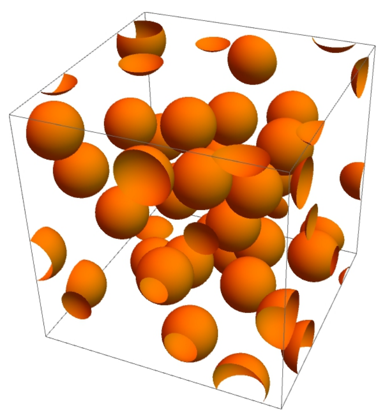
Numerical homogenization: periodic microstructures



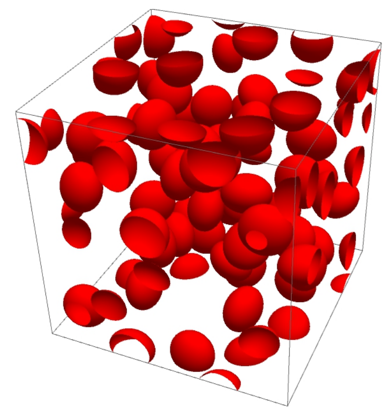
unit cell



Finitely polydisperse spherical pores



Monodisperse spherical pores



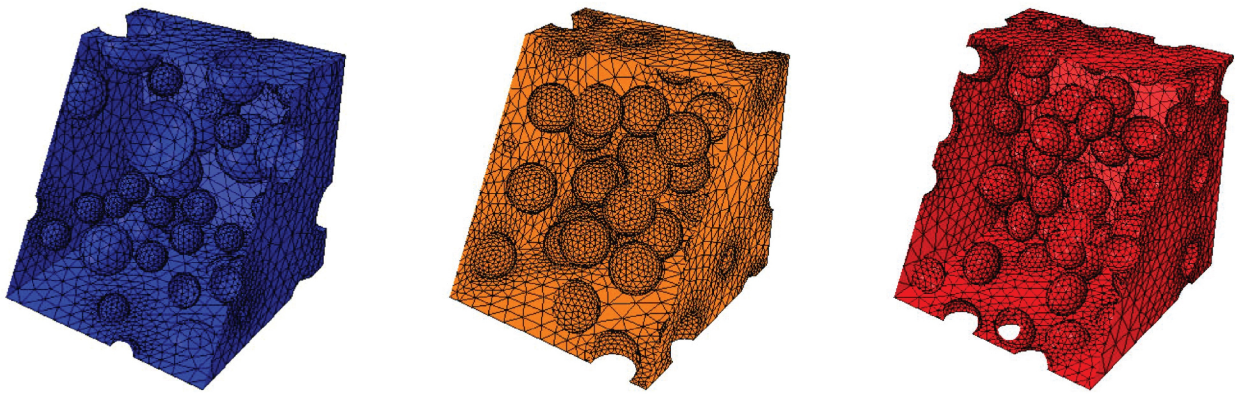
Monodisperse oblate spheroidal pores

Numerical homogenization: periodic microstructures...

- **Hybrid formulation in order to account for the (near) incompressibility of elastomeric matrix**

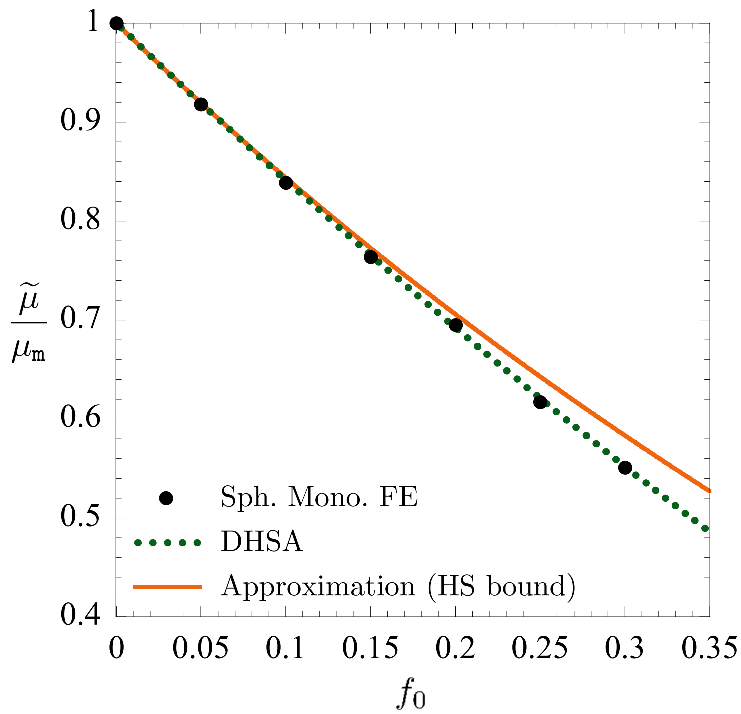
$$\boxed{\overline{W}\left(\overline{\mathbf{F}}, f_0\right) = \min_{\mathbf{u}\in\mathcal{U}} \max_{p\in\mathcal{P}} \int_Y \left\{ p[\det \mathbf{F}(\mathbf{u}) - 1] - \widehat{W}^*(\mathbf{X}, \mathbf{F}(\mathbf{u}), p) \right\} \mathrm{d}\mathbf{X}}$$

- **Conforming hybrid quadratic finite element discretizations**

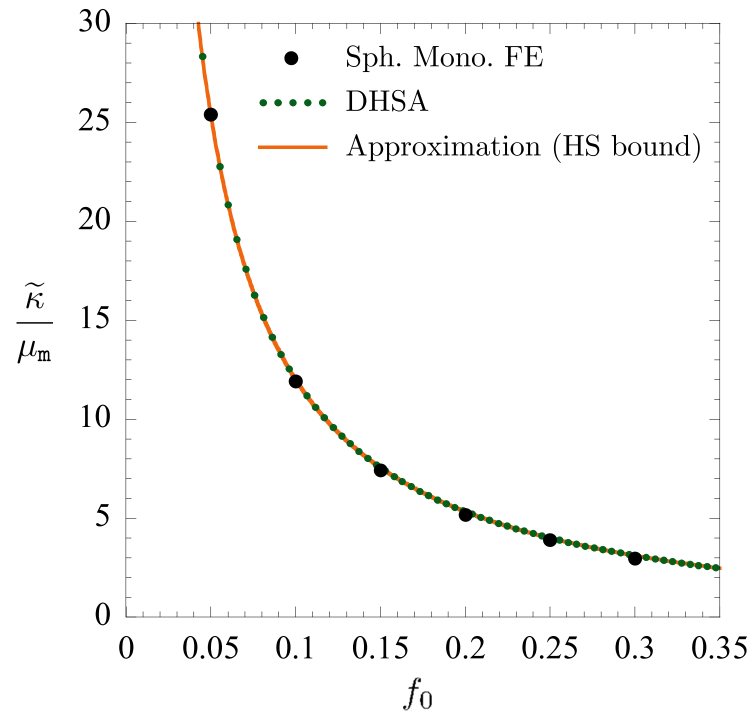


$N_{\mathrm{p}} = 5\,(N = 40)$ polydisperse spherical pores — $(N = 30)$ monodisperse spherical pores — $(N = 60)$ monodisperse spheroidal pores

Numerical homogenization: small deformation limit

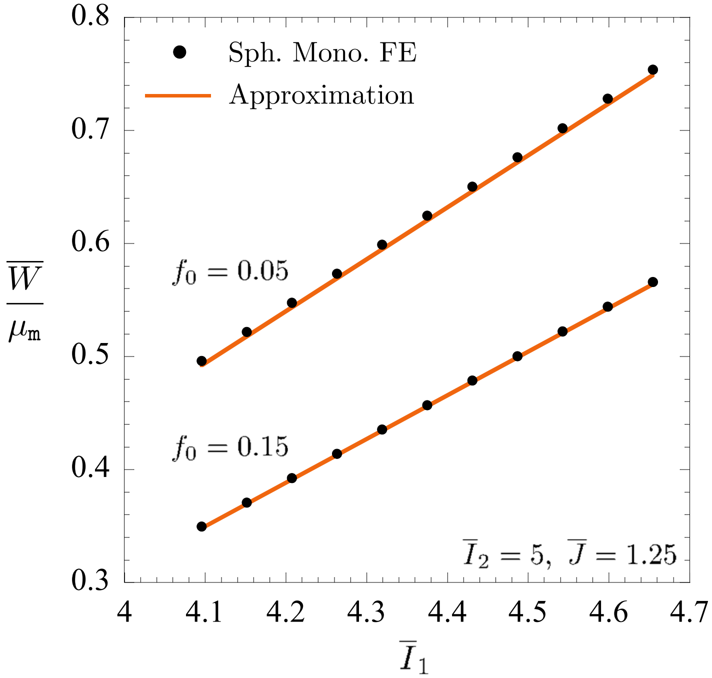


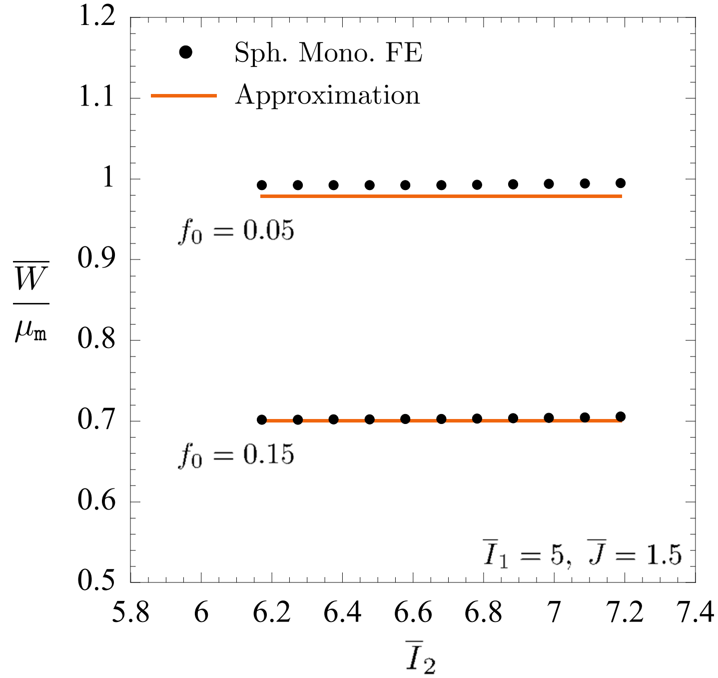
Effective initial shear modulus



Effective initial bulk modulus

Numerical homogenization: sample FE results





- **The solution is roughly linear in $\bar{I}_1(\overline{\mathbf{F}})$**
- **The solution is independent of $\bar{I}_2(\overline{\mathbf{F}})$**

$$\overline{W} = \overline{W}_{\bar{I}_1}\left(\bar{I}_1, f_0\right) + \overline{W}_{\bar{J}}\left(\bar{J}, f_0\right)$$

Porous Neo-Hookean elastomers: Explicit result

$$\overline{W}(\overline{\mathbf{F}}, f_0) = \frac{3(1-f_0)\mu_{\mathtt{m}}}{2\,(3+2f_0)}\left[\overline{I}_1 - 3\right] + \frac{3\mu_{\mathtt{m}}}{2\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1-f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3}+2f_0\right)}{3+2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\left(2\overline{J}+f_0-2\right)}{\left(\overline{J}-1+f_0\right)^{1/3}}\right]$$

Limit of small macroscopic deformations ($\overline{\mathbf{F}} \to \mathbf{I}$)

$$\overline{W}(\overline{\mathbf{F}}, f_0) = \frac{\widetilde{\mu}}{2}\left[\overline{I}_1 - 3\right] - \widetilde{\mu}\left(\overline{J}-1\right) + \frac{1}{2}\left(\widetilde{\kappa} + \frac{\widetilde{\mu}}{3}\right)\left(\overline{J}-1\right)^2 + \mathcal{O}\left(\|\overline{\mathbf{F}} - \mathbf{I}\|^3\right),$$

$$\widetilde{\mu} = \frac{3(1-f_0)}{3+2f_0}\mu\ ; \qquad \widetilde{\kappa} = \frac{4(1-f_0)}{3f_0}\mu.$$

Porous Neo-Hookean elastomers: Asymptotic limits

Isotropic deformations ($\overline{\mathbf{F}} = \overline{J}^{1/3}\mathbf{I}$)

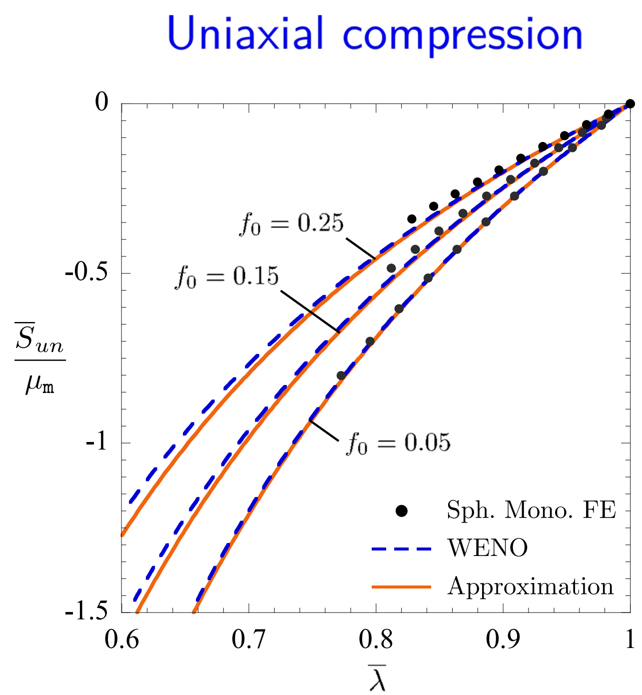
$$\overline{W}\left(\overline{J}^{1/3}\mathbf{I}, f_0\right) = \frac{3\mu}{2}\left[\frac{2\overline{J}-1}{\overline{J}^{1/3}} - \frac{2\overline{J}+f_0-2}{(\overline{J}+f_0-1)^{1/3}} f_0^{1/3} - (1-f_0)\right]$$

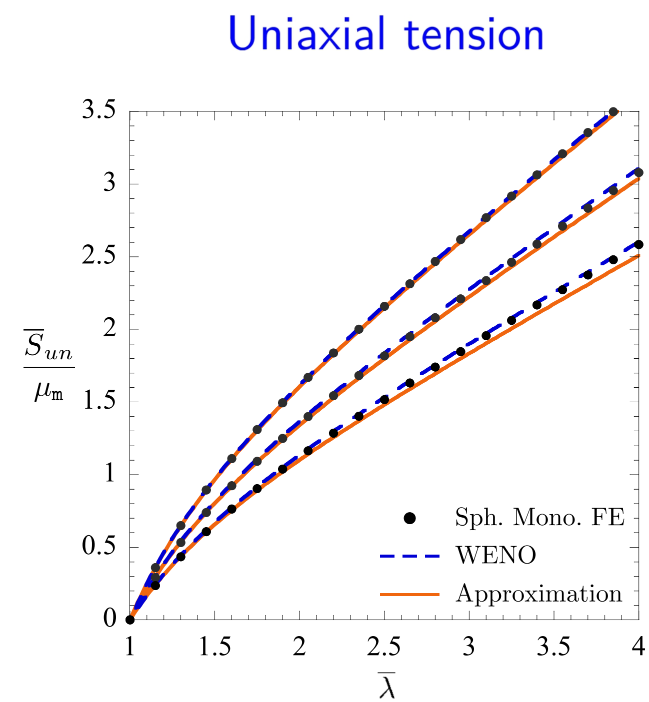
- In spite of corresponding to a solution for a different type of microstructure, the above result agrees identially with the result of Hashin for a porous Neo-Hookean elastomer with the HSA microstructure under hydrostatic loading

Limit of of infinitely large tensile deformations $\|\overline{\mathbf{F}}\| \to +\infty$

$$\overline{W}(\overline{\mathbf{F}}, f_0) = \frac{(1-f_0)\mu}{2+f_0}\,\overline{I}_1 + \frac{3(1+f_0^{1/3})(1-f_0^{1/3})^3\mu}{2+f_0}\,\overline{J}^{2/3}.$$

Porous Neo-Hookean elastomers: Finite deformation results





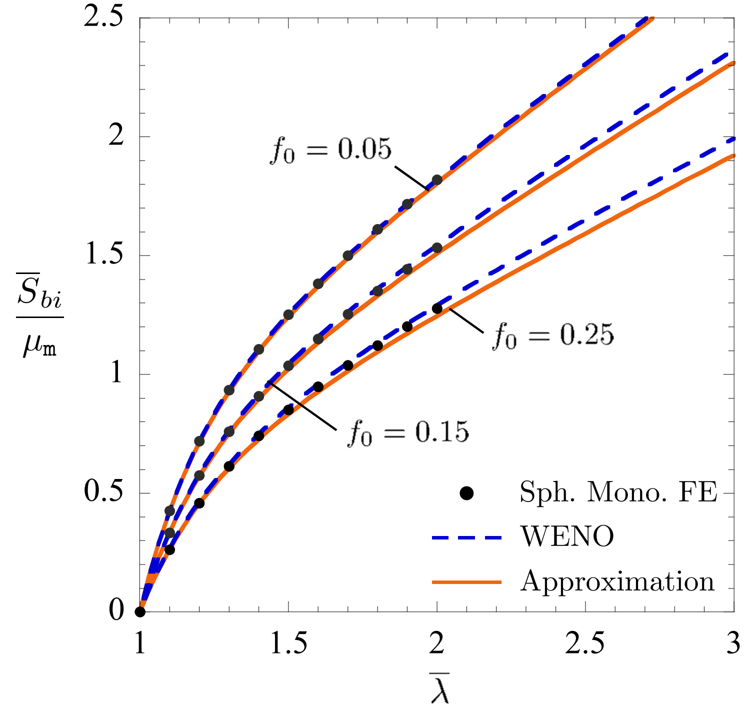
$$\overline{\lambda}_1 = \overline{\lambda},\ \overline{\lambda}_2 = \overline{\lambda}_3 = \overline{\lambda}_l$$

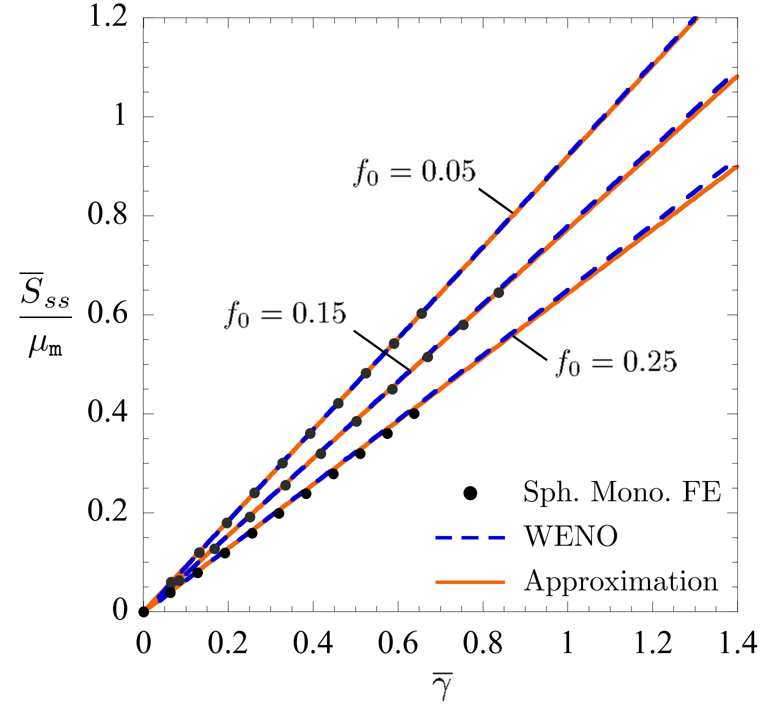
$$\overline{S}_{un} = \frac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}}(\overline{\lambda}, \overline{\lambda}_l, \overline{\lambda}_l, f_0),\ \frac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}_l}(\overline{\lambda}, \overline{\lambda}_l, \overline{\lambda}_l, f_0) = 0$$

Porous Neo-Hookean elastomers: Finite deformation results

Biaxial tension
$$\begin{cases} \overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda}, \ \overline{\lambda}_3 = \overline{\lambda}_l \\ \overline{S}_{bi} = \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}}(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}_l, f_0), \ \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}_l}(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}_l, f_0) = 0 \end{cases}$$

Simple shear
$$\begin{cases} \overline{\lambda}_1 = (\overline{\gamma} + \sqrt{\overline{\gamma}^2 + 4})/2, \ \overline{\lambda}_2 = \overline{\lambda}_1^{-1}, \ \overline{\lambda}_3 = 1 \\ \overline{S}_{ss} = \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\gamma}}(\overline{\lambda}_1, \overline{\lambda}_2, 1, f_0) \end{cases}$$





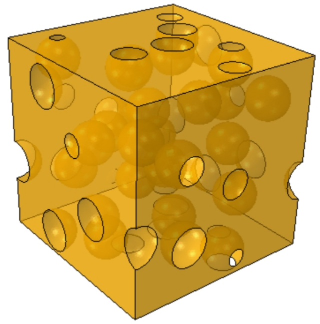
From Neo-Hookean to Non-Gaussian

$$\overline{W}(\overline{\mathbf{F}}, f_0) \doteq \min_{\mathbf{F}\in\mathcal{K}} \frac{1}{|\Omega_0|}\int_{\Omega_0} W\left(\mathbf{X},\mathbf{F}\right)\mathrm{d}\mathbf{X}$$

$$\overline{W} = \min_{\mathbf{F}\in\mathcal{K}} \frac{1}{|\Omega_0|}\int_{\Omega_0} \left(W + W_0 - W_0\right)\mathrm{d}\mathbf{X}$$

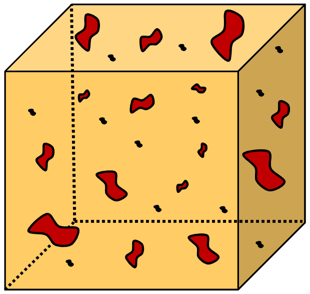
$$\overline{W} \geq \min_{\mathbf{F}\in\mathcal{K}} \frac{1}{|\Omega_0|}\int_{\Omega_0} W_0\mathrm{d}\mathbf{X} + \min_{\mathbf{F}} \frac{1}{|\Omega_0|}\int_{\Omega_0} (W - W_0)\mathrm{d}\mathbf{X}$$

$W\left(\mathbf{X},\mathbf{F}\right)$



Actual porous elastomer

$W_0\left(\mathbf{X},\mathbf{F}\right)$



Comparison medium

From Neo-Hookean to Non-Gaussian...

Porous nonlinear comparison medium with the same microstructure as the actual material

$$W_0(\mathbf{F},\mathbf{X}) = [1-\theta_0(\mathbf{X})]V_{\mathtt{m}}(\mathbf{F}) \qquad \text{with} \qquad V_{\mathtt{m}}(\mathbf{F}) = \begin{cases} \phi_{\mathtt{m}}(I_1) & \text{if} \quad J=1 \\ +\infty & \text{otherwise} \end{cases}$$

$$\overline{W}\left(\overline{\mathbf{F}}, f_0\right) = \begin{cases} \sup\limits_{\phi_{\mathtt{m}}} \left\{ \overline{W}_0(\overline{\mathbf{F}}, f_0) + (1-f_0) \min\limits_{\mathcal{J}_1} [\Psi_{\mathtt{m}}(\mathcal{J}_1) - \phi_{\mathtt{m}}(\mathcal{J}_1)] \right\} & \text{if} \quad \Psi_{\mathtt{m}} - \phi_{\mathtt{m}} > -\infty \\ \inf\limits_{\phi_{\mathtt{m}}} \left\{ \overline{W}_0(\overline{\mathbf{F}}, f_0) + (1-f_0) \max\limits_{\mathcal{J}_1} [\Psi_{\mathtt{m}}(\mathcal{J}_1) - \phi_{\mathtt{m}}(\mathcal{J}_1)] \right\} & \text{if} \quad \Psi_{\mathtt{m}} - \phi_{\mathtt{m}} < \infty \end{cases}$$

$$\overline{W}_0(\overline{\mathbf{F}}, f_0) \doteq \min_{\mathbf{F}\in\mathcal{K}} \frac{1}{|\Omega_0|} \int_{\Omega_0} W_0\left(\mathbf{X},\mathbf{F}\right) \mathrm{d}\mathbf{X}$$

From Neo-Hookean to Non-Gaussian...

Making use of Neo-Hookean porous elastomer for the Comparison medium

$$W_0(\mathbf{F},\mathbf{X}) = [1-\theta_0(\mathbf{X})]V_{\mathtt{m}}(\mathbf{F}) \qquad \text{with} \qquad V_{\mathtt{m}}(\mathbf{F}) = \begin{cases} \dfrac{\mu_0}{2}[I_1 - 3] & \text{if} \quad J = 1 \\ +\infty & \text{otherwise} \end{cases}$$

$$\overline{W}(\overline{\mathbf{F}}, f_0) = (1 - f_0)\,\Psi_{\mathtt{m}}\left(\frac{\mathcal{I}_1}{1 - f_0} + 3\right)$$

$$\mathcal{I}_1 = \frac{3(1 - f_0)}{3 + 2f_0}\left[\overline{I}_1 - 3\right] + \frac{3}{\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1 - f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3} + 2f_0\right)}{3 + 2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\left(2\overline{J} + f_0 - 2\right)}{\left(\overline{J} - 1 + f_0\right)^{1/3}}\right]$$

From Neo-Hookean to Non-Gaussian...

$$\overline{W}(\overline{\mathbf{F}}, f_0) = (1 - f_0)\, \Psi_{\mathtt{m}} \left(\frac{\mathcal{I}_1}{1 - f_0} + 3 \right)$$

$$\mathcal{I}_1 = \frac{3(1-f_0)}{3+2f_0}\left[\overline{I}_1 - 3\right] + \frac{3}{\overline{J}^{1/3}}\left[2\overline{J} - 1 - \frac{(1-f_0)\,\overline{J}^{1/3}\left(3\overline{J}^{2/3} + 2f_0\right)}{3+2f_0} - \frac{f_0^{1/3}\overline{J}^{1/3}\left(2\overline{J} + f_0 - 2\right)}{\left(\overline{J} - 1 + f_0\right)^{1/3}}\right]$$

Remarks:

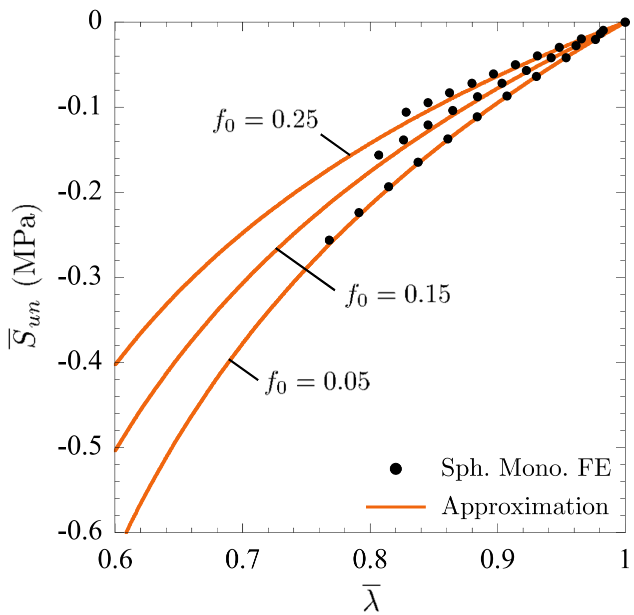
- **Small deformation limit:**
  - Exact for iterative microstructure
  - Accurate approximation for arbitrary microstructures with equi-axed pores

$$f = \frac{\overline{J} - 1}{\overline{J}} + \frac{f_0}{\overline{J}}$$

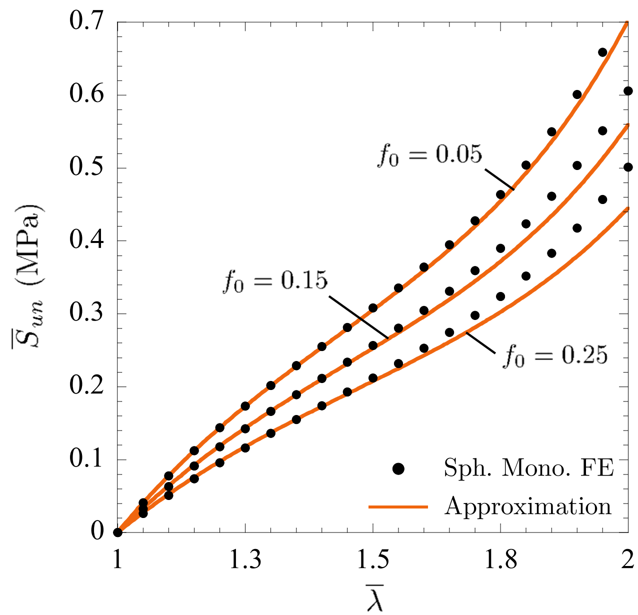
- **Finite deformations:**
  - Independent of $\overline{I}_2$
  - Exact evolution of the porosity
  - Accurate approximation for arbitrary loading conditions

Comparison with full field FE simulations

Uniaxial compression



Uniaxial tension



$$\overline{\lambda}_1 = \overline{\lambda},\ \overline{\lambda}_2 = \overline{\lambda}_3 = \overline{\lambda}_l$$

$$\overline{S}_{un} = \frac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}}(\overline{\lambda}, \overline{\lambda}_l, \overline{\lambda}_l, f_0),\ \ \frac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}_l}(\overline{\lambda}, \overline{\lambda}_l, \overline{\lambda}_l, f_0) = 0$$

Comparison with full field FE simulations...

Biaxial tension
$$\begin{cases} \overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda}, \ \overline{\lambda}_3 = \overline{\lambda}_l \\ \overline{S}_{bi} = \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}}(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}_l, f_0), \ \dfrac{\mathrm{d}\overline{W}}{\mathrm{d}\overline{\lambda}_l}(\overline{\lambda}, \overline{\lambda}, \overline{\lambda}_l, f_0) = 0 \end{cases}$$

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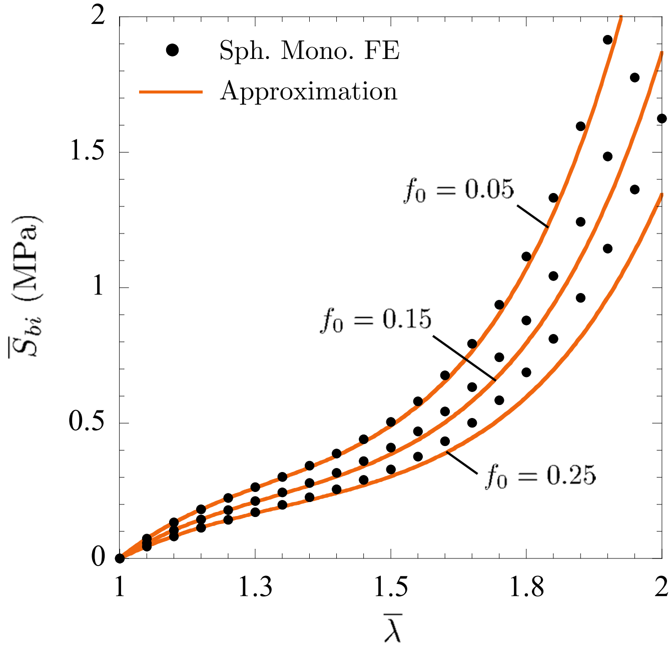

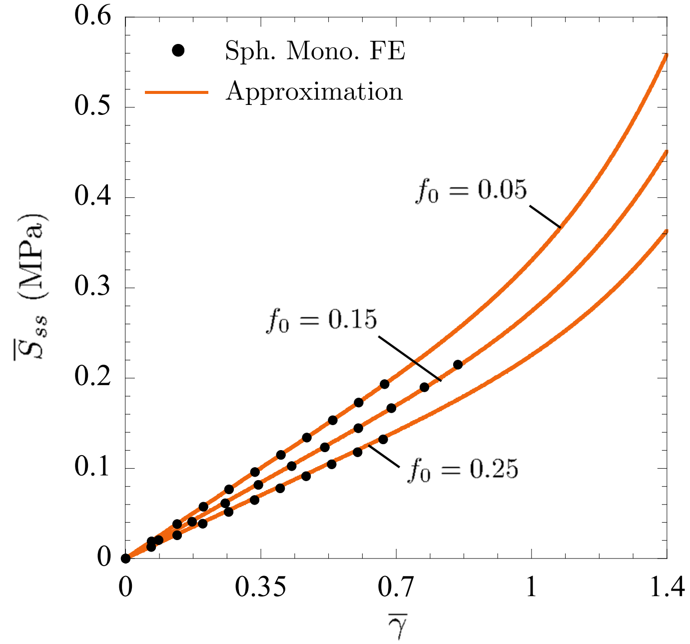

Remarks

Closing comments:

- **An explicit and accurate homogenization solution**
- **Readily implementable in existing FE codes (Abaqus (UHYPER), FEniCS)**

https://github.com/victorlefevre/UHYPER_Shrimali_Lefevre_Lopez-Pamies