CSE 401: Numerical Analysis
HW 5

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Solution 5: ODEs and Stability

Give ODE

$$y' = -5y \quad ; y(0) = 1$$

Here

$$f(t, \mathbf{y}) = -5y$$

(a) Stability of the System

$$J = \frac{\partial f}{\partial y}(t, \mathbf{y}) = -5 < 0$$

Hence the solution of the given ODE are stable.

(b) Stability of the Euler's Scheme

The Euler's method is illustrated via the following iterative scheme

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h_k \ \mathbf{f}(t, \mathbf{y}_k)$$

Plugging in the function in the above equation we get

$$y_{k+1} = y_k + h_k \cdot (-5y_k) \implies y_{k+1} = y_k(1 - 5h_k)$$

Plugging in the initial condition and reformulating the iterative scheme in terms of the initial value of the solution (assuming a constant step size h_0)

$$y_n = y_0 \cdot (1 - 5h_0)^n$$

Thus the Euler's scheme may be stable only if the magnitude of the growth factor is less than 1, i.e.

$$|1 - 5h_0| < 1 \implies -1 < 1 - 5h_0 < 1 \implies 0 < h_0 < \frac{2}{5}$$

Hence the Euler's scheme is stable only if the step size satisfies the following condition

$$\boxed{\mathbf{0 < h_0 < \frac{2}{5}}}$$

Hence in general the Euler's method is **not stable**. It has a limited stability region given above. For this case it is, therefore, $\boxed{\textbf{NOT STABLE}}$ since $\mathbf{h_0 = 0.5 > 0.4}$

(c) Approximate solution at t = 0.5s

Using the step size specified in the problem statement

$$y_1 = y_0 \cdot (1 - 5 \cdot 0.5) = y_0 \cdot (-1.5) = -1.5y_0 = \boxed{\mathbf{-1.5}}$$

(d) Stability of the backward Euler's Method

The iterative scheme for the backward Euler's method (corresponding to $y' = \lambda y$) is given as follows:

$$y_n = \left(\frac{1}{1 - h\lambda}\right)^k y_0$$

Substituting the value of λ and the step size h, we get the growth factor as follows

$$\frac{1}{1 - 0.5 \cdot (-5)} = \frac{1}{3.5} < 1$$

Thus the backward Euler scheme is stable for the given step size.

(e) Approximate solution at t = 0.5s

The approximate solution at $t = 0.5s$ using the backward euler scheme is given as

$$y_1 = \left(\frac{1}{3.5}\right) y_0 = \frac{2}{7}$$

Thus the approximate solution, using the backward euler scheme, at $t = 0.5s$ is given by

$$\boxed{y(t)|_{t=0.5} = \frac{\mathbf{2}}{\mathbf{7}}}$$
