CSE 401: Numerical Analysis
HW 5

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Solution 5: Chebyshev Quadrature

Give that all the weights are equal. Let the weights be denoted by w and the sampled points be denoted by x_1, x_2, x_3. Therefore we have a total of 4 parameters to be determined. Now for three point Chebyshev quadrature, the resulting rule should integrate polynomials upto degree 3 exactly.

Constant Polynomial:

We have

$$\int_{-1}^{1} 1 \; dx = 2 = w\,(1+1+1) \implies w = \frac{2}{3} \tag{1}$$

Linear Polynomial:

We have

$$\int_{-1}^{1} x \; dx = 0 = w\,(x_1 + x_2 + x_3) \implies x_1 + x_2 + x_3 = 0 \tag{2}$$

Quadratic Polynomial:

We have

$$\int_{-1}^{1} x^2 \; dx = \frac{2}{3} = w\left(x_1^2 + x_2^2 + x_3^2\right) \implies x_1^2 + x_2^2 + x_3^2 = 1 \tag{3}$$

Cubic Polynomial:

We have

$$\int_{-1}^{1} x^3 \; dx = 0 = w\left(x_1^3 + x_2^3 + x_3^3\right) \implies x_1^3 + x_2^3 + x_3^3 = 0 \tag{4}$$

In order to analytically solve the above four equations we proceed by making use of the algebraic identity:

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 &= (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1x_3) \\ &\implies x_1^3 + x_2^3 + x_3^3 = 3x_1x_2x_3 \;;\;\; \forall (x_1 + x_2 + x_3) = 0 \end{aligned} \tag{5}$$

Using equation (5) together equation (4) gives

$$x_1x_2x_3 = 0 \;; \tag{6}$$

This implies the following conditions:

- $x_1 = 0 \; ; x_2 \neq 0 \; ; x_3 \neq 0$
- $x_2 = 0 \; ; x_1 \neq 0 \; ; x_2 \neq 0$
- $x_3 = 0 \; ; x_1 \neq 0 \; ; x_2 \neq 0$

or

- $x_1 \; , \; x_2 = 0 \; ; \quad x_3 \neq 0$
- $x_2 \; , \; x_3 = 0 \; ; \quad x_1 \neq 0$
- $x_1 \; , \; x_3 = 0 \; ; \quad x_2 \neq 0$

This leads to the following conclusions:

- Now if any of the above three equations are satisfied then by equation (2) the remaining third variable would also be equal to zero and thus equation (3) would not hold true. Thus the only possibility is that only one of the sample points is zero.
- Note that due to the symmetry of the problem in x_1, x_2 and x_3 we can see that the solution corresponding to the first case, i.e. $x_1 = 0 \; ; x_2 \neq 0 \; ; x_3 \neq 0$ would be equivalent to the other two cases. Therefore we present the solution corresponding to the first case.

$$x_1 = 0 \; ; \quad x_2 \neq 0 \; ; \quad x_3 \neq 0$$

Then by (2) and (3)

$$x_2 = -x_3 \; ; \qquad 2x_3^2 = 1 \implies x_3 = \pm\frac{1}{\sqrt{2}} \; ; \quad x_2 = \mp\frac{1}{\sqrt{2}}$$

Again due to the symmetry of the problem in x_2 and x_3 it is sufficient to consider only one of the above possibilities as the other one is equivalent

$$x_2 = \frac{1}{\sqrt{2}} \; ; \qquad x_3 = -\frac{1}{\sqrt{2}}$$

Thus the solution of the system of nonlinear equations is given by

$$\boxed{w = \frac{2}{3} \; ; \quad x_1 = 0 \; ; \quad x_2 = \frac{1}{\sqrt{2}} \; ; \quad x_3 = -\frac{1}{\sqrt{2}}} \tag{7}$$

Remarks:

Since the above rule is for three points, it can integrate any polynomial up to and including degree 3 exactly. The degree of a quadrature rule, by definition, is the largest integer, $\mathbf{n}$, such that

$$\text{the rule exactly integrates , } \; x^k \; , \forall k \in [0, n]$$

Thus the degree of the resulting rule is three. ($\mathbf{n = 3}$). This can be verified by integrating a higher order polynomial, say degree 4 and verify if the quadrature rule works.

$$\int_{-1}^{1} x^4 \; dx = \frac{2}{5} \; ; \quad \text{but} \quad \frac{2}{3}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{3} \tag{8}$$

Hence the quadrature rule doesn't work. Thus the degree is 3.
