CSE 401: Numerical Analysis
HW 3

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Solution 2:

Given Matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

(a):

The characteristic polynomial of $\mathbf{A}$ is given by:

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} &= 0 \\ \implies (\lambda - 1)^2 - 4 &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \end{aligned}$$

(b):

Roots of the characteristic polynomial are

$$\lambda = -1, 3$$

(c):

The eigenvalues of the matrix $\mathbf{A}$ are the roots of the characteristic polynomial, i.e. $\lambda_1 = -1,\ \lambda_2 = 3$

(d):

The Eigen-vectors of $\mathbf{A}$ are calculated as follows:

· **Corresponding to $\lambda = -1$:**

Note that the eigen-vectors, in this case, have been normalized in the L^∞ norm.

$$(\mathbf{A} + \mathbf{I})\ \mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \mathbf{0}\ ; \quad \text{also} \quad \max_i \phi_i = 1$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix}$$

· **Corresponding to $\lambda = 3$**

$$(\mathbf{A} - 3\mathbf{I})\ \mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{Bmatrix} \phi_3 \\ \phi_4 \end{Bmatrix} = \mathbf{0}\ ; \quad \text{also} \quad \max_i \phi_i = 1$$

$$\begin{Bmatrix} \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.5 \end{Bmatrix}$$

(e):

Given Vector

$$\mathbf{x}_0 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Thus, the first iteration of the power iteration is as follows. Note that in this particular case, we normalize the Eigen-vector obtained below, in the L^∞ norm, and hence

$$\mathbf{A}\mathbf{x}_0 = \mathbf{x}_1 = \begin{Bmatrix} 1 \\ 0.4 \end{Bmatrix}$$

(f):

The power iteration will converge to the eigen-vector corresponding to the dominant eigenvalue of $\mathbf{A}$ ($\lambda = 3$), which is shown as below

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_2/\|\mathbf{x}_2\|_\infty = \frac{1}{2.6}\begin{Bmatrix} 2.6 \\ 1.4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.54 \end{Bmatrix}$$

$$\mathbf{A}\mathbf{x}_2 = \mathbf{x}_3/\|\mathbf{x}_3\|_\infty = \frac{1}{3.16}\begin{Bmatrix} 3.16 \\ 1.54 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.487 \end{Bmatrix}$$

Thus we can easily see that the eigenvector is converging to the one, corresponding to $\lambda = 3$ as obtained in the part (d). Here the procedure of normalization has been carried out, to implement power-iteration, with respect to the L^∞ norm of the vector. The same could be carried out using the L^2 norm as well, just that the L^∞ norm is less expensive to determine.

(g):

Using the Rayleigh quotient iteration:

$$\sigma = \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$
$$\implies \sigma = \frac{7}{2} = 3.5$$

(h):

The inverse iteration would converge to the eigenvector corresponding to the smallest (in absolute value) eigenvalue of $\mathbf{A}$ and hence it would converge to the eigenvector corresponding to $\lambda = -1$.

(i):

Inverse-iteration with shift $= 2$ would return the eigenvalue closest to the shift, i.e. 3.

(j):

Since the matrix is general (non-symmetric), using the Q-R iteration would result in A converging to a triangular matrix.