CSE 401: Numerical Analysis
HW 7

Bhavesh Shrimali

NetID: bshrima2

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Solution 4:

Given PDE

$$u_t = u_{xx} \tag{1}$$

Remarks:

- The first and the second plots correspond to the Explicit Finite-Difference Scheme, i.e., FTCS (Forward Time Centered Space). The curve corresponding to the first and the second cases are as follows:

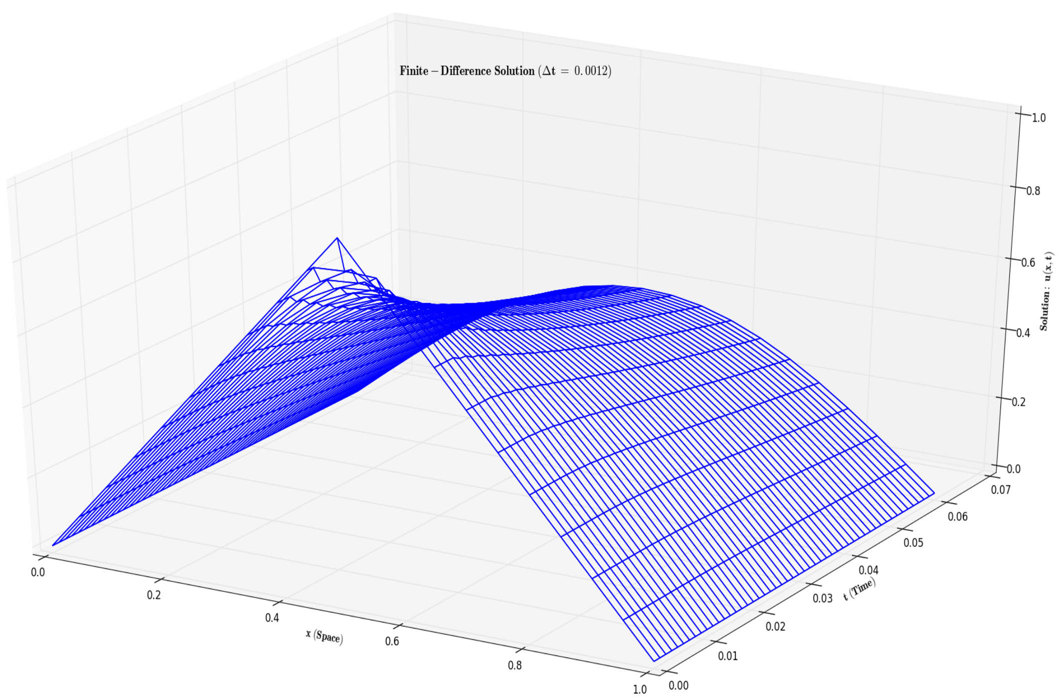

Figure 1: Forward-Time-Centered-Space (FTCS) Method or Euler's method

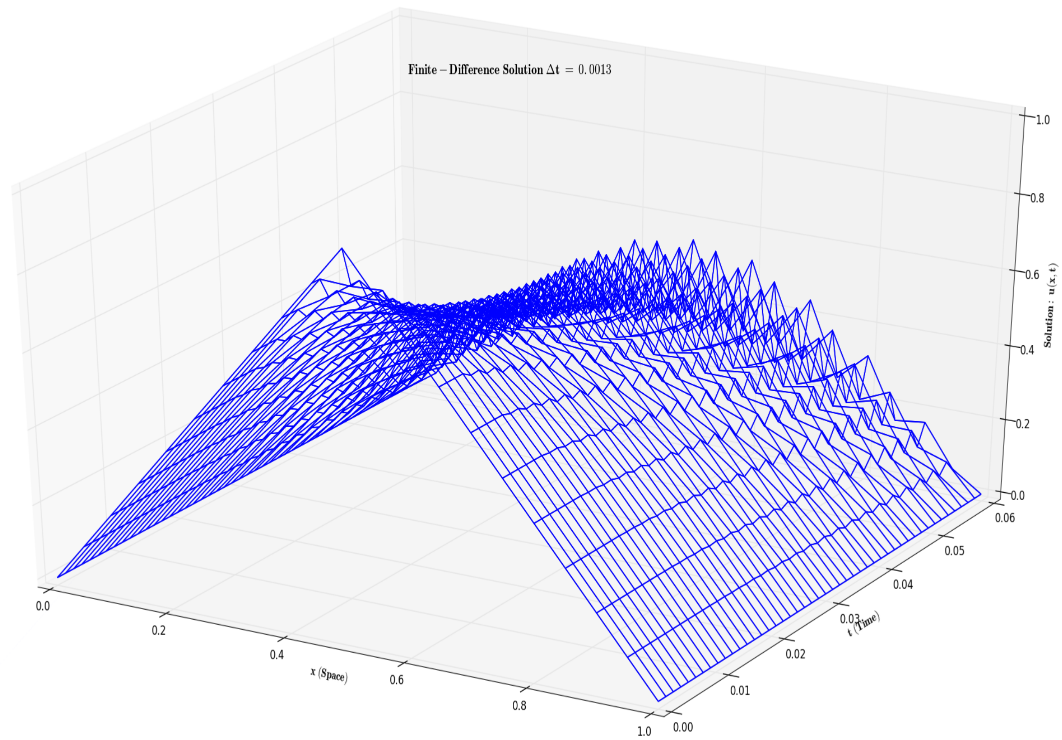

Figure 2: Forward-Time-Centered-Space (FTCS) Discretization - Coarser Step

- The second plot has a coarser step size, i.e. $\Delta t_2 > \Delta t_1$, and for an explicit Finite-Difference-Discretization, **the stability of the solution (which in this case can be shown to be unstable) decreases as we increase the step size**.

- Moreover this is evident by the fluctuations (spikes) towards the end of the time axis. Thus **increasing the step-size decreases the stability of the solution**.

- The stability of the Euler's method is ensured by the inequality

$$\Delta t \leq \frac{(\Delta x)^2}{2c}$$

  But in our case we have

$$\Delta t = 0.0013 \quad ; \quad \frac{(\Delta x)^2}{2c} = 1.25 \text{ x } 10^{-3} \quad ; \quad \Delta t \nleq \frac{(\Delta x)^2}{2c}$$

- Hence the explicit scheme in the second case, $\Delta t = 0.0013$,is not stable whereas the one in the first case, $\Delta t = 0.0012$ is stable.

- The third plot corresponds to an implicit Finite-Difference-Discretization. The implicit method,is unconditionally stable, and hence stays stable even for the coarser step.
- The implicit method involves, for one space dimension, involves **solving a tridiagonal linear system** at each step. Hence, by construction, it is unconditionally stable for coarser (step) sizes in time. This is reflected by the third and fourth plots given below.
- The **Backward Euler Method is first order accurate in time and second order in space.**

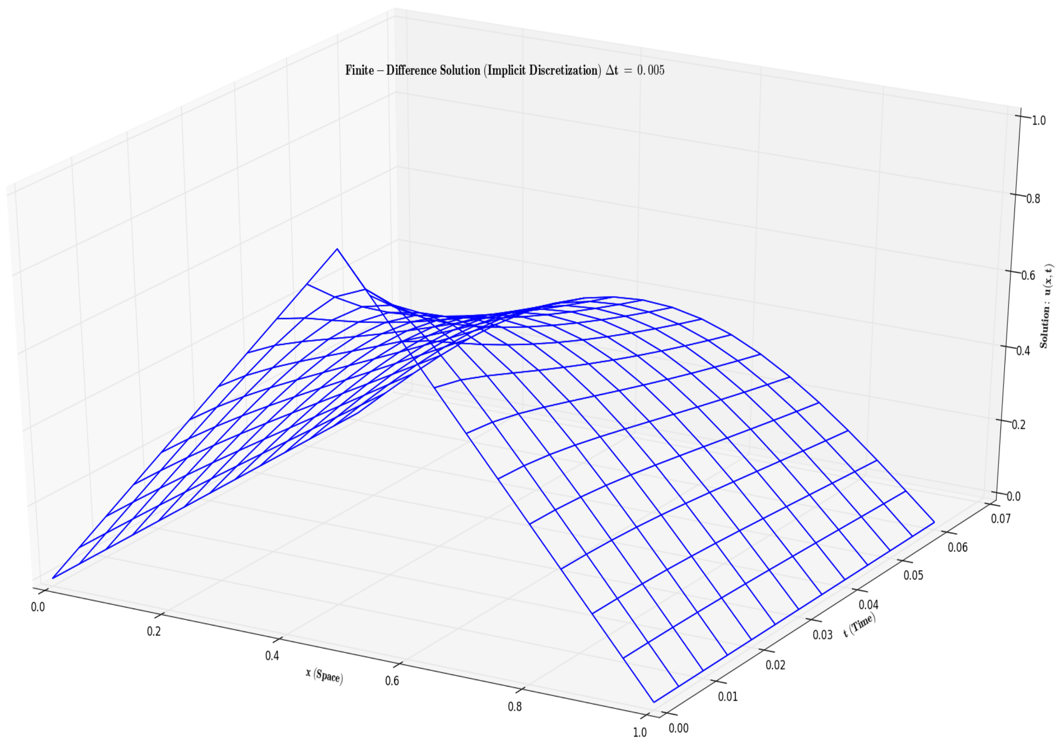

Figure 3: Backward-Time-Centered-Space (BTCS) or Backward Euler (Implicit FDM)

- **The Crank-Nicolson method**

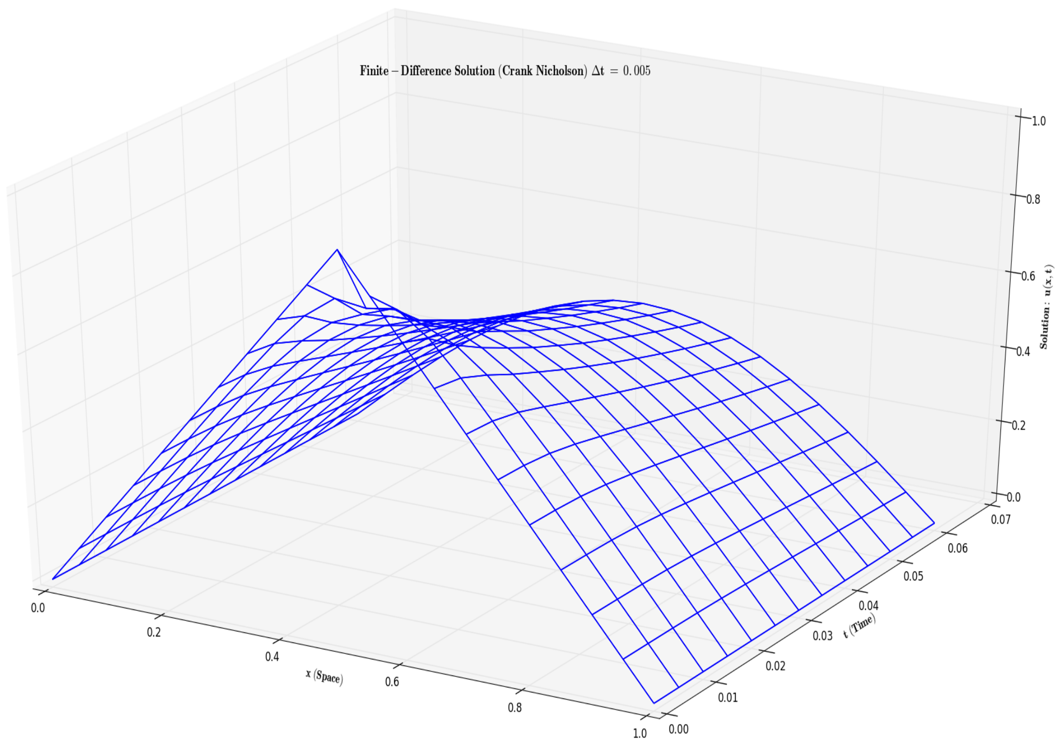

Figure 4: Crank Nicolson Discretization

- The Crank-Nicolson method is more stable than the first two cases (due to its unconditional stability).
- There are small numerical errors, in case of Crank-Nicolson, as can be seen from the plot, (it crosses the z-axis at several places, though very slightly) which make it slightly less accurate than Backward-Euler.
- **The Crank Nicolson method is second order accurate in both time and space, and hence more accurate than Backward Euler**
- The Crank-Nicolson method is a combination of the Forward and Backward Euler methods. Hence, for the coarser step that we have considered, there is a small irregularity (non-smoothness) in the solution in the first step.

- The fifth plot corresponds to the semi-discrete method. The plot differs from the previous plots in the step size employed to solve the system. Since a library routine is used to solve the system of linear ODEs that result from the semi-discrete form, the time step is chosen automatically by the library routine *scipy.integrate.odeint*.
- **The Semi-Discrete method results in a stiff ODE system.**
- The solution obtained from the Semi-Discrete Method is more stable than the explicit (Euler) method, however there is no considerable difference between the performance of the implicit methods (Backward-Euler and Crank-Nicolson).

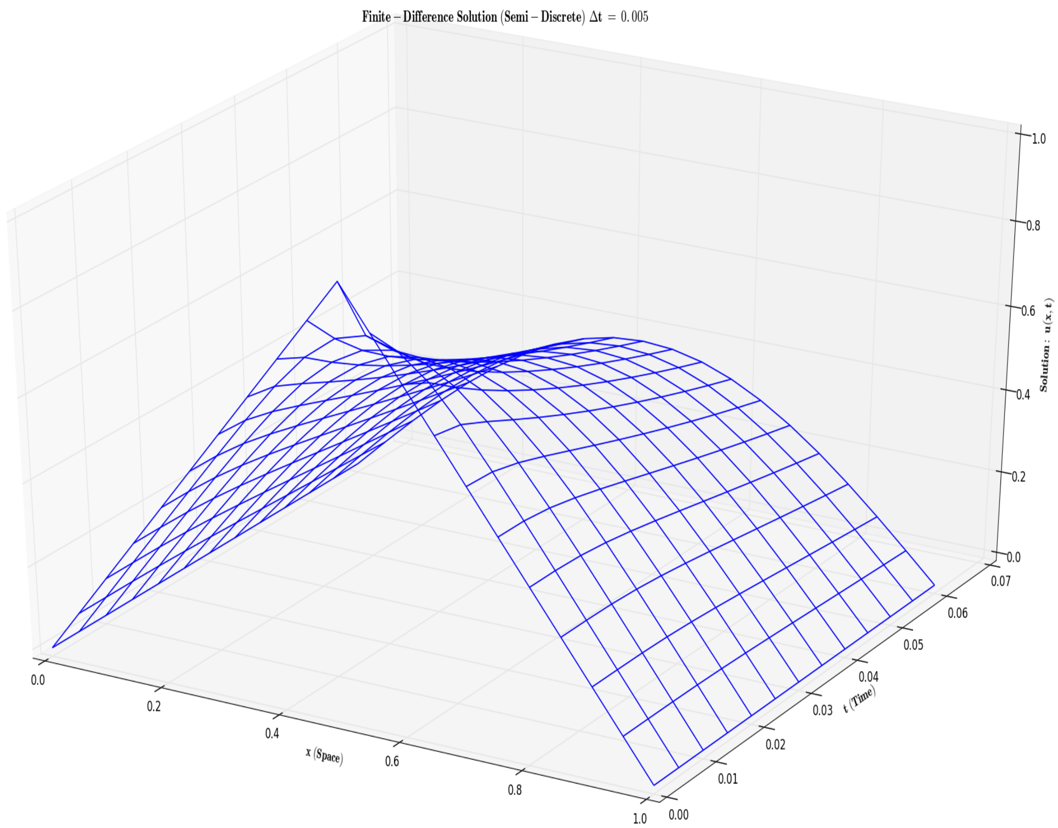

Figure 5: Semi-Discrete Method