CSE 401: Numerical Analysis
HW 4

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Solution 3:

(a):

Given function:

$$f(x,y) = x^2 - 4xy + y^2$$

We compute the critical points of the given function:

$$\boldsymbol{\nabla} f = \begin{Bmatrix} 2x - 4y \\ -4x + 2y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \implies x = 2y \ ; \quad y = 2x \implies x = y = 0$$

Computing the Hessian Matrix at the critical point $(0,0)$

$$\begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} 2-\lambda & -4 \\ -4 & 2-\lambda \end{vmatrix} = 0 \implies \lambda_1 = -2 \ ; \quad \lambda_2 = 6$$

The hessian matrix is non-singular but has both negative and positive eigen-values, thus it is indefinite. **Therefore,** $(0,0)$ **is a saddle point of** $f(x,y)$.

Comments on Global Maximum and Minimum:

We can infer that

- The given function does not have any global maxima or minima. In other words the function is **non-coercive**
- This can be observed by taking the direction $x = -y$, as along this direction the function shoots off to ∞ as $||y|| \to \infty$ and similarly shoots off, along $x = y$, to $-\infty$ as $||y|| \to \infty$

(b):

Given function:

$$f(x,y) = x^4 - 4xy + y^4$$

We compute the critical points of the given function:

$$\boldsymbol{\nabla} f = \begin{Bmatrix} 4x^3 - 4y \\ -4x + 4y^3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \implies x^3 = y \ ; \quad y^3 = x \implies (x,y) = \big[\{0,0\}\,, \quad \{1,1\}\,, \quad \{-1,-1\}\big]$$

Computing the Hessian Matrix $\mathbf{H}(x,y)$

$$\mathbf{H}(x,y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

Now we evaluate the Hessian Matrix at all the critical points.

· **H(0,0)**

$$\mathbf{H}(0,0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} -\lambda & -4 \\ -4 & -\lambda \end{vmatrix} = 0 \implies \lambda_1 = -4 \ ; \quad \lambda_2 = 4$$

The hessian matrix is non-singular but has both negative and positive eigen-values, thus it is indefinite. **Therefore,** $(0,0)$ **is a saddle point of** $f(x,y)$**.**

· **H(1,1) and H(-1,-1)**

$$\mathbf{H}(1,1) = \mathbf{H}(-1,-1) \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} 12-\lambda & -4 \\ -4 & 12-\lambda \end{vmatrix} = 0 \implies \lambda_1 = 8 \ ; \quad \lambda_2 = 16$$

The hessian matrix is non-singular and has only positive eigen-values, thus it is positive-definite. **Therefore,** $(1,1)$ **and** $(-1,-1)$ **are local minima of** $f(x,y)$ **at least. We will check now if they are global minima or not**

Comments on Global Maximum and Minimum:

We can infer that

- The given function has two global minima. This is because the function is **coercive**. The leading order term $x^4 + y^4$ always dominates the third term and shoots off to ∞ as $||x,y|| \to (\infty)$ or $-\infty$, and hence the function has two global minima and no global maxima.
- $f(1,1) = -2$ and $f(-1,-1) = -2$, thus both are global minima. Note that it is not possible to find any global maxima.
- We can see this by taking the direction $x = -y$, which shoots off to ∞ as $||x|| \to \infty$

(c):

Given function:

$$f(x,y) = 2x^3 - 3x^2 - 6xy(x-y-1)$$

We compute the critical points of the given function:

$$\boldsymbol{\nabla} f = \begin{Bmatrix} 6y(y-x+1) - 6xy - 6x + 6x^2 \\ 6xy + 6x(y-x+1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \implies (x,y) = \left[\{0,0\}, \quad \{0,-1\}, \quad \{-1,-1\}, \{1,0\}\right]$$

Computing the Hessian Matrix $\mathbf{H}(x,y)$

$$\mathbf{H}(x,y) = \begin{bmatrix} 12x - 12y - 6 & 12y - 12x + 6 \\ 12y - 12x + 6 & 12x \end{bmatrix}$$

Now we evaluate the Hessian Matrix at all the critical points.

· **H(0,0)**

$$\mathbf{H}(0,0) = \begin{bmatrix} -6 & 6 \\ 6 & 0 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} -6-\lambda & 6 \\ 6 & -\lambda \end{vmatrix} = 0 \implies \lambda_1 = -9.7082 \; ; \quad \lambda_2 = 3.7082$$

The hessian matrix is non-singular but has both negative and positive eigen-values, thus it is indefinite. **Therefore,** $(0,0)$ **is a saddle point of** $f(x,y)$.

· **H(-1,-1)**

$$\mathbf{H}(-1,-1) = \begin{bmatrix} -6 & 6 \\ 6 & -12 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} -6-\lambda & 6 \\ 6 & -12-\lambda \end{vmatrix} = 0 \implies \lambda_1 = -15.7082 \; ; \quad \lambda_2 = -2.2918$$

The hessian matrix is non-singular and has only negative eigen-values, thus it is negative-definite. **Therefore,** $(-1,-1)$ **is a local maximum of** $f(x,y)$**.**

· **H(0,-1)**

$$\mathbf{H}(0,-1) = \begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} 6-\lambda & -6 \\ -6 & -\lambda \end{vmatrix} = 0 \implies \lambda_1 = -3.7082 \; ; \quad \lambda_2 = 9.7082$$

The hessian matrix is non-singular and has both positive and negative eigen-values, thus it is indefinite. **Therefore,** $(0,-1)$ **is a saddle point of** $f(x,y)$**.**

· **H(1,0)**

$$\mathbf{H}(1,0) = \begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} 6-\lambda & -6 \\ -6 & 12-\lambda \end{vmatrix} = 0 \implies \lambda_1 = 2.2918 \; ; \quad \lambda_2 = 15.7082$$

The hessian matrix is non-singular and has positive eigen-values, thus it is positive-definite. **Therefore,** $(1,0)$ **is a local minimum of** $f(x,y)$**.**

Comments on Global Maximum and Minimum:

We can infer that

- The given function has two saddle points and one each — a local maximum and local minimum.
- Let us observe the function along the curve $y = x - 1$. Along this curve $f(x, y(x)) = 2x^3 - 3x^2$. This curve, at $x \to \infty$, shoots off to ∞ and, at $x \to -\infty$, shoots off to $-\infty$. Therefore there are no global maxima/minima. In other words the function is **non-coercive**

(d):

Given function:

$$f(x,y) = (x-y)^4 + x^2 - y^2 - 2x + 2y + 1$$

We compute the critical points of the given function:

$$\boldsymbol{\nabla} f = \begin{Bmatrix} 2x + 4(x-y)^3 - 2 \\ 2 - 4(x-y)^3 - 2y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \implies (x,y) = [\{1,1\}]$$

Computing the Hessian Matrix $\mathbf{H}(x,y)$

$$\mathbf{H}(x,y) = \begin{bmatrix} 12(x-y)^2 + 2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 - 2 \end{bmatrix}$$

Now we evaluate the Hessian Matrix at all the critical points.

· **H(1,1)**

$$\mathbf{H}(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Computing the eigenvalues of the hessian matrix

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0 \implies \lambda_1 = -2 \; ; \quad \lambda_2 = 2$$

The hessian matrix is non-singular but has both negative and positive eigen-values, thus it is indefinite. **Therefore,** $(1,1)$ **is a saddle point of** $f(x,y)$**.**

Comments on Global Maximum and Minimum:

We can infer that

- The given function has one saddle point.
- Since the leading term in the function is $(x-y)^4$, the function has no global maximum. This is illustrated via the following figure. As we can see that the function is increasing towards the bottom left corner $y \to \infty$.
- We can also consider the function along $x = y$, here $f(x, y(x)) = 1$ and thus $f(x, y)$ has no global minimizer. In other words the function is **non-coercive**.