MATH 553: Partial Differential Equations
Midterm Exam

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Problem: 1

Given quasi-linear equation:

$$u_t + u^{2106} u_x = 0 \tag{1}$$

subject to initial conditions

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

First, the characteristic equations are written for a general boundary condition ($u(x,0) = f(x)$)

$$\frac{\mathrm{d}x}{ds} = u^{2016} \ ; \quad \frac{\mathrm{d}t}{ds} = 1 \ ; \quad \frac{\mathrm{d}u}{ds} = 0 \ ; \quad \text{subject to} \quad \begin{cases} x(0) = \alpha \\ t(0) = 0 \\ u(0) = f(\alpha) \end{cases}$$

For the above ODEs the solution is given by (Note that k, c_1 and c_2 are constants)

$$u(s) = k \ ; \quad x(s) = k^{2016} s + c_1 \quad t(s) = s + c_2$$

Upon imposing the initial conditions for x and t

$$u(s) = k \ ; \quad x(s) = k^{2016} s + \alpha \ ; \quad t(s) = s$$

Now we consider each of the domains separately

- For $-\infty < x < 0$ we have the following. Note that the characteristics in this particular case are vertical lines

$$u(s) = 0 \ ; \quad x = \alpha \ ; \quad \text{or} \quad \boxed{u(x,t) = 0}$$

- For $0 \leq x \leq 1$ we have the following. Note that the characteristics in this particular case are inclined lines of slope 1.

$$u(s) = 1 \ ; \quad x = s + \alpha \ ; \quad \text{or} \quad \boxed{u(x,t) = 1}$$

- For $1 \leq x < \infty$ we have the following. Note that the characteristics in this particular case are vertical lines.

$$u(s) = 0 \ ; \quad x = \alpha \ ; \quad \text{or} \quad \boxed{u(x,t) = 0}$$

Now in order to fill the wedge between the first two regions, we make use of the fan solution given by

$$u(x,t) = G\left(\frac{x}{t}\right)$$

Substituting it in the differential equation (1) we get

$$-\frac{x}{t^2}G'\left(\frac{x}{t}\right) + G^{2016}\left(\frac{x}{t}\right)\frac{1}{t}G'\left(\frac{x}{t}\right) = 0 \implies G = \left(\frac{x}{t}\right)^{\frac{1}{2016}}$$

Additionally, for the transition between the second and the third region there is a jump discontinuity. To this end, we consider this to be an example of a conservation law, i.e.

$$u_t + (F(u))_x = 0 \quad \text{where} \quad F(u) = \frac{1}{2017}u^{2017}$$

The velocity of propagation of the shock is given by the Rankine-Hugoniot condition

$$\frac{\mathrm{d}x_s}{dt} = \frac{1}{2017}\frac{u_r^{2017} - u_l^{2017}}{u_r - u_l} = \frac{1}{2017}\left(\sum_{p=0}^{2016} u_l^p u_r^{2016-p}\right)$$

Substituting $u_r = 0$ and $u_l = 1$ we get

$$\frac{\mathrm{d}x_s}{dt} = \frac{1}{2017} \ ; \quad \text{or} \quad t = 2017(x-1) \quad \forall 1 \leq x \leq x_a$$

An important thing to note in this regard is that the shock is traveling much slowly as compared to the fan solution. And hence we can say that

$$\exists t_a \geq 0 \quad \text{such that} \quad 2017(x-1) = x \implies x_a = \frac{2017}{2016} \ ; \quad t_a = 2017(x_a - 1) = \frac{2017}{2016}$$

The expanding fan solution will catch up with the shock, at the above time. After the rarefaction catches up with the shock front, in principle there has to be another shock front the velocity of which is given by the Rankine-Hugoniot Condition:

$$\frac{\mathrm{d}x}{dt} = \frac{1}{2017}\frac{0 - u_l^{2017}}{0 - u_l} = \frac{1}{2017}(u_l)^{2016} = \frac{1}{2017}\left((x/t)^{\frac{1}{2016}}\right)^{2016} = \frac{1}{2017}\frac{x}{t}$$

Hence we have

$$\begin{aligned} \frac{\mathrm{d}x}{dt} &= \frac{1}{2017}\frac{x}{t} \\ \int\frac{dx}{x} &= \int\frac{1}{2017}\frac{dt}{t} + c \\ \implies x = \left(\frac{2017}{2016}\right)^{\frac{2016}{2017}} &\cdot t^{\frac{1}{2017}} \end{aligned}$$

To summarize, we have 1 rarefaction and two shockfronts. The first shockfront forms as a result of the jump condition at x=1 between $u = 1$ and $u = 0$ and the second shockfront forms as a result of a jump condition between the fan solution and $u = 0$ Therefore we can re-write the solution of the above equation in two intervals

$$\text{For} \quad 0 \leq t < \frac{2017}{2016}\ ; \quad u(x,t) = \begin{cases} 0 & -\infty < x < 0 \\ (x/t)^{1/2016} & 0 < x < t \\ 1 & t < x < 1 + \frac{t}{2017} \\ 0 & 1 + \frac{t}{2017} < x < \infty \end{cases}$$

and

$$\text{For} \quad \frac{2017}{2016} \leq t \ < \infty\ ; \quad u(x,t) = \begin{cases} 0 & -\infty < x < 0 \\ (x/t)^{1/2016} & 0 < x < \left(\frac{2017}{2016}\right)^{\frac{2016}{2017}} \cdot t^{\frac{1}{2017}} \\ 0 & \left(\frac{2017}{2016}\right)^{\frac{2016}{2017}} \cdot t^{\frac{1}{2017}} < x < \infty \end{cases}$$

To verify we check if the solution satisfies the conservation law

- For the first interval

$$\begin{aligned} \int_{-\infty}^{\infty} u(y,t)\ dy &= \int_0^t u(y,t)\ dy + \int_t^{1+t/2017} u(y,t)\ dy \\ &= \int_0^t \left(\frac{y}{t}\right)^{\frac{1}{2016}} dy + \int_t^{1+t/2017} 1 \cdot dy \\ &= \frac{2016}{2017 t^{\frac{1}{2016}}} t^{\frac{2017}{2016}} + 1 + \frac{t}{2017} - t \\ &= \frac{2016}{2017} t^{\left(\frac{2017}{2016} - \frac{1}{2016}\right)} + 1 + \frac{t}{2017} - t \\ &= \frac{2016}{2017} t + 1 + \frac{t}{2017} - t \\ &= 1 + t - t = 1 \qquad \boxed{\textbf{Satisfies!}} \end{aligned}$$

- For the second interval

$$\begin{aligned} \int_{-\infty}^{\infty} u(y,t)\ dy &= \int_0^{\left(\frac{2017}{2016}\right)^{\frac{2016}{2017}} \cdot t^{\frac{1}{2017}}} u(y,t)\ dy \\ &= \int_0^{\left(\frac{2017}{2016}\right)^{\frac{2016}{2017}} \cdot t^{\frac{1}{2017}}} \left(\frac{y}{t}\right)^{\frac{1}{2016}} dy \\ &= \frac{2016}{2017\ t^{\frac{1}{2016}}} \left[\left(\frac{2017}{2016}\right)^{\frac{2016}{2017}} \cdot t^{\frac{1}{2017}} \right]^{\frac{2017}{2016}} \\ &= \frac{2016}{2017} \frac{2017}{2016} t^{\left(-\frac{1}{2016} + \frac{1}{2016}\right)} = 1 \qquad \boxed{\textbf{Satisfies!}} \end{aligned}$$

Problem 2:

Given wave equation

$$u_{tt} - u_{xx} = 0 \; ; \qquad \text{subject to} \qquad u(x,0) = g(x) \; ; \quad u_t(x,0) = 0$$

The general solution can be written using the Duhamel's principle

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\; d\xi$$

(i)

For the given case $h(x) = 0$, thus

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t))$$

The problem is given in the infinite domain, i.e., $-\infty < x, t < \infty$. Now for the function $g(x)$, the transformation $x \to x - t$ and $x \to x + t$ just shifts the origin by an amount t rightward and leftward respectively ($t > 0$). Therefore it is appropriate to say that the L^∞ norm of the function $g(x+t)$ can never be greater than the L^∞ norm of the function $g(x)$ for each t.

$$\|g(x+t)\|_\infty \ngtr \|g(x)\|_\infty \; ; \qquad \|g(x-t)\|_\infty \ngtr \|g(x)\|_\infty$$

Thus

$$\|u(x,t)\|_\infty \le \frac{1}{2}\|g(x+t)\|_\infty + \frac{1}{2}\|g(x-t)\|_\infty \implies \|u(x,t)\|_\infty \le \frac{1}{2}\|g(x)\|_\infty + \frac{1}{2}\|g(x)\|_\infty$$

and hence

$$\boxed{\|u(x,t)\|_\infty \le \|g(x)\|_\infty}$$

Thus the above statement holds true.

(ii)

For the given case $g(x) = 0$

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\; d\xi$$

Take $h(x) = k$ (constant)

$$u(x,t) = \frac{k}{2} \cdot 2t = kt > \|g(x)\|_\infty = 0$$

That is, it is possible to have a solution that is nonzero $\forall t \neq 0$. Hence it does not satisfy the maximum principle. In particular, it is also possible to provide other counter examples such as

$$h(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

For this example, we will have again

$$u(x,t) = t > \|g(x)\|_\infty = 0 \ \forall t > 0 \quad \text{if} \begin{cases} x + t > 1 \\ x - t < -1 \end{cases}$$

or

$$u(x,t) = t \quad \text{if} \begin{cases} x > 1 - t \quad \textbf{and} \\ x < t - 1 \end{cases}$$

The attached graph depicts the region in which the above holds, and therefore the maximum principle does not hold. It is sufficient to provide just one region in which the maximum principle is violated, and hence I just provide the above region.

Problem 3:

Given nonlinear PDE

$$u_x + u_y + u_x u_y = 0 \quad ; \quad u(x,0) = x$$

Here

$$\mathcal{F} = p + q + pq$$

The corresponding characteristic equations are given by

$$\frac{\mathrm{d}x}{dt} = 1 + q \; ; \quad \frac{\mathrm{d}y}{dt} = 1 + p \; ; \quad \frac{\mathrm{d}z}{dt} = p + q + 2pq = pq$$
$$\frac{\mathrm{d}p}{dt} = 0 \; ; \quad \frac{\mathrm{d}q}{dt} = 0$$

We need a total of 5 initial conditions to close the problem. Let

$$p(0) = \phi(\alpha) \; ; \quad q(0) = \psi(\alpha)$$

then

$$x(0) = \alpha \; ; \quad y(0) = 0 \; ; \quad z(0) = f(\alpha) = \alpha \; ; \quad \phi(\alpha) + \psi(\alpha) + \phi(\alpha)\psi(\alpha) = 0$$
$$f'(\alpha) = 1 = 1 \cdot \phi(\alpha) + 0 = \phi(\alpha) \implies \phi(\alpha) = 1 \implies \psi(\alpha) = -\frac{1}{2}$$

Integrating the characteristic equations, we get

$$p = 1 \; ; \quad q = -\frac{1}{2} \; ; \quad x = \frac{t}{2} + \alpha \; ; \quad y = 2t \; ; \quad z = -\frac{t}{2} + \alpha$$

Using the last three equations from above to eliminate t and α we get,

$$\alpha = x - \frac{t}{2} \; ; \quad z = x - t$$

Thus the solution of the given PDE is

$$\boxed{u(x,y) = x - \frac{1}{2}y}$$

Problem 4:

Given heat equation

$$u_t - u_{xx} = 0 \; ; \qquad u(x,0) = g(x)$$
$$\begin{Bmatrix} u(0,t) \\ u_x(0,t) \end{Bmatrix} = \begin{Bmatrix} u(1,t) \\ u_x(1,t) \end{Bmatrix}$$

Since the boundary conditions are periodic, we make periodic extension of the initial condition $g(x)$, as follows

$$\tilde{g}(x) = g(x - \lfloor x \rfloor)$$

and the general solution can then be given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \; \tilde{g}(y) \; dy$$

For converting the integral back to the original domain $(0,1)$ let us consider a general domain going from k to $k+1$ where $k \in \mathbb{Z}$. Once we are able to obtain a closed form solution for the following integral, we can simply sum the integral to get our desired result.

$$\begin{aligned} \int_k^{k+1} e^{-(x-y)^2/4t} \; \tilde{g}(y) \; dy &= \int_k^{k+1} e^{-(x-y)^2/4t} \; g(y - \lfloor y \rfloor) \; dy \\ &= \int_k^{k+1} e^{-(x-y)^2/4t} \; g(y-k) \; dy \; ; \qquad (\textbf{Let} \quad y \to y-k) \\ &= \int_0^1 e^{-(x-(y+k))^2/4t} \; g(y) \; dy \end{aligned}$$

Now that we have transformed the integral back to the original domain, we can simply sum the above process from $k = -\infty$ to ∞, thus the solution to the above one dimensional heat equation with periodic boundary conditions is given by

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^1 \tilde{K}(x,y,t) g(y) \; dy \quad ; \quad \tilde{K}(x,y,t) = \sum_{k=-\infty}^{\infty} e^{-(x-(y+k))^2/4t}}$$

Extra Credit:

To this end, we consider the fourier series expansion of the solution. Using separation of variables, we have

$$u(x,t) = X(x)T(t)$$

substituting it back in the original differential equation gives back.

$$X(x)T'(t) - X''(x)T(t) = 0 \implies \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

since the left-hand side of the equation is only dependent on x, and the right hand side only dependent on t, the only possible way is that both of them are constants.

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda \; ; \quad \lambda > 0$$

Thus we get two equations

$$X''(x) + \lambda X(x) = 0$$
$$T'(t) + \lambda T(t) = 0$$

Using the periodic boundary conditions and substituting $\lambda = n^2\pi^2$, we solve the above two equations classically to get

$$X_n(x) = a_n \cos(n\pi x) + b_n \sin(n\pi x) \ ; \qquad n = 0, 1, 2, 3, ...$$
$$T_n(t) = e^{-n^2\pi^2 t} \ ; \qquad n = 0, 1, 2, 3, ...$$

Thus the solution looks like

$$u(x,t) = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t}(a_n \cos(n\pi x) + b_n \sin(n\pi x))$$
$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$
$$u(x,0) = g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

Now, using the orthogonality of the trigonometric functions we have

$$\int_0^1 g(x)\cos(m\pi x)\ dx = \int_0^1 a_0 \cos(m\pi x)\ dx + \int_0^1 \left(\sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)\right) \cos(m\pi x) dx$$

The first term and last term on the R.H.S vanish and the second term vanishes for all values of $m \neq n$

$$a_n = 2\int_0^1 g(x)\cos(n\pi x)\ dx$$

and similarly

$$b_n = 2\int_0^1 g(x)\sin(n\pi x)\ dx$$

Note that $b_{-p} = -b_p$ and $a_{-p} = a_p$. Substituting $n = -p$ in the above series and adding the two obtained series we get

$$2u(x,t) = a_0 + \sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}(a_n \cos(n\pi x) + b_n \sin(n\pi x))$$
$$u(x,t) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}(a_n \cos(n\pi x) + b_n \sin(n\pi x))$$

Now equating the Fourier series expansion and the solution formula

$$\frac{1}{\sqrt{4\pi t}}\int_0^1 \sum_{k=-\infty}^{\infty} e^{-(x-(y+k))^2/4t} g(y)\ dy = \frac{a_0}{2} + \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}(a_n \cos(n\pi x) + b_n \sin(n\pi x))$$

Thus we finally get

$$\sum_{n=-\infty}^{\infty}\left(\int_0^1 \frac{1}{\sqrt{4\pi t}} e^{-(x-(y+n))^2/4t} g(y)\ dy - e^{-n^2\pi^2 t}(\tilde{a_n}\cos(n\pi x) + \tilde{b_n}\sin(n\pi x))\right) = 0 \qquad (2)$$

where

$$\tilde{a_n} = \int_0^1 g(x)\cos(n\pi x)\ dx$$
$$\tilde{b_n} = \int_0^1 g(x)\sin(n\pi x)\ dx$$

Since the series sums to zero for all values of n, all the terms must be identically equal to zero. Substituting the above expressions in the equation (2)

$$\sum_{n=-\infty}^{\infty}\left(\int_0^1 \frac{1}{\sqrt{4\pi t}}e^{-(x-(y+n))^2/4t}g(y)\ dy - e^{-n^2\pi^2 t}\left(\int_0^1 g(y)\cos(n\pi y)\ dy\cos(n\pi x) + \int_0^1 g(y)\sin(n\pi y)\ dy\sin(n\pi x)\right)\right)$$

Taking $g(y)$ common factor and simplifying we get

$$\frac{1}{\sqrt{4\pi t}}e^{-(x-(y+n))^2/4t} - e^{-n^2\pi^2 t}\cos(n\pi(x-y)) = 0$$

Thus we get

$$\boxed{\frac{1}{\sqrt{4\pi t}}e^{-(x-(y+n))^2/4t} = e^{-n^2\pi^2 t}\cos(n\pi(x-y))\ \forall n \in \mathbb{Z}}$$