SOLUTIONS TO JANUARY-2020, SET-1 QP

Q1. (a) Test the convergence of the series $\sum \sqrt[3]{(n^3+1)} - n$.

Answer :

Jan.-2020, Set-1, Q1(a) M[8]

Given series is,

$$\sum \sqrt[3]{(n^3+1)} - n$$

Let, $u_n = \sqrt[3]{(n^3+1)} - n$

$$\Rightarrow \quad u_n = (n^3+1)^{\frac{1}{3}} - n$$

$$\Rightarrow \quad u_n = \left(n^3\left(1+\frac{1}{n^3}\right)\right)^{\frac{1}{3}} - n$$

$$\Rightarrow \quad u_n = (n^3)^{\frac{1}{3}}\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}} - n$$

$$\Rightarrow \quad u_n = n\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}} - n$$

$$\Rightarrow \quad u_n = n\left(1+\frac{1}{3}\left(\frac{1}{n^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{1}{n^3}\right)^3+\ldots\ldots\right)-n$$

$$\Rightarrow \quad u_n = n\left(1+\frac{1}{3n^3}+\left(\frac{-1}{9}\right)\frac{1}{n^6}+\frac{5}{81}\frac{1}{n^9}+\ldots\ldots\right)-n$$

$$\Rightarrow \quad u_n = \left(n+\frac{1}{3n^2}-\frac{1}{9n^5}+\frac{5}{81n^8}+\ldots\ldots\right)-n$$

$$\Rightarrow \quad u_n = \frac{1}{3n^2}-\frac{1}{9n^5}+\frac{5}{81n^8}\ldots\ldots$$

$$\Rightarrow \quad u_n = \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9n^3}+\frac{5}{81n^6}-\ldots\right]$$

$$v_n = \frac{\text{highest power of } n \text{ in numerator}}{\text{highest power of } n \text{ in denominator}}$$

$$\Rightarrow \quad v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9n^3}+\frac{5}{81n^6}\ldots\right]}{\frac{1}{n^2}}$$

$$= \lim_{\frac{1}{n}\to 0}\left(\frac{1}{3}-\frac{1}{9n^3}+\frac{5}{81n^6}\ldots\right)$$

$$= \frac{1}{3}-\frac{1}{9}(0)+\frac{5}{81}(0)\ldots\ldots$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{3} \neq 0$$

By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge together.

By P-series test, $\sum v_n = \sum \frac{1}{n^2}$ is convergent

$\therefore \quad \sum u_n$ is also convergent

Hence, the series $\sum \sqrt[3]{(n^3+1)} - n$ is convergent.

$$\underset{n\to\infty}{lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{lt}\frac{(3n+1)^2}{(3n+3)^2}$$

$$= \underset{n\to\infty}{lt}\frac{\left(n\left(3+\frac{1}{n}\right)\right)^2}{\left(n\left(3+\frac{3}{n}\right)\right)^2}$$

$$= \underset{n\to\infty}{lt}\frac{n^2\left(3+\frac{1}{n}\right)^2}{n^2\left(3+\frac{3}{n}\right)^2}$$

$$= \underset{\frac{1}{n}\to 0}{lt}\frac{\left(3+\frac{1}{n}\right)^2}{\left(3+\frac{3}{n}\right)^2}$$

$$= \frac{(3+0)^2}{(3+0)^2}$$

$$= \frac{(3)^2}{(3)^2} = 1$$

$\because \underset{n\to\infty}{lt}\frac{u_{n+1}}{u_n} = 1$, D-Alemberts ratio test fails

Consider,

$$\frac{u_n}{u_{n+1}} = \frac{(3n+3)^2}{(3n+1)^2}$$

Substracting '1' on both sides,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(3n+3)^2}{(3n+1)^2} - 1$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{(3n+3)^2-(3n+1)^2}{(3n+1)^2}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{9n^2+9+18n-9n^2-1-6n}{9n^2+6n+1}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{8+12n}{9n^2+6n+1}$$

Multiplying 'n' on both sides,

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{8+12n}{9n^2+6n+1}\right]$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{8n+12n^2}{9n^2+6n+1}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n^2\left(12+\frac{8}{n}\right)}{n^2\left(9+\frac{6}{n}+\frac{1}{n^2}\right)}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{12+\frac{8}{n}}{9+\frac{6}{n}+\frac{1}{n^2}}$$

$$\underset{n\to\infty}{lt}\, n\left[\frac{u_n}{u_{n+1}} - 1\right] = \underset{n\to\infty}{lt}\frac{12+\frac{8}{n}}{9+\frac{6}{n}+\frac{1}{n^2}}$$

$$= \underset{\frac{1}{n}\to 0}{lt}\frac{12+\frac{8}{n}}{9+\frac{6}{n}+\left(\frac{1}{n}\right)^2}$$

$$= \frac{12+8(0)}{9+6(0)+(0)^2}$$

$$= \frac{12+0}{9+0+0}$$

$$= \frac{12}{9} = \frac{4}{3}$$

$$\Rightarrow \underset{n\to\infty}{lt}\, n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{4}{3} > 1$$

By Raabe's test, the series $\sum u_n$ converges

Hence the series $\sum \frac{1^2.4^2.7^2....}{3^2.6^2.9^2....}$ is convergent.

(b) If f(x) = e^x and g(x) = e^{-x} in [a, b] then show that c is the average of a and b using Cauchy's mean value theorem.

Answer : **Jan.-2020, Set-1, Q2(b) M[7]**

For answer refer Unit-1, Q63.

Q3. (a) Solve the D.E r sinθ dθ + (r^3 – 2r^2 cosθ + cosθ)dr = 0.

Answer : **Jan.-2020, Set-1, Q3(a) M[8]**

Given differential equation is,

$$r\sin\theta\, d\theta + (r^3 - 2r^2\cos\theta + \cos\theta)dr = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$M\, d\theta + N\, dr = 0 \quad \text{... (2)}$$

Comparing equations (1) and (2),

$M = r\sin\theta$	$N = r^3 - 2r^2\cos\theta + \cos\theta$
$\frac{\partial M}{\partial r} = (1)\sin\theta$	$\frac{\partial N}{\partial \theta} = 0 - 2r^2(-\sin\theta) - \sin\theta$
$\Rightarrow \frac{\partial M}{\partial r} = \sin\theta$	$\Rightarrow \frac{\partial N}{\partial \theta} = 2r^2\sin\theta - \sin\theta$

$\because \frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, equation (1) is not an exact differential equation.

Substituting $t = x$ in equation (3),

$$P(x) = Ce^{rt}$$

$$\Rightarrow \quad 3P_0 = P_0 e^{x\left(\frac{\log 2}{50}\right)} \qquad [\because \text{ From equations (4), (5) \& (6)}]$$

$$\Rightarrow \quad 3 = e^{\frac{x\log 2}{50}}$$

$$\Rightarrow \quad \log 3 = \frac{x\log 2}{50}$$

$$\Rightarrow \quad x = 50\left(\frac{\log 3}{\log 2}\right)$$

$$\Rightarrow \quad x = 79.24$$

$$\Rightarrow \quad x \approx 79 \text{ years}$$

$\therefore$ The time required for the number of population to triple is 79 years approximately.

OR

Q4. (a) Solve the D.E

$$y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0.$$

Jan.-2020, Set-1, Q4(a) M[8

Answer :

Given differential equation is,

$$y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0 \quad \text{...(1}$$

Equation (1) is of the form,

$$Mdx + Ndy = 0 \quad \text{... (2}$$

Comparing equations (1) and (2),

$$M = y(x^2y^2 + 2);\ N = x(2 - 2x^2y^2)$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2y^3 + 2y) = 3x^2y^2 + 2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x - 2x^3y^2) = 2 - 6x^2y^2$$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation(1) is not an exact differential equation.

Integrating factor $(I.F) = \dfrac{1}{Mx - Ny}$

$$\Rightarrow \quad I.F = \frac{1}{y[x^2y^2 + 2]x - x[2 - 2x^2y^2]y}$$

$$\Rightarrow \quad I.F = \frac{1}{y^3x^3 + 2xy - 2xy + 2x^3y^3}$$

$$\Rightarrow \quad I.F = \frac{1}{3x^3y^3}$$

Multiplying equation (1) by $I.F$,

$$y[x^2y^2 + 2].\frac{1}{3x^3y^3}dx + x(2 - 2x^2y^2).\frac{1}{3x^3y^3}dy = 0$$

$$\Rightarrow \quad \left(\frac{x^2y^2 + 2}{3x^3y^2}\right)dx + \left(\frac{2 - 2x^2y^2}{3x^2y^3}\right)dy = 0 \quad \text{...(3}$$

Equation (1) is a linear differential equation of the form,

$$\frac{di}{dt} + Pi = Q \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$P = \frac{R}{L} \text{ and } Q = \frac{E}{L}\cos wt$$

Integrating factor $= e^{\int P dt}$

$$\Rightarrow \quad I.F = e^{\int \frac{R}{L} dt}$$

$$\Rightarrow \quad I.F = e^{\frac{Rt}{L}}$$

The general solution of equation (1) is given as,

$$i(I.F) = \int Q(I.F)\,dt + C$$

$$\Rightarrow \quad ie^{\frac{Rt}{L}} = \int \frac{E}{L}\cos wt\, e^{\frac{Rt}{L}} dt + C$$

$$\Rightarrow \quad ie^{\frac{Rt}{L}} = \frac{E}{L}\int e^{\frac{Rt}{L}} \cos wt\, dt + C$$

$$\Rightarrow \quad ie^{\frac{Rt}{L}} = \frac{E}{L}\left[\frac{e^{\frac{Rt}{L}}}{\left(\frac{R}{L}\right)^2 + w^2}\left(\frac{R}{L}\cos wt + w\sin wt\right)\right] + C \qquad \left[\because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$\Rightarrow \quad ie^{\frac{Rt}{L}} = \frac{E}{L}\left[\frac{e^{\frac{Rt}{L}}.L^2}{R^2 + w^2L^2}\left(\frac{R\cos wt + wL\sin wt}{L}\right)\right] + C$$

$$\Rightarrow \quad ie^{\frac{Rt}{L}} = \frac{E.e^{\frac{Rt}{L}}.L^2}{L^2(R^2 + w^2L^2)}(R\cos wt + w\sin wt) + C$$

$$\Rightarrow \quad ie^{\frac{Rt}{L}} = \frac{E.e^{\frac{Rt}{L}}}{R^2 + w^2L^2}(R\cos wt + w\sin wt) + C$$

$$\Rightarrow \quad ie^{\frac{Rt}{L}} = e^{\frac{Rt}{L}}\left[\frac{E}{R^2 + w^2L^2}(R\cos wt + w\sin wt) + ce^{\frac{-Rt}{L}}\right]$$

$$\Rightarrow \quad i = \frac{E}{R^2 + w^2L^2}(R\cos wt + w\sin wt) + Ce^{\frac{-Rt}{L}} \quad \text{... (3)}$$

Substituting $t = 0$ and $i = 0$ in equation (3),

$$0 = \frac{E}{R^2 + w^2L^2}(R\cos w(0) + w\sin w(0)) + Ce^{\frac{-R(0)}{L}}$$

$$\Rightarrow \quad 0 = \frac{E}{R^2 + w^2L^2}(R(1) + w(0)) + C(1)$$

$$\Rightarrow \quad 0 = \frac{E}{R^2 + w^2L^2}(R + 0) + C$$

$$\Rightarrow \quad 0 = \frac{ER}{R^2 + w^2L^2} + C$$

$$\Rightarrow \quad C = \frac{-ER}{R^2 + w^2L^2}$$

$\Rightarrow \quad P.I = P.I_1 - P.I_2 + P.I_3$ ... (5)

Consider,

$$P.I_1 = \frac{1}{2}\left(\frac{1}{D^2+1}\cos x\right)$$

Here X is of the form $\cos ax$,

Let, $D^2 = -a^2 = -1$

$$P.I_1 = \frac{1}{2}\left(\frac{1}{-1+1}\cos x\right)$$

$$\because \quad f(-a^2) = 0, P.I_1 = \frac{1}{2}\left(\frac{x}{2(1)}\sin x\right)$$

$$\Rightarrow \quad P.I_1 = \frac{x}{4}\sin x$$

Consider,

$$P.I_2 = \frac{1}{2}\left(\frac{1}{D^2+1}\cos 3x\right)$$

Here X is of the form $\cos ax$,

Let, $D^2 = -a^2 = -9$

$$P.I_2 = \frac{1}{2}\left(\frac{1}{-9+1}\cos 3x\right)$$

$$\Rightarrow \quad P.I_2 = \frac{1}{2}\left(\frac{-1}{8}\cos 3x\right)$$

$$\Rightarrow \quad P.I_2 = \frac{-1}{16}\cos 3x$$

Consider,

$$P.I_3 = \frac{1}{D^2+1}[e^x x^2]$$

$$= \frac{e^x}{(D+1)^2+1}x^2 \qquad \left[\because \frac{e^{ax}x}{D^2} = \frac{e^{ax}}{(D+a)^2}x\right]$$

$$= \frac{e^x}{D^2+2D+2}x^2$$

$$= \frac{e^x}{2\left(1+\frac{(D^2+2D)}{2}\right)}x^2$$

$$= \frac{e^x}{2}\left[1+\left(\frac{D^2+2D}{2}\right)\right]^{-1}x^2$$

$$= \frac{e^x}{2}\left[1-\left(\frac{D^2+2D}{2}\right)+\left(\frac{D^2+2D}{2}\right)^2-\ldots\ldots\right]x^2$$

$$= \frac{e^x}{2}\left[1-\frac{D^2}{2}-\frac{2D}{2}+\frac{D^4}{4}+\frac{4D^3}{4}+\frac{4D^2}{4}+\ldots\ldots\right]x$$

$$= \frac{e^x}{2}\left[x^2-\frac{D^2x^2}{2}-Dx^2+\frac{D^4}{4}(x^2)+D^3(x^2)+D^2(x^2)+\ldots\ldots\right]$$

$$= \frac{e^x}{2}\left[x^2-\frac{2}{2}-2x+0+0+2\right]$$

$$= \frac{e^x}{2}[x^2-1-2x+2]$$

$$= \frac{e^x}{2}[x^2-2x+1]$$

$$\Rightarrow \quad P.I_3 = \frac{e^x}{2}(x-1)^2$$

Substituting the corresponding values in equation (5),

$$P.I = \frac{x}{4}\sin x - \left(\frac{-1}{16}\cos 3x\right) + \frac{e^x}{2}(x-1)^2$$

$$\Rightarrow \quad P.I = \frac{x}{4}\sin x + \frac{1}{16}\cos 3x + \frac{e^x}{2}(x-1)^2 \quad \text{... (6)}$$

Substituting equations (4) and (6) in equation (3),

$$y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x + \frac{e^x}{2}(x-1)^2$$

OR

Q6. (a) Solve the D.E $(D^2 + 3D + 2)y = xe^x \sin x$.

Answer : **Jan.-2020, Set-1, Q6(a) M[8]**

For answer refer Unit-3, Q25.

(b) Solve the D.E $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x + x$.

Answer : **Jan.-2020, Set-1, Q6(b) M[7]**

Given differential equation is,

$$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x + x \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$f(D) = D^3 - 3D^2 + 4D - 2 \text{ and } X = e^x + \cos x + x$$

The general solution of a non-homogeneous linear differential equation is,

$$y = C.F + P.I \quad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e., $m^3 - 3m^2 + 4m - 2 = 0$

m = 1	1	−3	4	−2
	0	1	−2	2
	1	−2	2	0

$$\Rightarrow \quad (m-1)(m^2 - 2m + 2) = 0$$

$$\Rightarrow \quad m - 1 = 0; \; m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$\Rightarrow \quad m = 1 \; ; \quad m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow \quad m = 1 \; ; \quad m = \frac{2 \pm \sqrt{-4}}{2}$$

$$\Rightarrow \quad m = 1 \; ; \quad m = \frac{2 \pm 2i}{2}$$

$$\Rightarrow \quad m = 1 \; ; \quad m = 1 \pm i$$

The complementary function is given as,

$$C.F = c_1 e^x + e^x(c_1\cos x + c_2\sin x) \quad \text{... (4)}$$

Consider,

$$P.I_3 = \frac{x}{D^3 - 3D^2 + 4D - 2}$$

$$= \frac{x}{-2\left[1 - \left(\frac{D^3 - 3D^2 + 4D}{2}\right)\right]}$$

$$= \frac{-1}{2}\left[1 - \left(\frac{D^3 - 3D^2 + 4D}{2}\right)\right]^{-1} x$$

$$= \frac{-1}{2}\left[1 + \left(\frac{D^3 - 3D^2 + 4D}{2}\right) + \left(\frac{D^3 - 3D^2 + 4D}{2}\right)^2 \ldots\right] x$$

$$= \frac{-1}{2}\left[x + \frac{D^3}{2}(x) - \frac{3D^2}{2}(x) + \frac{4D}{2}(x)\right]$$

$$= \frac{-1}{2}\left[x + 0 - \frac{3}{2}(0) + 2(1)\right]$$

$$\Rightarrow \quad P.I_3 = \frac{1}{2}(x+2) \qquad \ldots (8)$$

Substituting equations (6), (7) and (8) in equation (5),

$$P.I = xe^x + \frac{1}{10}(3\sin x + \cos x) - \frac{1}{2}(x+2)$$

$$\Rightarrow \quad P.I = xe^x + \frac{1}{10}(3\sin x + \cos x) - \frac{x}{2} - 1 \qquad \ldots (9)$$

Substituting equations (4) and (9) in equation (3),

$$y = c_1 e^x + e^x(c_1 \cos x + c_2 \sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x) - \frac{x}{2} - 1$$

Q7. (a) If u = lx + my, v = mx - ly then show that

(i) $\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) = \frac{l^2}{l^2 + m^2}$

(ii) $\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) = \frac{l^2 + m^2}{l^2}$

Answer : **Jan.-2020, Set-1, Q7(a) M[8]**

Note: In question $\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$, $\left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$

are misprinted as $\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) = \frac{l^2}{l^2 + m^2}$, $\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) = \frac{l^2 + m^2}{l^2}$

Given equations are,

$$u = lx + my \qquad \ldots (1)$$

$$v = mx - ly \qquad \ldots (2)$$

(i) $\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$

Differentiating equation (1) partially with respect to x (keeping y as constant)

$$\left(\frac{\partial u}{\partial x}\right)_y = l(1) + 0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)_y = l \qquad \ldots (3)$$

Multiplying equations (1) and (2) with l and m respectively,

$$lu = l^2x + mly \qquad \ldots (4)$$

$$mv = m^2x - lmy \qquad \ldots (5)$$

Adding equations (4) and (5),

$$lu + mv = l^2x + mly + m^2x - lmy$$

$$\Rightarrow \quad lu + mv = l^2x + m^2x$$

$$\Rightarrow \quad lu + mv = (l^2 + m^2)x$$

$$\Rightarrow \quad x = \frac{lu + mv}{l^2 + m^2}$$

Differentiating above equation partially with respect to u (keeping v as constant),

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{l^2 + m^2}(l(1) + 0)$$

$$\Rightarrow \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2 + m^2} \qquad \ldots (6)$$

Multiplying equations (3) and (6),

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = l\left(\frac{l}{l^2 + m^2}\right)$$

$$= \frac{l^2}{l^2 + m^2}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$$

(ii) $\left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2}{l^2 + m^2}$

Differentiating equation (2) partially with respect to v (keeping x as constant)

$$\frac{\partial v}{\partial v} = 0 - l\left(\frac{\partial y}{\partial v}\right)_x$$

$$\Rightarrow \quad 1 = -l\left(\frac{\partial y}{\partial v}\right)_x$$

$$\Rightarrow \quad \left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{l} \qquad \ldots (7)$$

Multiplying equations (1) and (2) with m and l respectively,

$$mu = mlx + m^2y \qquad \ldots (8)$$

$$lv = lmx - l^2y \qquad \ldots (9)$$

Subtracting equations (8) and (9),

$$mu - lv = mlx + m^2y - lmx + l^2y$$

$$\Rightarrow \quad mu - lv = m^2y + l^2y$$

$$\Rightarrow \quad mu - lv = y(l^2 + m^2)$$

$$\Rightarrow \quad y = \frac{mu - lv}{l^2 + m^2}$$

Differentiating above equation partially with respect to y (keeping u as constant),

$$\left(\frac{\partial y}{\partial v}\right) = \frac{l}{l^2 + m^2}\left(0 - l\left(\frac{\partial v}{\partial y}\right)_u\right)$$

$$\Rightarrow \quad 1 = \frac{-l}{l^2 + m^2}\left(\frac{\partial v}{\partial y}\right)_u$$

$$\Rightarrow \quad \left(\frac{\partial v}{\partial y}\right)_u = \frac{-(l^2 + m^2)}{l} \qquad \ldots (10)$$

The auxiliary equation is given as,

$$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z) \quad \text{... (3)}$$

Substituting equations (1) and (2) in equation (3),

$$F(x, y, z) = x^m y^n z^p + \lambda(x + y + z - a) \quad \text{... (4)}$$

Partially differentiating equation (4) with respect to 'x', 'y' and z,

$$\frac{\partial F}{\partial x} = mx^{m-1} y^n z^p + \lambda$$

$$\frac{\partial F}{\partial y} = x^m ny^{n-1} z^p + \lambda$$

$$\frac{\partial F}{\partial z} = x^m y^n pz^{p-1} + \lambda$$

For maximum or minimum value, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$.

$$\frac{\partial F}{\partial x} = 0 \Rightarrow mx^{m-1} y^n z^p + \lambda = 0$$

$$\Rightarrow \frac{m}{x} x^m y^n z^p + \lambda = 0$$

$$\Rightarrow \frac{m}{x}(f) + \lambda = 0 \quad [\because \text{ From equation (1)}]$$

$$\Rightarrow \frac{mf}{x} = -\lambda$$

$$\Rightarrow x = \frac{-mf}{\lambda} \quad \text{... (5)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow x^m ny^{n-1} z^p + \lambda = 0$$

$$\Rightarrow \frac{n}{y} x^m y^n z^p + \lambda = 0$$

$$\Rightarrow \frac{n}{y}(f) + \lambda = 0 \quad [\because \text{ From equation (1)}]$$

$$\Rightarrow \frac{nf}{y} = -\lambda$$

$$\Rightarrow y = \frac{-nf}{\lambda} \quad \text{... (6)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow x^m y^n pz^{p-1} + \lambda = 0$$

$$\Rightarrow \frac{p}{z} x^m y^n z^p + \lambda = 0$$

$$\Rightarrow \frac{p}{z}(f) + \lambda = 0 \quad [\because \text{ From equation (1)}]$$

$$\Rightarrow \frac{pf}{z} = -\lambda$$

$$\Rightarrow z = \frac{-pf}{\lambda} \quad \text{... (7)}$$

Substituting equations (5), (6) and (7) in equation (2),

$$-\frac{mf}{\lambda} - \frac{nf}{\lambda} - \frac{pf}{\lambda} - a = 0$$

$$\Rightarrow -\frac{f}{\lambda}(m + n + p) = a$$

$$\lambda = -\frac{1}{a}(m + n + p)f$$

Substituting the value of 'λ' in equations (5), (6) and (7),

$$x = \frac{-mf}{\frac{-1}{a}(m+n+p)f}$$

$$\Rightarrow x = \frac{am}{m+n+p}$$

$$y = \frac{-nf}{\frac{-1}{a}(m+n+p)f}$$

$$\Rightarrow y = \frac{an}{m+n+p}$$

$$z = \frac{-pf}{\frac{-1}{a}(m+n+p)f}$$

$$\Rightarrow z = \frac{ap}{m+n+p}$$

Substituting the value of x, y and z in equation (1),

$$f(x, y, z) = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$= \left(\frac{a^m m^m . a^n . n^n . a^p . p^p}{(m+n+p)^m (m+n+p)^n (m+n+p)^p}\right)$$

$$= \left(\frac{a^{m+n+p} . m^m . n^n . p^p}{(m+n+p)^{m+n+p}}\right)$$

$\therefore$ The maximum value of $x^m y^n z^p$ is,

$$\frac{a^{m+n+p} . m^m . n^n . p^p}{(m+n+p)^{m+n+p}}$$

Q9. (a) Evaluate $\iint (x^2 + y^2)\,dxdy$ over the area bounded by the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Answer : **Jan.-2020, Set-1, Q9(a) M[8]**

For answer refer Unit-5, Q33.

$$= \frac{1}{2}\int_0^1\int_0^{1-x} [xy(1-x-y)^2]dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x} [xy[(1-x)^2 + y^2 - 2y(1-x)]]dy\,dx$$

$$= \frac{1}{2}\int_0^1\left[\int_0^{1-x} (xy(1-x)^2 + xy^3 - 2x(1-x)y^2)dy\right]dx$$

$$= \frac{1}{2}\int_0^1\left[x(1-x)^2.\frac{y^2}{2} + \frac{xy^4}{4} - \frac{2x(1-x)y^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1\left[\frac{x(1-x)^2(1-x)^2}{2} + \frac{x(1-x)^4}{4} - \frac{2x(1-x)(1-x)^3}{3} - \frac{x(1-x)^2(0)^2}{2}\right.$$

$$\left. - \frac{x(0)^4}{4} + \frac{2x(1-x)(0)^3}{3}\right]dx$$

$$= \frac{1}{2}\int_0^1\left[\frac{x(1-x)^4}{2} + \frac{x(1-x)^4}{4} - \frac{2x(1-x)^4}{3} - 0 - 0 + 0\right]dx$$

$$= \frac{1}{2}\int_0^1\left[\frac{x(1-x)^4}{2} + \frac{x(1-x)^4}{4} - \frac{2x(1-x)^4}{3}\right]dx$$

$$= \frac{1}{2}\int_0^1\left[\frac{6x(1-x)^4 + 3x(1-x)^4 - 8x(1-x)^4}{12}\right]dx$$

$$= \frac{1}{2\times 12}\int_0^1 [6x(1-x)^4 + 3x(1-x)^4 - 8x(1-x)^4]\,dx$$

$$= \frac{1}{24}\int_0^1 [x(1-x)^4]dx$$

$$= \frac{1}{24}\int_0^1 x(1 + 6x^2 - 4x - 4x^3 + x^4)\,dx$$

$$= \frac{1}{24}\int_0^1 (x + 6x^3 - 4x^2 - 4x^4 + x^5)\,dx$$

$$= \frac{1}{24}\left[\frac{x^2}{2} + \frac{6x^4}{4} - \frac{4x^3}{3} - \frac{4x^5}{5} + \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{24}\left[\frac{(1)^2}{2} + \frac{6(1)^4}{4} - \frac{4(1)^3}{3} - \frac{4(1)^5}{5} + \frac{(1)^6}{6} - \frac{(0)^2}{2} - \frac{6(0)^4}{4} + \frac{4(0)^3}{3} + \frac{4(0)^5}{5} - \frac{(0)^6}{6}\right]$$

$$= \frac{1}{24}\left[\frac{1}{2} + \frac{6}{4} - \frac{4}{3} - \frac{4}{5} + \frac{1}{6} - 0 - 0 + 0 + 0 - 0\right]$$

$$= \frac{1}{24}\left[\frac{1}{2} + \frac{6}{4} - \frac{4}{3} - \frac{4}{5} + \frac{1}{6}\right]$$

$$= \frac{1}{24}\left(\frac{1}{30}\right)$$

$$= \frac{1}{720}$$

$$\therefore \quad \iiint_V xyz\,dx\,dy\,dz = \frac{1}{720}$$

SOLUTIONS TO JANUARY-2020, SET-2 QP

Q1. (a) Test the convergence of the series $\sum\left(\frac{nx}{n+1}\right)^n$.

Answer : Jan.-2020, Set-2, Q1(a) M[8]

Given series is,

$$\sum\left(\frac{nx}{n+1}\right)^n$$

Let, $u_n = \left(\frac{nx}{n+1}\right)^n$

$$(u_n)^{\frac{1}{n}} = \left(\left(\frac{nx}{n+1}\right)^n\right)^{\frac{1}{n}}$$

$$\Rightarrow \quad (u_n)^{\frac{1}{n}} = \frac{nx}{n+1}$$

$$\underset{n\to\infty}{lt}\ (u_n)^{\frac{1}{n}} = \underset{n\to\infty}{lt}\left(\frac{nx}{n+1}\right)$$

$$= \underset{n\to\infty}{lt}\ \frac{nx}{n\left(1+\frac{1}{n}\right)}$$

$$= \underset{\frac{1}{n}\to 0}{lt}\ \frac{x}{1+\frac{1}{n}}$$

$$= \frac{x}{1+0}$$

$$\Rightarrow \quad \underset{n\to\infty}{lt}\ (u_n)^{\frac{1}{n}} = x$$

By Cauchy root test, $\sum u_n$

(i) Converges, if $x < 1$

(ii) Diverges, if $x > 1$

(iii) Test fails, if $x = 1$

If $x = 1$, $u_n = \left(\frac{n(1)}{n+1}\right)^n$

$$\Rightarrow \quad u_n = \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{n}{n\left(1+\frac{1}{n}\right)}\right)^n$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n = \frac{1}{e} \neq 0 \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

i.e., $\sum u_n$ is divergent, if $x = 1$

$\therefore$ The series $\sum\left(\frac{nx}{n+1}\right)^n$ converges if $x < 1$ and diverges if $x \geq 1$

Comparing equations (2) and (3),

$$M = \frac{x^2y^2+2}{3x^3y^2},\ N = \frac{2-2x^2y^2}{3x^2y^3}$$

$$\frac{\partial M}{\partial y} = \frac{3x^3y^2(2x^2y+0)-(x^2y^2+2)(6x^3y)}{(3x^3y^2)^2}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{6x^5y^3-6x^5y^3-12x^3y}{9x^6y^4}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{-12x^3y}{9x^6y^4} = \frac{-4}{3x^3y^3}$$

$$\frac{\partial N}{\partial x} = \frac{3x^2y^3(0-4xy^2)-(2-2x^2y^2)6xy^3}{(3x^2y^3)^2}$$

$$\Rightarrow \quad \frac{\partial N}{\partial x} = \frac{-12x^3y^5-12xy^3+12x^3y^5}{9x^4y^6}$$

$$\Rightarrow \quad \frac{\partial N}{\partial x} = \frac{-12xy^3}{9x^4y^6} = \frac{-4}{3x^3y^3}$$

$\because \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation (3) is an exact differential equation,

The general solution of equation (3) is given as,

$$\int_{(y\text{ constant})} M\,dx + \int_{(\text{terms independent of } x)} N\,dy = c$$

$$\Rightarrow \quad \int \frac{x^2y^2+2}{3x^3y^2}\,dx - \int \frac{2}{3y}\,dy = c$$

$$\Rightarrow \quad \int \frac{x^2y^2}{3x^3y^2}\,dx + \int \frac{2}{3x^3y^2}\,dx - \frac{2}{3}\int \frac{1}{y}\,dy = c$$

$$\Rightarrow \quad \frac{1}{3}\int \frac{1}{x}\,dx + \frac{2}{3y^2}\int \frac{1}{x^3}\,dx - \frac{2}{3}\int \frac{1}{y}\,dy = c$$

$$\Rightarrow \quad \frac{1}{3}\log x + \frac{2}{3y^2}\left(\frac{x^{-2}}{-2}\right) - \frac{2}{3}\log y = c$$

$$\Rightarrow \quad \frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$$

$\therefore$ The general solution is $\frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$.

(b) Solve the electrical circuit equation given by $L\frac{dI}{dt} + IR = E\cos wt$ with $I(0) = 0$.

Jan.-2020, Set-1, Q4(b) M[7]

Answer :

Given equation is,

$$L\frac{di}{dt} + iR = E\cos wt$$

$$\Rightarrow \quad \frac{di}{dt} + \frac{iR}{L} = \frac{E}{L}\cos wt \qquad \text{... (1)}$$

$i(0) = 0 \Rightarrow i = 0$ at $t = 0$

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Consider,

$$\frac{1}{M}\left(\frac{\partial N}{\partial \theta}-\frac{\partial M}{\partial r}\right)=\frac{1}{r\sin\theta}(2r^2\sin\theta-\sin\theta-\sin\theta)$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial \theta}-\frac{\partial M}{\partial r}\right)=\frac{2r^2\sin\theta-2\sin\theta}{r\sin\theta}$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial \theta}-\frac{\partial M}{\partial r}\right)=\frac{2\sin\theta(r^2-1)}{r\sin\theta}$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial \theta}-\frac{\partial M}{\partial r}\right)=2\left(r-\frac{1}{r}\right)$$

Integrating factor $(I.F)=e^{\int\frac{1}{M}\left(\frac{\partial N}{\partial \theta}-\frac{\partial M}{\partial r}\right)dr}$

$$\Rightarrow I.F=e^{\int 2\left(r-\frac{1}{r}\right)dr}$$

$$\Rightarrow I.F=e^{2\int\left(r-\frac{1}{r}\right)dr}$$

$$\Rightarrow I.F=e^{2\left(\frac{r^2}{2}-\log r\right)}$$

$$\Rightarrow I.F=e^{r^2-2\log r}$$

$$\Rightarrow I.F=e^{r^2}.e^{-2\log r}$$

$$\Rightarrow I.F=e^{r^2}.e^{\log r^{-2}}$$

$$\Rightarrow I.F=e^{r^2}.r^{-2}$$

$$\Rightarrow I.F=\frac{e^{r^2}}{r^2}$$

Multiplying equation (1) by $I.F$,

$$\frac{e^{r^2}.r}{r^2}\sin\theta\, d\theta+\frac{(r^3-2r^2\cos\theta+\cos\theta)e^{r^2}}{r^2}dr=0$$

$$\Rightarrow e^{r^2}r^{-1}\sin\theta\, d\theta+(r-2\cos\theta+r^{-2}\cos\theta)\, e^{r^2}dr=0 \quad \text{... (3)}$$

Comparing equations (2) and (3),

$$M=e^{r^2}r^{-1}\sin\theta,\ N=(r-2\cos\theta+r^{-2}\cos\theta)e^{r^2}$$

$$\frac{\partial M}{\partial r}=\sin\theta(e^{r^2}(-1(r^{-2}))+r^{-1}e^{r^2}.(2r))$$

$$\Rightarrow \frac{\partial M}{\partial r}=\sin\theta[-r^{-2}e^{r^2}+e^{r^2}2r.r^{-1}]$$

$$\Rightarrow \frac{\partial M}{\partial r}=\sin\theta[-e^{r^2}r^{-2}+2e^{r^2}]$$

$$\Rightarrow \frac{\partial M}{\partial r}=e^{r^2}\sin\theta[2-r^{-2}]$$

$$\frac{\partial N}{\partial \theta}=(0-2(-\sin\theta)+r^{-2}(-\sin\theta))e^{r^2}$$

$$\Rightarrow \frac{\partial N}{\partial \theta}=e^{r^2}(0+2\sin\theta-r^{-2}\sin\theta)$$

$$\Rightarrow \frac{\partial N}{\partial \theta}=e^{r^2}\sin\theta(2-r^{-2})$$

$\because \frac{\partial M}{\partial r}=\frac{\partial N}{\partial \theta}$ equation (3) is an exact differential equation

The general solution is given as,

$$\int_{(r\text{ constant})} M\, d\theta+\int_{(\text{terms independent of }\theta)} N\, dr=c$$

$$\Rightarrow \int e^{r^2}r^{-1}\sin\theta\, d\theta+\int re^{r^2}dr=c$$

$$\Rightarrow e^{r^2}.r^{-1}\int\sin\theta\, d\theta+\frac{e^{r^2}}{2}=c$$

$$\Rightarrow \frac{e^{r^2}}{r}(-\cos\theta)+\frac{e^{r^2}}{2}=c$$

$$\Rightarrow -2e^{r^2}\cos\theta+re^{r^2}=2rc$$

$$\Rightarrow 2e^{r^2}\cos\theta=re^{r^2}-2rc$$

$$\Rightarrow 2e^{r^2}\cos\theta=re^{r^2}+2ar \quad [\because a=-c=\text{constant}]$$

$\therefore$ The general solution is $2e^{r^2}\cos\theta=re^{r^2}+2ar$

(b) If the population of a country doubles in 50 years, in how many years will it triple, assuming that the rate of increase is proportional to the number of inhabitants?

Jan.-2020, Set-1, Q3(b) M[7]

Answer :

Let 'P' be the population in a country

Let P_0 be the initial population i.e., at $t=0$, $P(t)=P_0$

$$\Rightarrow P(0)=P_0 \quad \text{... (1)}$$

The population doubles in 50 years i.e., at $t=50$, $P(t)=2P_0$

$$\Rightarrow P(50)=2P_0 \quad \text{... (2)}$$

The rate of increase of population is proportional to number of inhabitants. According to law of natural growth and decay,

$$\frac{dP}{dt}\propto P$$

$$\Rightarrow \frac{dP}{dt}=kP$$

The solution of above equation is given as,

$$P(t)=Ce^{kt} \quad \text{... (3)}$$

Substituting $t=0$ in equation (3),

$$P(0)=Ce^{k(0)}$$

$$\Rightarrow P_0=C(1) \quad [\because \text{From equation (1)}]$$

$$\Rightarrow C=P_0 \quad \text{... (4)}$$

Substituting $t=50$ in equation (3),

$$P(50)=Ce^{k(50)}$$

$$\Rightarrow 2P_0=P_0e^{50k} \quad [\because \text{From equations (2) and (4)}]$$

$$\Rightarrow 2=e^{50k}$$

$$\Rightarrow 50k=\log 2$$

$$\Rightarrow k=\frac{\log 2}{50} \quad \text{... (5)}$$

Let the population triples in x years i.e., $t=x$

$$P(t)=3P_0$$

$$\Rightarrow P(x)=3P_0 \quad \text{... (6)}$$

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(b) If $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$

Answer : **Jan.-2020, Set-1, Q1(b) M[7]**

Let the function be,

$$f(x) = \log(1+x) \quad \text{... (1)}$$

Differentiating equation (1) with respect to x,

$$f'(x) = \frac{1}{1+x} \quad \text{... (2)}$$

Differentiating equation (2) with respect to x,

$$f''(x) = \frac{-1}{(1+x)^2} \quad \text{... (3)}$$

By Maclaurin's theorem, the expansion of $f(x)$ (with $n = 1$) is given as,

$$f(x) = f(0) + xf'(\theta x) \quad \text{... (4)}$$

From equation (1),

$$f(0) = \log(1+0)$$

$$\Rightarrow \quad f(0) = \log 1$$

$$\Rightarrow \quad f(0) = 0$$

From equation (2),

$$f'(\theta x) = \frac{1}{1+\theta x}$$

Substituting the corresponding values in equation (4),

$$\log(1+x) = 0 + x\left(\frac{1}{1+\theta x}\right)$$

$$\Rightarrow \quad \log(1+x) = \frac{x}{1+\theta x} \quad \text{... (5)}$$

Since $\theta > 0$ and $x > 0$,

$$1 + \theta x > 1$$

$$\Rightarrow \quad \frac{1}{1+\theta x} < 1$$

$$\Rightarrow \quad \frac{x}{1+\theta x} < x \quad \text{... (6)}$$

$$\Rightarrow \quad \log(1+x) < x \quad [\because \text{ From equation (5)}]$$

$$\Rightarrow \quad x > \log(1+x) \quad \text{... (7)}$$

By Maclaurin's theorem, the expansion of $f(x)$ upto 2 terms (with $n = 2$) is given as,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x) \quad \text{... (8)}$$

From equation (2),

$$f'(0) = \frac{1}{1+0}$$

$$\Rightarrow \quad f'(0) = 1$$

From equation (3),

$$f''(\theta x) = \frac{-1}{(1+\theta x)^2}$$

Substituting the corresponding values in equation (8),

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}\left(\frac{-1}{(1+\theta x)^2}\right)$$

$$\Rightarrow \quad \log(1+x) = 0 + x - \frac{x^2}{2(1+\theta x)^2}$$

$$\Rightarrow \quad \log(1+x) = x - \frac{x^2}{2(1+\theta x)^2} \quad \text{... (9)}$$

From equation (6),

$$\frac{x^2}{(1+\theta x)^2} < x^2$$

$$\Rightarrow \quad \frac{-x^2}{2(1+\theta x)^2} > \frac{-x^2}{2}$$

$$\Rightarrow \quad x - \frac{x^2}{2(1+\theta x)^2} > x - \frac{x^2}{2}$$

$$\Rightarrow \quad \log(1+x) > x - \frac{x^2}{2} \quad [\because \text{ From equation (9)}] \quad \text{... (10)}$$

From equations (7) and (10),

$$x > \log(1+x) > x - \frac{x^2}{2}$$

Hence if $x > 0$, $x > \log(1+x) > x - \frac{x^2}{2}$

OR

Q2. (a) Test the convergence of the series $\sum \frac{1^2.4^2.7^2.....}{3^2.6^2.9^2...}$

Answer : **Jan.-2020, Set-1, Q2(a) M[8]**

Given series is,

$$\sum \frac{1^2.4^2.7^2....}{3^2.6^2.9^2....}$$

Let,

$$u_n = \frac{1^2.4^2.7^2....}{3^2.6^2.9^2....}$$

$$\Rightarrow \quad u_n = \frac{1^2.4^2.7^2....(3n-2)^2}{3^2.6^2.9^2....(3n)^2}$$

$$u_{n+1} = \frac{1^2.4^2.7^2....(3n-2)^2(3(n+1)-2)^2}{3^2.6^2.9^2....(3n)^2(3(n+1))^2}$$

$$\Rightarrow \quad u_{n+1} = \frac{1^2.4^2.7^2....(3n-2)^2(3n+3-2)^2}{3^2.6^2.9^2....(3n)^2(3n+3)^2}$$

$$\Rightarrow \quad u_{n+1} = \frac{1^2.4^2.7^2....(3n-2)^2(3n+1)^2}{3^2.6^2.9^2....(3n)^2(3n+3)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{1^2.4^2.7^2....(3n-2)^2(3n+1)^2}{3^2.6^2.9^2....(3n)^2(3n+3)^2}}{\dfrac{1^2.4^2.7^2....(3n-2)^2}{3^2.6^2.9^2....(3n)^2}}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{[1^2.4^2.7^2....(3n-2)^2(3n+1)^2][3^2.6^2.9^2....(3n)^2]}{[3^2.6^2.9^2....(3n)^2(3n+3)^2)][1^2.4^2.7^2....(3n-2)^2]}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+1)^2}{(3n+3)^2}$$

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Multiplying equations (7) and (10),

$$\left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u = \frac{-1}{l}\left(\frac{-(l^2+m^2)}{l}\right)$$

$$= \frac{l^2+m^2}{l(l)}$$

$$= \frac{l^2+m^2}{l^2}$$

$$\therefore \quad \left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$$

(b) If the sum of three variables is a constant, then find the numbers when the product of three numbers is maximum.

Answer : **Jan.-2020, Set-1, Q7(b) M[7]**

Let, 'n' be any positive number and divided into three parts x, y, z such that,

$$x+y+z=n \quad \text{... (1)}$$

$$\Rightarrow \quad z=n-x-y$$

Then, the product of three parts $= xyz = xy(n-x-y)$

Let, $f = xy(n-x-y)$ ... (2)

Differentiating equation (2) partially with respect to x and y,

$$\frac{\partial f}{\partial x} = ny - 2xy - y^2$$

$$\frac{\partial f}{\partial y} = nx - x^2 - 2xy$$

For f to be maximum or minumum,

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

i.e., $ny - 2xy - y^2 = 0$ ... (3)

and $nx - x^2 - 2xy = 0$ ... (4)

Subtracting equations (3) and (4),

$$ny - 2xy - y^2 - nx + x^2 + 2xy = 0$$

$$\Rightarrow \quad n(y-x) + x^2 - y^2 = 0$$

$$\Rightarrow \quad -n(x-y) + (x+y)(x-y) = 0$$

$$\Rightarrow \quad (x-y)[x+y-n] = 0$$

$$\Rightarrow \quad x-y = 0, x+y-n = 0$$

$$\Rightarrow \quad x = y \text{ (or) } x+y=n$$

Since, $x+y=n$ is a contradiction of $x+y+z=n$, $x=y$ is considered

Substituting $x=y$ in equation (2),

$$nx - 2x^2 - x^2 = 0$$

$$\Rightarrow \quad nx - 3x^2 = 0$$

$$\Rightarrow \quad x(n-3x) = 0$$

$$\Rightarrow \quad n - 3x = 0 \quad [\because x \neq 0]$$

$$\Rightarrow \quad x = \frac{n}{3}$$

$$\therefore \quad x = y = \frac{n}{3}$$

Consider,

$$\frac{\partial^2 f}{\partial x^2} = 0 - 2y - 0$$

$$\Rightarrow \quad r = -2y$$

$$r\left(\frac{n}{3},\frac{n}{3}\right) = 2\left(\frac{n}{3}\right) = \frac{-2n}{3} < 0$$

$$\frac{\partial^2 f}{\partial x.\partial y} = s = n - 2x - 2y$$

$$\Rightarrow \quad s\left(\frac{n}{3},\frac{n}{3}\right) = n - 2\left(\frac{n}{3}\right) - 2\left(\frac{n}{3}\right)$$

$$\Rightarrow \quad s\left(\frac{n}{3},\frac{n}{3}\right) = \frac{-n}{3}$$

$$\frac{\partial^2 f}{\partial y^2} = 0 - 0 - 2x$$

$$\Rightarrow \quad t = -2x$$

$$t\left(\frac{n}{3},\frac{n}{3}\right) = -2\left(\frac{n}{3}\right) = \frac{-2n}{3}$$

$$rt - s^2 = \left(\frac{-2n}{3}\right)\left(\frac{-2n}{3}\right) - \left(\frac{n}{3}\right)^2 = \frac{n^2}{3} > 0$$

$\because$ $rt - s^2 > 0$ and $r < 0$ at $\left(\frac{n}{3}, \frac{n}{3}\right)$, f is maximum

From equation (1),

$$\frac{n}{3} + \frac{n}{3} + z = n$$

$$\Rightarrow \quad z = n - \frac{n}{3} - \frac{n}{3}$$

$$\Rightarrow \quad z = \frac{n}{3}$$

$$\therefore \quad \text{Maximum product} = \left(\frac{n}{3}\right)\left(\frac{n}{3}\right)\left(\frac{n}{3}\right) = \frac{n^3}{27}$$

Q8. (a) Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in powers of $(x-1)$ $(y-1)$ up to third degree and hence evaluate f(1.1, 0.9).

Answer : **Jan.-2020, Set-1, Q8(a) M[8]**

For answer refer Unit-4, Q42.

(b) Find maximum of $x^m y^n z^p$ given that $x+y+z=a$. Using Lagrange multiplier method.

Answer : **Jan.-2020, Set-1, Q8(b) M[7]**

Given function is,

$$f(x,y,z) = x^m y^n z^p \quad \text{... (1)}$$

And $\phi(x,y,z) = x+y+z = a$

$$\Rightarrow \quad \phi(x,y,z) = x+y+z-a \quad \text{... (2)}$$

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The particular integral is,

$$P.I = \frac{X}{f(D)}$$

$$= \frac{e^x + \cos x + x}{D^3 - 3D^2 + 4D - 2}$$

$$= \frac{e^x}{D^3 - 3D^2 + 4D - 2} + \frac{\cos x}{D^3 - 3D^2 + 4D - 2} + \frac{x}{D^3 - 3D^2 + 4D - 2} \quad \text{... (5)}$$

$$\Rightarrow \quad P.I = P.I_1 + P.I_2 + P.I_3$$

Consider,

$$P.I_1 = \frac{e^x}{D^3 - 3D^2 + 4D - 2}$$

Here, X is of the form e^{ax}

Let $D = a$ i.e., $D = 1$

$$P.I_1 = \frac{e^x}{(1)^3 - 3(1)^2 + 4(1) - 2}$$

$$\because \quad f(a) = 0, P.I_1 = \frac{xe^x}{f.'(D)}$$

$$\Rightarrow \quad P.I_1 = \frac{xe^x}{3D^2 - 6D + 4}$$

$$\Rightarrow \quad P.I_1 = \frac{xe^x}{3(1)^2 - 6(1) + 4}$$

$$\Rightarrow \quad P.I_1 = \frac{xe^x}{1}$$

$$\Rightarrow \quad P.I_1 = xe^x \quad \text{... (6)}$$

Consider,

$$P.I_2 = \frac{\cos x}{D^3 - 3D^2 + 4D - 2}$$

Here, X is of the form $\cos ax$

Let, $D^2 = -a^2$ i.e., $D^2 = -(1)^2 = -1$

$$P.I_2 = \frac{\cos x}{-1(D) - 3(-1) + 4D - 2}$$

$$= \frac{\cos x}{-D + 3 + 4D - 2}$$

$$= \frac{\cos x}{3D + 1} \times \frac{3D - 1}{3D - 1} \quad [\because \text{ Rationalizing the denominator}]$$

$$= \frac{3D(\cos x) - \cos x}{9D^2 - 1}$$

$$= \frac{3(-\sin x) - \cos x}{9(-1) - 1}$$

$$= \frac{-3\sin x - \cos x}{-10}$$

$$= \frac{-(3\sin x + \cos x)}{-10}$$

$$\Rightarrow \quad P.I_2 = \frac{3\sin x + \cos x}{10} \quad \text{... (7)}$$

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Substituting $C = \dfrac{-ER}{R^2 + w^2L^2}$ in equation (3),

$$i = \frac{E}{R^2 + w^2L^2}(R\cos wt + w\sin wt) - \frac{ER}{R^2 + w^2L^2}e^{\frac{-Rt}{L}}$$

$$\Rightarrow \quad i = \frac{E}{R^2 + w^2L^2}\left[R\cos wt + w\sin wt - Re^{\frac{-Rt}{L}}\right]$$

$\therefore$ The required solution is,

$$i(t) = \frac{E}{R^2 + w^2L^2}\left[R\cos wt + w\sin wt - Re^{\frac{-Rt}{L}}\right]$$

Q5. (a) Determine the charge on the capacitor at any time t > 0 in circuit in series having an EMF E(t) = 100 sin 60t, a resistor of 2 ohms, an inductor of 0.1 henries and capacitor of $\frac{1}{260}$ farads, if the initial current and charge on the capacitor are both zero.

Jan.-2020, Set-1, Q5(a) M[8]

Answer :

For answer refer Unit-3, Q49.

(b) Solve the D.E$(D^2 + 1)y = \sin x \sin 2x + e^x x^2$.

Jan.-2020, Set-1, Q5(b) M[7]

Answer :

Given differential equation is,

$$(D^2 + 1)y = \sin x \sin 2x + e^x x^2 \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$f(D) = D^2 + 1, X = \sin x \sin 2x + e^x x^2$$

The general solution is given as,

$$y = C.F + P.I \quad \text{... (3)}$$

The auxiliary equation is $f(m) = 0$

i.e., $m^2 + 1 = 0$

$\Rightarrow \quad m^2 = -1$

$\Rightarrow \quad m = \pm i$

The complementary function is,

$$C.F = c_1\cos x + c_2\sin x \quad \text{... (4)}$$

Particular integral is,

$$P.I = \frac{1}{f(D)}X$$

$$= \frac{1}{D^2+1}[\sin x \sin 2x + e^x x^2]$$

$$= \frac{1}{D^2+1}\left[\left[\frac{\cos(x-2x) - \cos(x+2x)}{2}\right] + e^x x^2\right] \qquad \left[\because \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}\right]$$

$$= \frac{1}{D^2+1}\left[\left(\frac{\cos x - \cos 3x}{2}\right) + e^x x^2\right]$$

$$= \frac{1}{2}\left(\frac{1}{D^2+1}\cos x\right) - \frac{1}{2}\left(\frac{1}{D^2+1}\cos 3x\right) + \frac{1}{D^2+1}(x^2e^x)$$

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(b) Test the convergence of the series $\frac{1}{6}-\frac{2}{11}+\frac{3}{16}-\frac{4}{21}+\ldots$

Answer : Jan.-2020, Set-2, Q1(b) M[7]

Given series is,

$$\frac{1}{6}-\frac{2}{11}+\frac{3}{16}-\frac{4}{21}+\ldots$$

The above series is an alternating series of the form $\sum(-1)^{n-1}\frac{n}{5n+1}$

Let, $u_n=\frac{n}{5n+1}$

$$u_{n+1}=\frac{n+1}{5(n+1)+1}$$

$$\Rightarrow \quad u_{n+1}=\frac{n+1}{5n+5+1}$$

$$\Rightarrow \quad u_{n+1}=\frac{n+1}{5n+6}$$

Consider,

$$u_n-u_{n+1}=\frac{n}{5n+1}-\frac{n+1}{5n+6}$$

$$\Rightarrow \quad u_n-u_{n+1}=\frac{(5n+6)n-(n+1)(5n+1)}{(5n+1)(5n+6)}$$

$$\Rightarrow \quad u_n-u_{n+1}=\frac{5n^2+6n-5n^2-n-5n-1}{(5n+1)(5n+6)}$$

$$\Rightarrow \quad u_n-u_{n+1}=\frac{-1}{(5n+1)(5n+6)}<0$$

$$\Rightarrow \quad u_n-u_{n+1}<0$$

$$\Rightarrow \quad u_n<u_{n+1}$$

$$\underset{n\to\infty}{lt}\, u_n=\underset{n\to\infty}{lt}\frac{n}{5n+1}$$

$$=\underset{n\to\infty}{lt}\frac{n}{n\left(5+\frac{1}{n}\right)}$$

$$=\underset{\frac{1}{n}\to 0}{lt}\frac{1}{5+\frac{1}{n}}$$

$$=\frac{1}{5+0}$$

$$\Rightarrow \quad \underset{n\to\infty}{lt}\, u_n=\frac{1}{5}\neq 0$$

According to Leibnitz test, the conditions for the convergence of series are not satisfied.

Hence, the series is not convergent and is oscillatory.

OR

Q2. (a) Test the convergence of the series $\sum\frac{4.7\ldots(3n+1)}{1.2.3\ldots n}x^n$.

Answer : Jan.-2020, Set-2, Q2(a) M[8]

Given series is,

$$\sum\frac{4.7\ldots\ldots(3n+1)}{1.2.3\ldots\ldots n}x^n$$

Let, $u_n=\frac{4.7\ldots\ldots(3n+1)}{1.2.3\ldots\ldots n}x^n$ ... (1)

$$u_{n+1}=\frac{4.7\ldots\ldots(3n+1)(3n+4)}{1.2.3\ldots\ldots n(n+1)}x^{n+1} \quad \ldots (2)$$

Dividing equation (2) by equation (1),

$$\frac{u_{n+1}}{u_n}=\frac{4.7\ldots\ldots(3n+1)(3n+4)}{1.2.3\ldots\ldots n(n+1)}x^{n+1}\times\frac{1.2.3\ldots n}{4.7\ldots\ldots(3n+1)}\cdot\frac{1}{x^n}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n}=\frac{(3n+1)(3n+4)}{(n+1)(3n+1)x^n}x^{n+1}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n}=\frac{(3n+4)}{(n+1)}x$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n}=\frac{n\left(3+\frac{4}{n}\right)}{n\left(1+\frac{1}{n}\right)}x$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n}=\frac{\left(3+\frac{4}{n}\right)}{\left(1+\frac{1}{n}\right)}x$$

Applying limit on both sides,

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{\left(3+\frac{4}{n}\right)}{\left(1+\frac{1}{n}\right)}x$$

$$=\underset{\frac{1}{n}\to 0}{lt}\frac{\left(3+\frac{4}{n}\right)}{\left(1+\frac{1}{n}\right)}x$$

$$=\frac{(3+0)}{(1+0)}x$$

$$\Rightarrow \quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=3x$$

By ratio test,

(i) Converges, if $3x<1$ i.e., $x<\frac{1}{3}$

(ii) Diverges, if $3x>1$ i.e., $x>\frac{1}{3}$

(iii) Ratio Test fails, if $3x=1$ i.e., $x=\frac{1}{3}$

At $x=\frac{1}{3}$, $\frac{u_{n+1}}{u_n}=\frac{(3n+4)}{(n+1)}\cdot\frac{1}{3}$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n}=\frac{(3n+4)}{(3n+3)}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}}=\frac{(3n+3)}{(3n+4)}$$

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(b) Find the area enclosed by the pair of curves $y = 4x - x^2$, $y = x$. **Jan.-2020, Set-1, Q10(**

Answer :

Given curves are,

$y = 4x - x^2$

And $y = x$

From equations (1) and (2),

$4x - x^2 = x$

$\Rightarrow \quad x^2 + x - 4x = 0$

$\Rightarrow \quad x^2 - 3x = 0$

$\Rightarrow \quad x(x-3) = 0$

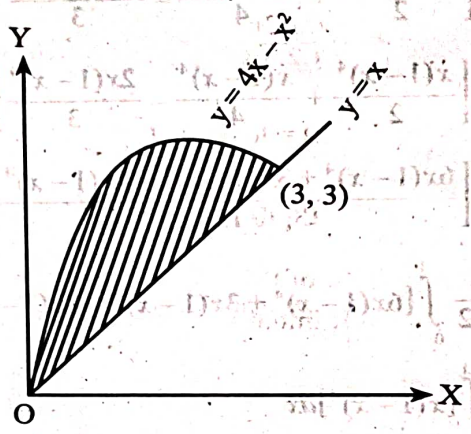
$\Rightarrow \quad x = 0, x = 3$

If $x = 0, y = 0$ i.e., (0, 0)

If $x = 3, y = 3$ i.e., (3, 3)

The point of intersection of equations (1) and (2) is (0, 0) and (3, 3)

The area bounded by the curves (1) and (2) is shown in figure below



Figure

From the figure, x varies from 0 to 3 and y varies from x to $4x - x^2$

$$\text{Required Area } (A) = \int_0^3 \int_x^{4x-x^2} dy\, dx$$

$$= \int_0^3 \left(\int_x^{4x-x^2} dy \right) dx$$

$$= \int_0^3 (y)_x^{4x-x^2} dx$$

$$= \int_0^3 (4x - x^2 - x)dx = \int_0^3 (3x - x^2)dx$$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{3} \right)_0^3 = \left(\frac{3(3)^2}{2} - \frac{3(0)^2}{2} - \frac{(3)^3}{3} + \frac{(0)^3}{3} \right)$$

$$= \left(\frac{27}{2} - 0 - \frac{27}{3} + 0 \right) = \frac{27}{2} - \frac{27}{3}$$

$$= \frac{27}{6} = \frac{9}{2}$$

$\therefore$ Area enclosed by the curves $y = 4x - x^2$ and $y = x$ is $\frac{9}{2}$ square units.

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(b) Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{(a^2-r^2)}{a}} r\,dr\,d\theta\,dz.$

Answer :

For answer refer Unit-5, Q10.

Jan.-2020, Set-1, Q9(b) M[7]

Q10. (a) Evaluate $\iiint_V xyz\,dx\,dy\,dz$ **where v is bounded by the co-ordinate planes and the plane** $x + y + z = 1$.

Answer :

Jan.-2020, Set-1, Q10(a) M[8]

Given planes are,

$x = 0, y = 0, z = 0$

And $x + y + z = 1$

$\Rightarrow \quad z = 1 - x - y$ ... (1)

$\therefore \quad z$ varies from 0 to $1 - x - y$

Substituting $z = 0$ in equation (1),

$x + y = 1$

$\Rightarrow \quad y = 1 - x$ ... (2)

$\therefore \quad y$ varies from 0 to $1 - x$

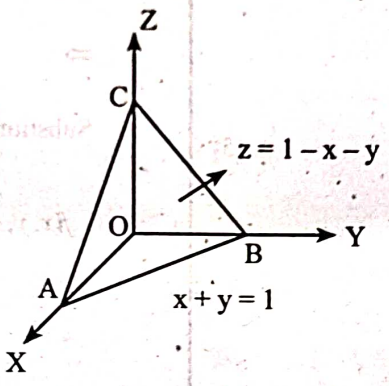
Substituting $y = 0$ in equation (2),

$0 = 1 - x$

$\Rightarrow \quad x = 1$

$\therefore \quad x$ varies from 0 to 1.

The region of integration is bounded by the planes $x + y + z = 1, x = 0, y = 0$ and $z = 0$ is shown in the figure below.



Figure

$$\iiint_V xyz\,dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} xyz\,dz\right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[xy.\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} [xy[(1-x-y)^2 - 0^2]]\,dy\,dx$$

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Subtracting '1' on both sides,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{3n+3}{3n+4} - 1$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{3n+3-3n-4}{3n+4}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{-1}{3n+4}$$

Multiplying 'n' on both sides,

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-n}{(3n+4)}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-n}{n\left(3+\frac{4}{n}\right)}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-1}{3+\frac{4}{n}}$$

Applying limit on both sides,

$$\underset{n\to\infty}{lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n\to\infty}{lt}\ \frac{-1}{3+\frac{4}{n}}$$

$$= \underset{\frac{1}{n}\to 0}{lt}\ \frac{-1}{3+\frac{4}{n}}$$

$$= \frac{-1}{3+0}$$

$$\Rightarrow \underset{n\to\infty}{Lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-1}{3} < 1$$

By Raabe's test the given series is divergent at $x = \frac{1}{3}$

$\therefore$ The given series is convergent if $x < \frac{1}{3}$ and divergent if $x \geq \frac{1}{3}$.

(b) Verify Rolle's mean value theorem for $f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right)$ in [a, b] where a > b > 0.

Answer :

Jan.-2020, Set-2, Q2(b) M[7]

For answer refer Unit-1, Q45.

Q3. (a) Solve the D.E $y^4\,dx = (x^{-3} - y^3x)dy = 0$.

Answer :

Jan.-2020, Set-2, Q3(a) M[8]

Note : In question $y^4dx - (x^{-3} - y^3x)dy = 0$ is misprinted as $y^4dx = (x^{-3/4} - y^3x)dy = 0$

Given equation is,

$$y^4dx - (x^{-3} - y^3x)dy = 0$$

$$\Rightarrow y^4dx + (y^3x - x^{-3})dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$M\,dx + N\,dy = 0 \quad \text{... (2)}$$

(b) A resistance of 100 ohms an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit as a function of t, if initially there is no current in the circuit.

Jan.-2020, Set-2, Q3(b) M[7]

Answer :

For answer refer Unit-2, Q52.

OR

Q4. (a) Solve the D.E $(r + \sin\theta - \cos\theta)dr + r(\sin\theta + \cos\theta)d\theta = 0$.

Jan.-2020, Set-2, Q4(a) M[7]

Answer :

Given differential equation is,

$$(r + \sin\theta - \cos\theta)dr + r(\sin\theta + \cos\theta)d\theta = 0$$

$$\Rightarrow \quad (r + \sin\theta - \cos\theta)dr + (r\sin\theta + r\cos\theta)d\theta = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$M\,d\theta + N\,dr = 0 \quad \text{... (2)}$$

Comparing equations (1) and (2),

$M = r\sin\theta + r\cos\theta$	$N = r + \sin\theta - \cos\theta$
$\frac{\partial M}{\partial r} = (1)\sin\theta + (1)\cos\theta$	$\frac{\partial N}{\partial \theta} = 0 + \cos\theta - (-\sin\theta)$
$\Rightarrow \frac{\partial M}{\partial r} = \sin\theta + \cos\theta$	$\Rightarrow \frac{\partial N}{\partial \theta} = \cos\theta + \sin\theta$

$\because \frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$ equation (1) is an exact differential equation

The general solution is given as,

$$\int_{(r\ \text{constant})} M\,d\theta + \int_{(\text{terms independent of }\theta)} N\,dr = c$$

$$\Rightarrow \int (r\sin\theta + r\cos\theta)d\theta + \int r\,dr = c$$

$$\Rightarrow r\int \sin\theta\,d\theta + r\int \cos\theta\,d\theta + \frac{r^2}{2} = c$$

$$\Rightarrow r(-\cos\theta) + r(\sin\theta) + \frac{r^2}{2} = c$$

$$\Rightarrow -r\cos\theta + r\sin\theta + \frac{r^2}{2} = c$$

$$\Rightarrow 2r(\sin\theta - \cos\theta) + r^2 = 2c$$

$$\Rightarrow 2r(\sin\theta - \cos\theta) + r^2 = C$$

$\therefore$ The general solution is $2r(\sin\theta - \cos\theta) + r^2 = C$

(b) Show that the family of curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self-orthogonal, where λ is a parameter.

Jan.-2020, Set-2, Q4(b) M[8]

Answer :

Given that,

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{... (1)}$$

Differentiating equation (1) with respect to x,

$$\frac{1}{a^2+\lambda}(2x) + \frac{1}{b^2+\lambda}(2y)\frac{dy}{dx} = 0$$

$$\Rightarrow 2\left[\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\cdot y'\right] = 0$$

$$\Rightarrow \frac{x}{a^2+\lambda} + \frac{yy'}{b^2+\lambda} = 0$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = D^2 + 1 \text{ and } X = x^2 \sin 2x$$

The general solution of a non-homogeneous linear differential equation is,

$$y = C.F + P.I \quad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e., $m^2 + 1 = 0$

$\Rightarrow \quad m^2 = -1$

$\Rightarrow \quad m = \pm i$

The roots are complex conjugate of the form $\alpha + i\beta$

Here $\alpha = 0, \beta = 1$

Hence the complementary function is,

$$C.F = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$= e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$\Rightarrow \quad C.F = c_1 \cos x + c_2 \sin x \quad \text{... (4)}$$

The particular integral is,

$$P.I = \frac{X}{f(D)}$$

$$= \frac{x^2 \sin 2x}{D^2 + 1}$$

$$= \text{I.P of } \frac{1}{D^2+1} x^2 e^{2ix} \qquad [\because \text{ I.P = Imaginary part}]$$

$$= \text{I.P of } e^{2ix} . \frac{1}{(D+2i)^2+1} x^2$$

$$= \text{I.P of } e^{2ix} . \frac{1}{D^2 - 4 + 4iD + 1} x^2 = \text{I.P of } e^{2ix} . \frac{1}{D^2 + 4iD - 3} x^2$$

$$= \text{I.P of } e^{2ix} . \frac{1}{-3\left(\frac{D^2+4iD}{-3} + 1\right)} x^2 = \text{I.P of } \frac{e^{2ix}}{-3}\left[\frac{1}{1 - \left(\frac{D^2+4iD}{3}\right)}\right] x^2$$

$$= \text{I.P of } \frac{e^{2ix}}{-3}\left[1 - \frac{(D^2+4iD)}{3}\right]^{-1} x^2 = \text{I.P of } \frac{e^{2ix}}{-3}\left[1 + \left(\frac{D^2+4iD}{3}\right) + \left(\frac{D^2+4iD}{3}\right)^2 + \ldots\right] x^2$$

$$= \text{I.P of } \frac{e^{2ix}}{-3}\left[1 + \left(\frac{D^2}{3} + \frac{4iD}{3}\right) + \left(\frac{D^4 + 16(-1)D^2 + 8iD^3}{9}\right) + \ldots\right] x^2$$

$$= \text{I.P of } \frac{e^{2ix}}{-3}\left[x^2 + \frac{D^2}{3}(x^2) + \frac{4iD}{3}(x^2) + \frac{D^4}{9}(x^2) - \frac{16D^2}{9}(x^2) + \frac{8iD^3}{9}(x^2) + \ldots\right]$$

$$= \text{I.P of } \frac{e^{2ix}}{-3}\left[x^2 + \frac{1}{3}(2) + \frac{4i}{3}(2x) + 0 - \frac{16(2)}{9} + 0 \ldots\right]$$

$$= \text{I.P of } \frac{e^{2ix}}{-3}\left[x^2 + \frac{2}{3} + \frac{8ix}{3} - \frac{32}{9}\right] = \text{I.P of } \frac{-1}{3}(\cos 2x + i \sin 2x)\left(x^2 + \frac{8ix}{3} - \frac{26}{9}\right)$$

$$\Rightarrow \quad y_P = \frac{-1}{3}\left[\left(x^2 - \frac{26}{9}\right)\sin 2x + \left(\frac{8}{3}\right)x \cos 2x\right] \quad \text{... (5)}$$

Substituting equations (4) and (5) in equation (3),

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{3}\left[\left(x^2 - \frac{26}{9}\right)\sin 2x + \left(\frac{8}{3}\right)x \cos 2x\right]$$

$$\Rightarrow \quad P.I = \frac{E_0 \sin wt}{LD^2 + RD + \frac{1}{C}}$$

$$= \frac{E_0 \sin wt}{L(-w^2) + RD + \frac{1}{C}}$$

$$= \frac{E_0 \sin wt}{-L\left(\frac{1}{LC}\right) + RD + \frac{1}{C}}$$

$$= \frac{E_0 \sin wt}{-\frac{1}{C} + RD + \frac{1}{C}} = \frac{E_0 \sin wt}{RD}$$

$$= \frac{E_0}{R}\int \sin wt\, dt = \frac{E_0}{R}\left[-\frac{\cos wt}{w}\right]$$

$$\Rightarrow \quad P.I = \frac{-E_0}{Rw}\cos wt \qquad \text{... (4)}$$

Substituting equations (3) and (4) in equation (2),

$$q = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos wt + c_2 \sin wt) - \frac{E_0}{Rw}\cos wt \qquad \text{... (5)}$$

Differentiating equation (5) with respect to 't',

$$\frac{dq}{dt} = i = \left(1 - \frac{Rt}{2L}\right)(c_1(-\sin wt)w + c_2(\cos wt)w) - \frac{R}{2L}(c_1 \cos wt + c_2 \sin wt) - \frac{E_0}{Rw}(-\sin wt)w$$

$$\Rightarrow \quad i = \left(1 - \frac{Rt}{2L}\right)(-c_1 \sin wt + c_2 \cos wt)w - \frac{R}{2L}(c_1 \cos wt + c_2 \sin wt) + \frac{E_0}{R}\sin wt \qquad \text{... (6)}$$

Substituting $t = 0$, $q = 0$ in equation (5),

$$0 = (1 - 0)(c_1 \cos(0) + c_2(0)) - \frac{E_0}{Rw}\cos(0)$$

$$\Rightarrow \quad 0 = c_1 - \frac{E_0}{Rw}$$

$$\Rightarrow \quad c_1 = \frac{E_0}{Rw}$$

When $q = 0$ Then $i = 0$

Substituting $t = 0$, $i = 0$ in equation (6),

$$0 = (1 - 0)(-c_1(0) + c_2\cos(0))w - \frac{R}{2L}(c_1\cos(0) + c_2(0)) + \frac{E_0}{R}\sin(0)$$

$$\Rightarrow \quad 0 = wc_2 - \frac{R}{2L}c_1$$

$$\Rightarrow \quad c_2 = \frac{R}{2Lw}c_1$$

$$= \frac{R}{2Lw}\left(\frac{E_0}{Rw}\right)$$

$$\Rightarrow \quad c_2 = \frac{E_0}{2Lw^2} \qquad \left[\because c_1 = \frac{E_0}{Rw}\right]$$

Substituting c_1 and c_2 values in equation (6),

$$i = \left(1 - \frac{Rt}{2L}\right)\left(-\frac{E_0}{Rw}\sin wt + \frac{E_0}{2Lw^2}\cos wt\right)w - \frac{R}{2L}\left(\frac{E_0}{Rw}\cos wt + \frac{E_0}{2Lw^2}\sin wt\right) + \frac{E_0}{R}\sin wt$$

$$= -\frac{E_0 w}{Rw}\sin wt + \frac{E_0 w}{2Lw^2}\cos wt + \frac{E_0 Rtw}{2LRw}\sin wt - \frac{E_0 Rtw}{4L^2w^2}\cos wt - \frac{RE_0}{2LRw}\cos wt - \frac{RE_0}{4L^2w^2}\sin wt + \frac{E_0}{R}\sin wt$$

$$= -\frac{E_0}{R}\sin wt + \frac{E_0}{2Lw}\cos wt + \frac{E_0 t}{2L}\sin wt - \frac{E_0 tR}{4L^2 w}\cos wt - \frac{E_0}{2Lw}\cos wt - \frac{RE_0}{4L^2w^2}\sin wt + \frac{E_0}{R}\sin wt$$

$$= \frac{E_0 t}{2L}\sin wt - \frac{E_0 tR}{4L^2 w}\cos wt - \frac{RE_0}{4L^2 w^2}\sin wt$$

$$\Rightarrow \quad i = \left(\frac{E_0 t}{2L}\right)\sin wt \qquad \left[\because \frac{R}{L} \text{ is small}\right]$$

Hence, the current in a circuit at time t is $i = \left(\frac{E_0 t}{2L}\right)\sin wt$.

OR

Consider,

$$P.I_2 = \frac{-xe^{-x}}{2(D^2-4)}$$

$$= \frac{-1}{2}\left[\frac{e^{-x}}{(D-1)^2-4}.x\right]$$

$$= \frac{-1}{2}\left[\frac{e^{-x}}{D^2+1-2D-4}.x\right] = \frac{-1}{2}\left[\frac{e^{-x}}{D^2-2D-3}.x\right]$$

$$= \frac{-e^{-x}}{2}\left[\frac{x}{-3\left[1+\frac{D^2-2D}{-3}\right]}\right] = \frac{-e^{-x}}{2(-3)}\left[\frac{x}{1-\left(\frac{D^2-2D}{3}\right)}\right]$$

$$= \frac{-e^{-x}}{-6}\left[1-\left(\frac{D^2-2D}{3}\right)\right]^{-1}x = \frac{e^{-x}}{6}\left[1+\left(\frac{D^2-2D}{3}\right)+\left(\frac{D^2-2D}{3}\right)^2+....\right]x$$

$$= \frac{e^{-x}}{6}\left[x+\frac{D^2}{3}(x)-\frac{2D}{3}(x)+\frac{D^4}{9}(x)+\frac{4D^2}{9}(x)-\frac{4D^3}{9}(x)+....\right]$$

$$= \frac{e^{-x}}{6}\left[x+0-\frac{2}{3}(1)+0+0-0+...\right]$$

$$\Rightarrow \quad P.I_2 = \frac{e^{-x}}{6}\left[x-\frac{2}{3}\right] \quad ...(7)$$

Consider,

$$P.I_3 = \frac{54x}{D^2-4}$$

$$= \frac{54}{-4\left(1-\frac{D^2}{4}\right)}.x = \frac{-54}{4}\left[1-\frac{D^2}{4}\right]^{-1}x$$

$$= \frac{-27}{2}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+....\right]x = \frac{-27}{2}\left[x+\frac{D^2}{4}(x)+\frac{D^4}{16}(x)+....\right]$$

$$= \frac{-27}{2}[x+0+0+...]$$

$$\Rightarrow \quad P.I_3 = \frac{-27}{2}x \quad ...(8)$$

Consider,

$$P.I_4 = \frac{8}{D^2-4}$$

Here, X is of the form k,

Let $D = 0$,

$$P.I_4 = \frac{8}{(0)^2-4}$$

$$\Rightarrow \quad P.I_4 = \frac{8}{-4}$$

$$\Rightarrow \quad P.I_4 = -2 \quad ...(9)$$

Substituting equations (6), (7), (8) and (9) in equation (5),

$$P.I = \frac{-e^x}{6}\left(x+\frac{2}{3}\right)+\frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)-\frac{27}{2}x-2$$

$$\Rightarrow \quad P.I = \frac{-xe^x}{6}-\frac{2}{3}\left(\frac{e^x}{6}\right)+\frac{xe^{-x}}{6}-\frac{2}{3}\left(\frac{e^{-x}}{6}\right)-\frac{27}{2}x-2$$

$$\Rightarrow \quad P.I = \frac{-xe^x+xe^{-x}}{6}-\frac{2}{3}\left(\frac{e^x+e^{-x}}{6}\right)-\frac{27}{2}x-2$$

$$\Rightarrow \quad P.I = \frac{-x}{3}\left(\frac{e^x-e^{-x}}{2}\right)-\frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right)-\frac{27}{2}x-2$$

$$\Rightarrow \quad P.I = \frac{-x}{3}\sinh x-\frac{2}{9}\cosh x-\frac{27}{2}x-2 \quad ...(10)$$

Dividing above equation by sinθ,

$$\frac{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}{\sin\theta} = \frac{-6r^n\sin\theta(2\cos^2\theta-\sin^2\theta)}{\sin\theta}$$

$$\Rightarrow \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n(2\cos^2\theta-\sin^2\theta)$$

$$\Rightarrow \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n(2\cos^2\theta+\cos^2\theta-1)$$

$$\Rightarrow \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n(3\cos^2\theta-1)$$

$$\Rightarrow \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6u \quad \text{... (3)}$$

Consider,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial u}{\partial\theta}\right)=0$$

$\Rightarrow$ $n(n+1)u-6u=0$ [$\because$ From equations (2) and (3)]

$\Rightarrow$ $u(n(n+1)-6)=0$

$\Rightarrow$ $u(n^2+n-6)=0$

$\Rightarrow$ $n^2+n-6=0$ [$\because u\neq 0$]

$\Rightarrow$ $n^2+3n-2n-6=0$

$\Rightarrow$ $n(n+3)-2(n+3)=0$

$\Rightarrow$ $(n+3)(n-2)=0$

$\Rightarrow$ $n=-3, n=2$

$\therefore$ For the value of $n=-3$ or 2,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial u}{\partial\theta}\right)=0$$

(b) Find the extreme values of sinx siny sin(x + y).

Answer :

For answer refer Unit-4, Q53.

Jan.-2020, Set-2, Q7(b) M[7]

OR

Q8. (a) Expand e^x cosy near $\left(1,\frac{\pi}{4}\right)$.

Answer :

Jan.-2020, Set-2, Q8(a) M[7]

Given function is,

$$f(x,y)=e^x\cos y \quad \text{... (1)}$$

$$(x_0,y_0)=\left(1,\frac{\pi}{4}\right)$$

From the definition of Taylor's series,

$$f_x(x,y)=f(x_0,y_0)+\frac{1}{1!}[(x-x_0)f_x(x_0,y_0)+(y-y_0)\ f_0(x_0,y_0)]+\frac{1}{2!}[(x-x_0)^2f_{xx}(x_0,y_0)$$

$$+2(x-x_0)(y-y_0)f_{xy}(x_0,y_0)+(y-y_0)^2f_{yy}(x_0,y_0)] \quad \text{... (2)}$$

$$\Rightarrow \quad f_{xy}(x, y) = e^x(-\sin y)$$

$$f_{xy}(x_0, y_0) = f_{xy}\left(1, \frac{\pi}{4}\right) = e^1\left(-\sin\frac{\pi}{4}\right) = e.\frac{-1}{\sqrt{2}} = \frac{-e}{\sqrt{2}}$$

Substituting the corresponding values in equation (2),

$$f(x, y) = \frac{e}{\sqrt{2}} + \frac{1}{1!}\left[(x-1)\left(\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{y}\right)\left(\frac{-e}{\sqrt{2}}\right)\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{e}{\sqrt{2}}\right) + 2(x-1)\left(y - \frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right)\right] + \left(y - \frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right) + \ldots$$

$$= \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{(x-1)^2}{2!} - (x-1)\left(y - \frac{\pi}{4}\right) - \frac{\left(y - \frac{\pi}{4}\right)^2}{2!} + \ldots\right]$$

$$\therefore \quad f(x, y) = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{(x-1)^2}{2!} - (x-1)\left(y - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right)^2 + \ldots\right]$$

(b) If $x = r\cos\theta$, $y = r\sin\theta$. Evaluate $J = \frac{\partial(x,y)}{\partial(r,\theta)}$ and $J' = \frac{\partial(r,\theta)}{\partial(x,y)}$

Jan.-2020, Set-2, Q8(b) M[8]

Answer :

For answer refer Unit-4, Q28.

Q9. (a) Evaluate $\iint r^3 dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Jan.-2020, Set-2, Q9(a) M[8]

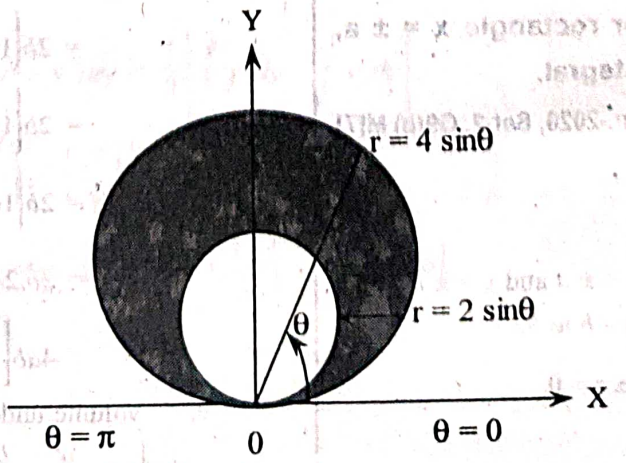
Answer :

Given integral is,

$$\iint r^3\, dr\, d\theta$$

And circles are $r = 2\sin\theta$ and $r = 4\sin\theta$

Thus, r varies from $2\sin\theta$ to $4\sin\theta$ and θ covers the complete region i.e, from 0 to π as shown in figure.



Figure

Q10. (a) By change of integration evaluate $\int_0^1 \int_{x^2}^{x} xy\, dy\, dx$

Answer :

Jan.-2020, Set-2, Q10(a) M[7]

Given integral is,

$$\int_0^1 \int_{x^2}^{x} xy\, dy\, dx$$

From the limits of integration, x varies from 0 to 1 and y varies from x^2 to x,

(a) $y = x^2$

If $x = 0, y = (0)^2 \Rightarrow y = 0$ i.e., (0, 0)

If $x = 1, y = (1)^2 \Rightarrow y = 1$ i.e., (1, 1)

(b) $y = x$

If $x = 0, y = 0$ i.e., (0, 0)

If $x = 1, y = 1$ i.e., (1, 1)

By changing the order of integration, y varies from 0 to 1,

$y = x^2 \Rightarrow x = \sqrt{y}$ i.e., x varies y to $\sqrt{y}$

Then, $\int_0^1 \int_{x^2}^{x} xy\, dy\, dx = \int_0^1 \int_y^{\sqrt{y}} xy\, dx\, dy$

$$= \int_0^1 \left[\int_y^{\sqrt{y}} xy\, dx\right] dy$$

$$= \int_0^1 \left[y \int_y^{\sqrt{y}} x\, dx\right] dy$$

$$= \int_0^1 \left[y\left(\frac{x^2}{2}\right)_y^{\sqrt{y}}\right] dy$$

$$= \int_0^1 \left[\frac{y}{2}((\sqrt{y})^2 - y^2)\right] dy$$

$$= \int_0^1 \left[\frac{y}{2}(y - y^2)\right] dy$$

$$= \frac{1}{2}\int_0^1 (y^2 - y^3)\, dy = \frac{1}{2}\left[\int_0^1 y^2\, dy - \int_0^1 y^3\, dy\right]$$

$$= \frac{1}{2}\left[\left(\frac{y^3}{3}\right)_0^1 - \left(\frac{y^4}{4}\right)_0^1\right] = \frac{1}{2}\left[\left(\frac{1^3}{3} - \frac{0^3}{3}\right) - \left(\frac{1^4}{4} - \frac{0^4}{4}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - 0\right) - \left(\frac{1}{4} - 0\right)\right] = \frac{1}{2}\left(\frac{1}{3} - 0 - \frac{1}{4} - 0\right)$$

$$= \frac{1}{2}\left(\frac{1}{12}\right)$$

$$= \frac{1}{24}$$

$$\therefore \int_0^1 \int_{x^2}^{x} xy\, dy\, dx = \int_0^1 \int_y^{\sqrt{y}} xy\, dx\, dy = \frac{1}{24}.$$

SOLUTIONS TO JANUARY-2020, SET-3

Q1. (a) Test the convergence of the series $\sum\sqrt{(n^4+1)-(n^4-1)}$.

Answer :

Jan.-2020, Set-3, Q1(a) M[8]

Note : In question $\sum\sqrt{n^4+1}-\sqrt{n^4-1}$ is misprinted as $\sum\sqrt{(n^4+1)-(n^4-1)}$

For answer refer Unit-1, Q9.

(b) Verify the Lagrange's mean value theorem for f(x) = x(x – 1)(x – 2) in [0, 0.5]

Answer :

Jan.-2020, Set-3, Q1(b) M[7]

For answer refer Unit-1, Q57.

OR

Q2. (a) Test the convergence of the series $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\ldots$

Answer :

Jan.-2020, Set-3, Q2(a) M[8]

Given series is,

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\ldots$$

$$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\frac{n^n}{n!}x^n$$

$$u_n=\frac{n^n}{n!}x^n$$

$$u_{n+1}=\frac{(n+1)^{n+1}}{(n+1)!}x^{n+1}$$

$$\frac{u_n}{u_{n+1}}=\frac{\frac{n^n}{n!}.x^n}{\frac{(n+1)^{n+1}}{(n+1)!}.x^{n+1}}$$

$$=\frac{n^n}{n!}x^n\times\frac{(n+1)!}{(n+1)^{n+1}.x^{n+1}}$$

$$=\frac{n^n x^n.(n+1)n!}{n!(n+1)^n(n+1)x^n x}=\frac{n^n}{(n+1)^n}\cdot\frac{1}{x}$$

$$=\frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$$\Rightarrow\quad\frac{u_n}{u_{n+1}}=\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

Applying limit on both sides,

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\frac{1}{x}$$

$$=\frac{1}{x}.\frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=\frac{1}{x}\left(\frac{1}{e}\right)\qquad\left[\because\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\right]$$

$$\Rightarrow\quad\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{ex}$$

(b) Determine the current i(t) in an L-C-R circuit with E.M.F $E(t) = E_0 \sin wt$ in case the circuit is tuned to resonance so that $w^2 = \frac{1}{LC}$ and $\frac{R}{L}$ is so small and assuming that $q(0) = i(0) = 0$.

Answer : **Jan.-2020, Set-2, Q5(b) M7**

Given differential equation is,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0 \sin wt \quad \text{...(1)}$$

Initial condition is,

At, $t = 0, q = 0$

Equation (1) can be written as,

$$LD^2q + RDq + \frac{q}{C} = E_0 \sin wt$$

$$\Rightarrow \quad (LD^2 + RD + \frac{1}{C})q = E_0 \sin wt \quad \text{...(2)}$$

The complete solution is given as,

$$q = C.F + P.I$$

Complementary Function

The auxiliary equation is,

$$f(m) = 0$$

$$\Rightarrow \quad Lm^2 + Rm + \frac{1}{C} = 0$$

$$\Rightarrow \quad m = \frac{-R \pm \sqrt{R^2 - 4L\left(\frac{1}{C}\right)}}{2L}$$

$$= \frac{1}{2L}\left[-R \pm \sqrt{R^2 - \frac{4L}{C}}\right]$$

$$= \frac{-R}{2L} \pm \sqrt{\frac{R^2}{(2L)^2} - \frac{4L}{(2L)^2 C}}$$

$$= -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{4L}{4L^2 C}}$$

$$= \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Since, $\frac{R}{L}$ is small,

$$m = \frac{-R}{2L} \pm i\frac{1}{\sqrt{(LC)}}$$

$$= \frac{-R}{2L} \pm i(w) \qquad \left[\because w^2 = \frac{1}{LC}\right]$$

$$\Rightarrow \quad m = -\frac{R}{2L} \pm iw$$

Roots are complex conjugate

$$C.F = e^{\frac{-R}{2L}t}(c_1 \cos wt + c_2 \sin wt)$$

Neglecting the terms in $\left(\frac{R}{L}\right)^2$,

$$C.F = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos wt + c_2 \sin wt) \quad \text{...(3)}$$

Particular Integral

The particular integral is,

$$P.I = \frac{E_0 \sin wt}{f(D)}$$

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$$\Rightarrow \frac{yy'}{b^2+\lambda} = \frac{-x}{a^2+\lambda}$$

$$\Rightarrow (a^2+\lambda)yy' = -x(b^2+\lambda)$$

$$\Rightarrow a^2yy' + \lambda yy' = -xb^2 - x\lambda$$

$$\Rightarrow \lambda yy' + \lambda x = -b^2x - a^2yy'$$

$$\Rightarrow \lambda(x+yy') = -(b^2x + a^2yy')$$

$$\Rightarrow \lambda = -\frac{(b^2x+a^2yy')}{x+yy'} \quad \text{... (2)}$$

Substituting equation (2) in equation (1),

$$\frac{x^2}{a^2+\left(\frac{-(b^2x+a^2yy')}{x+yy'}\right)} + \frac{y^2}{b^2+\left(\frac{-(b^2x+a^2yy')}{x+yy'}\right)} = 1$$

$$\Rightarrow \frac{x^2}{a^2-\frac{(b^2x+a^2yy')}{x+yy'}} + \frac{y^2}{b^2-\frac{(b^2x+a^2yy')}{x+yy'}} = 1$$

$$\Rightarrow \frac{x^2}{\frac{a^2(x+yy')-(b^2x+a^2yy')}{x+yy'}} + \frac{y^2}{\frac{b^2(x+yy')-(b^2x+a^2yy')}{x+yy'}} = 1$$

$$\Rightarrow \frac{x^2(x+yy')}{a^2x-a^2yy'-b^2x-a^2yy'} + \frac{y^2(x+yy')}{b^2x+b^2yy'-b^2x-a^2yy'} = 1$$

$$\Rightarrow \frac{x^2(x+yy')}{a^2x-b^2x} + \frac{y^2(x+yy')}{b^2yy'-a^2yy'} = 1$$

$$\Rightarrow \frac{x^2(x+yy')}{x(a^2-b^2)} + \frac{y^2(x+yy')}{yy'(b^2-a^2)} = 1$$

$$\Rightarrow \frac{x^2(x+yy')}{(a^2-b^2)x} - \frac{y^2(x+yy')}{(a^2-b^2)yy'} = 1$$

$$\Rightarrow \frac{x(x+yy')}{a^2-b^2} - \frac{y(x+yy')}{(a^2-b^2)y'} = 1$$

$$\Rightarrow \frac{(x+yy')}{a^2-b^2}\left[x-\frac{y}{y'}\right] = 1$$

$$\Rightarrow (x+yy')\left(x-\frac{y}{y'}\right) = a^2-b^2 \quad \text{... (3)}$$

For orthogonal trajectories, replace y' with $\frac{-1}{y'}$ in equation (3),

$$\left(x+y\left(\frac{-1}{y'}\right)\right)\left(x-\frac{y}{\left(\frac{-1}{y'}\right)}\right) = a^2-b^2$$

$$\Rightarrow \left(x-\frac{y}{y'}\right)(x+yy') = a^2-b^2 \quad \text{... (4)}$$

Equations (3) and (4) are same, i.e., D.E of the given family and the D.E of its orthogonal trajectories are same. Therefore, the given family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda} = 1$ is self-orthogonal.

Q5. (a) Solve the D.E $(D^2+1)y = x^2 \sin 2x$.

Answer :

Jan.-2020, Set-2, Q5(a) M[8]

Given differential equation is,

$$(D^2+1)y = x^2 \sin 2x \quad \text{... (1)}$$

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Comparing equations (1) and (2),

$$M = y^4 \qquad N = y^3x - x^{-3}$$

$$\frac{\partial M}{\partial y} = 4y^3 \qquad \frac{\partial N}{\partial x} = y^3 - (-3)x^{-4}$$

$$\Rightarrow \frac{\partial N}{\partial x} = y^3 + 3x^{-4}$$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation (1) is not an exact differential equation

Consider,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y^3x - x^{-3}}(4y^3 - y^3 - 3x^{-4})$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3y^3 - 3x^{-4}}{xy^3 - x^{-3}}$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3(y^3 - x^{-4})}{x(y^3 - x^{-4})}$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3}{x}$$

Integrating factor $(I.F) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$

$$\Rightarrow I.F = e^{\int \frac{3}{x}dx}$$

$$\Rightarrow I.F = e^{3\int \frac{1}{x}dx}$$

$$\Rightarrow I.F = e^{3\log x}$$

$$\Rightarrow I.F = e^{\log x^3}$$

$$\Rightarrow I.F = x^3$$

Multiplying equation (1) by $I.F$,

$$y^4x^3dx + x^3(y^3x - x^{-3})dy = 0$$

$$\Rightarrow y^4x^3dx + (x^4y^3 - 1)dy = 0 \qquad \text{... (3)}$$

Comparing equations (2) and (3),

$$M = y^4x^3 \qquad N = x^4y^3 - 1$$

$$\frac{\partial M}{\partial y} = 4y^3x^3 \qquad \frac{\partial N}{\partial x} = 4x^3y^3 - 0$$

$$\Rightarrow \frac{\partial N}{\partial x} = 4x^3y^3$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation (3) is an exact differential equation

The general solution of equation (3) is given as,

$$\int_{(y \text{ constant})} M\,dx + \int_{(\text{terms independent of } x)} N\,dy = c$$

$$\Rightarrow \int y^4x^3dx + \int(-1)dy = c$$

$$\Rightarrow y^4\int x^3\,dx - \int dy = c$$

$$\Rightarrow \frac{y^4x^4}{4} - y = c$$

$$\Rightarrow y^4x^4 - 4y = 4c$$

$$\Rightarrow y^4x^4 - 4y = C$$

$\therefore$ The general solution is $y^4x^4 - 4y = C$

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Differentiating equation (1) with respect to x,

$$f_x(x, y) = \frac{d}{dx}[e^x \cos y]$$

$$= \cos y \frac{d}{dx} e^x$$

$$\Rightarrow \quad f_x(x, y) = e^x \cos y$$

$$f_x(x_0, y_0) = f_x\left(1, \frac{\pi}{4}\right) = e^1 \cos\left(\frac{\pi}{9}\right) = e.\frac{1}{\sqrt{2}} = \frac{e}{\sqrt{2}}$$

Differentiating equation (1) with respect to y,

$$f_y(x, y) = \frac{d}{dy}[e^x \cos y]$$

$$= e^x \frac{d}{dy} \cos y$$

$$\Rightarrow \quad f_y(x, y) = e^x(-\sin y)$$

$$f_y(x_0, y_0) = f_y\left(1, \frac{\pi}{4}\right) = e^1\left(-\sin\frac{\pi}{4}\right) = e.\frac{-1}{\sqrt{2}} = \frac{-e}{\sqrt{2}}$$

Differentiating equation (3) with respect to x,

$$f_{xx}(x, y) = \frac{d}{dx}[e^x \cos y]$$

$$= \cos y \frac{d}{dx}[e^x]$$

$$\Rightarrow \quad f_{xx}(x, y) = e^x \cos y$$

$$f_{xx}(x_0, y_0) = f_{xx}\left(1, \frac{\pi}{4}\right) = e^1 \cos\frac{\pi}{4} = e.\frac{1}{\sqrt{2}} = \frac{e}{\sqrt{2}}$$

Differentiating equation (4) with respect to y,

$$f_{yy}(x, y) = \frac{d}{dy}[e^x(-\sin y)]$$

$$= e^x \frac{d}{dy}(-\sin y)$$

$$\Rightarrow \quad f_{yy}(x, y) = -e^x \cos y$$

$$f_{yy}(x_0, y_0) = f_{yy}\left(1, \frac{\pi}{4}\right) = -e^1 \cos\left(\frac{\pi}{4}\right) = -e.\frac{1}{\sqrt{2}} = \frac{-e}{\sqrt{2}}$$

Differentiating equation (3) with respect to 'y',

$$f_{xy}(x, y) = \frac{d}{dy}.[e^x \cos y]$$

$$= e^x \frac{d}{dy} \cos y$$

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Substituting equations (4) and (10) in equation (3),

$$y = c_1e^{-2x} + c_2e^{2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x - \frac{27}{2}x - 2$$

(b) Solve the D.E $(D^2 + 4)y = 4\tan 2x$ by the method of variation of parameters.

Answer : Jan.-2020, Set-2, Q6(b) M[8]

For answer refer Unit-3, Q28.

Q7. (a) If $u = r^n(3\cos^2\theta - 1)$ then prove that $\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u}{\partial\theta}\right] = 0.$

Answer : Jan.-2020, Set-2, Q7(a) M[8]

Given that,

$$u = r^n(3\cos^2\theta - 1) \quad \text{... (1)}$$

Differentiating equation (1) partially with respect to r,

$$\frac{\partial u}{\partial r} = n.r^{n-1}(3\cos^2\theta - 1)$$

Multiplying above equation by r^2,

$$r^2\frac{\partial u}{\partial r} = r^2.n.r^{n-1}(3\cos^2\theta - 1)$$

$$\Rightarrow \quad r^2\frac{\partial u}{\partial r} = n.r^{n-1+2}(3\cos^2\theta - 1)$$

$$\Rightarrow \quad r^2\frac{\partial u}{\partial r} = n.r^{n+1}(3\cos^2\theta - 1)$$

Differentiating above equation partially with respect to r,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = n(3\cos^2\theta - 1).\frac{\partial}{\partial r}r^{n+1}$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = n(3\cos^2\theta - 1)(n+1)r^{n+1-1}$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = n(3\cos^2\theta - 1)(n+1)r^n$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = n(n+1)[(3\cos^2\theta - 1)r^n]$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = n(n+1)u \ [\because \text{From equation (1)}] \text{ ... (2)}$$

Differentiating equation (1) partially with respect to θ,

$$\frac{\partial u}{\partial\theta} = r^n(3(2\cos\theta(-\sin\theta)) - 0)$$

$$\Rightarrow \quad \frac{\partial u}{\partial\theta} = r^n(-6\cos\theta\sin\theta)$$

Multiplying above equation by $\sin\theta$,

$$\sin\theta\frac{\partial u}{\partial\theta} = -6r^n\cos\theta\sin^2\theta$$

Differentiating above equation partially with respect to θ,

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n.\frac{\partial}{\partial\theta}(\cos\theta\sin^2\theta)$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n(\sin^2\theta(-\sin\theta) + \cos\theta.2\sin\theta.\cos\theta)$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = -6r^n\sin\theta(2\cos^2\theta - \sin^2\theta)$$

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Q6. (a) Solve the D.E $(D^2 - 4)y = x \sinh x + 54x + 8$.

Jan.-2020, Set-2, Q6(a) M[7]

Answer :

Given differential equation is,

$$(D^2 - 4)y = x \sinh x + 54x + 8 \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$f(D) = D^2 - 4 \text{ and } X = x \sinh x + 54x + 8$$

The general solution of a non-homogeneous linear differential equation is,

$$y = C.F + P.I \quad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e., $m^2 - 4 = 0$

$\Rightarrow \quad (m+2)(m-2) = 0$

$\Rightarrow \quad m_1 = -2, m_2 = 2$

The roots are real and distinct

The complementary function is given as,

$$C.F = c_1 e^{-2x} + c_2 e^{2x} \quad \text{... (4)}$$

The particular integral is,

$$P.I = \frac{X}{f(D)}$$

$$= \frac{x \sinh x + 54x + 8}{D^2 - 4}$$

$$= \frac{x\left(\frac{e^x - e^{-x}}{2}\right) + 54x + 8}{D^2 - 4}$$

$$= \frac{xe^x}{2(D^2 - 4)} + \left(\frac{-xe^{-x}}{2(D^2 - 4)}\right) + \frac{54x}{D^2 - 4} + \frac{8}{D^2 - 4}$$

$$\Rightarrow \quad P.I = P.I_1 + P.I_2 + P.I_3 + P.I_4 \quad \text{... (5)}$$

Consider,

$$P.I_1 = \frac{xe^x}{2(D^2 - 4)}$$

$$= \frac{1}{2}\left[\frac{e^x}{(D+1)^2 - 4} \cdot x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{D^2 + 2D + 1 - 4} \cdot x\right] = \frac{1}{2}\left[\frac{e^x}{D^2 + 2D - 3} \cdot x\right]$$

$$= \frac{e^x}{2}\left[\frac{x}{-3\left[1 + \frac{D^2 + 2D}{-3}\right]}\right]$$

$$= \frac{e^x}{2(-3)}\left[\frac{x}{1 - \left(\frac{D^2 + 2D}{3}\right)}\right] = \frac{-e^x}{6}\left[1 - \left(\frac{D^2 + 2D}{3}\right)\right]^{-1} x$$

$$= \frac{-e^x}{6}\left[1 + \left(\frac{D^2 + 2D}{3}\right) + \left(\frac{D^2 + 2D}{3}\right)^2 + \ldots\right]x = \frac{-e^x}{6}\left[x + \frac{D^2}{3}(x) + \frac{2D}{3}(x) + \frac{D^4}{9}(x) + \frac{4D^2}{9}(x) + \frac{4D^3}{9}(x) + \ldots\right]$$

$$= \frac{-e^x}{6}\left[x + 0 + \frac{2}{3}(1) + 0 + 0 + 0 + \ldots\right]$$

$$\Rightarrow \quad P.I_1 = \frac{-e^x}{6}\left[x + \frac{2}{3}\right]$$

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By D'Alembert's ratio test, $\sum u_n$

(i) Converges if $\frac{1}{ex} > 1$ i.e., $x < \frac{1}{e}$

(ii) Diverges if $\frac{1}{ex} < 1$ i.e., $x > \frac{1}{e}$

(iii) Test fails when $\frac{1}{ex} = 1$ i.e., $x = \frac{1}{e}$

When $x = \frac{1}{e}$, $\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot e$

Applying log on both sides,

$$\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left[\frac{e}{\left(1+\frac{1}{n}\right)^n}\right]$$

$$\Rightarrow \log\left(\frac{u_n}{u_{n+1}}\right) = \log e - \log\left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \log\left(\frac{u_n}{u_{n+1}}\right) = 1 - n\log\left(1+\frac{1}{n}\right) \qquad [\because \log_e e = 1]$$

Multiplying n on both sides,

$$n\log\left(\frac{u_n}{u_{n+1}}\right) = n\left[1 - n\log\left(1+\frac{1}{n}\right)\right]$$

$$\Rightarrow n\log\left(\frac{u_n}{u_{n+1}}\right) = n - n^2\log\left(1+\frac{1}{n}\right)$$

Applying limit on both sides,

$$\lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \lim_{n\to\infty}\left(n - n^2\log\left(1+\frac{1}{n}\right)\right)$$

$$= \lim_{n\to\infty}\left[n - n^2\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right)\right]$$

$$\left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

$$= \lim_{n\to\infty}\left[n - \frac{n^2}{n} + \frac{n^2}{2n^2} - \frac{n^2}{3n^3} + \ldots\right]$$

$$= \lim_{n\to\infty}\left[n - n + \frac{1}{2} - \frac{1}{3n} + \ldots\right]$$

$$= \lim_{\frac{1}{n}\to 0}\left[\frac{1}{2} - \frac{1}{3n} + \ldots\right]$$

$$= \frac{1}{2} - \frac{1}{3}(0) + \ldots$$

$$\Rightarrow \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1}{2} < 1$$

By logarithmic test, the series is divergent at $x = \frac{1}{e}$

Thus, the series converges if $x < \frac{1}{e}$ and diverges if $x \geq \frac{1}{e}$

(b) Test the convergence of the series $\sum \frac{(-1)^{n-1}}{\sqrt{n(n+1)(n+2)}}$

Answer : Jan.-2020, Set-3, Q2(b)

Given series is,

$$\sum \frac{(-1)^{n-1}}{\sqrt{n(n+1)(n+2)}}$$

Let,

$$u_n = \frac{1}{\sqrt{n(n+1)(n+2)}}$$

$$u_{n+1} = \frac{1}{\sqrt{(n+1)(n+2)(n+3)}}$$

$$u_n - u_{n+1} = \frac{1}{\sqrt{n(n+1)(n+2)}} - \frac{1}{\sqrt{(n+1)(n+2)(n+3)}}$$

$$\Rightarrow u_n - u_{n+1} = \frac{1}{\sqrt{(n+1)(n+2)}}\left[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+3}}\right]$$

$$\Rightarrow u_n - u_{n+1} = \frac{\sqrt{n+3} - \sqrt{n}}{\sqrt{(n+1)(n+2)n(n+3)}} > 0$$

$$\Rightarrow u_n - u_{n+1} > 0$$

$$\Rightarrow u_n > u_{n+1}, \forall n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}}$$

$$\Rightarrow \lim_{n\to\infty} u_n = \frac{1}{\infty} = 0$$

$\therefore$ From Leibnitz test,

$$\sum \frac{(-1)^{n-1}}{\sqrt{n(n+1)(n+2)}} \text{ is convergent.}$$

Q3. (a) Solve the D.E $(1+x^2)\frac{dy}{dx} - 2xy = (1+x^2)$; $y(0) = 1$

Answer : Jan.-2020, Set-3, Q3(a)

Given differential equation is,

$$(1+x^2)\frac{dy}{dx} - 2xy = 2x(1+x^2)$$

$$\Rightarrow \frac{dy}{dx} - \frac{2xy}{1+x^2} = 2x$$

Equation (1) represents a linear differential equation of the form,

$$\frac{dy}{dx} + Py = Q$$

Comparing equations (1) and (2),

$$P = \frac{-2x}{1+x^2} \; ; Q = 2x$$

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(b) Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$ taken over the volume bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$ and $z = 0$, $z = 1$.

Answer : Jan.-2020, Set-2, Q10(b) M

Given integral is,

$$\iiint_V (x+y+z)\,dx\,dy\,dz$$

And the planes are $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

$$\iiint_V (x+y+z)\,dx\,dy\,dz = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1}(x+y+z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1\left[\int_0^1 (x+y+z)\,dz\right]dy\,dx$$

$$= \int_0^1\int_0^1\left[xz+yz+\frac{z^2}{2}\right]_0^1 dy\,dx$$

$$= \int_0^1\int_0^1\left(x(1)+y(1)+\frac{(1)^2}{2}-x(0)-y(0)-\frac{(0)^2}{2}\right)dy\,dx$$

$$= \int_0^1\int_0^1\left[x+y+\frac{1}{2}-0-0-0\right]dy\,dx$$

$$= \int_0^1\int_0^1\left(x+y+\frac{1}{2}\right)dy\,dx$$

$$= \int_0^1\left[\int_0^1\left(x+y+\frac{1}{2}\right)dy\right]dx$$

$$= \int_0^1\left[xy+\frac{y^2}{2}+\frac{1}{2}y\right]_0^1 dx$$

$$= \int_0^1\left(x(1)+\frac{(1)^2}{2}+\frac{1}{2}(1)-x(0)-\frac{0^2}{2}-\frac{1}{2}(0)\right)dx$$

$$= \int_0^1\left[x+\frac{1}{2}+\frac{1}{2}-0-0-0\right]$$

$$= \int_0^1 (x+1)\,dx$$

$$= \left(\frac{x^2}{2}+x\right)_0^1$$

$$= \frac{(1)^2}{2}+1-\frac{0^2}{2}-0$$

$$= \frac{3}{2}$$

$$\therefore \iiint_V (x+y+z)\,dx\,dy\,dz = \frac{3}{2}$$

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$$\iint r^3\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=2\sin\theta}^{r=4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\frac{1}{4}\left[(4\sin\theta)^4-(2\sin\theta)^4\right]d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\frac{1}{4}(256\sin^4\theta-16\sin^4\theta)\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\frac{1}{4}\left[240\sin^4\theta\right]d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}60\sin^4\theta\,d\theta$$

$$= 60\int_{\theta=0}^{\theta=\pi}\sin^4\theta\,d\theta$$

$$= 60\left[2\int_{\theta=0}^{\theta=\frac{\pi}{2}}\sin^4\theta\,d\theta\right]$$

$$= 120\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\theta$$

$$\left[\because \int\sin^n\theta\,d\theta = \frac{n-1}{n}I_{n-2}\right]$$

$$= 120\times\frac{1\times3}{4\times2}\times\frac{\pi}{2}$$

$$= \frac{45\pi}{2}$$

$$\therefore \iint r^3\,dr\,d\theta = \frac{45\pi}{2}$$

(b) Find the volume under the paraboloid $x^2 + y^2 + z = 16$ over rectangle $x = \pm a$, $y = \pm b$ using triple integral.

Answer : **Jan.-2020, Set-2, Q9(b) M[7]**

Given paraboloid is,

$$x^2+y^2+z=16 \quad \ldots(1)$$

Dimensions of rectangle are $x = \pm a$ and $y = \pm b$ i.e., x ies from $-a$ to a and y varies from $-b$ to b

Over the rectangle on xy-plane, $z = 0$

From equation (1),

$$z = 16 - x^2 - y^2$$

$\therefore$ z varies from 0 to $16 - x^2 - y^2$

The volume under the paraboloid is given as,

$$V = \int_{-a}^{a}\int_{-b}^{b}\int_0^{16-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-a}^{a}\int_{-b}^{b}\left[\int_0^{16-x^2-y^2} dz\right]dy\,dx$$

$$= \int_{-a}^{a}\int_{-b}^{b}[z]_0^{16-x^2-y^2}\,dy\,dx$$

$$= \int_{-a}^{a}\int_{-b}^{b}(16-x^2-y^2-0)\,dy\,dx$$

$$= \int_{-a}^{a}\int_{-b}^{b}(16-x^2-y^2)\,dy\,dx$$

$$= \int_{-a}^{a}\left[\int_{-b}^{b}(16-x^2-y^2)\,dy\right]dx$$

$$= \int_{-a}^{a}\left[\int_{-b}^{b}16\,dy-\int_{-b}^{b}x^2\,dy-\int_{-b}^{b}y^2\,dy\right]dx$$

$$= \int_{-a}^{a}\left[16\int_{-b}^{b}dy-x^2\int_{-b}^{b}dy-\left(\frac{y^3}{3}\right)_{-b}^{b}\right]dx$$

$$= \int_{-a}^{a}\left[16(y)_{-b}^{b}-x^2(y)_{-b}^{b}-\frac{1}{3}(b^3-(-b)^3)\right]dx$$

$$= \int_{-a}^{a}\left[16(b+b)-x^2(b+b)-\frac{1}{3}(b^3+b^3)\right]dx$$

$$= \int_{-a}^{a}\left[16(2b)-x^2(2b)-\frac{2b^3}{3}\right]dx$$

$$= 2b\int_{-a}^{a}\left(16-x^2-\frac{b^2}{3}\right)dx$$

$$= 2b\left[\int_{-a}^{a}16\,dx-\int_{-a}^{a}x^2dx-\int_{-a}^{a}\frac{b^2}{3}dx\right]$$

$$= 2b\left[16\int_{-a}^{a}dx-\left(\frac{x^3}{3}\right)_{-a}^{a}-\frac{b^2}{3}\int_{-a}^{a}dx\right]$$

$$= 2b\left[16(x)_{-a}^{a}-\frac{1}{3}(a^3-(-a)^3)-\frac{b^2}{3}(x)_{-a}^{a}\right]$$

$$= 2b\left[16(a+a)-\frac{1}{3}(a^3+a^3)-\frac{b^2}{3}(a+a)\right]$$

$$= 2b\left[16(2a)-\frac{1}{3}(2a^3)-\frac{b^2}{3}(2a)\right]$$

$$= 2b.2a\left[16-\frac{a^2}{3}-\frac{b^2}{3}\right]$$

$$= 4ab\left[16-\frac{a^2}{3}-\frac{b^2}{3}\right]$$

$\therefore$ Volume under the paraboloid and the given rectangle is $4ab\left[16-\frac{a^2}{3}-\frac{b^2}{3}\right]$ cubic units

OR

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The general solution of a linear differential equation is given as,

$$y(I.F) = \int Q(I.F)\,dx + c \quad \text{... (3)}$$

Integrating factor $(I.F) = e^{\int P\,dx}$

$$= e^{\int \frac{-2x}{1+x^2}dx}$$

$$= e^{-\log(1+x^2)}$$

$$= e^{\log(1+x^2)^{-1}}$$

$$= (1+x^2)^{-1}$$

$$\Rightarrow \quad I.F = \frac{1}{1+x^2}$$

Substituting the corresponding values in equation (3),

$$y\left(\frac{1}{1+x^2}\right) = \int 2x\left(\frac{1}{1+x^2}\right)dx + c$$

$$\Rightarrow \quad \frac{y}{1+x^2} = \int \frac{2x}{1+x^2}\,dx + c$$

$$\Rightarrow \quad \frac{y}{1+x^2} = \log(1+x^2) + c \quad \text{... (4)}$$

Initial condition is,

$y(0) = 1$ i.e., $x = 0, y = 1$

Substituting $x = 0, y = 1$ in equation (4),

$$\frac{1}{1+0} = \log(1+0) + c$$

$$\Rightarrow \quad 1 = \log 1 + c$$

$$\Rightarrow \quad 1 = 0 + c$$

$$\Rightarrow \quad c = 1$$

Substituting $c = 1$ in equation (4),

$$\frac{y}{1+x^2} = \log(1+x^2) + 1$$

$\therefore$ The general solution is,

$$\frac{y}{1+x^2} = \log(1+x^2) + 1$$

(b) Find the orthogonal trajectories of the confocal and coaxial parabolas $r = \frac{2a}{1+\cos\theta}$

Answer : **Jan.-2020, Set-3, Q3(b) M[7]**

For answer refer Unit-2, Q46.

OR

Q4. (a) Solve the D.E $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0$

Answer : **Jan.-2020, Set-3, Q4(a) M[7]**

Given differential equation is,

$$(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$Mdx + Ndy = 0 \quad \text{... (2)}$$

(b) Find the current in electrical circuit is given by $L\frac{di}{dt} + iR = E$ where E = 6 volts, R = 100 ohms, L = 0.1 henry and how long will be it before the current has reached one-half its maximum value.

Answer : Jan.-2020, Set-3, Q4(b) M[8]

Given differential equation is,

$$L\frac{di}{dt} + iR = E \quad \text{... (1)}$$

$E = 6$ volts

$R = 100$ ohms

$L = 0.1$ henry

Equation (1) can be written as,

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \quad \text{... (2)}$$

Equation (2) represents a linear differential equation of the form,

$$\frac{dy}{dx} + Py = Q \quad \text{... (3)}$$

Comparing equations (2) and (3),

$$P = \frac{R}{L}, Q = \frac{E}{L}$$

The solution of equation (2) is given as,

$$y(I.F) = \int Q(I.F)\,dx + c \quad \text{... (4)}$$

Where,

$$I.F = e^{\int P\,dx}$$
$$= e^{\int P\,dt}$$
$$= e^{\int \frac{R}{L}dt}$$
$$= e^{\frac{R}{L}t}$$

Substituting the corresponding values in equation (4),

$$i\left(e^{\frac{R}{L}t}\right) = \int \frac{E}{L}\left(e^{\frac{R}{L}t}\right)dt + c$$

$$\Rightarrow \quad ie^{\frac{R}{L}t} = \frac{E}{L}\int e^{\frac{R}{L}t}\,dt + c$$

$$\Rightarrow \quad ie^{\frac{R}{L}t} = \frac{E}{L}\cdot\frac{e^{\frac{R}{L}t}}{\frac{R}{L}} + c$$

$$\Rightarrow \quad ie^{\frac{R}{L}t} = \frac{E}{R}e^{\frac{R}{L}t} + c$$

$$\Rightarrow \quad i = \frac{E}{R}\frac{e^{\frac{R}{L}t}}{e^{\frac{R}{L}t}} + \frac{c}{e^{\frac{R}{L}t}}$$

$$\Rightarrow \quad i = \frac{E}{R} + ce^{-\frac{R}{L}t}$$

$$\therefore \quad i = \frac{E}{R} + ce^{-\frac{R}{L}t} \quad \text{... (5)}$$

If t tends to infinity, then,

$$i = \frac{E}{R} + ce^{-\infty}$$

$$\Rightarrow \quad i = \frac{E}{R}$$

$\therefore$ The maximum value of i is $\frac{E}{R}$

Substituting $t = 0$, $i = 0$ in equation (5),

$$0 = \frac{E}{R} + ce^{0}$$

$$\Rightarrow \quad 0 = \frac{E}{R} + c$$

$$\Rightarrow \quad c = -\frac{E}{R}$$

Substituting $c = -\frac{E}{R}$ in equation (5),

$$i = \frac{E}{R} + \left(-\frac{E}{R}\right)e^{-\frac{R}{L}t}$$

$$= \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\therefore \quad i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad \text{... (6)}$$

Let T represents the time when current i has reached half of its maximum value

$$\Rightarrow \quad i = \frac{E}{2R} \quad \text{... (7)}$$

Let $t = T$ when $i = \frac{E}{R}\left(1 - e^{-\frac{RT}{L}}\right)$ ... (8)

Comparing equations (7) and (8),

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{RT}{L}}\right)$$

$$\Rightarrow \quad \frac{1}{2} = 1 - e^{-\frac{R}{L}T}$$

$$\Rightarrow \quad e^{-\frac{R}{L}T} = 1 - \frac{1}{2}$$

$$\Rightarrow \quad e^{-\frac{R}{L}T} = \frac{1}{2}$$

Applying logarithm on both sides,

$$\ln\left(e^{-\frac{R}{L}T}\right) = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{-R}{L}T = \ln 1 - \ln 2$$

$$\Rightarrow \quad \frac{-R}{L}T = -\ln 2$$

$$\Rightarrow \quad T = \frac{L}{R}\ln 2$$

$$\therefore \quad T = \frac{L}{R}\ln 2 \quad \text{... (9)}$$

Substituting the corresponding values in equation (9),

$$T = \frac{0.1}{100}\ln 2$$

$$= 0.001(0.6931)$$

$$= 0.0006931$$

$$\therefore \quad T = 0.0006931 \text{ sec.}$$

$$= \frac{-e^{2x}}{4}\left[1-\frac{D^2-3D}{4}\right]^{-1}(1+x)$$

$$= \frac{-e^{2x}}{4}\left[1+\left(\frac{D^2-3D}{4}\right)+\left(\frac{D^2-3D}{4}\right)^2+.....\right](1+x) \qquad [\because (1-D)^{-1}=1+D+D^2+....]$$

$$= \frac{-e^{2x}}{4}\left[1+\frac{D^2}{4}-\frac{3D}{4}+\frac{D^4}{4}+\frac{9D^2}{4}-\frac{6D^3}{4}.....\right](1+x)$$

$$= \frac{-e^{2x}}{4}\left[(1+x)+\frac{D^2}{4}(1+x)-\frac{3D}{4}(1+x)+\frac{D^4}{4}(1+x)+\frac{9D^2}{4}(1+x)-\frac{6D^3}{4}(1+x)....\right]$$

$$= \frac{-e^{2x}}{4}\left[1+x+\frac{1}{4}[D^2(1)+D^2(x)]-\frac{3}{4}[D(1)+D(x)]+\frac{1}{4}(D^4(1)+D^4(x))+\frac{9}{4}(D^2(1)+D^2(x))-\frac{6}{4}(D^3(1)+D^3(x))....\right]$$

$$= \frac{-e^{2x}}{4}\left[1+x+\frac{1}{4}(0+0)-\frac{3}{4}(0+1)+\frac{1}{4}(0+0)+\frac{9}{4}(0+0)-\frac{6}{4}(0+0)\right]$$

$$= \frac{-e^{2x}}{4}\left[1+x+0-\frac{3}{4}+0+0-0\right]$$

$$= \frac{-e^{2x}}{4}\left[1+x-\frac{3}{4}\right]$$

$$= \frac{-e^{2x}}{4}\left[\frac{4+4x-3}{4}\right]$$

$$\Rightarrow \quad y_p = \frac{-e^{2x}}{16}(4x+1) \qquad ...(5)$$

Substituting equations (4) and (5) in equation (3),

$$y = c_1e^x + c_2e^{6x} - \frac{e^{2x}}{16}(4x+1)$$

(b) Solve the D.E $y'' - 2y' + y = e^x \log x$ by the Method of Variation of parameters.

Answer : **Jan.-2020, Set-3, Q6(b) M[8]**

Given differential equation is,

$$y'' - 2y' + y = e^x \log x$$

$$\Rightarrow \quad D^2y - 2Dy + y = e^x \log x$$

$$\Rightarrow \quad (D^2 - 2D + 1)y = e^x \log x \qquad ...(1)$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$[f(D)]y = Q(x) \qquad ...(2)$$

Comparing equations (1) and (2),

$$f(D) = D^2 - 2D + 1, \; Q(x) = e^x \log x$$

The general solution is given as,

$$y = C.F + P.I \qquad ...(3)$$

The auxiliary equation is, $f(m) = 0$

i.e., $m^2 - 2m + 1 = 0$

$$\Rightarrow \quad (m-1)^2 = 0$$

$$\Rightarrow \quad m = 1, 1$$

Roots are real and equal

The condition for maximum or minimum value is,

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x - 4x^3 = 0$$

$$\Rightarrow 4x(1 - x^2) = 0$$

$$\Rightarrow 4x(1 - x)(1 + x) = 0$$

$$\Rightarrow \quad 4x = 0 \;;\quad 1 - x = 0 \;;\quad 1 + x = 0$$

$$\Rightarrow \quad x = 0 \;;\quad x = 1 \;;\quad x = -1$$

Consider,

$$\frac{\partial f}{\partial y} = 0 \Rightarrow -4y + 4y^3 = 0$$

$$\Rightarrow 4y(y^2 - 1) = 0$$

$$\Rightarrow 4y(y - 1)(y + 1) = 0$$

$$\Rightarrow \quad 4y = 0 \;;\quad y - 1 = 0 \;;\quad y + 1 = 0$$

$$\Rightarrow \quad y = 0 \;;\quad y = 1 \;;\quad y = -1$$

The critical points are (0, 0), (0, 1), (0, –1), (1, 0), (1, 1), (1, – 1), (–1, 0), (–1, 1), (–1, –1)

Consider,

$$rt - s^2 = (4 - 12x^2)(-4 + 12y^2) - (0)^2$$

$$= 4(1 - 3x^2)\,4(3y^2 - 1) - 0$$

$$\Rightarrow \quad rt - s^2 = 16(1 - 3x^2)(3y^2 - 1)$$

$$rt - s^2|_{(0,0)} = 16(1 - 3(0)^2)(3(0)^2 - 1) = -16 < 0$$

$\Rightarrow$ (0, 0) is not an extreme value

$$rt - s^2|_{(0,1)} = 16(1 - 3(0)^2)(3(1)^2 - 1) = 32 > 0$$

$$r = 4 - 12(0)^2 = 4 > 0$$

$\Rightarrow$ f has minimum value at (0, 1)

$$rt - s^2|_{(0,-1)} = 16(1 - 3(0)^2)(3(-1)^2 - 1) = 32 > 0$$

$$r = 4 - 12(0)^2 = 4 > 0$$

$\Rightarrow$ f has minimum value at (0 – 1)

$$rt - s^2|_{(1,0)} = 16(1 - 3(1)^2)(3(0)^2 - 1) = 32 > 0$$

$$r = 4 - 12(1)^2 = -8 < 0$$

$\Rightarrow$ f has maximum value at (1, 0)

$$rt - s^2|_{(1,1)} = 16(1 - 3(1)^2)(3(1)^2 - 1) = -64 < 0$$

$\Rightarrow$ (1, 1) is not an extreme value

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

Substituting the corresponding values in above equation,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(2z) + \frac{\partial u}{\partial s}(2y)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = 2z\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s} \quad \text{...(4)}$$

Similarly,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

Substituting the corresponding values in above equation,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(2y) + \frac{\partial u}{\partial s}(2x)$$

$$\Rightarrow \quad \frac{\partial u}{\partial z} = 2y\frac{\partial u}{\partial r} + 2x\frac{\partial u}{\partial s} \quad \text{... (5)}$$

Consider,

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = (y^2 - zx)\left[2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s}\right] + (x^2 - yz)\left[2z\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s}\right]$$

$$+ (z^2 - xy)\left[2y\frac{\partial u}{\partial r} + 2x\frac{\partial u}{\partial s}\right] \quad [\because \text{ From equations (3), (4) and (5)}]$$

$$= 2x(y^2 - zx)\frac{\partial u}{\partial r} + 2z(y^2 - zx)\frac{\partial u}{\partial s} + 2z(x^2 - yz)\frac{\partial u}{\partial r} + 2y(x^2 - yz)\frac{\partial u}{\partial s}$$

$$+ 2y(z^2 - xy)\frac{\partial u}{\partial r} + 2x(z^2 - xy)\frac{\partial u}{\partial s}$$

$$= [2x(y^2 - zx) + 2z(x^2 - yz) + 2y(z^2 - xy)]\frac{\partial u}{\partial r}$$

$$+ [2z(y^2 - zx) + 2y(x^2 - yz) + 2x(z^2 - xy)]\frac{\partial u}{\partial s}$$

$$= [2xy^2 - 2x^2z + 2x^2z - 2yz^2 + 2yz^2 - 2xy^2]\frac{\partial u}{\partial r}$$

$$+ [2y^2z - 2xz^2 + 2x^2y - 2y^2z + 2xz^2 - 2x^2y]\frac{\partial u}{\partial s}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(0)$$

$$= 0$$

$$\therefore \quad (y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$$

(b) If u = x + y + z, uv = y + z, uvw = z then find $J\left(\frac{x, y, z}{u, v, w}\right)$

Answer : **Jan.-2020, Set-3, Q8(b) M[7]**

For answer refer Unit-4, Q31.

Q9. (a) Evaluate $\int_0^2 \int_0^{\sqrt{2x - x^2}} (x^2 + y^2)\,dx\,dy$ by changing in to polar co-ordinates.

Answer : **Jan.-2020, Set-3, Q9(a) M[7]**

For answer refer Unit-5, Q18.

(b) Evaluate $\iiint_V dx\,dy\,dz$ where V is the volume bounded by the planes x = 0, y = 0, z = 0, z = 1 and the cylinder $x^2 + y^2 = 1$

Answer : **Jan.-2020, Set-3, 9(b) M[8]**

Given planes are,

$x = 0, y = 0, z = 0, z = 1$

The required volume is given as,

$$\iiint_V dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r\left[\int_0^{r\cos\theta+2} dz\right] dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r[z]_0^{r\cos\theta+2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r(r\cos\theta+2-0)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r(r\cos\theta+2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 (r^2\cos\theta+2r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_1^2 (r^2\cos\theta+2r)\,dr\right] d\theta$$

$$= \int_0^{2\pi}\left[\cos\theta\int_1^2 r^2\,dr + 2\int_1^2 r\,dr\right] d\theta$$

$$= \int_0^{2\pi}\left[\cos\theta\left(\frac{r^3}{3}\right)_1^2 + 2\left(\frac{r^2}{2}\right)_1^2\right] d\theta$$

$$= \int_0^{2\pi}\left[\cos\theta\left(\frac{2^3}{3}-\frac{1^3}{3}\right) + 2\left(\frac{2^2}{2}-\frac{1^2}{2}\right)\right] d\theta$$

$$= \int_0^{2\pi}\left[\cos\theta\left(\frac{8}{3}-\frac{1}{3}\right) + 2\left(2-\frac{1}{2}\right)\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{7}{3}\cos\theta+3\right] d\theta$$

$$= \frac{7}{3}\int_0^{2\pi}\cos\theta\,d\theta + 3\int_0^{2\pi} d\theta$$

$$= \frac{7}{3}(\sin\theta)_0^{2\pi} + 3(\theta)_0^{2\pi}$$

$$= \frac{7}{3}(\sin 2\pi - \sin 0) + 3(2\pi - 0)$$

$$= \frac{7}{3}(0-0) + 3(2\pi)$$

$$= 0 + 6\pi = 6\pi$$

$\therefore$ Volume $= 6\pi$ cubic units.

(b) If 0 < a < b, then prove that $\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$.

Answer : Jan.-2020, Set-4, Q1(b) M[8]

For answer refer Unit-1, Q59 (Excluding (i) and (ii)).

OR

Q2. (a) Test the convergence of the series $x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \ldots$

Answer : Jan.-2020, Set-4, Q2(a) M[7]

Given series is,

$$x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \ldots$$

The above series is a power series of the form $\frac{(-1)^{n-1}x^n}{n^2}$

Let, $u_n = (-1)^{n-1} \cdot \frac{x^n}{n^2}$

$$u_{n+1} = (-1)^{n-1+1} \frac{x^{n+1}}{(n+1)^2}$$

$$\Rightarrow \quad u_{n+1} = \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{(-1)^n x^{n+1}}{(n+1)^2}}{\frac{(-1)^{n-1}x^n}{n^2}}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(-1)^n x^n . x . n^2}{(-1)^n . (-1) x^n . (n+1)^2}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{x . n^2}{(-1) n^2 \left(1+\frac{1}{n}\right)^2}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{-x}{\left(1+\frac{1}{n}\right)^2}$$

$$\underset{n\to\infty}{lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{lt} \frac{-x}{\left(1+\frac{1}{n}\right)^2}$$

$$= \underset{\frac{1}{n}\to 0}{lt} \frac{-x}{\left(1+\frac{1}{n}\right)^2}$$

$$= \frac{-x}{(1+0)^2}$$

$$\Rightarrow \quad \underset{n\to\infty}{lt} \frac{u_{n+1}}{u_n} = -x$$

The series converges for $|x| < 1$, diverges for $|x| > 1$ and test fails for $x = \pm 1$

If $x = 1$, $u_n = \frac{(-1)^{n-1}.(1)^n}{n^2}$

$$\Rightarrow \quad u_n = \frac{(-1)^{n-1}}{n^2}$$

The above series represents the alternating series of form $\sum \frac{(-1)^{n-1}}{n^2}$

Let, $v_n = \frac{1}{n^2}$

$$v_{n+1} = \frac{1}{(n+1)^2}$$

$$v_n - v_{n+1} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$= \frac{(n+1)^2 - n^2}{n^2(n+1)^2}$$

$$= \frac{n^2+1+2n-n^2}{n^2(n+1)^2}$$

$$= \frac{1+2n}{n^2(n+1)^2}$$

$$\Rightarrow \quad v_n - v_{n+1} > 0$$

$$\Rightarrow \quad v_n > v_{n+1}$$

$$\underset{n\to\infty}{lt} v_n = \underset{n\to\infty}{lt} \frac{1}{n^2}$$

By P-test, v_n converges.

$\therefore$ The series converges if $x = 1$

If $x = -1$, $u_n = \frac{(-1)^{n-1}(-1)^n}{n^2}$

$$\Rightarrow \quad u_n = \frac{(-1)^{2n-1}}{n^2}$$

$$\Rightarrow \quad |u_n| = \frac{1}{n^2}$$

$$\underset{n\to\infty}{lt} |u_n| = \underset{n\to\infty}{lt} \frac{1}{n^2}$$

By P-test, $|u_n|$ converges

$\therefore$ The series converges if $x = -1$, since it is absolutely convergent.

Hence, the given series is convergent for $-1 \le x \le 1$.

(b) Expand $\tan^{-1}(x)$ about origin.

Answer : Jan.-2020, Set-4, Q2(b) M[8]

Given function is,

$$f(x) = \tan^{-1}x \quad \ldots (1)$$

Differentiating equation (1) with respect to x,

$$f'(x) = \frac{1}{1+x^2}$$

$$\Rightarrow \quad (1+x^2) f'(x) = 1. \quad \ldots (2)$$

$$f(0) = \tan^{-1}0 = 0$$

$$f'(0) = \frac{1}{1+0} = 1$$

Differentiating equation (2) with respect to 'x',

$$2x f'(x) + (1+x^2) f''(x) = 0 \quad \ldots (3)$$

Substituting $x = 0$ in equation (3),

$$2(0)f'(0) + (1+0^2)f''(0) = 0$$

$$\Rightarrow \quad 0 + f''(0) = 0$$

$$\Rightarrow \quad f''(0) = 0$$

Differentiating equation (3) with respect to 'x',

$$2xf''(x) + 2.f'(x) + (2x)f''(x) + (1+x^2)f'''(x) = 0 \quad \ldots (4)$$

Substituting $x = 0$ in equation (4),

$$2(0)f''(0) + 2f'(0) + 2(0)f''(0) + (1+0^2)f'''(0) = 0$$

$$\Rightarrow \quad 0 + 2(1) + 0 + f'''(0) = 0$$

$$\Rightarrow \quad f'''(0) = -2$$

Integrating on both sides,

$$\int \frac{d\theta}{\theta - 30} = -k\int dt$$

$$\Rightarrow \quad \log(\theta - 30) = -kt + c \quad \text{... (1)}$$

Initial condition is, $\theta = 100$ at $t = 0$

From equation (1),

$$\log(100 - 30) = -k(0) + c$$

$$\Rightarrow \quad c = \log 70$$

Substituting the value of 'c' in equation (1),

$$\log(\theta - 30) = -kt + \log 70$$

$$\Rightarrow \quad kt = \log 70 - \log(\theta - 30) \quad \text{... (2)}$$

The temperature reduces to 70°C after 15 minutes i.e., $t = 15$ $\theta = 70$

From equation (2),

$$k(15) = \log 70 - \log(70 - 30)$$

$$\Rightarrow \quad 15k = \log 70 - \log 40$$

$$\Rightarrow \quad 15k = \log\left(\frac{70}{40}\right)$$

$$\Rightarrow \quad k = \frac{1}{15}\log\left(\frac{7}{4}\right)$$

Substituting the value of 'k' in equation (2),

$$\frac{1}{15}\log\left(\frac{7}{4}\right)t = \log 70 - \log(\theta - 30)$$

At $\theta = 40°$,

$$\frac{1}{15}\log\left(\frac{7}{4}\right)t = \log 70 - \log(40 - 30)$$

$$\Rightarrow \quad t = \frac{\log 70 - \log(40 - 30)}{\frac{1}{15}\log\left(\frac{7}{4}\right)}$$

$$\Rightarrow \quad t = \frac{15(\log 70 - \log 10)}{\log\left(\frac{7}{4}\right)}$$

$$\Rightarrow \quad t = 15\frac{\log\left(\frac{70}{10}\right)}{\log\left(\frac{7}{4}\right)}$$

$$\Rightarrow \quad t = 15\frac{\log 7}{\log\left(\frac{7}{4}\right)}$$

$$\Rightarrow \quad t = \frac{15 \times 0.845}{0.243}$$

$$\Rightarrow \quad t = 52.16$$

$$\Rightarrow \quad t \approx 52.2$$

$$\therefore \quad t = 52.2 \text{ minutes}$$

OR

Q4. (a) Solve the D.E $\left(y + \frac{x^2}{2} + \frac{y^3}{3}\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0$.

Answer : **Jan.-2020, Set-4, Q4(a) M[8]**

Given differential equation is,

$$\left(y + \frac{x^2}{2} + \frac{y^3}{3}\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$Mdx + Ndy = 0 \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$M = y + \frac{x^2}{2} + \frac{y^3}{3} \qquad N = \frac{1}{4}(x + xy^2)$$

$$\frac{\partial M}{\partial y} = 1 + 0 + \frac{3y^2}{3} \qquad \frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2(1))$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 1 + y^2 \qquad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$$

$\because \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation (1) is not an exact differential equation.

Consider,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x + xy^2}\left(1 + y^2 - \frac{1}{4}(1 + y^2)\right)$$

$$= \frac{4}{x + xy^2}\left(1 + y^2 - \frac{1}{4} - \frac{y^2}{4}\right)$$

$$= \frac{4}{x + xy^2}\left(\frac{3}{4} + \frac{3y^2}{4}\right)$$

$$= \frac{4}{x(1 + y^2)} \quad \frac{(1 + y^2)3}{4}$$

$$\Rightarrow \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3}{x}$$

Integrating factor, $(I.F) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$

$$\Rightarrow \quad I.F = e^{\int \frac{3}{x}dx}$$

$$\Rightarrow \quad I.F = e^{3\int \frac{1}{x}dx}$$

$$\Rightarrow \quad I.F = e^{3\log x}$$

$$\Rightarrow \quad I.F = e^{\log x^3}$$

$$\Rightarrow \quad I.F = x^3$$

Multiplying equation (1) by $I.F$

$$x^3\left(y + \frac{x^2}{2} + \frac{y^3}{3}\right)dx + \frac{x^3}{4}(x + xy^2)dy = 0$$

$$\Rightarrow \quad \left(x^3y + \frac{x^5}{2} + \frac{x^3y^3}{3}\right)dx + \left(\frac{x^4 + x^4y^2}{4}\right)dy = 0 \quad \text{... (3)}$$

(b) A particle of mass 4 gm executing simple harmonic motion has velocities 8 cm/sec. and 6 cm/sec. respectively. When it is at distance 3 cm and 4 cm from the centre of its path. Find its period and amplitude. Find also the force acting on the particle when it is at distance 1 cm from the centre.

Answer : **Jan.-2020, Set-4, Q5(b) M[7]**

Given that,

Mass of particle $(m) = 4$ gm

Velocity $(v_1) = 8$ cm/sec

Velocity $(v_2) = 6$ cm/sec

Distance $(x_1) = 3$ cm

Distance $(x_2) = 4$ cm

Distance $(x) = 1$ cm

If a particle executes SHM with velocities v_1 and v_2 at distances x_1 and x_2 respectively, then

$$\frac{1}{\omega} = \sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

Substituting the corresponding values in above equation,

$$\frac{1}{\omega} = \sqrt{\frac{(3)^2 - (4)^2}{(6)^2 - (8)^2}}$$

$$\Rightarrow \frac{1}{\omega} = \sqrt{\frac{9-16}{36-64}}$$

$$\Rightarrow \frac{1}{\omega} = \sqrt{\frac{-7}{-28}}$$

$$\Rightarrow \frac{1}{\omega} = \sqrt{\frac{1}{4}}$$

$$\Rightarrow \frac{1}{\omega} = \frac{1}{2}$$

$$\Rightarrow \omega = 2$$

Time period $(T) = \frac{2\pi}{\omega}$

$$\Rightarrow T = \frac{2\pi}{2}$$

$$\Rightarrow T = \pi$$

$\therefore$ Time period $(T) = \pi$ sec.

The velocity of a particle when it is at a distance of x cm is given as,

$$v^2 = \omega^2 (a^2 - x^2) \quad \text{... (1)}$$

where a is amplitude.

Substituting v_1 and x_1 values in equation (1),

$$(8)^2 = \omega^2 (a^2 - 3^2)$$

$$\Rightarrow 64 = \omega^2 (a^2 - 9) \quad \text{... (2)}$$

Substituting v_2 and x_2 values in equation (1),

$$(6)^2 = \omega^2 (a^2 - 4^2)$$

$$\Rightarrow 36 = \omega^2 (a^2 - 16) \quad \text{... (3)}$$

Dividing equation (2) by equation (3),

$$\frac{64}{36} = \frac{\omega^2(a^2 - 9)}{\omega^2(a^2 - 16)}$$

$$\Rightarrow \frac{16}{9} = \frac{a^2 - 9}{a^2 - 16}$$

$$\Rightarrow 16a^2 - 256 = 9a^2 - 81$$

$$\Rightarrow 16a^2 - 9a^2 = -81 + 256$$

$$\Rightarrow 7a^2 = 175$$

$$\Rightarrow a^2 = \frac{175}{7}$$

$$\Rightarrow a^2 = 25$$

$$\Rightarrow a = 5$$

$\therefore$ Amplitude $(a) = 5$ cm

The force acting on particle m at distance x is given as,

$$F = m\omega^2 x$$

Substituting the corresponding values in above equation,

$$F = 4(2)^2(1)$$

$$\Rightarrow F = 16$$

$\therefore$ Force $(F) = 16$ dynes.

OR

Q6. (a) Solve the D.E $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$.

Answer : **Jan.-2020, Set-4, Q6(a) M[7]**

Given differential equation is,

$$(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = D^3 - 5D^2 + 7D - 3, X = e^{2x}\cosh x$$

The general solution of a non-homogeneous linear differential equation is given as,

$$y = C.F + P.I \quad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e., $m^3 - 5m^2 + 7m - 3 = 0$

By trial and error method,

m = 1	1	−5	7	−3
	0	1	−4	3
	1	−4	3	0

$$\Rightarrow (m-1)(m^2 - 4m + 3) = 0$$

$$\Rightarrow (m-1)(m^2 - 3m - m + 3) = 0$$

$$\Rightarrow (m-1)(m(m-3) - 1(m-3)) = 0$$

$$\Rightarrow (m-1)(m-1)(m-3) = 0$$

$$\Rightarrow m = 1, 1, 3$$

Q5. (a) Solve the D.E $(D^2 + 2D + 1)y = x\cos x$ **Jan.-2020, Set-3, Q5(a)**

Answer :

For answer refer Unit-3, Q23.

(b) $(D^2 - 3D + 2)y = \sin(e^{-x})$ **Jan.-2020, Set-3, Q5(b)**

Answer :

For answer refer Unit-3, Q33.

OR

Q6. (a) Solve the D.E $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}(1 + x)$ **Jan.-2020, Set-3, Q6(a)**

Answer :

Given differential equation is,

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}(1+x)$$

The above equation can be written in symbolic form as,

$$(D^2 - 7D + 6)\,y = e^{2x}(1+x)$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$[f(D)]y = X$$

Comparing equations (1) and (2),

$$f(D) = D^2 - 7D + 6,\ X = e^{2x}(1+x)$$

The general solution of non-homogeneous linear differential equation is given as,

$$y = y_c + y_p$$

Auxiliary equation is, $f(m) = 0$

i.e., $m^2 - 7m + 6 = 0$

$\Rightarrow$ $m^2 - 6m - m + 6 = 0$

$\Rightarrow$ $m(m-6) - 1(m-6) = 0$

$\Rightarrow$ $(m-1)(m-6) = 0$

$\Rightarrow$ $m = 1, 6$

Roots are real and distinct

Complementary function is given as,

$$y_c = c_1e^x + c_2e^{6x}$$

Particular integral is given as,

$$y_p = \frac{X}{f(D)}$$

$$= \frac{e^{2x}(1+x)}{D^2 - 7D + 6}$$

$$= e^{2x}\frac{1}{(D+2)^2 - 7(D+2) + 6}(1+x)$$

$$= e^{2x}\frac{1}{D^2 + 4 + 4D - 7D - 14 + 6}(1+x)$$

$$= e^{2x}\frac{1}{D^2 - 3D - 4}(1+x)$$

$$= e^{2x}\frac{1}{-4\left[1 - \frac{D^2 - 3D}{4}\right]}(1+x)$$

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Comparing equations (1) and (2),

$$M = 3xy^2 - y^3 \qquad N = -(2x^2y - xy^2)$$

$$\frac{\partial M}{\partial y} = 6xy - 3y^2 \qquad \frac{\partial N}{\partial x} = -(4xy - y^2)$$

$\because \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation (1) is not an exact differential equation.

Integrating factor,

$$(I.F) = \frac{1}{Mx + Ny}$$

$$= \frac{1}{(3xy^2 - y^3)x + (-2x^2y + xy^2)y}$$

$$= \frac{1}{3x^2y^2 - xy^3 - 2x^2y^2 + xy^3}$$

$$\Rightarrow \quad I.F = \frac{1}{x^2y^2}$$

Multiplying equation (1) with $I.F$,

$$\frac{1}{x^2y^2}[(3xy^2 - y^3)dx - (2x^2y - xy^2)dy] = 0$$

$$\Rightarrow \quad \frac{3xy^2 - y^3}{x^2y^2}dx + \frac{[-2x^2y + xy^2]}{x^2y^2}dy = 0$$

$$\Rightarrow \quad \left(\frac{3xy^2}{x^2y^2} - \frac{y^3}{x^2y^2}\right)dx + \left[\frac{-2x^2y}{x^2y^2} + \frac{xy^2}{x^2y^2}\right]dy = 0$$

$$\Rightarrow \quad \left[\frac{3}{x} - \frac{y}{x^2}\right]dx + \left[\frac{-2}{y} + \frac{1}{x}\right]dy = 0$$

Comparing equations (2) and (3),

$$M = \frac{3}{x} - \frac{y}{x^2} \qquad N = \frac{-2}{y} + \frac{1}{x}$$

$$\frac{\partial M}{\partial y} = 0 - \frac{1}{x^2} \qquad \frac{\partial N}{\partial x} = 0 - \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{-1}{x^2} \qquad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{-1}{x^2}$$

$\because \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation (3) is an exact differential equation

The general solution of equation (3) is given as,

$$\int_{(y \text{ constant})} M\,dx + \int_{(\text{Terms independent of } x)} N\,dy = c$$

$$\Rightarrow \quad \left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \int \frac{-2}{y}\,dy = c$$

$$\Rightarrow \quad 3\int \frac{1}{x}\,dx - y\int \frac{1}{x^2}\,dx - 2\int \frac{1}{y}\,dy = c$$

$$\Rightarrow \quad 3\log x - y\left(\frac{-1}{x}\right) - 2\log y = c$$

$$\Rightarrow \quad 3\log x + \frac{y}{x} - 2\log y = c$$

$\therefore$ The general solution is $3\log x + \frac{y}{x} - 2\log y = c$

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$rt - s^2|_{(1,-1)} = 16(1 - 3(1)^2)(3(-1)^2 - 1) = -64 < 0$

$\Rightarrow$ $(1, -1)$ is not an extreme value

$rt - s^2|_{(-1,0)} = 16(1 - 3(-1)^2)(3(0)^2 - 1) = 32 > 0$

$r = 4 - 12(1)^2 = -8 < 0$

$\Rightarrow$ f has maximum value at $(-1, 0)$

$rt - s^2|_{(-1,1)} = 16(1 - 3(-1)^2)(3(1)^2 - 1) = -64 < 0$

$\Rightarrow$ $(-1, 1)$ is not an extreme value

$rt - s^2|_{(-1,-1)} = 16(1 - 3(-1)^2)(3(-1)^2 - 1) = -64 < 0$

$\Rightarrow$ $(-1, -1)$ is not an extreme value

$\therefore$ The extreme points are $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$

OR

Q8. (a) If $u = f(x^2 + 2yz, y^2 + 2zx)$ prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

Answer : **Jan.-2020, Set-3, Q8(a)**

Given function is,

$$u = f(x^2 + 2yz, y^2 + 2zx)$$

Let,

$$r = x^2 + 2yz$$

$$s = y^2 + 2zx$$

Partially differentiating equations (1) and (2) with respect to x,

$$\frac{\partial r}{\partial x} = 2x + 0, \quad \frac{\partial s}{\partial x} = 0 + 2z(1)$$

$$\Rightarrow \quad \frac{\partial r}{\partial x} = 2x; \quad \frac{\partial s}{\partial x} = 2z$$

Partially differentiating equations (1) and (2) with respect to y,

$$\frac{\partial r}{\partial y} = 0 + 2z(1); \quad \frac{\partial s}{\partial y} = 2y + 0$$

$$\Rightarrow \quad \frac{\partial r}{\partial y} = 2z; \quad \frac{\partial s}{\partial y} = 2y$$

Partially differentiating equations (1) and (2) with respect to z,

$$\frac{\partial r}{\partial z} = 0 + 2y(1); \quad \frac{\partial s}{\partial z} = 0 + 2x(1)$$

$$\Rightarrow \quad \frac{\partial r}{\partial z} = 2y; \quad \frac{\partial s}{\partial z} = 2x$$

From chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$$

Substituting the corresponding values in above equation,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(2x) + \frac{\partial u}{\partial s}(2z)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s}$$

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SOLUTIONS TO JANUARY-2020, SET-4 QP

Q1. (a) Test the convergence of the series $\frac{2}{3}x + \frac{3^2x^2}{4^2} + \frac{4^3x^3}{5^3} + \ldots\ldots$

Answer : Jan.-2020, Set-4, Q1(a) M[7]

Given series is,

$$\frac{2}{3}x + \frac{3^2x^2}{4^2} + \frac{4^3x^3}{5^3} + \ldots..$$

$$\Rightarrow \sum\left(\frac{n+1}{n+2}\right)^n x^n$$

Let, $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$

$$(u_n)^{1/n} = \left(\left(\frac{n+1}{n+2}\right)^n x^n\right)^{\frac{1}{n}}$$

$$\Rightarrow (u_n)^{1/n} = \left(\frac{n+1}{n+2}\right)^{n/n} x^{\frac{n}{n}}$$

$$\Rightarrow (u_n)^{1/n} = \left(\frac{n+1}{n+2}\right)x$$

$$\underset{n\to\infty}{lt}\,(u_n)^{1/n} = \underset{n\to\infty}{lt}\left(\frac{n+1}{n+2}\right)x$$

$$= \underset{n\to\infty}{lt}\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}x = \underset{\frac{1}{n}\to 0}{lt}\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)}x$$

$$= \frac{(1+0)}{(1+0)}x$$

$$\Rightarrow \underset{n\to\infty}{lt}\,(u_n)^{1/n} = x$$

By Cauchy's n^{th} root test, $\sum u_n$

(i) Converges, if $x < 1$

(ii) Diverges, if $x > 1$

(iii) Test fails, if $x = 1$

If $x = 1$, $u_n = \left(\frac{n+1}{n+2}\right)^n (1)^n$

$$\Rightarrow u_n = \left(\frac{n+1}{n+2}\right)^n$$

$$\underset{n\to\infty}{lt}\,u_n = \underset{n\to\infty}{lt}\left(\frac{n+1}{n+2}\right)^n$$

$$= \underset{n\to\infty}{lt}\frac{\left(n\left(1+\frac{1}{n}\right)\right)^n}{\left(n\left(1+\frac{2}{n}\right)\right)^n} = \underset{n\to\infty}{lt}\frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n\left(1+\frac{2}{n}\right)^n}$$

$$= \underset{n\to\infty}{lt}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{\underset{n\to\infty}{lt}\left(1+\frac{1}{n}\right)^n}{\underset{n\to\infty}{lt}\left(1+\frac{2}{n}\right)^n}$$

$$= \frac{e}{e^2}$$

$$\Rightarrow \underset{n\to\infty}{lt}\,u_n = \frac{1}{e} \neq 0$$

i.e., $\sum u_n$ is divergent, if $x = 1$

$\therefore$ The series $\left(\frac{n+1}{n+2}\right)^n x^n$ converges if $x < 1$ and diverges if $x \geq 1$.

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And cylinder, is $x^2 + y^2 = 1$ ... (1)

$\Rightarrow \quad y^2 = 1 - x^2$

$\Rightarrow \quad y = \sqrt{1 - x^2}$

From equation (1),

$x^2 + 0^2 = 1 \qquad [\because y = 0]$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = \pm 1$

$\therefore$ x varies from 0 to 1, y varies from 0 to $\sqrt{1 - x^2}$ and z varies from 0 to 1

The volume bounded by the planes and the cylinder is given as,

$$\iiint_V dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx\,dz$$

$$= \int_0^1 \int_0^1 \left[\int_0^{\sqrt{1-x^2}} dy\right] dx\,dz$$

$$= \int_0^1 \int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx\,dz$$

$$= \int_0^1 \int_0^1 (\sqrt{1-x^2} - 0)\,dx\,dz$$

$$= \int_0^1 \int_0^1 \sqrt{1-x^2}\,dx\,dz$$

$$= \int_0^1 \left[\int_0^1 \sqrt{1-x^2}\,dx\right] dz$$

$$= \int_0^1 \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 dz$$

$$= \int_0^1 \left[\frac{1}{2}\sqrt{1-1^2} + \frac{1}{2}\sin^{-1}(1) - \frac{0}{2}\sqrt{1-0^2} - \frac{1}{2}\sin^{-1}(0)\right] dz$$

$$= \int_0^1 \left[\frac{1}{2}(0) + \frac{1}{2}\left(\frac{\pi}{2}\right) - 0 - \frac{1}{2}(0)\right] dz$$

$$= \int_0^1 \left(0 + \frac{\pi}{4} - 0 - 0\right) dz$$

$$= \frac{\pi}{4}\int_0^1 dz$$

$$= \frac{\pi}{4}[z]_0^1$$

$$= \frac{\pi}{4}(1 - 0)$$

$$= \frac{\pi}{4}$$

$\therefore$ Volume $= \dfrac{\pi}{4}$ cubic units

OR

Q10. (a) By change of order of integration evaluate $\int_0^a \int_x^a (x^2 + y^2)\,dx\,dy$

Answer : **Jan.-2020, Set-3, Q10(a) M[7]**

For answer refer Unit-5, Q13.

(b) Evaluate $\iiint_V dV$, where V is the region that lies below the plane $z = x + 2$, above xy plane and between the cylinder $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Answer : **Jan.-2020, Set-3, Q10(b) M[8]**

Given equations are,

$x^2 + y^2 = 1$ ... (1)

$x^2 + y^2 = 4$ ... (2)

$z = x + 2$ ... (3)

The cylindrical coordinates are given as,

$x = r\cos\theta,$

$y = r\sin\theta,$

$z = z$

$\Rightarrow \quad dx\,dy\,dz = r\,dr\,d\theta\,dz$

and $x^2 + y^2 = r^2$ ... (4)

From equations (1) and (4),

$r^2 = 1$

$\Rightarrow \quad r = \pm 1$

From equations (2) and (4),

$r^2 = 4$

$\Rightarrow \quad r = \pm 2$

From equation (3),

$z = r\cos\theta + 2$

As the region of integration lies above the xy-plane (i.e., $z = 0$), below the plane $z = r\cos\theta + 2$ and between the cylinders, r varies from 1 to 2, θ varies from 0 to 2π and z varies from 0 to $r\cos\theta + 2$

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The complementary function is given as,

$$C.F = (c_1 + c_2x)e^x \quad \text{... (4)}$$

i.e., $C.F = c_1f_1 + c_2f_2$

$$W(x) = \begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix}$$

$$\Rightarrow \quad W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix}$$

$$\Rightarrow \quad W(x) = |e^x.e^x(1+x) - e^x.xe^x|$$

$$\Rightarrow \quad W(x) = |e^{2x} + xe^{2x} - xe^{2x}| = e^{2x}$$

The Particular Integral is given as,

$$P.I = Pf_1 + Rf_2 \quad \text{... (5)}$$

$$P = -\int \frac{Q(x)f_2}{W(x)}dx$$

$$= -\int \frac{e^x \log x.\, xe^x}{e^{2x}}dx$$

$$= -\int \frac{e^{2x}x\log x}{e^{2x}}dx$$

$$= -\int x\log x\, dx$$

$$= -\left[\log x\int x\,dx - \int \frac{d}{dx}(\log x)\int x\,dx.\,dx\right]$$

$$= -\left[\log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}.dx\right]$$

$$= -\left[\log x\frac{x^2}{2} - \frac{1}{2}\int x\,dx\right]$$

$$= -\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]$$

$$\Rightarrow \quad P = -\frac{x^2}{2}\log x + \frac{x^2}{4}$$

$$R = \int \frac{Q(x)f_1}{W(x)}dx$$

$$= \int \frac{e^x\log x.\,e^x}{e^{2x}}dx$$

$$= \int \frac{e^{2x}\log x}{e^{2x}}dx$$

$$= \int \log x\,dx$$

$$= \log x\int 1\,dx - \int \frac{d}{dx}(\log x)\int 1\,dx\,dx$$

$$= \log x(x) - \int \frac{1}{x}.x.dx$$

$$= x\log x - \int 1\,dx$$

$$\Rightarrow \quad R = x\log x - x$$

Substituting the corresponding values in equation (5)

$$P.I = \left(\frac{-x^2}{2}\log x + \frac{x^2}{4}\right)e^x + (x\log x - x)xe^x$$

$$= \frac{-e^xx^2}{2}\log x + \frac{e^xx^2}{4} + x^2e^x\log x - x^2e^x$$

$$\Rightarrow \quad P.I = \frac{x^2e^x\log x}{2} - \frac{3x^2e^x}{4} \quad \text{... (6)}$$

Substituting equations (4) and (6) in equation (3),

$$y = (c_1 + c_2x)e^x + \frac{x^2e^x\log x}{2} - \frac{3}{4}x^2e^x$$

$\therefore$ The solution is $y = (c_1 + c_2x)e^x + \dfrac{x^2e^x\log x}{2} - \dfrac{3}{4}x^2e^x$

Q7. (a) If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ then find $x^2\dfrac{\partial^2u}{\partial x^2} + 2xy\dfrac{\partial^2u}{\partial x\partial y} + y^2\dfrac{\partial^2u}{\partial y^2}$ using Euler's theorem.

Answer : **Jan.-2020, Set-3, Q7(a) M[illegible]**

For answer refer Unit-4, Q11.

(b) Find the extreme values of $2(x^2 - y^2) - x^4 + y^4$

Answer : **Jan.-2020, Set-3, Q7(b) M[illegible]**

Given function is,

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4 \quad \text{... (1)}$$

Differentiating equation (1) partially with respect to 'x'

$$\frac{\partial f}{\partial x} = 2[2x - 0] - 4x^3 + 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 4x - 4x^3 \quad \text{... (2)}$$

Differentiating equation (1) partially with respect to 'y'

$$\frac{\partial f}{\partial y} = 2(0 - 2y) - 0 + 4y^3$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = -4y + 4y^3 \quad \text{... (3)}$$

Differentiating equation (2) partially with respect to x

$$\frac{\partial^2 f}{\partial x^2} = 4(1) - 4(3x^2)$$

$$\Rightarrow \quad r = 4 - 12x^2$$

Differentiating equation (3) partially with respect to x

$$\frac{\partial^2 f}{\partial x\partial y} = -0 + 0$$

$$\Rightarrow \quad s = 0$$

Differentiating equation (3) partially with respect to y

$$\frac{\partial^2 f}{\partial y^2} = -4(1) + 4(3y^2)$$

$$\Rightarrow \quad t = -4 + 12y^2$$

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The complementary function is given as,

$$y_c = (c_1 + c_2x)e^x + c_3e^{3x} \quad \text{... (4)}$$

The particular integral is,

$$P.I = \frac{X}{f(D)}$$

$$= \frac{e^{2x}\cosh x}{D^3 - 5D^2 + 7D - 3}$$

$$= \frac{e^{2x}\left(\frac{e^x + e^{-x}}{2}\right)}{D^3 - 5D^2 + 7D - 3}$$

$$= \frac{e^{3x} + e^x}{2(D^3 - 5D^2 + 7D - 3)}$$

$$= \frac{1}{2}\left[\frac{e^{3x}}{D^3 - 5D^2 + 7D - 3} + \frac{e^x}{D^3 - 5D^2 + 7D - 3}\right]$$

$$\Rightarrow \quad P.I = \frac{1}{2}[P.I_1 + P.I_2] \quad \text{... (5)}$$

Consider,

$$P.I_1 = \frac{e^{3x}}{D^3 - 5D^2 + 7D - 3}$$

$$\Rightarrow \quad P.I_1 = \frac{e^{3x}}{(3)^3 - 5(3)^2 + 7(3) - 3}$$

$$\because \quad f(a) = 0, P.I_1 = \frac{xe^{3x}}{3D^2 - 10D + 7}$$

$$\Rightarrow \quad P.I_1 = \frac{xe^{3x}}{3(3)^2 - 10(3) + 7}$$

$$\Rightarrow \quad P.I_1 = \frac{xe^{3x}}{4} \quad \text{... (6)}$$

Consider,

$$P.I_2 = \frac{e^x}{D^3 - 5D^2 + 7D - 3}$$

$$\Rightarrow \quad P.I_2 = \frac{e^x}{(1)^3 - 5(1)^2 + 7(1) - 3}$$

$$\because \quad f(a) = 0, P.I_2 = \frac{xe^x}{3D^2 - 10D + 7}$$

$$\Rightarrow \quad P.I_2 = \frac{xe^x}{3(1)^2 - 10(1) + 7}$$

$$\because \quad f'(a) = 0, P.I_2 = \frac{x^2e^x}{6D - 10}$$

$$\Rightarrow \quad P.I_2 = \frac{x^2e^x}{6(1) - 10}$$

$$\Rightarrow \quad P.I_2 = -\frac{x^2e^x}{4} \quad \text{... (7)}$$

Substituting equations (6) and (7) in equation (5),

$$P.I = \frac{1}{2}\left[\frac{xe^{3x}}{4} - \frac{x^2e^x}{4}\right]$$

$$\Rightarrow \quad P.I = \frac{xe^{3x}}{8} - \frac{x^2e^x}{8} \quad \text{... (8)}$$

Substituting equations (4) and (8) in equation (3),

$$y = (c_1 + c_2x)e^x + c_3e^{3x} + \frac{xe^{3x}}{8} - \frac{x^2e^x}{8}$$

$\therefore$ The solution is,

$$y = (c_1 + c_2x)e^x + c_3e^{3x} + \frac{xe^{3x}}{8} - \frac{x^2e^x}{8}$$

(b) Solve the D.E $(D^2 - 1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$ by the Method of Variation of parameters.

Answer : Jan.-2020, Set-4, Q6(b) M[8]

Given differential equation is,

$$(D^2 - 1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x}) \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = Q(x) \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$f(D) = D^2 - 1, Q(x) = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$$

Auxiliary equation is $f(m) = 0$

i.e., $m^2 - 1 = 0$

$\Rightarrow \quad m^2 = 1$

$\Rightarrow \quad m = \pm 1$

Roots are real and distinct.

The complementary function is given as,

$$y_c = c_1e^x + c_2e^{-x} \quad \text{... (3)}$$

$$y_c = c_1f_1 + c_2f_2 \quad \text{... (4)}$$

Comparing equations (3) and (4),

$$f_1 = e^x, f_2 = e^{-x}$$

$$W(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$\Rightarrow \quad W(x) = -(e^0) - (e^0) = -2 \quad [\because e^0 = 1]$$

The particular integral is given as,

$$y_p = Pf_1 + Rf_2 \quad \text{... (5)}$$

$$P = -\int \frac{Q(x)f_2}{W(x)}dx$$

$$= -\int \frac{(e^{-x}\sin(e^{-x}) + \cos(e^{-x}))e^{-x}}{-2}dx$$

$$= \frac{1}{2}\int (e^{-x}e^{-x}\sin(e^{-x}) + e^{-x}\cos(e^{-x}))dx$$

$$\Rightarrow \quad P = \frac{1}{2}\int (e^{-2x}\sin(e^{-x}) + e^{-x}\cos(e^{-x}))dx$$

Let, $e^{-x} = t$

$\Rightarrow \quad -e^{-x}dx = dt$

$\Rightarrow \quad -tdx = dt$

$\Rightarrow \quad dx = \frac{-dt}{t}$

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Comparing equations (2) and (3),

$$M = x^3y + \frac{x^5}{2} + \frac{x^3y^3}{3} \qquad N = \frac{x^4 + x^4y^2}{4}$$

$$\frac{\partial M}{\partial y} = x^3(1) + 0 + \frac{x^3}{3}(3y^2) \qquad \frac{\partial N}{\partial x} = \frac{4x^3 + 4x^3y^2}{4}$$

$$\Rightarrow \frac{\partial M}{\partial y} = x^3 + x^3y^2 \qquad \Rightarrow \frac{\partial N}{\partial x} = x^3 + x^3y^2$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation (3) is an exact differential equation

The general solution of equation (3) is,

$$\int_{(y \text{ constant})} M\,dx + \int_{(\text{terms independent of } x)} N\,dy = c$$

$$\Rightarrow \int\left(x^3y + \frac{x^5}{2} + \frac{x^3y^3}{3}\right)dx + \int 0\,dy = c$$

$$\Rightarrow y\int x^3dx + \frac{1}{2}\int x^5\,dx + \frac{y^3}{3}\int x^3dx + 0 = c$$

$$\Rightarrow \frac{yx^4}{4} + \frac{1}{2}\left(\frac{x^6}{6}\right) + \frac{y^3}{3}\left(\frac{x^4}{4}\right) = c$$

$$\Rightarrow \frac{yx^4}{4} + \frac{x^6}{12} + \frac{y^3x^4}{12} = c$$

$$\Rightarrow \frac{3x^4y + x^6 + x^4y^3}{12} = c$$

$\therefore$ The general solution is $\dfrac{3x^4y + x^6 + x^4y^3}{12} = c$

(b) A voltage Ee^{-at} is applied at t = 0 to a circuit containing inductance L and resistance R. Then find the current i(t) at any time.

Answer : **Jan.-2020, Set-4, Q4(b) M[7]**

For answer refer Unit-2, Q53.

Q5. (a) Solve the D.E $(D^2 + 4)y = x \sin x$.

Answer : **Jan.-2020, Set-4, Q5(a) M[8]**

Given differential equation is,

$$(D^2 + 4)y = x\sin x \qquad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \qquad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = D^2 + 4 \text{ and } X = x\sin x$$

The general solution of a non-homogeneous linear differential equation is given as,

$$y = C.F + P.I \qquad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e, $m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m = \pm 2i$

The roots are complex conjugate of the form $\alpha \pm i\beta$

Here, $\alpha = 0, \beta = 2$.

The complementary function is given as,

$$C.F = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

$$\Rightarrow C.F = e^{0x}(c_1\cos 2x + c_2\sin 2x)$$

$$\Rightarrow C.F = c_1\cos 2x + c_2\sin 2x \qquad \text{... (4)}$$

The particular integral is given as,

$$y_p = \frac{X}{f(D)}$$

$$= \frac{1}{D^2+4}x\sin x$$

$$= I.P \text{ of } \frac{1}{(D^2+4)}xe^{ix}$$

$$= I.P \text{ of } e^{ix}.\frac{x}{[(D+i)^2+4]}$$

$$= I.P \text{ of } e^{ix}\frac{x}{D^2+i^2+2Di+4}$$

$$= I.P \text{ of } e^{ix}\frac{x}{D^2-1+2Di+4}$$

$$= I.P \text{ of } e^{ix}\frac{x}{D^2+2Di+3}$$

$$= I.P \text{ of } e^{ix}\frac{x}{3\left(1+\frac{D^2+2Di}{3}\right)}$$

$$= I.P \text{ of } \frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}\right)^{-1}x$$

$$= I.P \text{ of } \frac{e^{ix}}{3}\left[1-\frac{D^2}{3}-\frac{2}{3}Di+\ldots\right]x$$

$$= I.P \text{ of } \frac{e^{ix}}{3}\left[x+\frac{D^2}{3}(x)-\frac{2}{3}iD(x)\right]$$

$$= I.P \text{ of } \frac{e^{ix}}{3}\left[x+\frac{1}{3}(0)-\frac{2}{3}i\right]$$

$$= I.P \text{ of } \frac{e^{ix}}{3}\left[x-\frac{2}{3}i\right]$$

$$= I.P \text{ of } \frac{(\cos x + i\sin x)}{3}\left(x-\frac{2}{3}i\right)$$

$$= I.P \text{ of } \frac{\left(x\cos x+\frac{2}{3}\sin x\right)+i\left(-\frac{2}{3}\cos x+x\sin x\right)}{3}$$

$$= I.P \text{ of } \frac{\left(x\cos x+\frac{2}{3}\sin x\right)}{3}+\frac{i}{3}\left(-\frac{2}{3}\cos x+x\sin x\right)$$

$$= \frac{1}{3}\left(-\frac{2}{3}\cos x + x\sin x\right)$$

$$= -\frac{2}{9}\cos x + \frac{x}{3}\sin x$$

$$\Rightarrow y_p = \frac{x}{3}\sin x - \frac{2}{9}\cos x \qquad \text{... (5)}$$

Substituting equations (4) and (5) in equation (3),

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x.$$

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By continuing the process,

$$f^{iv}(0) = 0$$

$$f^{v}(0) = 24$$

Applying Maclaurin's theorem,

$$f(x) = f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots\ldots$$

$$= 0 + \frac{x}{1}(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-2) + \frac{x^4}{24}(0) + \frac{x^5}{120}(24) + \ldots\ldots = 0 + x + 0 - \frac{x^3}{3} + 0 + \frac{x^5}{5} + \ldots\ldots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots.$$

$$\therefore \quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots..$$

Q3. (a) Solve the D.E [cosx tany + cos(x + y)]dx + [sinx sec²y + cos(x + y)]dy = 0.

Answer :

Jan.-2020, Set-4, Q3(a) M[7]

Given differential equation is,

$$[\cos x \tan y + \cos(x+y)]dx + [\sin x \sec^2 y + \cos(x+y)]dy = 0 \quad \ldots (1)$$

Equation (1) is of the form,

$$Mdx + Ndy = 0 \quad \ldots (2)$$

Comparing equations (1) and (2),

$$M = \cos x \tan y + \cos(x+y)$$

$$\frac{\partial M}{\partial y} = \cos x \sec^2 y + (-\sin(x+y))(1)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y)$$

$$N = \sin x \sec^2 y + \cos(x+y)$$

$$\frac{\partial N}{\partial x} = \sec^2 y \cos x + (-\sin(x+y))(1)$$

$$\Rightarrow \frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation (1) is an exact differential equation.

The general solution is given as,

$$\underset{(y \text{ constant})}{\int M\,dx} + \underset{(\text{terms independent of } x)}{\int N\,dy} = c$$

$$\Rightarrow \int (\cos x \tan y + \cos(x+y))dx + \int 0\,dy = c$$

$$\Rightarrow \tan y \int \cos x\,dx + \int \cos(x+y)dx + 0 = c$$

$$\Rightarrow \tan y(-\sin x) - \sin(x+y)(1) = c$$

$$\Rightarrow -\tan y \sin x - \sin(x+y) = c$$

$$\Rightarrow \sin x \tan y + \sin(x+y) = -c$$

$$\Rightarrow \sin x \tan y + \sin(x+y) = a$$

$\therefore$ The required solution is, $\sin x \tan y + \sin(x+y) = a$ $\quad [\because -c = a = \text{constant}]$

(b) If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C.

Answer :

Jan.-2020, Set-4, Q3(b) M[8]

Given that,

The body is kept at the temperature of 30°C.

Let θ be the temperature of the body at time 't'.

From Newton's law of cooling,

$$\frac{-d\theta}{dt} \propto (\theta - 30)$$

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta - 30)$$

$$\Rightarrow \frac{d\theta}{\theta - 30} = -k\,dt$$

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Then, $P = \frac{1}{2}\int (t^2 \sin t + t\cos t)\left(\frac{-dt}{t}\right)$

$\Rightarrow \quad P = \frac{-1}{2}\int \frac{t^2 \sin t + t\cos t}{t}\,dt$

$= \frac{-1}{2}\int (t\sin t + \cos t)\,dt = \frac{-1}{2}\left[\int t\sin t\,dt + \int \cos t\,dt\right]$

$= \frac{-1}{2}\left[t\int \sin t\,dt - \int \frac{d}{dt}(t)\int \sin t\,dt\,dt + (\sin t)\right]$

$= \frac{-1}{2}\left[t(-\cos t) - \int (1)(-\cos t)\,dt + \sin t\right] = \frac{-1}{2}\left[-t\cos t + \int \cos t\,dt + \sin t\right]$

$= \frac{-1}{2}[-t\cos t + \sin t + \sin t] = \frac{-1}{2}[-t\cos t + 2\sin t]$

$\Rightarrow \quad P = \frac{1}{2}[t\cos t - 2\sin t]$

$\Rightarrow \quad P = \frac{1}{2}[e^{-x}\cos(e^{-x}) - 2\sin(e^{-x})]$

$R = \int \frac{Q(x)f_1}{W(x)}dx$

$\Rightarrow \quad R = \int \frac{(e^{-x}\sin(e^{-x}) + \cos(e^{-x}))e^{x}}{-2}dx$

$\Rightarrow \quad R = \frac{-1}{2}\int (e^{-x}.e^{x}\sin(e^{-x}) + e^{x}\cos(e^{-x}))dx$

$\Rightarrow \quad R = \frac{-1}{2}\int (\sin(e^{-x}) + e^{x}\cos(e^{-x}))dx$

Let $e^{-x} = t$ (or) $e^{x} = 1/t$

$\Rightarrow \quad -e^{-x}dx = dt$

$\Rightarrow \quad -t\,dx = dt$

$\Rightarrow \quad dx = \frac{-dt}{t}$

Then, $R = \frac{-1}{2}\int \left(\sin t + \frac{1}{t}\cos t\right)\left(\frac{-dt}{t}\right)$

$= \frac{1}{2}\int \frac{\left(\sin t + \frac{1}{t}\cos t\right)}{t}dt$

$= \frac{1}{2}\int \left(\frac{\sin t}{t} + \frac{1}{t^2}\cos t\right)dt = \frac{1}{2}\left[\int \frac{\sin t}{t}dt + \int \frac{1}{t^2}\cos t\,dt\right]$

$= \frac{1}{2}\left[\frac{1}{t}\int \sin t\,dt - \int \frac{d}{dt}\left(\frac{1}{t}\right)\int \sin t\,dt\,dt + \int \frac{1}{t^2}\cos t\,dt\right]$

$= \frac{1}{2}\left[\frac{1}{t}(-\cos t) - \int \left(\frac{-1}{t^2}\right)(-\cos t)\,dt + \int \frac{1}{t^2}\cos t\,dt\right]$

$= \frac{1}{2}\left[\frac{-\cos t}{t} - \int \frac{1}{t^2}\cos t\,dt + \int \frac{1}{t^2}\cos t\,dt\right]$

$= \frac{1}{2}\left(\frac{-\cos t}{t}\right)$

$= \frac{-1}{2}\,\frac{\cos(e^{-x})}{e^{-x}}$

$\Rightarrow \quad R = \frac{-e^{x}}{2}\cos(e^{-x})$

Differentiating equation (1) partially with respect to x, y, z,

$$\frac{\partial u}{\partial x} = 2xe^{-y}\cosh z$$

$$\frac{\partial u}{\partial y} = -x^2e^{-y}\cosh z$$

$$\frac{\partial u}{\partial z} = x^2e^{-y}\sinh z$$

Differentiating equation (2) partially with respect to x, y, z,

$$\frac{\partial v}{\partial x} = 2xe^{-y}\sinh z$$

$$\frac{\partial v}{\partial y} = -x^2e^{-y}\sinh z$$

$$\frac{\partial v}{\partial z} = x^2e^{-y}\cosh z$$

Differentiating equation (3) partially with respect to x, y, z,

$$\frac{\partial w}{\partial x} = 12x^3e^{-2y}$$

$$\frac{\partial w}{\partial y} = -6x^4e^{-2y}$$

$$\frac{\partial w}{\partial z} = 0$$

The Jacobian of u, v, w with respect to x, y, z is,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Substituting the corresponding values in above equation,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 2xe^{-y}\cosh z & -x^2e^{-y}\cosh z & x^2e^{-y}\sinh z \\ 2xe^{-y}\sinh z & -x^2e^{-y}\sinh z & x^2e^{-y}\cosh z \\ 12x^3e^{-2y} & -6x^4e^{-2y} & 0 \end{vmatrix}$$

$$= 2xe^{-y}\cosh z\,(0\,(-x^2e^{-y}\sinh z) - x^2e^{-y}\cosh z\,(-6x^4e^{-2y})) - (-x^2e^{-y}\cosh z)(0(2xe^{-y}\sinh z)$$
$$- 12x^3e^{-2y}(x^2e^{-y}\cosh z)) + x^2e^{-y}\sinh z\,(-6x^4e^{-2y}(2xe^{-y}\sinh z) - 12x^3e^{-2y}(-x^2e^{-y}\sinh z))$$

$$= 2xe^{-y}\cosh z(0 + 6x^6e^{-3y}\cosh z) + x^2e^{-y}\cosh z\,(0 - 12x^5e^{-3y}\cosh z) + x^2e^{-y}\sinh z\,(-12x^5e^{-3y}\sinh z$$
$$+ 12x^5e^{-3y}\sinh z)$$

$$= 2xe^{-y}\cosh z(6x^6e^{-3y}\cosh z) + x^2e^{-y}\cosh z\,(-12x^5e^{-3y}\cosh z) + x^2e^{-y}\sinh z\,(0)$$

$$= 12x^7e^{-4y}\cosh^2 z - 12x^7e^{-4y}\cosh^2 z + 0$$

$$= 0$$

$\because$ $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$, the functions u, v and w are functionally dependent.

Relation between u, v and w

From equation (1),

$$u^2 = (x^2e^{-y}\cosh z)^2$$

$$\Rightarrow \quad u^2 = x^4e^{-2y}\cosh^2 z \qquad \text{... (4)}$$

From equation (2),

$$v^2 = (x^2e^{-y}\sinh z)^2$$

$$\Rightarrow \quad v^2 = x^4e^{-2y}\sinh^2 z \qquad \text{.... (5)}$$

Substracting equations (4) and (5),

$$u^2 - v^2 = x^4e^{-2y}\cosh^2 z - x^4e^{-2y}\sinh^2 z$$

$$\Rightarrow \quad u^2 - v^2 = x^4e^{-2y}(\cosh^2 z - \sinh^2 z)$$

$$\Rightarrow \quad u^2 - v^2 = x^4e^{-2y}(1) \qquad [\because \cosh^2 z - \sinh^2 z = 1]$$

$$\Rightarrow \quad u^2 - v^2 = \frac{w}{3} \qquad [\because \text{ From equation (3)}]$$

$$\Rightarrow \quad 3(u^2 - v^2) = w$$

$\therefore$ Relation between u, v and w is $3(u^2 - v^2) = w$.

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x+y+z) + \frac{\partial}{\partial y}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right)$$

$$\Rightarrow \quad \frac{\partial F}{\partial y} = 1+\lambda\left(\frac{-b}{y^2}\right)$$

$$\Rightarrow \quad \frac{\partial F}{\partial y} = \frac{y^2-b\lambda}{y^2}$$

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(x+y+z) + \lambda\frac{\partial}{\partial z}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right)$$

$$\Rightarrow \quad \frac{\partial F}{\partial z} = 1+\lambda\left(\frac{-c}{z^2}\right)$$

$$\Rightarrow \quad \frac{\partial F}{\partial z} = \frac{z^2-c\lambda}{z^2}$$

The condition for minimum value is,

$\frac{\partial F}{\partial x}=0$	$\frac{\partial F}{\partial y}=0$	$\frac{\partial F}{\partial z}=0$
$\Rightarrow \frac{x^2-a\lambda}{x^2}=0$	$\Rightarrow \frac{y^2-b\lambda}{y^2}=0$	$\Rightarrow \frac{z^2-c\lambda}{z^2}=0$
$\Rightarrow x^2=a\lambda$	$\Rightarrow y^2=b\lambda$	$\Rightarrow z^2=c\lambda$
$\Rightarrow x=\pm\sqrt{a\lambda}$	$\Rightarrow y=\pm\sqrt{b\lambda}$	$\Rightarrow z=\pm\sqrt{c\lambda}$

Substituting the corresponding values in equation (2),

$$\frac{a}{\sqrt{a\lambda}}+\frac{b}{\sqrt{b\lambda}}+\frac{c}{\sqrt{c\lambda}}=1$$

$$\Rightarrow \quad \frac{\sqrt{a}}{\sqrt{\lambda}}+\frac{\sqrt{b}}{\sqrt{\lambda}}+\frac{\sqrt{c}}{\sqrt{\lambda}}=1$$

$$\Rightarrow \quad \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{\lambda}}=1$$

$$\Rightarrow \quad \sqrt{\lambda}=\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

$$\Rightarrow \quad \lambda=\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2$$

Substituting the value of λ in x, y and z

$$x = \pm\sqrt{\lambda a} = \pm\left(\sqrt{(\sqrt{a})^2(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right)$$

$$\Rightarrow \quad x = \pm(\sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c}))$$

$$y = \pm\sqrt{\lambda b} = \pm\left(\sqrt{(\sqrt{b})^2(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right)$$

$$\Rightarrow \quad y = \pm(\sqrt{b}(\sqrt{a}+\sqrt{b}+\sqrt{c}))$$

$$z = \pm\sqrt{\lambda c} = \pm\left(\sqrt{(\sqrt{c})^2(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\right)$$

$$\Rightarrow \quad z = \pm(\sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c}))$$

At the stationary point, $(x, y, z) = (\sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c}, \sqrt{b}(\sqrt{a}+\sqrt{b}+\sqrt{c}), \sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c}))$ f has minimum value

Substituting the vaues of x, y and z in equation (1),

$$\text{Minimum value of } f = \sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{a}+\sqrt{b}+\sqrt{c})+\sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c})$$

$$= (\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}+\sqrt{c})$$

$$= (\sqrt{a}+\sqrt{b}+\sqrt{c})^2$$

$$\therefore \quad \text{Minimum value} = (\sqrt{a}+\sqrt{b}+\sqrt{c})^2$$

(b) Find the volume of region bounded by the surface $z = x^2 + y^2$ and $z = 2x$. Using triple integral.

Answer : **Jan.-2020, Set-4, Q9(b) M[8]**

For answer refer Unit-5, Q38.

OR

Q10. (a) Find using double integral the volume bounded by cylinder $x^2 + y^2 = 4$ the plane $y + z = 3$; $z = 0$.

Answer : **Jan.-2020, Set-4, Q10(a) M[7]**

Given that,

Cylinder, $x^2 + y^2 = 4$ ... (1)

Plane, $y + z = 3$ ... (2)

And $z = 0$

From equation (1),

$x^2 = 4 - y^2$

$\Rightarrow \quad x = \pm\sqrt{4 - y^2}$

$\therefore \quad x$ varies from $-\sqrt{4 - y^2}$ to $\sqrt{4 - y^2}$

If $x = 0$, $4 - y^2 = 0$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

$\therefore \quad y$ varies from -2 to 2

The base of a cylinder represents a circle in xy-plane of radius 2 units with origin O(0, 0) as centre

From equation (2),

$z = 3 - y$

$\therefore$ The volume bounded by the cylinder and the planes is given as,

$$V = \iint z\,dx\,dy$$

$$= \int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (3 - y)\,dx\,dy = \int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (3 - y)\,dx\,dy$$

$$= \int_{y=-2}^{2}(3 - y)\left[\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\right]dy = \int_{y=-2}^{2}(3 - y)(x)_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\,dy$$

$$= \int_{2}^{2}(3 - y)\left(\sqrt{4 - y^2} - (-\sqrt{4 - y^2})\right)dy = \int_{2}^{2}(3 - y).2\sqrt{4 - y^2}\,dy$$

$$= 2\int_{-2}^{2}(3 - y)\sqrt{4 - y^2}\,dy = 2\left[\int_{-2}^{2}3\sqrt{4 - y^2}\,dy - \int_{-2}^{2}y\sqrt{4 - y^2}\,dy\right]$$

$$= 6\int_{-2}^{2}\sqrt{4 - y^2}\,dy - 2\int_{-2}^{2}y\sqrt{4 - y^2}\,dy$$

$$= 6\int_{y=-2}^{2}\sqrt{4^2 - (y)^2}\,dy - 2(0) \qquad \left[\because \int_{-2}^{2}y\sqrt{4 - y^2} \text{ is an odd function}\right]$$

$$= 6\left[\frac{y\sqrt{4^2 - y^2}}{2} + \frac{(2)^2}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]_{-2}^{2} - 0 \qquad \left[\because \int\sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= 6\left[\left(\frac{2\sqrt{4 - 4}}{2} + 2\sin^{-1}\left(\frac{2}{2}\right)\right) - \left(\frac{(-2)\sqrt{4 - 4}}{2} + 2\sin^{-1}\left(\frac{-2}{2}\right)\right)\right]$$

$$= 6[0 + 2\sin^{-1}(1) + 0 - 2\sin^{-1}(-1)] = 6[2\sin^{-1}(1) + 2\sin^{-1}(1)]$$

$$= 6\left[2\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\right] = 6[2\pi] = 12\pi.$$

$\therefore$ Volume bounded by the cylinder and the planes $= 12\pi$ cubic units.

Q9. (a) Evaluate $\int_0^\infty \int_0^\infty \frac{dx\,dy}{(x^2+y^2+a^2)^2}$ using polar co-ordinates. **Jan.-2020, Set-4, Q9(a) MD**

Answer :

Given integral is,

$$\int_0^\infty \int_0^\infty \frac{dx\,dy}{(x^2+y^2+a^2)^2}$$

$$\Rightarrow \int_{x=0}^\infty \int_{y=0}^\infty \frac{dx\,dy}{(x^2+y^2+a^2)^2}$$

From the limits of integration, x varies from 0 to ∞ and y varies from 0 to ∞ such that the region of integration lies in the first quadrant of xy-plane

Let,

$x = r\cos\theta$,

$y = r\sin\theta$

$\Rightarrow \quad dxdy = r\,dr\,d\theta$

And $x^2+y^2 = r^2$

Limits are from $r = 0$ to ∞ and $\theta = 0$ to $\frac{\pi}{2}$ in the first quadrant.

$$\int_0^\infty \int_0^\infty \frac{dx\,dy}{(x^2+y^2+a^2)^2} = \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^\infty \frac{r}{(r^2+a^2)^2} dr\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^\infty \frac{1}{2} du . \frac{1}{u^2}\right] d\theta$$

$$\left[\begin{array}{l} \because \text{Let, } u = r^2+a^2 \\ du = 2rdr \\ \Rightarrow rdr = \frac{1}{2} du \end{array}\right]$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\int_0^\infty u^{-2} du\right] d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{u^{-2+1}}{-2+1}\right]_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{u^{-1}}{-1}\right)_0^\infty d\theta = \frac{-1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{u}\right)_0^\infty d\theta$$

$$= \frac{-1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{r^2+a^2}\right)_0^\infty d\theta = \frac{-1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\infty+a^2} - \frac{1}{0^2+a^2}\right) d\theta$$

$$= \frac{-1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\infty} - \frac{1}{a^2}\right) d\theta = \frac{-1}{2}\int_0^{\frac{\pi}{2}} \left(0 - \frac{1}{a^2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{a^2}\right) d\theta = \frac{1}{2a^2}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2a^2}[\theta]_0^{\pi/2}$$

$$= \frac{1}{2a^2}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{4a^2}$$

$$\therefore \int_0^\infty \int_0^\infty \frac{dx\,dy}{(x^2+y^2+a^2)^2} = \frac{\pi}{4a^2}$$

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OR

Q8. (a) If u = f(r, s), r = x + y, s = x – y show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$

Jan.-2020, Set-4, Q8(a) M[7]

Answer :

Given that,

$$u = f(r, s),$$

$$r = x + y$$

And $s = x - y$

Partially differentiating r, s with respect to x and y,

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial r}{\partial y} = 1$$

$$\frac{\partial s}{\partial x} = 1, \quad \frac{\partial s}{\partial y} = -1$$

Applying chain rule of partial differentiation,

i.e., $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \text{...(1)}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(-1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \quad \text{...(2)}$$

Adding equations (1) and (2),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$$

$$= 2\frac{\partial u}{\partial r}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$$

(b) Find the minimum values of x + y + z subject to $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ using Lagrange's multiplier method.

Jan.-2020, Set-4, Q8(b) M[8]

Answer :

Given function is,

$$f(x, y, z) = x + y + z \quad \text{...(1)}$$

$$\phi(x, y, z) = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$$

$$\Rightarrow \phi(x, y, z) = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} - 1 \quad \text{...(2)}$$

According to Lagrange's function,

$$F(x, y, z) = f(x + y + z) + \lambda[\phi(x, y, z)]$$

$$\Rightarrow F(x, y, z) = (x + y + z) + \lambda\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} - 1\right) \quad [\because \text{From equations (1) and (2)}] \quad \text{...(3)}$$

Differentiating equation (3) with respect to x, y, z

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x + y + z) + \lambda\frac{\partial}{\partial x}\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} - 1\right)$$

$$\Rightarrow \frac{\partial F}{\partial x} = 1 + \lambda\left[\frac{-a}{x^2}\right]$$

$$\Rightarrow \frac{\partial F}{\partial x} = \frac{x^2 - a\lambda}{x^2}$$

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Substituting the corresponding values in equation (5),

$$y_p = \frac{1}{2}[e^{-x}\cos(e^{-x}) - 2\sin(e^{-x})](e^{x}) + \left(\frac{-e^{x}}{2}\cos(e^{-x})\right)e^{-x}$$

$$\Rightarrow \quad y_p = \frac{1}{2}[e^{x}e^{-x}\cos(e^{-x}) - 2e^{x}\sin(e^{-x}) - e^{-x}e^{x}\cos(e^{-x})]$$

$$\Rightarrow \quad y_p = \frac{1}{2}[-2e^{x}\sin(e^{-x})]$$

$$\Rightarrow \quad y_p = -e^{x}\sin(e^{-x}) \qquad \text{... (6)}$$

The general solution is given as,

$$y = y_c + y_p$$

Substituting equations (3) and (6) in above equation,

$$y = c_1e^{x} + c_2e^{-x} + (-e^{x}\sin(e^{-x}))$$

$$\Rightarrow \quad y = c_1e^{x} + c_2e^{-x} - e^{x}\sin(e^{-x})$$

$\therefore$ The solution is $y = c_1e^{x} + c_2e^{-x} - e^{x}\sin(e^{-x})$

Q7. (a) If $u = (x^2 + y^2)^{\frac{1}{3}}$ then find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ using Euler's theorem.

Jan.-2020, Set-4, Q7(a) M[7]

Answer :

Given function is,

$$u = (x^2 + y^2)^{1/3}$$

$$\Rightarrow \quad f(x, y) = (x^2 + y^2)^{1/3}$$

Then, $f(kx, ky) = [(kx)^2 + (ky)^2]^{1/3}$

$$= [k^2(x^2 + y^2)]^{1/3}$$

$$= k^{2/3}(x^2 + y^2)^{1/3}$$

$$\Rightarrow \quad f(kx, ky) = k^{2/3}f(x, y)$$

$\therefore$ Degree of homogeneous function $(n) = \frac{2}{3}$.

According to Euler's theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{2}{3}\left(\frac{2}{3} - 1\right)(x^2 + y^2)^{1/3}$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{2}{3}\left(\frac{-1}{3}\right)(x^2 + y^2)^{1/3}$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-2}{9}(x^2 + y^2)^{1/3}$$

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-2}{9}(x^2 + y^2)^{1/3}$$

(b) Prove that the functions $u = x^2e^{-y}\cosh z$, $v = x^2e^{-y}\sinh z$ and $w = 3x^4e^{-2y}$ are functionally dependent and hence find the relation between them.

Jan.-2020, Set-4, Q7(b) M[

Answer :

Given functions are,

$$u = x^2e^{-y}\cosh z$$

$$v = x^2e^{-y}\sinh z$$

$$w = 3x^4e^{-2y}$$

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(b) Evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dx\,dy\,dz$

Jan.-2020, Set-4, Q10(b)

Answer :

Given integral is,

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dx\,dy\,dz$$

Let,

$x = r\cos\theta,$

$y = r\sin\theta,$

$z = z$

$\Rightarrow \quad dx\,dy\,dz = r\,dr\,d\theta\,dz$

And $x^2 + y^2 = r^2$

The region of integration is the right half of the circle with radius 1 unit.

$\therefore$ r varies from 0 to 1 and θ varies from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$

Since $\sqrt{x^2+y^2} = r$, z varies from r^2 to r.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dx\,dy\,dz = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{r^2}^{r} r\,dz\,dr\,d\theta$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r\left[\int_{r^2}^{r} dz\right] dr\,d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r[z]_{r^2}^{r}\,dr\,d\theta$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r(r-r^2)\,dr\,d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} (r^2-r^3)\,dr\,d\theta$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left[\int_{0}^{1}(r^2-r^3)\,dr\right]d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left[\frac{(1)^3}{3}-\frac{(1)^4}{4}-\frac{(0)^3}{3}+\frac{(0)^4}{4}\right]d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1}{3}-\frac{1}{4}-0+0\right)d\theta$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1}{12}\right)d\theta = \frac{1}{12}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{12}(\theta)_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{12}\left(\frac{\pi}{2}-\left(\frac{-\pi}{2}\right)\right)$$

$$= \frac{1}{12}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= \frac{1}{12}(\pi) = \frac{\pi}{12}$$

$$\therefore \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dx\,dy\,dz = \frac{\pi}{12}$$

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