IDC 402 - Non-linear dynamics of the duffing oscillator

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1 Abstract

In this term paper we analyzed the dynamics of the duffing oscillator and studied the transition to chaos, by observing the time series analysis and subsequent phase space trajectories for a range of initial conditions and parameter values such as harmonic excitation, the point at which chaos is observed is approximately determined. Furthermore by plotting poincare sections we observe the dynamics of the strange attractor in the chaotic regime.

2 Introduction

Duffing Equation, first developed by a German engineer and researcher Georg Wilhelm Christian Casper Duffing (1861- 1944), is an externally forced and damped oscillator equation that exhibits a range of interesting dynamic behavior in its solutions. The general duffing equation can be written as-

$$\mathrm{m}\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = F(t) \tag{2.1}$$

where each term has it's own significance

- There is periodic external forcing that comes from the term $F(t) = \gamma cos(\omega t)$. The parameter γ is the strength of the driving force and ω is the frequency of forcing.
- αx is a classical restoring force that follows Hooke's Law and βx^3 represents a cubic restoring force that controls the nonlinear response of the system.
- $\delta\dot{x}$ represents the damping in the system.

It's straightforward to analyze the solutions to by converting the nonlinear equation to a system of first order differential equations. Under the variable transformation $v = \dot{x}$, which results in the system,

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\delta v - \alpha x - \beta x^3 + \gamma\cos(\omega t) \end{aligned} \tag{2.2}$$

2.1 Analysis in the unforced regime

The equation for total energy in free motion ($\delta = 0, \gamma = 0$)can be integrated over time by multiplying by $\dot{x}$.

$$E(t) \equiv \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 = \text{const}$$

Therefore, in this case, the Duffing equation is a Hamiltonian system. $E(t)$ is a single-well potential for $\beta > 0$, and it is a double-well potential for $\beta < 0$. The trajectory moves on the constant surface of $E(t)$.

With damping in effect ,the $E(t)$ decreases till the trajectories decays to the equilibrium points. It satisfies,

$$\frac{dE(t)}{dt} = -\delta\dot{x}^2 \leq 0$$

Meaning, $E(t)$ is is locally asymptotically stable about the origin (for $\alpha, \beta > 0$), hence it's a lyapunov function. Stability of the equlibrium points for non zero α, β can be determined by finding out the eigenvalues of the jacobian.The eqb. points are-

$$\alpha x + \beta x^3 = x\left(\alpha + \beta x^2\right) = 0$$

Thus x $= \pm\sqrt{-\frac{\alpha}{\beta}}\left(\text{for } \frac{\alpha}{\beta} < 0\right)$ and $x = 0$

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\delta\dot{x} - \alpha x - \beta x^3 \end{pmatrix}$$

The jacobioan can be written as

$$J(x) = \begin{pmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & -\delta \end{pmatrix}$$

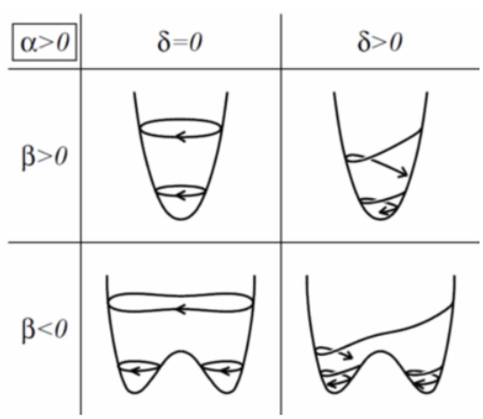
By calculating the eigenvalues, one can determine the stability of the system at the fixed point.
at fixed point $x = 0$,unstable equilibrium for $\alpha < 0$ and stable eqb for $\alpha \leq 0$

$$\lambda_{\text{eig}} = \frac{-\delta \pm \sqrt{\delta^2 - 4\alpha}}{2} > 0$$

at fixed point x $= \pm\sqrt{-\frac{\alpha}{\beta}}$ stable equilibrium for $\alpha < 0, \beta > 0$ and unstable eqb for $\alpha > 0, \beta < 0$

$$\lambda_{eig} = \frac{-\delta \pm \sqrt{\delta^2 + 8\alpha}}{2} \leq 0$$

A bifurcation point can be experimentally observed from plotting displacement against time and comparing



(a) The shape of $E(t)$ and schematic trajectories of the Duffing oscillator .

the periods as parameter values are changed, or determined from the eigenvalue of the Jacobian matrix, whereby a negative eigenvalue (purely imaginary) implies definite presence of bifurcation. At the point at which the period of the forcing is doubled, or the point of bifurcation, the phase space diagram no longer exhibits periodic or quasi-periodic behavior, rather it has a chaotic motion, or a strange attractor.

3 Numerical analysis and results

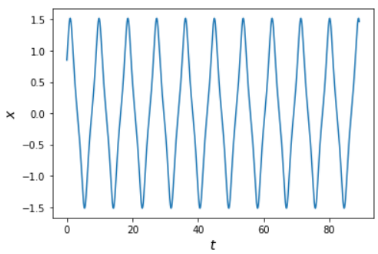
To compute phase space trajectories we numerically solve the differential equation using RK4 method and also compute displacement and velocity vs time.By approximating each successive derivative from previously computed derivative and taking each computed value at its weight, the correctness at the step size increases.

Compared to Euler's Method, Newton's Method, or other numerical approximation methods for nonlinear differential equations, Runge-Kutta method poses advantage in accuracy and error bounds.

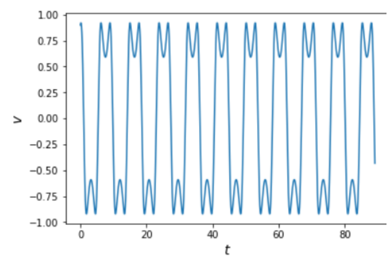
Now without further ado we show phase space plots for different cases(for all cases $\alpha = -1, \beta = 1, \omega = 1.4$) and step size = 0.01-

- Case 1- Free Motion(unforced and undamped) $\boldsymbol{\delta = 0, \gamma = 0, x(0) = 0.85, \dot{x}(0) = 0.9}$

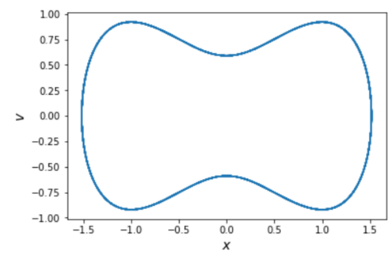
  Under absence of damping and excitation, the system remains to be in periodic oscillator motion, showing a closed curve attractor (limit cycle) in the phase space diagram. The fixed points (equilibrium) are located at three points, -1, 0, and 1, where 0 is the unstable one.



(a) displacement vs time



(b) velocity vs time

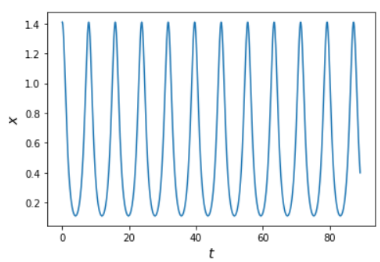


(c) Phase space plot.

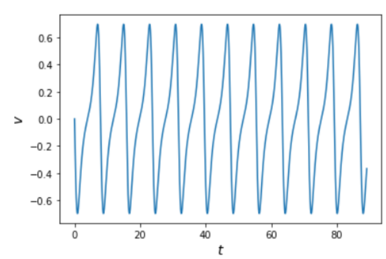
  The total energy, the sum of kinetic and potential, is conserved due to no dissipative or exciting forces applied to the system.

- Case 2-Free Motion(different initial condition) $\boldsymbol{\delta = 0, \gamma = 0, x(0) = 1.41, \dot{x}(0) = 0}$

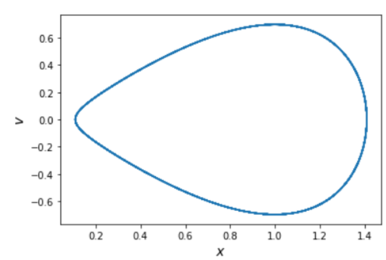
  With varied initial position, x, the system exhibits a slight change in the phase space diagram. The fixed point (equilibrium) is now located at $x = 1$ only, inducing the phase space diagram to look as such.



(a) displacement vs time



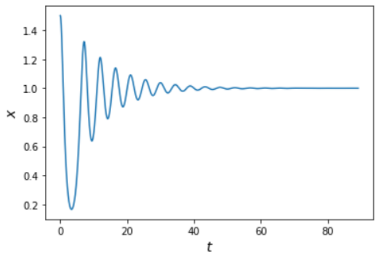
(b) velocity vs time



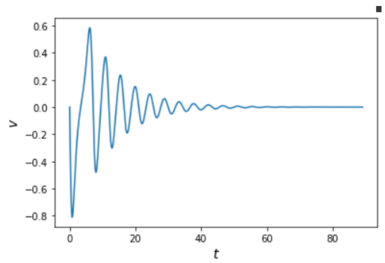
(c) Phase space plot.

- Case 3-damped motion $\boldsymbol{\delta = 0.2, \gamma = 0, x(0) = 1.5, \dot{x}(0) = 0}$

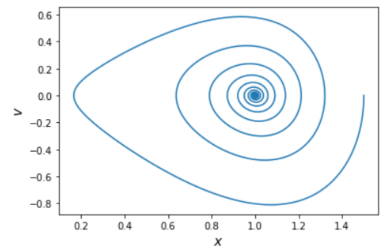
  Damping coefficient is slightly increased , and the system becomes a dissipative one, as the total energy decaying away over time. There are fixed points at $x = 1$, and multiple experiments indicate that any initial conditions in this system lead to similar phase plot. The point to which the phase converges is the stable equilibrium point which is the attractor point or a stable spiral.



(a) displacement vs time



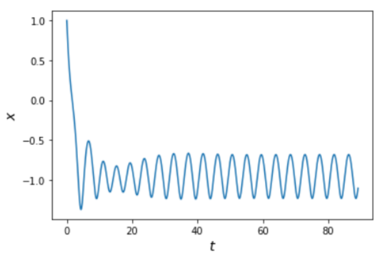
(b) velocity vs time



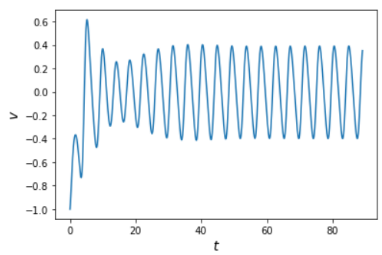
(c) Phase space plot.

- Case 4-<u>damped motion and driven motion</u> $\boldsymbol{\delta = 0.1, \gamma = 0.1, x(0) = 1, \dot{x}(0) = -1}$

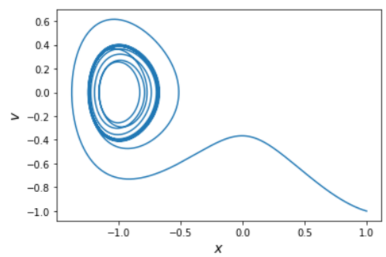
  As an external excitation is applied to the system,the motion of the mass becomes slightly chaotic and becomes no longer predictable, since any slight changes in the model parameters results in a very different movement of the particle. After initial transient state (aperiodic), it approaches a stable period around the fixed point at $x = -1$ in a closed orbit, exhibiting a limit cycle.



(a) displacement vs time



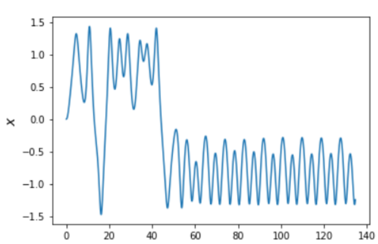
(b) velocity vs time



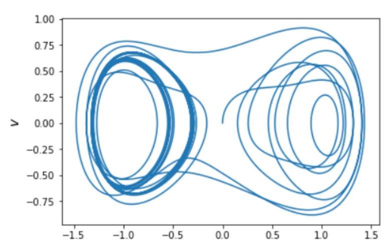
(c) Phase space plot with x=-1 as the final attractor.

- <u>Damped and driven motion(Further driving), onset of period doubling</u> $\boldsymbol{\delta = 0.1, \gamma = 0.32, x(0) = 0, x(\dot{0}) = 0}$

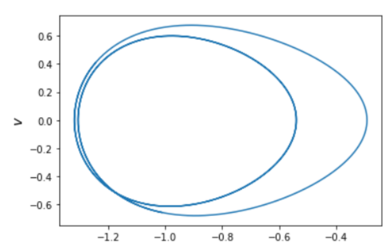
  Initially multiple periodic orbits are observed due to initial transient period. The trajectory seems to move in an irregular manner between $x = -1$ and $x = 1$ fixed points, but after some time, it stabilizes in a periodic orbit, settling at $x = -1$ point. The final phase space trajectory for the last 1000 points shows the final 2 orbits it has settled to showing <u>period doubling with period $\frac{2\pi}{\omega}$</u> causing a bifurcation .The final point it settles to depends on the initial condition.



(a) displacement vs time



(b) full phase space trajectory

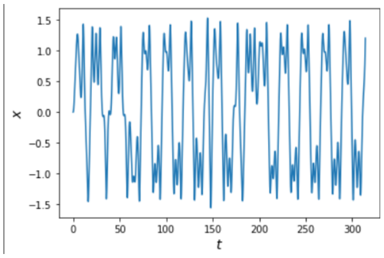


(c) phase space trajectory for final 1000 points.

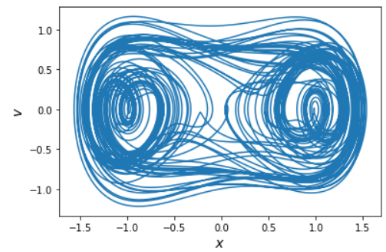
- <u>Onset of chaos</u> $\boldsymbol{\delta = 0.1, \gamma = 0.34, x(0) = 0, \dot{x}(0) = 0}$

  Chaotic motion is observed from phase space. The periodic behavior is no longer exhibited throughout the time series. It undergoes fixed points at $x = -1, 0, +1$ repeatedly without much pattern. The final parameter values reveal that the system has transitioned from the phase space where bifurcation occurs into a region of chaos.

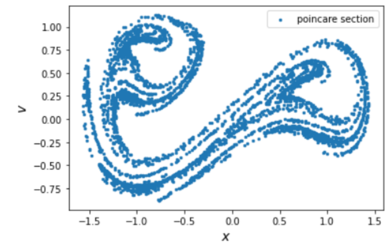
This does not allow for the system to eventually reach a stable, fixed orbit but instead the solutions continue to move through the phase-space in an unpredictable fashion.



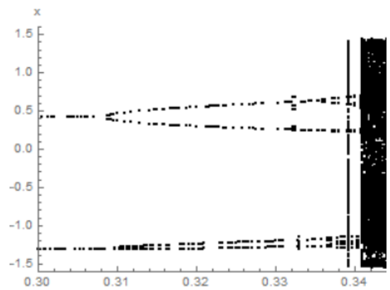
(a) displacement vs time



(b) full phase space trajectory



(c) Poincare section with 5000 iterations.



(d) Bifurcation plot for displacement vs driving amplitude showing the onset of chaos between $\gamma = 0.338 - 0.35$.

This can be further analyzed by plotting the poincare sections,which is a snapshot of the phase space trajectory at fixed multiples of the driving period $T = \frac{2N\pi}{\omega}$, N $= 1, 2, 3 \ldots$ For simple harmonic motion, this plot would be a single point, or a few discrete points. Instead, a fractal is observed for the chaotic duffing oscillator. This is called the infamous behaviour of the strange attractor, it is the limiting set of points to which the trajectory tends (after the initial transient) every period of the driving force.

Bifurcation diagram can also be plotted between displacement and driving amplitude by scanning 200 values of γ in the range from 0.3 to 0.35, with $\gamma = 0.1$ and $\omega = 1.4$. It starts with $v = 0$ and $x = 1$, at rest in the $x = 1$ minimum. The equations are integrated for 1500 periods of the driving force with the first 1000 being ignored and the value of x plotted at the end of each of the remaining 500 periods. From this period doubling route through chaos is verified.

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4 Conclusion

- We analyzed and visualized the dynamics of the duffing oscillator as it eventually transitions to chaos. The oscillator is unable to exhibit chaos when the oscillator is undamped and unforced due to energy conservation. To observe chaotic trajectories, energy conservation is eliminated by including a damping and driving term.
- For a host of initial conditions and parameter values of driving amplitude and damping, the point at which chaos starts to be observed is approximately determined to be between $\gamma = 0.338$ and $\gamma = 0.35$.
- Moreover, the onset of chaos is saliently captured through bifurcation diagram,phase space trajectories and poincare section.
- Henceforth we can see that, minute perturbations in the parameters of a nonlinear system results in a drastic change in its dynamical behavior.

5 Code

The code for the above plots can be found at the following github repository
https://github.com/bhavikorange/IDC-402–Duffing-oscillator.git