The Quasi-Normal Modes of a Scalar Field on a Schwarzschild Black Hole

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Abstract

This project involves studying the effects of a scalar field in the presence of a Schwarzschild black hole (which is static and spherically symmetric), determining the quasi-normal modes. Quasi-normal modes (QNMs) are the characteristic oscillations of a black hole's response to perturbations in spacetime. We begin by deriving the wave equation for this case and then use numerical methods to solve it to determine the QNMs. To derive the wave equation we will use the covariant derivative and the Schwarzschild metric. For the numerical part, we use a time integrator, Runge-Kutta 4, and second-order finite difference to compute the spatial derivatives of each point. We also verify numerical stability and test convergence. Lastly, we will analyze the wave's temporal evolution at a fixed spatial point, so we can determine the QNms, by using the least-squares method to fit this time evolution.
From our results, we obtained $\omega = (6.51 \pm 0.01) \times 10^{-2}$ for the quasi-normal mode and $\tau = 161.29 \pm 10.41$ for the time decay.

I. Introduction

In general relativity, a scalar field is invariant under coordinate transformations. Let us assume it does not interact with any other field. Then, applying the d'Alembertian to a scalar field Φ, we have $\Box\Phi = 0$, which is the Klein-Gordon equation with mass zero. [1]
In flat space, d'Alembertian becomes $\Box = \partial_\mu \partial^\mu$, where $\partial_\mu = \partial/\partial x^\mu$ (this notation will be used throughout this text), where Einstein's convention (summation of repeated indices) was used. However, in the presence of a black hole, the curvature of space-time becomes relevant and therefore the partial derivatives will now have to be promoted to covariant derivatives. Then, $\Box\Phi = \nabla_\mu \nabla^\mu = 0$, where ∇_μ is the covariant derivative.

I.1. Covariant Derivatives of Vectors

Let us consider a basis $\{\vec{e}_\gamma\}$, a vector field $\vec{A} = A^\gamma \vec{e}_{(\gamma)}$ and its derivative,

$$\nabla_\beta \vec{A} = \frac{\partial A^\alpha}{\partial x^\beta} \vec{e}_{(\alpha)} + A^\alpha \frac{\vec{e}_{(\alpha)}}{\partial x^\beta}. \tag{1}$$

Since the first term (on the right-hand side) is a linear combination of the basis vectors, it can be computed easily. On the other hand, for the second term we need to compute how the basis vectors change on different points of a manifold M. For that, we are going to consider two points, $\mathbf{x}$ and $\mathbf{x}'$. Note that $\vec{e}_\alpha(\mathbf{x})$ and $\vec{e}_\alpha(\mathbf{x}')$ are each part of their respective tangent spaces, $T_\mathbf{x}(M)$ and $T_{\mathbf{x}'}(M)$ such that $T_\mathbf{x}(M) \neq T_{\mathbf{x}'}(M)$. So, to define the derivative of a vector field on a manifold, we need a rule to compare vectors from different tangent spaces. This rule is called a connection. Firstly, let us consider the Minkowski spacetime, which is flat. If we consider a point $\mathbf{x}$, there is a coordinate basis $\{\vec{e}_{M(\alpha)}(\mathbf{x})\}$ which belongs to the tangent space $T_\mathbf{x}(M)$. In this case, we can impose that $\vec{e}_{M(\alpha)}(\mathbf{x}) = \vec{e}_{M(\alpha)}(\mathbf{x}')$, for any other given point $\mathbf{x}'$, since the space is flat. [2]
Thus, the basis vectors are constant, and consequently

$$\frac{\partial \vec{e}_{M(\alpha)}}{\partial x^\beta} = \vec{0}. \tag{2}$$

This is the connection in Minkowski spacetime. Let us now recall the equivalence principle, *in any and every locally Lorentz (inertial) frame, the laws of special relativity must hold.* [3]
This implies that, in a general spacetime, the rules of Minkowski spacetime must hold locally. Therefore, we must impose the basis vectors are constant in a local inertial frame. This is the connection in this case. So, it is convenient to do a coordinate transformation $\Lambda_\alpha^{\alpha'}$ from the basis $\{\vec{e}_{(\alpha)}\}$ to a local basis $\{\vec{e}_{M(\alpha')}\}$,

$$\vec{e}_{(\alpha)} = \Lambda_\alpha^{\alpha'} \vec{e}_{M(\alpha')} \Rightarrow \frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = \left(\frac{\partial}{\partial x^\beta} \Lambda_\alpha^{\alpha'} \right) \vec{e}_{M(\alpha')}, \tag{3}$$

since $\vec{e}_{M(\alpha')}$ are constant, as we saw in Equation (2). Note that the right-hand side is a linear combination of the basis vectors $\{\vec{e}_{M(\alpha')}\}$ and therefore, the left-hand side of the equation is a vector. [2]
Then, we can express it as a linear combination of our initial basis $\{\vec{e}_\gamma\}$ such that,

$$\frac{\partial \vec{e}_{(\alpha)}}{\partial x_\beta} = \Gamma^\gamma_{\alpha\beta} \vec{e}_{(\gamma)}, \tag{4}$$

where $\Gamma^\gamma_{\alpha\beta}$ are called affine connection or Christoffel Symbols. Here, α implies the basis vector being differentiated and β implies on which coordinate the differentiation is being applied. Now, we can rewrite Equation (1) as,

$$\nabla_\beta \vec{A} = \frac{\partial A^\alpha}{\partial x^\beta} \vec{e}_{(\alpha)} + A^\alpha \Gamma^\gamma_{\alpha\beta} \vec{e}_{(\gamma)}, \tag{5}$$

or with the dummy indices were relabeled,

$$\nabla_\beta \vec{A} = \left(\frac{\partial A^\alpha}{\partial x^\beta} + A^\gamma \Gamma^\alpha_{\gamma\beta}\right) \vec{e}_{(\alpha)}. \tag{6}$$

Finally, let us introduce the following notation:

$$A^\alpha_{,\beta} = \frac{\partial A^\alpha}{\partial x^\beta}, \qquad A^\alpha_{;\beta} = \frac{\partial A^\alpha}{\partial x^\beta} + A^\mu \Gamma^\alpha_{\mu\beta}. \tag{7}$$

I.2. Covariant Derivative of Tensors

For the covariant derivative of rank 1 tensors, let us consider a scalar field $\tilde{\Omega} = \omega_\alpha A^\omega$, with arguments ω_α and A^α. Then, $\nabla_\beta \tilde{\Omega} = \omega_{\alpha;\beta} A^\alpha + \omega_\alpha A^\alpha_{,\beta}$. Using Equation (7), we get

$$\nabla_\beta \tilde{\Omega} = \omega_{\alpha,\beta} A^\alpha + \omega_\alpha \left(A^\alpha_{;\beta} - A^\gamma \Gamma^\alpha_{\gamma\beta}\right). \tag{8}$$

Relabeling the dummy indices, this equation becomes

$$\nabla_\beta \tilde{\Omega} = \left(\omega_{\alpha,\beta} - \omega_\gamma \Gamma^\gamma_{\alpha\beta}\right) A^\alpha + \omega_\gamma A^\alpha_{;\beta}. \tag{9}$$

From this we can intuitively extract the covariant derivative of a rank 1 tensor,

$$\omega_{\alpha;\beta} = \omega_{\alpha,\beta} - \omega_\gamma \Gamma^\gamma_{\alpha\beta}. \tag{10}$$

To compute the covariant derivative of rank 2 tensors. Let us now consider a scalar field $\Omega = T_{\mu\nu} A^\mu B^\nu$, with components A^μ and B^ν. Then,

$$\nabla_\beta \Omega = T_{\mu\nu,\beta} A^\mu B^\nu + T_{\mu\nu}(A^\mu_{,\beta}\omega_\mu + A^\mu B^\nu_{,\beta}). \tag{11}$$

Using Equation (7) the second term of the right-hand side of the equation becomes

$$T_{\mu\nu} A^\mu_{,\beta} B^\nu = T_{\mu\nu}\left(A^\mu_{;\beta} - A^\gamma \Gamma^\mu_{\gamma\beta}\right) B^\nu, \tag{12}$$

and the third becomes

$$T_{\mu\nu} A^\mu B^\nu_{,\beta} = T_{\mu\nu} A^\mu \left(B^\nu_{;\beta} - B^\gamma \Gamma^\nu_{\gamma\beta}\right). \tag{13}$$

Relabeling the dummy indices and putting back in Equation (11), $\nabla_\beta \Omega$ is given by,

$$C\left(T_{\mu\nu,\beta} - T_{\gamma\nu}\Gamma^\gamma_{\mu\beta} - T_{\mu\gamma}\Gamma^\gamma_{\nu\beta}\right) + T_{\mu\nu} C_{;\beta} \tag{14}$$

with $C = A^\mu B^\nu$. From this equation, we can intuitively get the covariant derivative of $T_{\mu\nu}$,

$$T_{\mu\nu;\beta} = T_{\mu\nu,\beta} - T_{\gamma\nu}\Gamma^\gamma_{\mu\beta} - T_{\mu\gamma}\Gamma^\gamma_{\nu\beta}. \tag{15}$$

I.3. Metric Tensor

Let us now define the metric tensor, but first we need introduce the concept of inner product.

Definition 1. Inner Product
Let V be a vector space and $\vec{u}, \vec{v}, \vec{w} \in V$ three vectors. The inner product between two vectors, denoted by $\cdot$ is a scalar that follows the following properties, for any scalars α, β:
(1) (Conjugate) Symmetry: $\vec{x} \cdot \vec{y} = \overline{\vec{y} \cdot \vec{x}}$.
(2) Linearity: $\vec{x} \cdot (\alpha\vec{y} + \beta\vec{w}) = \alpha\,(\vec{x} \cdot \vec{y}) + \beta\,(\vec{x} \cdot \vec{w})$.

Definition 2. Metric Tensor
Let $\mathbf{g}$ be a rank 2 tensor such that for two arbitrary vectors, $\vec{x}$ and $\vec{y}$, we have, $g(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$, where $\cdot$ denotes the inner product between these two vectors.

Note that from the first property of Definition 1, the metric tensor is symmetric, $g(\vec{x}, \vec{y}) = g(\vec{y}, \vec{x})$, for any given $\vec{x}, \vec{y}$. Now, if we replace $\vec{x}, \vec{y}$ with the basis vectors, we get $g_{\mu\nu} = g(\vec{e}_{(\mu)}, \vec{e}_{(\nu)}) = \vec{e}_{(\mu)} \cdot \vec{e}_{(\nu)}$, to compute the inner product using any space and coordinates, $\vec{x} \cdot \vec{y} = g(x^\mu \vec{e}_{(\mu)}, y^\nu \vec{e}_{(\nu)}) = x^\mu y^\nu g_{\mu\nu}$. We can use the metric tensor to upper and lower the indices by $x_\mu = g_{\mu\nu} x^\nu$ and $x_\nu = g^{\mu\nu} x_\mu$, respectively. Let us now apply the covariant derivative on the metric tensor. Using the equivalence principle, we can choose a coordinate system where $g_{\mu\nu}$ reduces to the Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, which is constant. Then, using Equation (15), we get,

$$g_{\mu\nu;\beta} = \eta_{\mu\nu;\beta} = \frac{\partial \eta_{\mu\nu}}{\partial x^\beta} - \Gamma^\gamma_{\mu\beta}\eta_{\gamma\nu} - \Gamma^\gamma_{\mu\beta}\eta_{\mu\beta} = 0. \tag{16}$$

With this we can conclude that the metric tensor is always zero.

I.4. Christoffel Symbols

Let us consider a scalar field Φ. We can use the usual derivative and we get a rank 1 tensor ($\Phi_{,\alpha}$). Then, if apply the covariant derivative, we get a rank 2 tensor, $\Phi_{,\alpha;\beta}$. If we consider the Minkowski space, this derivative reduces to $\Phi_{,\alpha,\beta}$. Since they are usual derivatives, they can commute and therefore $\Phi_{,\alpha,\beta} = \Phi_{,\beta,\alpha} \Rightarrow \Phi_{,\alpha;\beta} = \Phi_{,\beta;\alpha}$. Then, this tensor is symmetric. Therefore, using Equation (10), we get $\Phi_{,\alpha;\beta} = \Phi_{,\beta;\alpha} \Leftrightarrow \Phi_{,\alpha,\beta} - \Phi_{,\gamma}\Gamma^\gamma_{\alpha\beta} = \Phi_{,\beta,\alpha} - \Phi_{,\gamma}\Gamma^\gamma_{\beta\alpha}$ and then we can see that the Christoffel symbols are symmetric,

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}. \tag{17}$$

Let us now express the Christoffel symbols in terms of the metric. Recall that the covariant derivative of the metric tensor is always zero. Therefore, using Equation (15), we get

$$g_{\mu\nu;\beta} = 0 \Leftrightarrow g_{\mu\nu,\beta} - g_{\gamma\nu}\Gamma^{\gamma}_{\mu\beta} - g_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta} = 0. \tag{18}$$

Then, we have

$$\begin{aligned} g_{\mu\nu,\beta} &= g_{\gamma\nu}\Gamma^{\gamma}_{\mu\beta} + g_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta}, \\ g_{\mu\beta,\nu} &= g_{\gamma\beta}\Gamma^{\gamma}_{\mu\nu} + g_{\mu\gamma}\Gamma^{\gamma}_{\beta\nu}, \\ -g_{\nu\beta,\mu} &= -g_{\gamma\beta}\Gamma^{\gamma}_{\nu\mu} - g_{\nu\gamma}\Gamma^{\gamma}_{\beta\mu}. \end{aligned} \tag{19}$$

Considering that the metric tensor is symmetric ($g_{ij} = g_{ji}$) and summing these three expressions, we get

$$\begin{aligned} g_{\mu\nu,\beta} + g_{\mu\beta,\nu} - g_{\nu\beta,\mu} = g_{\gamma\nu}(\Gamma^{\gamma}_{\mu\beta} - \Gamma^{\gamma}_{\beta\mu}) + \\ + g_{\gamma\beta}(\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\gamma}_{\nu\mu}) + g_{\mu\gamma}(\Gamma^{\gamma}_{\nu\beta} + \Gamma^{\gamma}_{\beta\nu}). \end{aligned} \tag{20}$$

Since the Christoffel symbols are symmetric as we seen in Equation (17), it follows that only the last term on the right-hand side of the previous equation does not cancel out and then,

$$g_{\mu\nu,\beta} + g_{\mu\beta,\nu} - g_{\nu\beta,\mu} = 2g_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta}. \tag{21}$$

Lastly, dividing both sides of the equation by two and applying the inverse metric $g^{\mu\sigma}$ on left on both sides of the equation, we get

$$\begin{aligned} g^{\mu\sigma}g_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta} &= \frac{1}{2}g^{\mu\sigma}\left(g_{\mu\nu,\beta} + g_{\mu\beta,\nu} - g_{\nu\beta,\mu}\right) \\ \Leftrightarrow \Gamma^{\sigma}_{\nu\beta} &= \frac{1}{2}g^{\mu\sigma}\left(g_{\mu\nu,\beta} + g_{\mu\beta,\nu} - g_{\nu\beta,\mu}\right), \end{aligned} \tag{22}$$

since $g^{\mu\sigma}g_{\mu\gamma} = \delta^{\sigma}_{\gamma}$.

II. The Wave Equation

II.1. Schwarzschild Metric

To apply d'Alembertian to a scalar field, we must now choose a metric. We are going to choose the Schwarzschild metric, since our black hole is static and spherically symmetric. [4]
Therefore, our metric is defined by,

$$g_{\mu\nu} = \text{diag}\left(-1 + \frac{r_s}{r}, \frac{1}{1 - \frac{r_s}{r}}, r^2, r^2\sin^2\theta\right), \tag{23}$$

in spherical coordinates and with $r_s = 2GM/c^2$, where G is the gravitational constant, M is the mass of the black hole and c is the speed of light in vacuum. The inverse metric is given by,

$$g^{\mu\nu} = \text{diag}\left(\frac{r}{r_s - r}, \frac{r_s - r}{r}, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right). \tag{24}$$

Since the matrix is diagonal, the determinant of $g_{\mu\nu}$, g, is simply the product of the entries in the main diagonal,

$$g = -r^4\sin^2\theta. \tag{25}$$

We are now in a position to solve our problem, $\Box\Phi = \nabla_\mu\nabla^\mu\Phi = 0$. Since Φ is a scalar field, $\nabla^\mu\Phi = \partial^\mu\Phi$. Since $\partial^\mu\Phi$ is a vector (or a contravariant tensor), we apply Equation (6) to calculate the covariant derivative we get:

$$0 = \Box\Phi = \nabla_\mu(\partial^\mu\Phi) = \partial_\mu(\partial^\mu\Phi) + (\partial^\nu\Phi)\Gamma^{\mu}_{\nu\mu}. \tag{26}$$

Let us now consider the following propositions. [5]

Proposition 1. Let A be a matrix of order n with scalar entries and λ_i $(0 \leq i \leq n)$ its eigenvalues. Then, $\text{tr}(A) = \sum_i \lambda_i$ and $\det(A) = \Pi_i.\lambda_i$.

Proof. If we factor the matrix as $A = SJS^{-1}$, with J the Jordan canonical form of A, we have $\text{tr}(A) = \text{tr}(SJS^{-1}) = \text{tr}(S^{-1}SJ) = \text{tr}(J) = \sum_i \lambda_i$, since J has all eigenvalues of A along the main diagonal. Here we used the fact that the trace is invariant under circular shifts, since it is defined as the sum of the entries along the main diagonal. Now, note that when writing the characteristic polynomial we get $p(\lambda) = \Pi_i(\lambda - \lambda_i)$. Since $p(\lambda) = \det(A - \lambda I_n)$ (with I_n the identity matrix), $p(0) = \det(A)$ equals the independent term, $\Pi_i\lambda_i$. □

Proposition 2. Let A be an invertible matrix of order n with scalar entries. Then, $\log(\det(A)) = \text{tr}(\log(A))$.

Proof. Let λ_i $(0 \leq i \leq n)$ be the eigenvalues of A (counting their multiplicities). Using Proposition 1, we have $\exp(\sum_i \lambda_i) = e^{\text{tr}(A)}$. On the other hand, $\exp(\sum_i \lambda_i) = \Pi_i e^{\lambda_i} = \det(e^A)$. Then, $\det(e^A) = e^{\text{tr}(A)}$. Doing $e^A \to A$ which implies that $A \to \log(A)$, we have $\det(A) = e^{\text{tr}(\log(A))}$. Finally, applying the logarithm on both sides we get our result. □

Taking the the derivative on both sides, we get (with $|A| \equiv \det A$),

$$\frac{\partial_k A}{|A|} = \text{tr}\left(A^{-1}\partial_k A\right) \Rightarrow \frac{1}{T}\partial_k T = T^{\mu\nu}\partial_k T_{\mu\nu},. \tag{27}$$

where in the last term we apply tensor notation, with $T_{\mu\nu}$ is a rank 2 tensor and T its determinant.

Proposition 3. Let g be the determinant of the metric tensor $g_{\mu\nu}$. Then, $\Gamma^{i}_{ij} = \partial_j \log\sqrt{|g|}$.

Proof. Using Equation (22), we have

$$\Gamma^i_{ij} = \frac{1}{2} g^{\mu i} \left(g_{\mu i,j} + g_{\mu j,i} - g_{ij,\mu} \right), \tag{28}$$

where the two last terms cancel out by relabeling the (dummy) indices. Finally, applying the Equation (27), Γ^i_{ij} becomes

$$\frac{1}{2} g^{\mu i} \partial_j g_{\mu i} = \frac{\partial_j g}{2g} = \frac{1}{2} \partial_j \left(\log |g| \right) = \partial_j \log \sqrt{|g|}. \tag{29}$$

□

Returning to Equation (26) and using the symmetry of Christoffel symbols, Equation (17), we get

$$0 = \partial_\mu \partial^\mu \Phi + \partial^\nu \Phi \left(\partial_\nu \log \sqrt{-g} \right), \tag{30}$$

where we used Proposition 3 and the fact that $g < 0$. Then, with some manipulation, we can obtain

$$\begin{aligned} 0 &= \partial_\mu \partial^\mu \Phi + \partial^\nu \Phi \frac{\partial_\nu \sqrt{-g}}{\sqrt{-g}} = \\ &= \frac{1}{\sqrt{-g}} \left(\sqrt{-g} \partial^\mu \partial_\mu \Phi + (\partial^\nu \Phi)(\partial_\nu \sqrt{-g}) \right) = \\ &= \frac{\sqrt{-g} \partial_\mu (g^{\mu\nu} \partial_\nu \Phi) + (g^{\mu\nu} \partial_\mu \Phi)(\partial_\nu \sqrt{-g})}{\sqrt{-g}}. \end{aligned} \tag{31}$$

Finally, after relabeling the dummy indices and simplifying this equation, it follows that

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) = 0. \tag{32}$$

II.2. Derivation of the Wave Equation

We now derive the wave equation. To do this, we continue by differentiating Equation (32), replacing the metric tensor, its inverse and its determinant, Equations (23), (24) and (25) (respectively), and using the fact $g_{\mu\nu}$ is diagonal, we have
Since $g_{\mu\nu}$ is diagonal,

$$\begin{aligned} 0 &= \frac{1}{r^2 \sin^2\theta} \Big(\partial_0 \left(g^{00} \partial_0 \Phi \right) + \partial_1 \left(g^{11} \partial_1 \Phi \right) + \\ &\quad + \partial_2 \left(g^{22} \partial_2 \Phi \right) + \partial_3 \left(g^{33} \partial_3 \Phi \right) \Big) = \\ &= \frac{r}{r_s - r} \partial_0^2 \Phi + \frac{2r - r_s}{r^2} \partial_1 \Phi + \frac{r - r_s}{r} \partial_1^2 \Phi + \\ &\quad + \frac{1}{r^2} \left(\cot\theta \partial_2 \Phi + \partial_2^2 \Phi + \frac{1}{\sin^2\theta} \partial_3^2 \Phi \right). \end{aligned} \tag{33}$$

Then, our wave equation equation can be written as

$$-\frac{1}{c^2} \frac{r(\partial_t^2 \Phi)}{r_s - r} = \frac{2r - r_s}{r^2} \partial_r \Phi + \frac{r - r_s}{r} \partial_r^2 \Phi + \frac{\tilde{\nabla} \Phi}{r^2}, \tag{34}$$

with

$$\tilde{\nabla}^2 = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2}. \tag{35}$$

We will now introduce the tortoise coordinates (r_*) as,

$$\frac{\partial r_*}{\partial r} = \left(1 - \frac{r_s}{r} \right)^{-1} = \frac{r}{r - r_s} \Rightarrow \frac{\partial r}{\partial r_*} = \frac{r - r_s}{r}. \tag{36}$$

Replacing $\partial_r \to \partial_{r_*}$ in Equation (34), we get

$$\begin{aligned} 0 &= \frac{1}{c^2} \frac{r}{r_s - r} \partial_t^2 \Phi + \frac{2r - r_s}{r^2} \frac{r}{r - r_s} \partial_{r_*} \Phi + \\ &\quad + \frac{r - rs}{r} \partial_r \left(\frac{r}{r - r_s} \partial_{r_*} \Phi \right) + \frac{1}{r^2} \tilde{\nabla} \Phi = \\ &= -\frac{1}{c^2} \partial_t^2 \Phi + \frac{r - r_s}{r^3} \tilde{\nabla} \Phi + \frac{2}{r} \frac{r - r_s}{r} \partial_{r_*} \Phi + \partial_{r_*}^2 \Phi. \end{aligned} \tag{37}$$

We now want this equation to resemble the known wave equation in Minkowski space, linear and second order in space and time. For that, we are now going to rescale our field $\Phi \to \Psi$ via $\Phi = f(r)\Psi$, with $f(r) \equiv f$ a function we want to find. Then we define $\gamma = \gamma(r) = (r - r_s)/r$, $\Psi' = \partial_{r_*} \Psi$, f' and γ' (where $'$ denotes the derivative with respect to r). Note that $\partial r / \partial r_* = \gamma$. We now impose, a rule such that there are no first order derivatives. Therefore, using $2(r - r_s)/r^2 = 2\gamma/r$,

$$\begin{aligned} \frac{2}{r} \frac{r - r_s}{r} \partial_{r_*} \Phi + \partial_{r_*}^2 \Phi &= \frac{2\gamma}{r} \left(\gamma f' \Psi + f \Psi' \right) + \\ &\quad + 2\gamma f' \Psi' + f \Psi'' + \gamma\gamma' f' \Psi + \gamma^2 f'' \Psi = \\ &= \Psi'' f + \Psi \left(\frac{2\gamma^2 f'}{r} + \gamma\gamma' f' + \gamma^2 f'' \right) \\ &\quad + \Psi' \left(\frac{2\gamma f}{r} + 2\gamma f' \right). \end{aligned} \tag{38}$$

Since we want only second order derivatives, the last term must be zero and then,

$$\frac{2\gamma f}{r} + 2\gamma f' = 0 \Leftrightarrow \frac{f}{r} + f' = 0 \Leftrightarrow f + r f' = 0, \tag{39}$$

which is an ordinary differential equation with solution

$$\frac{d}{dr}(r f) = 0 \Leftrightarrow r f(r) = k \Leftrightarrow f(r) = \frac{k}{r}, \tag{40}$$

with k an arbitrary constant. We will choose $k = 1$ for simplicity. Recall that there is an additional term that multiplies by Ψ, $2\gamma^2 f'/r + \gamma' f' + \gamma^2 f'' = -(1 - r_s/r)(r_s/r^4)$.

Therefore, Equation (38) yields to,

$$-\left(1-\frac{r_s}{r}\right)\frac{r_s}{r^4}\Psi+\frac{1}{r}\partial^2_{r_*}\Psi. \qquad (41)$$

Replacing this in our wave equation and multiplying it by r, Equation (37), we have

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}=\frac{r-r_s}{r^3}\tilde{\nabla}\Psi-\left(1-\frac{r_s}{r}\right)\frac{r_s}{r^3}\Psi+\frac{\partial^2\Psi}{\partial r_*^2}. \quad (42)$$

Let us now separate our field $\Psi\equiv\Psi(t,r,\theta,\varphi)$ in two parts such that,

$$\Psi=\sum_{\ell,m}\psi_{\ell m}(t,r)Y^{\ell m}(\theta,\varphi)\equiv\psi_{\ell m}Y^{\ell m}, \qquad (43)$$

from which we get,

$$\begin{aligned}0=&-\frac{1}{c^2}\frac{\partial^2\psi_{\ell m}}{\partial t^2}Y^{\ell m}+\frac{\partial^2\psi_{\ell m}}{\partial r_*^2}Y^{\ell m}-\\&-\left(1-\frac{r_s}{r}\right)\frac{r_s}{r^3}\psi_{\ell m}Y^{\ell m}+\frac{r-r_s}{r^3}\psi_{\ell m}\tilde{\nabla}Y^{\ell m}.\end{aligned} \qquad (44)$$

This way, the radial and angular parts separate, which simplifies our problem. Here is why: our black hole is spherically symmetric, therefore $Y^{\ell m}$ corresponds to spherical harmonics that respect the eigenvalue equation [6],

$$\tilde{\nabla}^2Y^{\ell m}=-\ell(\ell+1)Y^{\ell m}. \qquad (45)$$

Then, the last term of Equation (44) yields to

$$-\frac{r-r_s}{r^3}\left(\ell(\ell+1)\right)\psi_{\ell m}Y^{\ell m}, \qquad (46)$$

using Equation (45).
Finally, by replacing this on Equation (44), we can divide our wave equation by $Y^{\ell m}$, from which we get a second order partial differential equation (PDE) on two variables (time and space), similar to the wave equation in Minkowski space, except our equation has an additional term $V(r)$, which we call an effective potential,

$$-\frac{1}{c^2}\frac{\partial^2\psi_{\ell m}}{\partial t^2}+\frac{\partial^2\psi_{\ell m}}{\partial r_*^2}-V(r)\psi_{\ell m}=0, \qquad (47)$$

such that $V(r)$,

$$V(r)=\left(1-\frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}+\frac{r_s}{r^3}\right). \qquad (48)$$

III. Numerical Methods

To solve our PDE, we are going to use numerical methods. Here we will see the methods used to obtain the solution and determine the QNMs. However, the integral code is available on my personal GitHub, written in C++. [7]
We will also use natural units such that $c=M=G=1$.

III.1. Preliminaries

To solve our equation we need to discretize it. First, we select a grid size. Since we are not able to make a grid that goes to infinity, we are going to choose a range for r_* that goes from $-r_*^{\max}$ to $+r_*^{\max}$, with a large value for the latter. We now need to select a time step and a space step. But first, let us consider the Courant-Friedrich-Lewy (or simply CFL) condition: $c\Delta t\leq\Delta r_*$, with Δt the time step and Δr_* the space step. This condition holds because we want our numerical domain of dependence to be larger than the physical domain of dependence, so the numerical solution converges to the exact solution, since we want our domain to cover the whole grid. [8]

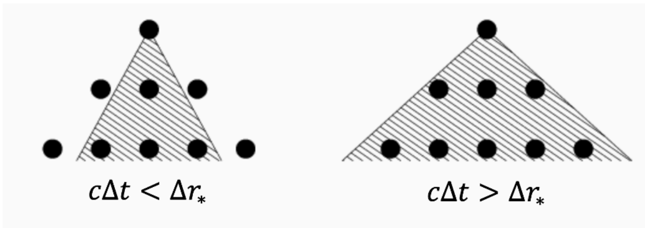

Figure 1: Illustration of the CFl Condition. Image from [8] (adapted).

Therefore, our space and time steps are (respectively),

$$\Delta r_*=\frac{2r_*^{\max}}{N+1}, \qquad \Delta t=CFL\cdot\Delta r_*, \qquad (49)$$

with N the number of spacial points and CFL a fixed number lower than one, in order to respect the CFL condition. We are going to choose $CFl=0.5$.

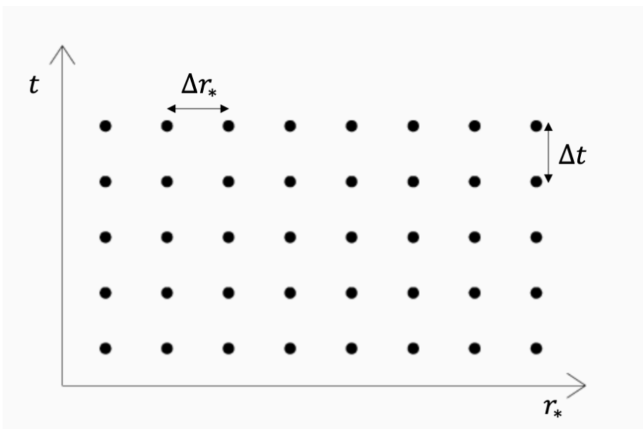

Figure 2: Grid Points. Image from [8] (adapted).

Note: the larger the number of points, the lower the error, and thus the more accurate the numerical solution.

III.2. Tortoise Coordinates

Earlier we introduced tortoise coordinates in Equation (36). Integrating with respect to r, we get

$$r_* = \int 1 + \frac{1}{\frac{r}{r_s} - 1}\, dr = r + r_s \log\left|\frac{r}{r_s} - 1\right|. \tag{50}$$

Rewriting these terms, we get

$$\exp\left(\frac{r_* - r}{r_s}\right) = \frac{r - r_s}{r_s} \Leftrightarrow \frac{r}{r_s} = 1 + W(x), \tag{51}$$

where we used the Lambert W function, $W(x)$, with argument $x = \exp(r_*/r_s - 1)$. [9]
This is useful, since we have a library (*boost*) in C++ that computes the value of $W(x)$. For robustness, we implement two exceptions:

(1) If $r_* \ll r_s$, r becomes really small and we consider $r \approx r_s x$, since $W(x) \approx x$ for small x.
(2) If $r_* \gg r_s$, we consider $r \approx r_*$, because a large r_* implies a large r and therefore, the logarithm in Equation (50) is really small compared to r.

For reasonable values of r_*, we compute $W(x)$ using the command *boost::math::lambert_w0(x)*, with $x = \exp(r_*/r_s - 1)$ and $r = r_s(1 + W(x))$.

III.3. Numerical Solution

III.3.1. Initial Data

To solve any PDE, we need to choose initial data. We will choose a Gaussian pulse with an initial velocity of zero.
This means that for a field $\psi_0 \equiv \psi_{\ell m}(t = 0, r_*)$, we have

$$\psi_0 = A\exp\left(\frac{(r_* - \mu)^2}{2\sigma^2}\right), \qquad \dot{\psi}_0 = 0, \tag{52}$$

with A the amplitude, μ the location of the peak and σ the width of the pulse, and $\dot{\psi}_0$ denotes the time derivative of $\psi_{\ell m}(t = 0, r_*)$.

III.3.2. The Bulk

We are going to start by solving the equation on the interior spatial points (the bulk), from 1 to $N - 1$, with N spatial points, and then worry about boundary conditions. To solve our problem, we are going to use Runge (RK4) for time integration and second order finite difference to compute the spatial derivatives of each point. For that, let us introduce the following notation: for a given scalar field $\psi(t, r_*)$, $\psi_{i+1}^j = \psi(t + \Delta t, r_*)$ and $\psi_i^{j+1} = \psi(t, r_* + \Delta r_*)$, where Δt is the time step and Δr_* is the space step.
Starting with the second order finite difference, if we expand in Taylor ψ_i^{j-1} and ψ_i^{j+1} around the point (i, j), we get (respectively),

$$\begin{aligned} \psi_i^{j-1} &\approx \psi_i^j - \frac{\partial\psi}{\partial r_*}\Delta r_* + \frac{1}{2}\frac{\partial^2\psi}{\partial r_*^2}(\Delta r_*)^2 \\ \psi_i^{j+1} &\approx \psi_i^j + \frac{\partial\psi}{\partial r_*}\Delta r_* + \frac{1}{2}\frac{\partial^2\psi}{\partial r_*^2}(\Delta r_*)^2. \end{aligned} \tag{53}$$

The sum of these two expressions yield to

$$\begin{aligned} &\psi_i^{j-1} + \psi_i^{j+1} = 2\psi_i^j + \frac{\partial^2\psi}{\partial r_*^2}(\Delta r_*)^2 \\ \Leftrightarrow\ &\frac{\partial^2\psi}{\partial r_*^2} = \frac{\psi_i^{j-1} - 2\psi_i^j + \psi_i^{j+1}}{(\Delta r_*)^2}, \end{aligned} \tag{54}$$

which we call the second order central finite difference approximation for space. [8]
This approximation cannot be applied at the boundaries because it would require ghost points ($j = -1$ and $j = N + 1$) outside the grid domain. We will discuss a solution for this in the next subsection. We now need to discuss our time integrator, RK4. [10]
This method works for first order differential equations with the form $y' = f(t, y)$, with f some arbitrary function. Therefore, we get our solution by solving the following system of equations, for some scalar field $\pi(t, r_*)$ (with $c = 1$),

$$\begin{cases} \dfrac{\partial\psi}{\partial t} = \pi \\ \dfrac{\partial^2\psi}{\partial t^2} = \dfrac{\partial\pi}{\partial t} \end{cases} \Leftrightarrow \begin{cases} \dfrac{\partial\psi}{\partial t} = \pi \\ \dfrac{\partial^2\psi}{\partial r_*^2} - V(r) = \dfrac{\partial\pi}{\partial t} \end{cases}. \tag{55}$$

From the initial data, we know $\pi(0, r_*) = 0$. Then, we will create eight vectors with N dimension (k^α and z^α, with $\alpha \in 1, 2, 3, 4$) and apply the following algorithm, repeating for each $i \in [0, T - 1]$ (where T are the time points):

First Stage: repeat for each $j \in [1, N - 1]$,
$k_j^1 = \pi_i^j \Delta t.$
$z_j^1 = \left(D(\psi_i^{j-1}, \psi_i^j, \psi_i^{j+1}) - V_j u_i^j\right)\Delta t.$
Second Stage: repeat for each $j \in [1, N - 1]$,
$k_j^2 = (\pi_i^j + 0.5 z_j^1)\Delta t.$
$z_j^2 = \left(D(\psi_i^{j-1} + 0.5k_{j-1}^1, \psi_i^j + 0.5k_j^1, \psi_i^{j+1} + 0.5k_{j+1}^1) - V_j(\psi_i^j + 0.5k_{j-1}^1)\right)\Delta t.$
Third Stage: repeat for each $j \in [1, N - 1]$,
$k_j^3 = (\pi_i^j + 0.5 z_j^2)\Delta t.$
$z_j^3 = \left(D(\psi_i^{j-1} + 0.5k_{j-1}^2, \psi_i^j + 0.5k_j^2, \psi_i^{j+1} + 0.5k_{j+1}^2) - V_j(u_i^j + 0.5k_{j-1}^2)\right)\Delta t.$
Fourth Stage: repeat for each $j \in [1, N - 1]$,
$k_j^4 = (\pi_i^j + 0.5 z_j^3)\Delta t.$
$z_j^4 = \left(D(\psi_i^{j-1} + k_{j-1}^3, \psi_i^j + k_j^3, \psi_i^{j+1} + k_{j+1}^3) - V_j(\psi_i^j + k_{j-1}^3)\right)\Delta t.$
Fifth Stage: repeat for each $j \in [1, N - 1]$,
$\psi_{i+1}^j = \psi_i^j + (k_j^1 + 2k_j^2 + 2k_j^3 + k_j^4)/6.$
$\pi_{i+1}^j = \pi_i^j + (z_j^1 + 2z_j^2 + 2z_j^3 + z_j^4)/6.$

Figure 3: Algorithm to solve our equation.

Here the function D gives us the result of the second space derivative, using the (central) finite difference method on Equation (54), and V_j the potential evaluated on the j-th point. Remember that V is a function of r and not r_* and therefore we will use the method discussed in Subsection III.2. to convert r_* to r and then compute V.

However, there is a problem with this algorithm: it does not consider the boundary conditions. We will now discuss how to fix this.

III.3.3. Boundary Conditions

Since we do not want our wave to reflect, we are going to need boundary conditions. Then, when it reaches the left boundary, the wave should disappear, since it is being absorbed by the black hole, $r_* \to -\infty$ (purely ingoing wave). The same applies for the right boundary, but this time we do not want it to reflect because it should continue to travel in free space, $r_* \to +\infty$ (purely outgoing wave). To implement this, we will use the following boundary conditions (with $c = 1$),

$$\frac{\partial \psi_{\ell m}}{\partial r_*} \pm \frac{\partial \psi_{\ell m}}{\partial t} = 0 \Rightarrow \frac{\partial^2 \psi_{\ell m}}{\partial t \partial r_*} \pm \frac{\partial^2 \psi_{\ell m}}{\partial t^2} = 0, \quad (56)$$

where the last step we applied the time derivative to minimize the reflections. Here we can approximate our potential, Equation (48), since for $r \to r_s$ (left boundary where $r_* \to -\infty$) and $r \to +\infty$ (right boundary where $r_* \to +\infty$), $V(r) \to 0$.

To implement this, we used RK4 once again to compute the time derivatives and finite difference to compute the spatial derivatives. For the time derivatives, the method is very similar to the one we used for the bulk. However, computing spatial derivatives requires additional points not available in our grid. Therefore, for second order accuracy, we will use the first three points (0, 1 and 2) for the right boundary and the last three points ($N-2$, $N-1$ and N) for the left boundary, where N is the number of spatial points of our grid. [10], [11]

If we expand in Taylor around (i, j) once again, as we did in Equation (53), but for ψ_i^{j-2} and ψ_i^{j-1}, we get

$$\begin{aligned} \psi_i^{j-2} &\approx \psi_i^j - 2\frac{\partial \psi}{\partial r_*}\Delta r_* + 4\frac{1}{2}\frac{\partial^2 \psi}{\partial r_*^2}(\Delta r_*)^2 \\ \psi_i^{j-1} &\approx \psi_i^j - \frac{\partial \psi}{\partial r_*}\Delta r_* + \frac{1}{2}\frac{\partial^2 \psi}{\partial r_*^2}(\Delta r_*)^2. \end{aligned} \quad (57)$$

We now want, for some a, b, c,

$$\begin{aligned} \frac{\partial \psi}{\partial r_*} &= a\psi_i^{j-2} + b\psi_i^{j-1} + c\psi_i^j = (a+b+c)\psi_i^j - \\ &-(2a+b)\frac{\partial \psi}{\partial r_*}(\Delta r_*) + (4a+b)\frac{\partial^2 \psi}{\partial r_*^2}(\Delta r_*)^2 \\ &\Leftrightarrow \begin{cases} a+b+c = 0 \\ (-2a-b)(\Delta r_*) = 1 \\ (4a+b)(\Delta r_*)^2 = 0 \end{cases}. \end{aligned} \quad (58)$$

Solving this system, we get $2\Delta r_*(a, b, c) = (1, -4, 3)$. Then,

$$-\frac{\partial \psi}{\partial r_*} = -\frac{\psi_i^{j-2} - 4\psi_i^{j-1} + \psi_i^j}{2\Delta r_*}, \quad (59)$$

which we call the backward finite difference approximation for space and it is our left boundary condition (notice the minus sign). Similarly, we can do a forward finite difference approximation for space using $a\psi_i^j + b\psi_i^{j+1} + \psi_i^{j+2}$, from which we get,

$$\frac{\partial \psi}{\partial r_*} = \frac{-\psi_i^j + 4\psi_i^{j+1} - 3\psi_i^{j+2}}{2\Delta r_*}, \quad (60)$$

such that $(a, b, c) = (-1, 4, -3)$.

Finally, let us go back to our algorithm in the previous section and apply the following changes (repeating for each $i \in [0, T-1]$), respecting our boundary conditions, Equation (56).

Zeroth Stage:
Apply the previous algorithm from First to Forth Stage for: $j = 0$, changing the operator $D \to f$ and $j = N$, changing the operator $D \to b$.
Fifth Stage (Update): repeat for each $j \in [0, N]$,
$u_{i+1}^j = u_i^j + (k_j^1 + 2k_j^2 + 2k_j^3 + k_j^4)/6$.
$\pi_{i+1}^j = \pi_i^j + (z_j^1 + 2z_j^2 + 2z_j^3 + z_j^4)/6$.

Figure 4: Algorithm to solve our equation (revisited).

Here, the functions f and b compute (respectively) the spatial derivatives, using the forward and backward finite difference approximation, Equations (59) and (60).

III.3.4. Solving the Equation

We now proceed to solve our equation. Let us recall that we are using $c = G = M = 1$ (natural units). Then, we now pick a $r_*^{\max} = 2000$. Remember that our initial pulse is a Gaussian: we now choose an amplitude $A = 100$, the location of the peak $\mu = 100$ and the width $\sigma = 25$. We also need to choose the size of our grid: the spatial points are

going to be $N = 8000$ and the time points are going to be $T = 5000$. This way, using Equation (49), we get our steps. Finally, we need to pick an angular momentum ℓ, since our field depends directly with ℓ, as we can see in Equation (47). We are going to choose $\ell = 4$ for better visualization.
This way, we use our numerical methods to solve our equation, to store our solution in a $T \times N$ matrix.

III.3.5. Convergence Test

To determine whether our numerical solution converges to the exact solution, we perform a convergence test. For that, we are going to solve our problem three different times, each one with different spatial points and, consequently, different space steps. Then, for given N spatial points and Δr_* space step, we have

(1) ψ_H which corresponds to N and Δr_*.
(2) ψ_M which corresponds to $N/2$ and $\Delta r_*/2$.
(3) ψ_L which corresponds to $N/4$ and $\Delta r_*/4$.

This way, we have three different resolutions and we can compute the errors between (ψ_H, ψ_M) and (ψ_M, ψ_L) via

$$\varepsilon_1 = |\psi_H - \psi_M|\,, \quad \varepsilon_2 = |\psi_M - \psi_L|\,, \tag{61}$$

such that $\varepsilon_2 \approx 4\varepsilon_1$, which is the idea of our convergence test: if this equality holds, then our numerical solution is converging to the exact solution. In other words means that the high resolution errors need to be four times greater than the low resolution error. [11]
Figure 5 shows the time and spatial steps we are going to use to run our convergence test.

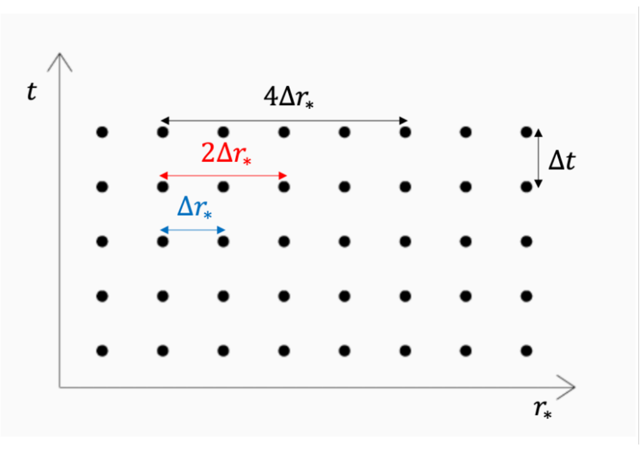

Figure 5: Spatial Steps for Convergence Test. Image from [8] (adapted).

In our case, we solved the problem starting with $(N, \Delta r_*) = (8000, 2r_*^{\max}/(8000+1))$ for ψ_H, then $(N/2, \Delta r_*/2)$ for ψ_M and finally with $(N/4, \Delta r_*/4)$ for ψ_L, using Equation (61). We also chose $(A, \mu, \sigma) = (100, 500, 25)$ and $t = 500$. We then get,

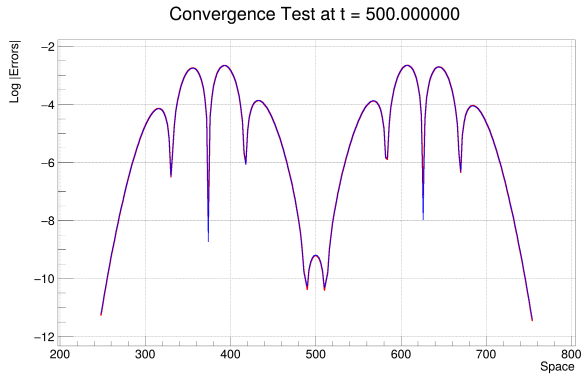

Figure 6: Convergence Test (Linear Log Scale). Red Line: $4\varepsilon_1$ and Blue Line: ε_2.

As we can see, the two curves practically lie on top of each other such that $\varepsilon_2 \approx 4\varepsilon_1$. Therefore, the numerical solution converges to the exact solution.

IV. Quasi-Normal Modes

We are now in a position to determine the QNMs. For that, we are going to pick a point in space which we will call our observer. We chose to pick $r_* = 500$ and get the vector solution at this position from our solution matrix.
We then fit the data using

$$f(t) = A \exp\left(-\frac{t - t_p}{\tau}\right) \sin\left(\omega(t - t_p) + \varphi\right), \tag{62}$$

with A, τ, ω and φ are the fit parameters and t_p corresponds to the time of the maximum peak. We use a linear-log scale for better visualization. Since we are focusing on the QNMs, this will not interfere with the result. We now can make a plot t vs. $\log|\psi_{\ell m}(t, 500)|$ and, applying the logarithm to Equation (62), we can fit our data, from which we get

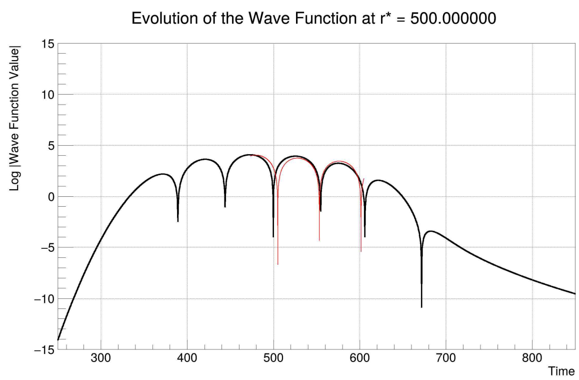

Figure 7: Quasi-Normal Modes at $r_* = 500$.

The numerical results are,

(1) Amplitude: $B = (40.83 \pm 0.01) \times 10^{-1}$.
(2) Decay Time: $tau = 161.29 \pm 10.41$.
(3) Frequency: $\omega = (6.51 \pm 0.01) \times 10^{-2}$.
(4) Phase: $\varphi = (10.52 \pm 0.03) \times 10^{-1}$.

with t_p occurring at 472.94 and $B = \log|A|$ and a reduced chi-squared of $\chi^2_{red} = 294/524 \approx 0.50$. As we can see, the damping timescale ($\tau = 161.29 \pm 10.41$) indicates a slowly decaying ring-down, while the frequency ($\omega = (6.51 \pm 0.01) \times 10^{-2}$) indicates considerable oscillations. Finally, the reduced χ^2 value suggests a good fit. However, since $\chi^2 < 1$, it is possible that the uncertainties may have been overestimated.

V. Conclusion

In this project, we studied the behavior of a massless scalar field propagating in the curved spacetime of a Schwarzschild black hole. We began by introducing the mathematical foundation, focusing on covariant derivatives for vectors and tensors, which are essential for curved spacetimes. Then we derived the wave equation in the Schwarzschild spacetime. To simplify the equation, we introduced tortoise coordinates that pushes the radial coordinate to negative infinity, when we are close to the event horizon. The resulting wave equation contains an effective potential that depends on both the radial coordinate and the angular momentum ℓ. This potential is essential to determine the black hole's quasi-normal modes. To solve the equation numerically, we used a stable time evolution scheme. Runge-Kutta 4 method was used as a time integrator, while second-order finite differences were applied for spatial derivatives. Absorbing boundary conditions were imposed to prevent reflections from the boundaries, and a convergence test were performed to check if our numerical solution converges to the exact solution. From our results, we extracted the dominant quasi-normal mode and decay timescale of the scalar field, $\omega = (6.51 \pm 0.01) \times 10^{-2}$ and $\tau = 161.29 \pm 10.41$ (respectively). Finally, other directions for this study could include exploring higher angular momentum modes (larger ℓ), studying the case of a scalar field with mass and other black hole spacetimes.

VI. Acknowledgments

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