Summary of Thesis: Some Zero Sum Problems in Combinatorial Number Theory

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1 Background and Motivation

Zero-sum problems form a central topic in additive combinatorics and combinatorial number theory, focusing on subsequences of group elements that sum to zero. Rooted in the classic Erdős-Ginzburg-Ziv (EGZ) Theorem, this field explores invariants such as:

- **Davenport's constant (D(G))**: The minimal length required so that any sequence over a finite abelian group G has a non-empty zero-sum subsequence.
- **E(G) (EGZ constant)**: The least length ensuring a subsequence of size $|G|$ with zero sum.
- $\mathbf{s}(\mathbf{G})$,”$(\mathbf{G})$: Constants requiring zero-sum subsequences of length tied to the group's exponent.
- **Weighted zero-sum invariants (DA(G), EA(G))**: Extensions where group elements are combined with coefficients from a fixed set A.

These invariants are important for understanding non-unique factorizations, factorization algorithms (like the quadratic sieve), and broader combinatorial structures.

2 Contributions of the Thesis

The thesis presents three major results, each in its own chapter, along with background and preliminaries.

2.1 *Chapter 2: Bounds on Davenport's Constant*

For a finite abelian group $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ (with invariants $n_1|n_2|\cdots|n_r$), it is conjectured (Śliwa) that:

$$D(G) \leqslant \sum_{i=1}^{r} n_i$$

The thesis improves upper bounds by using Alon-Dubiner constants ($c(r)$):

$$D(G) \leqslant n_r + n_{r-1} + (c(3)-1)n_{r-2} + \cdots + (c(r)-1)n_1 + 1$$

Applications include links between Davenport's constant and smooth numbers in the quadratic sieve (used in integer factorization).

2.2 *Chapter 3: Higher-Dimensional EGZ Theorem*

Extends the EGZ theorem to groups of higher rank. For cyclic and rank-2 groups, $s(G)$ and $\eta(G)$ are well-studied, but higher ranks remained open.

Main result: For groups $\mathbb{Z}_{nm}^r$, under certain constraints,

$$s(\mathbb{Z}_{nm}^r) = (a_r + 1)(nm - 1) + 1$$

where a_r is a constant depending on r, and bounds involve the Alon-Dubiner constant.

Provides progress towards conjectures such as:

$$s(\mathbb{Z}_n^3) = \begin{cases} 8n - 7 & \text{if } n \text{ is even} \\ 9n - 8 & \text{if } n \text{ is odd.} \end{cases}$$

2.3 *Chapter 4: Weighted Zero-Sum Problems*

Introduces weighted versions of Davenport and EGZ constants:

- $D_A(G)$: Minimum t such that any sequence of t elements has a weighted zero-sum subsequence with coefficients in A.
- $E_A(G)$: Analogous for subsequences of length $|G|$.

Focuses on $A = \{x^2 : x \in (\mathbb{Z}/n\mathbb{Z})^*\}$, i.e., quadratic residues modulo n.

Main results provide exact or sharp bounds for $D_{R_n}(n)$ and $E_{R_n}(n)$ (where R_n is the set of quadratic residues).

Techniques combine Yuan-Zeng's results, Chowla's theorem, and Kneser's theorem.

3 Key Theorems

Some highlighted contributions include:

1. New bounds on $D(G)$: Tighter than previously known general results.
2. Link between quadratic sieve and zero-sum constants: Showing Davenport constants govern smooth-number subsequence requirements.
3. Exact formula for $s(G)$ in structured cases: Extending knowledge beyond rank-2 groups.
4. Weighted zero-sum constants: Explicit formulas for quadratic residues modulo n.

4 Significance

- Advances understanding of Davenport's constant, one of the core invariants in additive number theory.
- Establishes connections between zero-sum theory and computational number theory (e.g., factoring methods).
- Extends classical EGZ-type results to higher dimensions and weighted settings, broadening the scope of additive combinatorics.
- Provides techniques (like use of Alon-Dubiner bounds and Kneser's theorem) that are applicable to further open problems.

5 Structure of the Thesis

1. Introduction: Overview of EGZ theorem, Davenport constant, Kneser's theorem, and weighted zero-sum ideas.
2. On Davenport's Constant: New upper bounds and applications.
3. Higher-Dimensional Analogue of EGZ Theorem: Results on $s(G), \eta(G)$ for higher rank groups.
4. Weighted Zero-Sum Theorems: Bounds for quadratic-residue weighted invariants.
5. Bibliography: Extensive references including Alon, Dubiner, Reiher, Gao, Geroldinger, and others.

6 Conclusion

The thesis contributes substantially to zero-sum problems in combinatorial number theory, offering:

- Stronger bounds for Davenport's constant,
- New results for higher-dimensional EGZ analogues,
- Progress in weighted zero-sum problems with quadratic residues.

These results advance the state of knowledge and open avenues for further research on precise values of constants for higher rank groups and weighted settings.