Summary of Thesis: *Some Zero Sum Problems in Combinatorial Number Theory*

Author: Bhavin K. Moriya **Institution:** Harish-Chandra Research Institute (HRI), Allahabad **Supervisor:** Prof. S. D. Adhikari

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1. Background and Motivation

Zero-sum problems form a central topic in **additive combinatorics** and **combinatorial number theory**, focusing on subsequences of group elements that sum to zero. Rooted in the classic **Erdős–Ginzburg–Ziv (EGZ) Theorem**, this field explores invariants such as:

- **Davenport's constant (D(G))** – the minimal length required so that any sequence over a finite abelian group G has a non-empty zero-sum subsequence.
- **E(G) (EGZ constant)** – the least length ensuring a subsequence of size |G| with zero sum.
- **s(G), η(G)** – constants requiring zero-sum subsequences of length tied to the group's exponent.
- **Weighted zero-sum invariants (DA(G), EA(G))** – extensions where group elements are combined with coefficients from a fixed set A.

These invariants are important for understanding **non-unique factorizations**, **factorization algorithms (like the quadratic sieve)**, and broader combinatorial structures.

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2. Contributions of the Thesis

The thesis presents three major results, each in its own chapter, along with background and preliminaries.

Chapter 2: Bounds on Davenport's Constant

- For a finite abelian group $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ (with invariants $n_1 | n_2 | \cdots | n_r$), it is conjectured (Śliwa) that:

  $$ D(G) \leq \sum_{i=1}^r n_i $$

- The thesis improves upper bounds by using **Alon–Dubiner constants (c(r))**:

  $$ D(G) \leq n_r + n_{r-1} + (c(3)-1)n_{r-2} + \cdots + (c(r)-1)n_1 + 1 $$

- Applications include links between Davenport's constant and **smooth numbers** in the **quadratic sieve** (used in integer factorization).

Chapter 3: Higher-Dimensional EGZ Theorem

- Extends the EGZ theorem to groups of higher rank.

- For cyclic and rank-2 groups, $s(G)$ and $η(G)$ are well-studied, but higher ranks remained open.

- Main result: for groups $\mathbb{Z}^r_{nm}$, under certain constraints,

  $$ s(\mathbb{Z}^r_{nm}) = (a_r + 1)(nm - 1) + 1 $$

  where a_r is a constant depending on r, and bounds involve the Alon–Dubiner constant.

- Provides progress towards conjectures such as:

  $$ s(\mathbb{Z}^3_n) = \begin{cases} 8n - 7 & \text{if n is even}\ 9n - 8 & \text{if n is odd.} \end{cases} $$

Chapter 4: Weighted Zero-Sum Problems

- Introduces **weighted versions** of Davenport and EGZ constants:
  - $D_A(G)$: minimum t such that any sequence of t elements has a weighted zero-sum subsequence with coefficients in A.
  - $E_A(G)$: analogous for subsequences of length |G|.
- Focuses on $A = {x^2 : x \in (\mathbb{Z}/n\mathbb{Z})^* }$, i.e., quadratic residues modulo n.
- Main results provide exact or sharp bounds for $D_{R_n}(n)$ and $E_{R_n}(n)$ (where R_n is the set of quadratic residues).
- Techniques combine **Yuan–Zeng's results**, **Chowla's theorem**, and **Kneser's theorem**.

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3. Key Theorems

Some highlighted contributions include:

1. **New bounds on D(G):** tighter than previously known general results.
2. **Link between quadratic sieve and zero-sum constants:** showing Davenport constants govern smooth-number subsequence requirements.
3. **Exact formula for s(G) in structured cases:** extending knowledge beyond rank-2 groups.
4. **Weighted zero-sum constants:** explicit formulas for quadratic residues modulo n.

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4. Significance

- Advances **understanding of Davenport's constant**, one of the core invariants in additive number theory.
- Establishes connections between **zero-sum theory and computational number theory** (e.g., factoring methods).
- Extends classical EGZ-type results to **higher dimensions and weighted settings**, broadening the scope of additive combinatorics.
- Provides techniques (like use of Alon–Dubiner bounds and Kneser's theorem) that are applicable to further open problems.

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5. Structure of the Thesis

1. **Introduction** – overview of EGZ theorem, Davenport constant, Kneser's theorem, and weighted zero-sum ideas.
2. **On Davenport's Constant** – new upper bounds and applications.
3. **Higher-Dimensional Analogue of EGZ Theorem** – results on $s(G)$, $\eta(G)$ for higher rank groups.
4. **Weighted Zero-Sum Theorems** – bounds for quadratic-residue weighted invariants.
5. **Bibliography** – extensive references including Alon, Dubiner, Reiher, Gao, Geroldinger, and others.

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6. Conclusion

The thesis contributes substantially to **zero-sum problems in combinatorial number theory**, offering:

- Stronger bounds for Davenport's constant,
- New results for higher-dimensional EGZ analogues,
- Progress in weighted zero-sum problems with quadratic residues.

These results advance the state of knowledge and open avenues for further research on precise values of constants for higher rank groups and weighted settings.
