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WEEK 2 – Solving Systems of linear equation – Elimination

1. Manipulating equation
   (1) Multiplying by an constant
   (2) Adding two equation.
   → non-singular (Unique sol^n)

2. Solving singular Systems of linear equations

(i) Redundant

System
$a+b=10$
$2a+2b=10$

divide by coeff. of a
$a+b=10$
$a+b=10$
↳ From this eq^n we can't get info. about b or a

solved system
$a = \underline{x}$ → Degree of freedom
$b = \underline{10-x}$
∴ solved system
$a+b=10$

(ii) Contradictory
$a+b=10$
$2a+2b=24$ → $a+b=10$
$a+b=12$
↳ $0=2$
contradiction
⟹ Therefore no sol^n to this system of eq^n.

3. Matrix Row reduction (Gaussian – Elimination)

Row echelon form (REF) is a form of matrix using row operations to make solving linear eq^n easier

Cond^n for a matrix to be REF
(1) All zero rows must be at bottom
(2) Each leading entry (first non-zero element in a row) must appear to the right of the leading entry in the row above it.
(3) All entries below each leading entry must be zero.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Gaussian elimination is a method used to solve systems of linear eq^n by transforming the system augmented matrix into REF using elementary row operations

For 2x2 matrix → $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4 Row operations that preserve singularity

$\begin{bmatrix}5 & 1\\4 & 3\end{bmatrix}$ ① switching rows $\Rightarrow$ $\begin{bmatrix}4 & 3\\5 & 1\end{bmatrix}$

$D = 15-4 = 11 \neq 0$ (non-singular)

$D = 4-15 = -11 \neq 0$ (non-singular)

② Multiplying a row by a (non-zero) scalar

$\begin{bmatrix}5 & 1\\4 & 3\end{bmatrix} \Rightarrow [5\ \ 1] \times \boxed{10} = [50\ \ 10] \Rightarrow \begin{bmatrix}50 & 10\\4 & 3\end{bmatrix}$

$D = 5\cdot3 - 4\cdot1 = 11$

$D = 5(10\cdot3) - 1(10\cdot4)$
$= \boxed{10}\cdot 11$

③ Adding row to another row

$\begin{bmatrix}5 & 1\\4 & 3\end{bmatrix}$ $R_1 = R_1 + R_2 \Rightarrow \begin{bmatrix}9 & 4\\4 & 3\end{bmatrix}$ $D = 27 - 16 = 11$

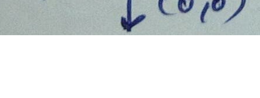
$D = 11$

5. Rank of a matrix which in some way measures how much information that matrix or its corresponding system of linear equations is carrying

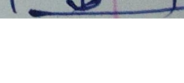
Compressing Images — Reducing rank

$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

System 1	System 2	System 3
$a+b=0$ $a+2b=0$ $\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$	$a+b=0$ $2a+2b=0$ $\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$	$0a+0b=0$ $0a+0b=0$
Two eq^n	Two eq^n	Two eq^n
Two pieces of info	One piece of info	Zero piece of information
Rank = 2	Rank = 1	Rank = 0
Dimension of sol^n space = 0	Dimension of sol^n space = 1	Dimension of sol^n space = 2

point.  (0,0)

(line) 

 (plane)

(the no. of rows in matrix)

$\downarrow$

Rank = 2 − (Dimension of sol^n space)

Matrix is non-singular if and only if it has full Rank

↳ Rank is equal to the no. of rows.

6. Rank of a matrix – in general case

7. Row Echelon form

original matrix

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \rightarrow \text{non-singular matrix}$$

$\downarrow$ divide each row by the leftmost coefficient

$$\begin{bmatrix} 1 & 0.2 \\ 1 & -0.75 \end{bmatrix} \xrightarrow[R_2 = R_1 - R_2]{} \begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix} \xrightarrow[\text{by } -0.95]{\text{divide the 2nd row}} \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \rightarrow \text{singular matrix}$$

$\downarrow$

$$\begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

Row echelon form, singularity, & Rank

$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ → non-singular
→ Rank = 2 (2 ones in the diagonal)

$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$ → singular
→ Rank = 1 (1 one in the diagonal)

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ → singular
Rank = 0 (0 ones in the diagonal)

8. Row echelon form – general case

System

$$\begin{aligned} a+b+2c&=12\\ 3a-3b-c&=3\\ 2a-b+6c&=24 \end{aligned} \longrightarrow$$

System

$$\begin{aligned} a+b+2c&=12\\ -6b-7c&=-33\\ 6c&=18 \end{aligned}$$

matrix

$$\begin{bmatrix} 1 & 1 & 2\\ 3 & -3 & -1\\ 2 & -1 & 6 \end{bmatrix} \longrightarrow \underset{\text{REF}}{\begin{bmatrix} 1 & 1 & 2\\ 0 & -6 & -7\\ 0 & 0 & 6 \end{bmatrix}}$$

REF in general

Rank = 5

$$\begin{matrix} 2 & * & * & * & *\\ 0 & 1 & * & * & *\\ 0 & 0 & 3 & * & *\\ 0 & 0 & 0 & -5 & *\\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Rank = 3

$$\begin{matrix} 3 & * & * & * & *\\ 0 & 0 & 1 & * & *\\ 0 & 0 & 0 & -4 & *\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

① zero rows at the bottom
② Each row has a pivot (leftmost non-zero entry)

$*$ → any no. (zero/+)

③ Every pivot is to the right of the pivots on the rows above

Rank = no. of Pivots

Note: ① In general, pivots diff than 1 are allowed
② For this lect, pivots are 1
This makes no mathematical difference.

9. Reduced Row echelon form

Original system	Intermediate system	Solved sys
$5a+b=17$	$a+0.2b=3.4$	$a=3$
$4a-3b=6$	$b=2$	$b=2$

Original matrix	REF	RREF
$\begin{bmatrix} 5 & 1\\ 4 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.2\\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$

Diagonal matrix

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Cond^n for RREF

① Satisfies all REF rules
② Each leading entry is 1
③ Each leading 1 is the only non-zero entry in its column (all other elements in that column must be zero - above & below it)

10. The Gaussian Elimination Algorithm
① Augmented Matrix

$2a - b + c = 1$
$2a + 2b + 4c = -2$
$4a + b = -1$

1st Pivot

Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & -1 \end{array}\right]$$

From this

① find the pivot on the diagonal of the matrix to start → select the top left cell as pivot
→ use row operation to set that one & then the down ones to zeroes

turn 1st Pivot to 1

$R_1 \to \frac{1}{2}R_1$ $\begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & -1 \end{bmatrix}$

Set all the values below pivot to 0

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$ $\begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 3 & 3 & -3 \\ 0 & 3 & -2 & -3 \end{bmatrix}$

Set 2nd Pivot to 1

$R_2 \to \frac{R_2}{3}$ $\begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & -2 & -3 \end{bmatrix}$

Set all the values below Pivot 2 to 0

$R_3 \to R_3 - 3R_2$ $\begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & 0 \end{bmatrix}$

Set Pivot 3 to 1

$R_3 \to \frac{R_3}{(-5)}$ $\begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Backsubstitution

You will start from the bottom row and work your way to the top.

You'll use the pivot from each row to cancel the values in cells above it (make it 0)

top

bottom

$$\begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & \boxed{1} & 0 \end{bmatrix} \xRightarrow{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Downarrow$ $R_1 \leftarrow R_1 - \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xLeftarrow[R_1 \leftarrow R_1 + \frac{R_2}{2}]{\text{For 2nd row Pivot}} \begin{bmatrix} 1 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This reduced the system of equation to solution to the system

$$\begin{aligned} 2a - b + c &= 1 \\ 2a + 2b + 4c &= -2 \\ 4a + b &= -1 \end{aligned} \longrightarrow \begin{aligned} a &= 0 \\ b &= -1 \\ c &= 0 \end{aligned}$$ to this

Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What if the system is singular?

If the matrix is singular then the RREF will have a row that is all zeroes.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & 4 & 5 & 1 \\ 3 & 6 & 4 & 6 \end{array}\right] \xrightarrow[\text{reduction}]{\text{After row}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 0 & -7 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ Here the Gaussian algorithm stops

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Check if it has infinitely many or no soln.
Just look the column of constants if the constant value in the rows of zeroes is also zero the row just says $0a + 0b + 0c = 0$ (Infinitely many soln)

If it is not zero. then $0a + 0b + 0c = 10$ that is not possible (no soln)

Gaussian Elimination Summary

1. Create the augmented matrix
2. Get the matrix into RREF
3. Complete back substitution
4. Stop if you encounter a row of 0s