Mathematics for CS II (CS202: Logic)

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What is logic?

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- *logic* puzzles...

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- *logic* circuits...
- program *logic*...
- logic in conversation: tere baat mein koi *logic* nahin hai...

ponder...

What is this course about?

Mathematics?

What is this course about?

Mathematics + **Applications**

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Mathematics + **Applications**

Mathematical Logic + **Computational Logic**

Course Logistics

Course Component	Percentage
Assignments	30%
Quizzes	$2 \times 10\%$
End-Sem Exam	50%

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- *Start assignments early*: they might look easy but will take some doing. No deadline extensions.
- *Attend classes and keep in sync with the lectures*: logic is an easy course, but can get difficult if you don't give it enough time.
- *Seek help from me and TAs*: we are there to help you.
- *No unfair means*: CSE policy on unfair means is quite strict; the same will be followed in this course.

Course Logistics

Course webpage:

https://web.cse.iitk.ac.in/users/subhajit/courses/CS335_Jul2018/CS202.php

(link from my homepage)

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Assugnment submission through: *moodle.cse.iitk.ac.in*

Books

- Michael Huth and Mark Ryan. Logic in Computer Science: Modelling and Reasoning about Systems. 2nd Edition.
- Mordechai Ben-Ari. Mathematical Logic for Computer Science. 3rd ed. 2012 Edition. [Acknowledgement: some figures are taken from the teaching material provided]
- Herbert B. Enderton. A Mathematical Introduction to Logic. 2nd Edition.
- Madhavan Mukund and S P Suresh. Introduction to Logic.

Logic: The language of mathematics

"The aim of logic in computer science is to *develop languages to model the situations* we encounter as computer science professionals, in such a way that *we can reason about them formally*."

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Formal Reasoning

Construct arguments that can be defended rigorously or checked by a machine.

Why this formalism?

- Premise: *Some cars* rattle.
- Premise: My car is *some car*.
- Conclusion: My car rattles.

– Smullyan (1978, p. 183)

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Informal natural reasoning can be flawed!

Propositional Logic

If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late. Therefore, there were taxis at the station.

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Is the above argument correct?

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If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late. Therefore, there were taxis at the station.

If it is raining and Jane does not have her umbrella with her, then she will get wet. Jane is not wet. It is raining. Therefore, Jane has her umbrella with her.

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Compare the above two statements.

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Same structure!

If **p** *and not* **q**, *then* **r**. *Not* **r**. **p** . *Therefore*, **q**.

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Propositions

- $\mathbf{p} \equiv$ *the train arrives late*
- $\mathbf{q} \equiv$ *there are taxis at the station*
- $\mathbf{r} \equiv$ *John is late for his meeting*

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Operators

- $\wedge \equiv$ *and*
- $\neg \equiv$ *not*
- $\rightarrow \equiv$ *then*

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$\mathbf{p} \wedge \neg\mathbf{q} \rightarrow \mathbf{r}$ *;* $\neg\mathbf{r}$. **p** *.* *Therefore*, **q**.

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We can *compose* these operators to create arguments of arbitrary complexity!

Understanding the language formally

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Syntax + **Semantics**

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Semantics

Defines the **meaning** of strings in the language

Propositional Logic: Syntax

Countably infinite set of atomic propositions, $\mathcal{P} = \{p_0, p_{1,} \ldots\}$ and three logical connectives $\neg, \vee, \wedge$

Propositional Formulas

The set Φ of formulas of propositional logic is the smallest set satisfying the following conditions:

- Every atomic proposition $p \in \Phi$
- If $\alpha \in \Phi$, then $(\neg\alpha) \in \Phi$
- If $\alpha, \beta \in \Phi$, then $\alpha \vee \beta \in \Phi$
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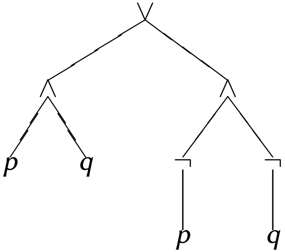
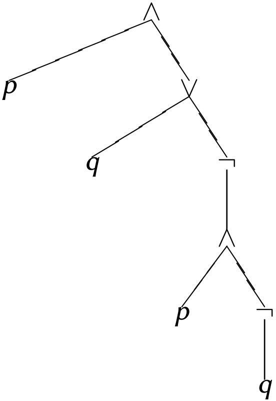
Backus–Naur (BNF) form

$\Phi ::= p \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi$

(Abstract) Syntax Trees





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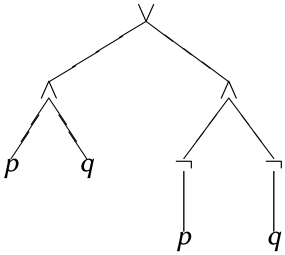

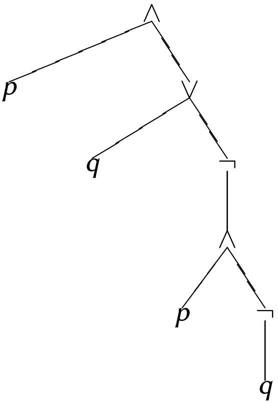

What are the formula (strings) corresponding to the two trees?

Selecting ASTs

Precedence: What operator binds “tigher”? [$\neg$, $\wedge$, $\vee$]

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Associativity: How do the operators associate?

Additional Symbols

- Implication ($\rightarrow$)
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Excercise: Modify the syntax to include these additional symbols

Formal Semantics

Interpretation

An **interpretation** for a formula $A \in \mathscr{F}$ is a *total function* $\mathscr{I}_\mathrm{A} \mapsto \{T, F\}$ that assigns a truth value to *every* atom in the set of propositions $\mathscr{P}_A$.

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An **interpretation** can be lifted to a set of formulas $S = \{A_1, \ldots\}$ over propositions $\mathscr{P}_S = \bigcup_i \mathscr{P}_{A_i}$ as a function $\mathscr{I}_A \mapsto \{T, F\}$

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Valuation

The **valuation** of $A \in \mathscr{F}$ is defined as a total function $v_{\mathscr{I}_A} \mapsto \{T, F\}$ that assigns a truth value to A under $\mathscr{I}_A$. Note that $v_{\mathscr{I}_A}$ can be defined inductively on the structure of A.

Semantics (as truth table)

A	$v(A_1)$	$v(A_2)$	$v(A)$
$\neg A_1$	T		F
$\neg A_1$	F		T
$A_1 \vee A_2$	F	F	F
$A_1 \vee A_2$	otherwise		T
$A_1 \wedge A_2$	T	T	T
$A_1 \wedge A_2$	otherwise		F

Satisfiability, Validity

Satisfiability

$A \in \mathscr{F}$ is satisfiable iff $v_{\mathscr{I}_A}(A) = T$ for *some* interpretation of $\mathscr{I}$
A satisfying interpretation is called a **model** for A

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Unsatisfiable

$A \in \mathscr{F}$ is unsatisfiable iff $v_{\mathscr{I}}(A) = F$ for *all* interpretations of $\mathscr{I}$
Also referred to as a **contradiction**
[*not satisfiable*]

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Falsifiable

$A \in \mathscr{F}$ is falsifiable (denoted $\not\models A$) iff $v_{\mathscr{I}}(A) = F$ for *some* interpretations of $\mathscr{I}$
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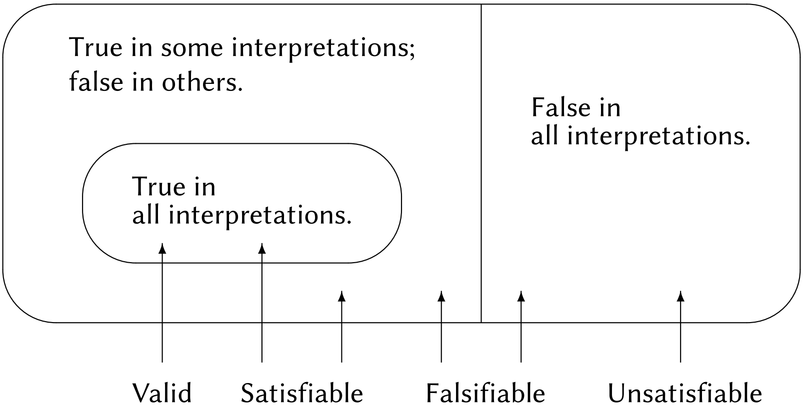
All these concepts can be lifted to a set of formulas $A_i \in \mathscr{F}$ instead of a single formula.

For example, a set of formulas $A_i \in \mathscr{F}$ is satisfiable iff $v_{\mathscr{I}}(A_i) = T$ on *all* A_i for *some* interpretation of $\mathscr{I}$

Satisfiability, Validity



Solving computer science problems using SAT solving

Graph Coloring

Given a graph G and (atmost) k colors, assign colors to the nodes such that no two neighbours share the same color.

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Encoding in SAT

- Variables:
  e_{ij}: an edge exists between the vertices i and j
  a_{ij}: the i^{th} vertex is assigned the j^{th} color
- Constraints (for k colors):
  - Each node v is assigned a color: $\forall_v.(a_{v1} \vee a_{v2} \vee \cdots \vee a_{vk})$
  - Each node v is given exactly one color: $\forall_v \neg(a_{vi} \wedge a_{vj})$
  - Two neighboring nodes don't have same color: $\forall_{v,w} \neg(e_{vw} \wedge a_{vi} \wedge a_{wi})$

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Example: $(p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee p_5) \wedge \ldots$

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Can be seen as a list of list of literals!

Disjunctive Normal Form (CNF)

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Negation Normal Form (NNF)

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Not a cannonical form

Converting to Conjunctive Normal Form

Conversion

1. Convert to NNF: push negations into the formula (De Morgan's Law):
   $\neg(p \vee q)$ to $(\neg p) \wedge (\neg q)$
   $\neg(p \wedge q)$ to $(\neg p) \vee (\neg q)$
2. Repeatedly apply the distributive law where a disjunction occurs over a conjunction:
   $p \vee (q \wedge r)$ to $(p \vee q) \wedge (p \vee r)$

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What can be the blowup in the worst case?

SAT solvers

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  - SAT: a model
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- How to check validity?
- It accepts the formula as a CNF formula in the DIMACS representation:

```
c
c start with comments
c
c
p cnf 5 3
1 -5 4 0
-1 5 3 4 0
-3 -4 0
```

Assignment 1: Sudoku

Write an encoding for:

- Sudoku solving, and
  - Additional constraint: The numbers in the main diagonals are also unique
- Sudoku generation
  - Use the above encoding as a black box

Implement it using the minisat solver.

Logical Equivalence $\equiv$

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$A_1 \equiv A_2$ if and only if $v_{\mathscr{I}}(A_1) = v_{\mathscr{I}}(A_2)$ for all interpretations $\mathscr{I}$.

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- Assume an **induction hypothesis**
- Prove the **base case**
- Prove the **inductive step** for *k+1, assuming induction hypothesis on k*

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Example: Prove that the sum of n numbers is $\frac{n(n+1)}{2}$

Structural Induction

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To show that some property ϕ holds for all logical formula ψ
Base case: Show that ϕ holds for all propositions p
Induction Hypothesis: Assume that the formula holds for formula A_1 and A_2
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Proof: By mathematical induction of the height of the syntax trees of $\neg A_1$, $A_1 \wedge A_2$ and $A_1 \vee A_2$

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prove larger formula by induction on its subformulas

Structural Induction

Theorem

For every well-formed propositional logic formula, the number of left brackets is equal to the number of right brackets.
$\Phi ::= p \mid (\neg\Phi) \mid (\Phi \wedge \Phi) \mid (\Phi \vee \Phi)$

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Proof: Base Case: For the case of propositions p, #left(p)=#right(p)=0
Inductive Step: Consider the case for $(\Phi_1 \wedge \Phi_2)$, #left(Φ_i)=#right(Φ_i)=n, i=1,2 (induction hypothesis)
So, #left$(\Phi_1 \wedge \Phi_2)$= #left(Φ_1) + #left(Φ_2) + 1 = $n_1 + n_2 + 1$
And, #right$(\Phi_1 \wedge \Phi_2)$= #right(Φ_1) + #right(Φ_2) + 1 = $n_1 + n_2 + 1$
So, proved for all conjunctions. SImilarly prove for negation and disjunction.

Proof System

Proof System (Deductive System)

- a set (possibly infinite) set of formulas, called *axioms*
- a set of *inference rules*

used to deduce **tautologies**

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Proofs

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- A proof can be denoted:
  - As a proof tree, where each node is an axiom (leaf node) or inference rule (non-leaf node)
  - As a sequence of valid sequents $\gamma_1, \gamma_2, \ldots \gamma_i, \ldots, \gamma_n$. For a sequent γ_i to be valid, its premise(s) is either some ϕ_i, an axiom, or a conclusion proven in some preceeding γ_j, $j < i$

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used to deduce **tautologies**

Proofs

- A sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1, \psi_2, \ldots, \psi_m$ is valid, if a proof for it can be found. ϕ_i are premises/antecedents, ψ_i are conclusions/consequents.
- A proof can be denoted:
  - As a proof tree, where each node is an axiom (leaf node) or inference rule (non-leaf node)
  - As a sequence of valid sequents $\gamma_1, \gamma_2, \ldots \gamma_i, \ldots, \gamma_n$. For a sequent γ_i to be valid, its premise(s) is either some ϕ_i, an axiom, or a conclusion proven in some preceeding γ_j, $j < i$

Natural Deduction

Natural Deduction (ND) is one of the proof systems for propositional logic

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$$\frac{premises}{conclusion}\, rule\_name$$

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$$\frac{premises}{conclusion}\,rule\_name$$

To Prove: $p \wedge q, r \vdash q \wedge r$

$$\dfrac{\dfrac{p \wedge q}{q}\wedge e_2 \quad r}{q \wedge r}\wedge i$$ *proof tree*	1. $p \wedge q$ (premise) 2. r (premise) 3. q ($\wedge e_2$ 1) 4. $q \wedge r$ ($\wedge i$ 3,2) *flattened* proof

Natural Deduction

Natural Deduction (ND) is one of the proof systems for propositional logic

$$\frac{premises}{conclusion} rule\_name$$

To Prove: $p \land q, r \vdash q \land r$

$$\cfrac{\cfrac{p \land q}{q} \land e_2 \quad r}{q \land r} \land i$$

proof tree

1. $p \land q$ (premise)
2. r (premise)
3. q ($\land e_2$ 1)
4. $q \land r$ ($\land i$ 3,2)

flattened proof

How to check a proof:

Natural Deduction

Natural Deduction (ND) is one of the proof systems for propositional logic

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How to check a proof:

- *Check that each line corresponds to a valid rule, axiom on a set of premises which have already been established*

Natural Deduction

Conjunction

$$\frac{\phi \quad \psi}{\phi \land \psi} \land i \qquad \frac{\phi \land \psi}{\phi} \land e_1 \qquad \frac{\phi \land \psi}{\psi} \land e_2$$

Natural Deduction

Conjunction

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\wedge i \qquad \frac{\phi \wedge \psi}{\phi}\wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi}\wedge e_2$$

Procedural Interpretation:

1. $\wedge i$: first prove ϕ and ψ, and use them for $\phi \wedge \psi$
2. $\wedge e$: to prove ϕ, prove $\phi \wedge \psi$ (strengthen proof); it is useful when $\phi \wedge \psi$ is already lying around

Natural Deduction

Disjunction

$$\frac{\phi}{\phi \vee \psi}\vee i_1 \qquad \frac{\psi}{\phi \vee \psi}\vee i_2 \qquad \frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \dots\\ \gamma\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \dots\\ \gamma\end{array}}}{\gamma}\vee e$$

Natural Deduction

Disjunction

$$\frac{\phi}{\phi \vee \psi}\vee i_1 \qquad \frac{\psi}{\phi \vee \psi}\vee i_2 \qquad \frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi \\ \dots \\ \gamma\end{array}} \quad \boxed{\begin{array}{c}\psi \\ \dots \\ \gamma\end{array}}}{\gamma}\vee e$$

Procedural Interpretation:

1. $\vee i$: to prove $\phi \vee \psi$, try proving ϕ (or ψ)
2. $\vee e$: to prove γ from $\phi \wedge \psi$ try proving it from both ϕ and ψ

Natural Deduction

Disjunction

$$\frac{\phi}{\phi \lor \psi}\lor i_1 \qquad \frac{\psi}{\phi \lor \psi}\lor i_2 \qquad \frac{\phi \lor \psi \quad \boxed{\begin{array}{c}\phi\\ \cdots\\ \gamma\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \cdots\\ \gamma\end{array}}}{\gamma}\lor e$$

Procedural Interpretation:

1. $\lor i$: to prove $\phi \lor \psi$, try proving ϕ (or ψ)
2. $\lor e$: to prove γ from $\phi \land \psi$ try proving it from both ϕ and ψ

Note the *scope* (shown by the box): the scope contains *local assumptions*, and hence the derived formulas are not valid outside the proof

Natural Deduction

Negation

$$\frac{\boxed{\begin{array}{c}\phi\\ \cdots\\ \bot\end{array}}}{\neg\phi}\neg i \qquad \frac{\phi \quad \neg\phi}{\bot}\neg e \qquad \frac{\neg\neg\phi}{\phi}\neg\neg$$

Natural Deduction

Negation

$$\frac{\boxed{\begin{array}{c}\phi \\ \cdots \\ \bot\end{array}}}{\neg\phi}\neg i \qquad \frac{\phi \quad \neg\phi}{\bot}\neg e \qquad \frac{\neg\neg\phi}{\phi}\neg\neg$$

Procedural Interpretation:

1. $\neg\phi$: to prove $\neg\phi$, assume ϕ and try to derive a contradiction (proof by contradiction)

Natural Deduction

$\perp$

$\perp$ (no intro rule) $\dfrac{\perp}{\phi}\perp$

Natural Deduction

$\bot$

$\bot$ (no intro rule) $\dfrac{\bot}{\phi}\bot$

Procedural Interpretation:

1. $\bot$: anything goes in a mad world (a world with contradictions)

Natural Deduction

$\rightarrow$

$$\frac{\boxed{\begin{array}{c}\phi \\ \ldots \\ \psi\end{array}}}{\phi \rightarrow \psi} \rightarrow i_1 \qquad \frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e^a \qquad \frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi} \rightarrow e^b$$

[a] modus ponens
[b] modus tollens

Natural Deduction

$\rightarrow$

$$\dfrac{\boxed{\begin{array}{c}\phi \\ \dots \\ \psi\end{array}}}{\phi \rightarrow \psi} \rightarrow i_1 \qquad \dfrac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e^a \qquad \dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi} \rightarrow e^b$$

[a] modus ponens
[b] modus tollens

Procedural Interpretation:

1. $\rightarrow i$: to prove $\phi \rightarrow \psi$, assume ϕ and prove ψ

Natural Deduction

Bi-implication

$$\dfrac{\boxed{\begin{array}{c}\phi \\ \dots \\ \psi\end{array}} \quad \boxed{\begin{array}{c}\psi \\ \dots \\ \phi\end{array}}}{\phi \leftrightarrow \psi} \rightarrow i_1 \qquad \dfrac{\phi \quad \phi \leftrightarrow \psi}{\psi} \rightarrow e1 \qquad \dfrac{\psi \quad \phi \leftrightarrow \psi}{\phi} \rightarrow e2$$

ND Proofs

To Prove: $(q \longrightarrow r) \vdash ((\neg q \longrightarrow \neg p) \longrightarrow (p \longrightarrow r))$

1. $q \longrightarrow r$	(premise)
$((\neg q \longrightarrow \neg p) \longrightarrow (p \longrightarrow r))$	

Proofs

1. $q \longrightarrow r$ (premise)
2. $\neg q \longrightarrow \neg p$ (assumption)
3. p (assumption)

7. r ($\longrightarrow$e 1, 6)
8. $p \longrightarrow r$ ($\longrightarrow$i 3–7)
9. $(\neg q \longrightarrow \neg p) \longrightarrow (p \longrightarrow r)$ ($\longrightarrow$i 2–8)

Proofs

1.	$q \longrightarrow r$	(premise)
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How to go about a proof

1. This "structure" of the proof can be deduced from the formula to be proved.
2. The left out part needs "cleverness".

Proofs

1.	$q \longrightarrow r$	(premise)
2.	$\neg q \longrightarrow \neg p$	(assumption)
3.	p	(assumption)
4.	$\neg\neg p$	($\neg\neg$i 3)
5.	$\neg\neg q$	(MT 2, 4)
6.	q	($\neg\neg$e 5)
7.	r	($\longrightarrow$e 1, 6)
8.	$p \longrightarrow r$	($\longrightarrow$i 3–7)
9.	$(\neg q \longrightarrow \neg p) \longrightarrow (p \longrightarrow r)$	($\longrightarrow$i 2–8)

How to go about a proof

1. This "structure" of the proof can be deduced from the formula to be proved.
2. The left out part needs "cleverness".

Premises versus Assumptions

- The proof is subject to the provided premises; they are globally valid and can be assumed everywhere.
- The assumptions are made during the proof (eg. to do case-split for $\vee$); lemmas proved with assumptions are valid only within the respective scope.

ND Proofs

General heuristic:

1. Start with applying the introduction rules, backward from the conclusion. Ex., to prove $A \longrightarrow B$, add A as an assumption to get to B. To prove $A \bigwedge B$, use $\wedge i$ to prove A, and then prove B. (often dictated by the structure of the formula to be proved)
2. When you run out, use elimination rules to go forward. Ex. If you have $A \longrightarrow B$ and A, derive B. If you have $A \bigvee B$, split on cases, considering A in one case and B in the other.

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Structure of the proof

1. The structure of the proof can be formed from the syntactic structure of the formula to be proved
2. The "gaps" in this structure needs to be deduced in a clever way

Practice on certain proofs

1. $\neg q \rightarrow \neg p \vdash p \rightarrow \neg\neg q$
2. $p \rightarrow (q \rightarrow r) \vdash p \wedge q \rightarrow r$
3. $p \rightarrow q \vdash p \wedge r \rightarrow q \wedge r$
4. $(p \vee q) \vee r \vdash p \vee (q \vee r)$
5. $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$
6. $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$
7. $p \rightarrow \neg p \vdash \neg \mathrm{p}$

Semantics: Logical Consequence

- For a set of formulas U and a formula A, A is *logical consequence* of U (denoted $U \vDash A$) iff every model of U is a model of A.
  - A is true in all interpretations that U is true (A may or may not be true on interpretations that do not satisfy U)
  - Example: $\{p, \neg q\} \models A$

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  - Example: $\{p, \neg q\} \models A$
- Note the difference between $\vdash$ and $\vDash$
  - $p_1, \ldots, p_n \vdash q$: you can derive a formula of syntactic structure q from formulas $p_1, \ldots, p_n$ (derives, syntactic)
  - $p_1, \ldots, p_n \models q$: the formuala q evaluates true on all satisfying interpretations (models) of $p_1, \ldots, p_n$ (models, semantic)

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  - $p_1, \ldots, p_n \models q$: the formuala q evaluates true on all satisfying interpretations (models) of $p_1, \ldots, p_n$ (models, semantic)
- Note the difference between $\models$ and $\rightarrow$ (also $\equiv$ and $\leftrightarrow$)
  - $\rightarrow$ and $\leftrightarrow$ are symbols in the language of propositional logic
  - $\models$ and $\equiv$ are symbols in the matalanguage—the language used to describe properties of the language of propositional language

Theories

Closure under $\models$

A set of formulas $\mathscr{T}$ is closed under logical consequence iff for all formulas A, if $\mathscr{T} \models A$.

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Theory

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Theories are constructed by selecting *axioms* (A) and deducing their logical consequence.

Axiomatization

Axiomatizable

A theory $\mathscr{T}$ is *axiomatizable* iff there exist axioms U such that $\mathscr{T} = \{\text{formulas } A \text{ such that } U \models A\}$. If U is finite, $\mathscr{T}$ is *finitely* axoiomatizable.

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- Tautologies are not interesting. It is more interesting to construct theories (logical consequences under a given set of axioms).
- Arithmetic is axiomatizable with *Peano's axioms*. However, it is *not finitely axiomatizable* as the induction axiom is not a single axiom but a axiom schema (includes one axiom per predicate definable in the first-order language of Peano arithmetic).

Decision Procedures in Propositional Logic

Decision Procedures

An *algorithm* is a decision procedure for a given set of formulas $\mathscr{U} \subseteq \mathscr{F}$ if, given an arbitrary formula $A \in \mathscr{F}$ it *always terminates*, and returns answer **yes** if $A \in \mathscr{U}$ and **no** if $A \notin \mathscr{U}$.

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- A decision procedure for satisfiability can be used for validity by checking the negation of the formula for satisfiability. Such a procedure is called *refutation procedure* (we are proving validity by refuting its negation).
- Truth table construction is an (inefficient) decision procedure.

Truth Tables as Decision Procedure

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Truth Tables as Decision Procedure

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

It can also be tabulated as follows:

$(p$	$\rightarrow$	$q)$	$\leftrightarrow$	$(\neg$	q	$\rightarrow$	$\neg$	$p)$
T		F			F			T
T		F		T	F			T
T		F		T	F		F	T
T		F		T	F	F	F	T
T	F	F		T	F	F	F	T
T	F	F	T	T	F	F	F	T

Truth Tables as Decision Procedure

a compact tabulation

Truth Tables as Decision Procedure

a compact tabulation

p	q	$(p$	$\rightarrow$	$q)$	$\leftrightarrow$	$(\neg$	q	$\rightarrow$	$\neg$	$p)$
T	T	T	T	T	T	F	T	T	F	T
T	F	T	F	F	T	T	F	F	F	T
F	T	F	T	T	T	F	T	T	T	F
F	F	F	T	F	T	T	F	T	T	F

Is $p \lor q \equiv q \lor p$?

Is $p \vee q \equiv q \vee p$?

p	q	$p \vee q$	$\neg p$	$\neg q$	$(p \vee q) \wedge \neg p \wedge \neg q$
T	T	T	F	F	F
T	F	T	F	T	F
F	T	T	T	F	F
F	F	F	T	T	F

Is $p \vee q \equiv q \vee p$?

p	q	$p \vee q$	$\neg p$	$\neg q$	$(p \vee q) \wedge \neg p \wedge \neg q$
T	T	T	F	F	F
T	F	T	F	T	F
F	T	T	T	F	F
F	F	F	T	T	F

Don't be fooled by the simplicity of truth tables; it is incredibly powerful tool!

Semantic tableaux

An efficient decision procedure (for satisfiability)

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Basic Algorithm

- Recursively *decompose* the formula down to literals

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- Check for *contradiction*: existance of positive and negative literal on the same node

Semantic tableaux

An efficient decision procedure (for satisfiability)

Basic Algorithm

- Recursively *decompose* the formula down to literals
- Check for *contradiction*: existance of positive and negative literal on the same node
- Formula is satisfiable if for atleast one leaf node:
  - there does *not exist a pending* subformula (pending subformula is one which has never been picked for decomposition)
  - there does n *ot exist a contradiction* on the leaf

Formula Decomposition

- Formulas classified into α and β formulas
  - α *formulas* require both the components to be satisfied; both components from the decomposition get added to the current list of subformulas
  - β *formulas* require any one of components to be satisfied; leads to splitting of the current path, each path inheriting one of the components from the decomposition

Semantic tableau for $p\wedge (\ngg q\vee \ngg p)$

$$
\begin{array}{ccc}
 & p \wedge (\neg q \vee \neg p) & \\
 & \downarrow & \\
 & p, \neg q \vee \neg p & \\
\swarrow & & \searrow \\
p, \neg q & & p, \neg p \\
\odot & & \times
\end{array}
$$

Semantic tableau for $p\wedge (\ngg q\vee \ngg p)$

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p, \neg q & & p, \neg p \\
\odot & & \times
\end{array}
$$

Model: $p, \neg q$

Semantic tableaux $(p \lor q) \land (\neg p \land \neg q)$

$$(p \lor q) \land (\neg p \land \neg q)$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

$$\downarrow$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

$$\begin{array}{c} (p \vee q) \wedge (\neg p \wedge \neg q) \\ \checkmark \\ \downarrow \\ p \vee q, \neg p \wedge \neg q \end{array}$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

$$\checkmark$$

$$\downarrow$$

$$p \vee q, \neg p \wedge \neg q$$

$$\downarrow$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

$$
\begin{array}{c}
(p \vee q) \wedge (\neg p \wedge \neg q) \\
\checkmark \\
\downarrow \\
p \vee q, \neg p \wedge \neg q \\
\checkmark \\
\downarrow \\
p \vee q, \neg p, \neg q
\end{array}
$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

$$\checkmark$$

$$\downarrow$$

$$p \vee q, \neg p \wedge \neg q$$

$$\checkmark$$

$$\downarrow$$

$$p \vee q, \neg p, \neg q$$

$$\swarrow \qquad \searrow$$

Semantic tableaux $(p \lor q) \land (\neg p \land \neg q)$

$$
\begin{array}{c}
(p \lor q) \land (\neg p \land \neg q) \\
\checkmark \\
\downarrow \\
p \lor q, \neg p \land \neg q \\
\qquad\qquad \checkmark \\
\downarrow \\
p \lor q, \neg p, \neg q \\
\checkmark \qquad\qquad\qquad \\
\begin{array}{cc}
\swarrow & \searrow \\
p, \neg p, \neg q & q, \neg p, \neg q
\end{array}
\end{array}
$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

$$
\begin{array}{c}
(p \vee q) \wedge (\neg p \wedge \neg q) \\
\checkmark \\
\downarrow \\
p \vee q, \neg p \wedge \neg q \\
\qquad\qquad\checkmark \\
\downarrow \\
p \vee q, \neg p, \neg q \\
\checkmark\qquad\qquad \\
\swarrow \qquad\qquad \searrow \\
\begin{array}{cc}
p, \neg p, \neg q & q, \neg p, \neg q \\
\times & \times
\end{array}
\end{array}
$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

Semantic tableaux may **not** be unique!

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

Semantic tableaux may **not** be unique!

Another tableaux for $(p \vee q) \wedge (\neg p \wedge \neg q)$

$$
\begin{array}{ccc}
 & (p \vee q) \wedge (\neg p \wedge \neg q) & \\
 & \downarrow & \\
 & p \vee q, \neg p \wedge \neg q & \\
 \swarrow & & \searrow \\
p, \neg p \wedge \neg q & & q, \neg p \wedge \neg q \\
\downarrow & & \downarrow \\
p, \neg p, \neg q & & q, \neg p, \neg q \\
\times & & \times
\end{array}
$$

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

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\swarrow & & \searrow \\
p, \neg p \wedge \neg q & & q, \neg p \wedge \neg q \\
\downarrow & & \downarrow \\
p, \neg p, \neg q & & q, \neg p, \neg q \\
\times & & \times
\end{array}
$$

Which one is better?

Semantic tableaux $(p \vee q) \wedge (\neg p \wedge \neg q)$

Semantic tableaux may **not** be unique!

Another tableaux for $(p \vee q) \wedge (\neg p \wedge \neg q)$

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 & \downarrow & \\
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 \swarrow & & \searrow \\
p, \neg p \wedge \neg q & & q, \neg p \wedge \neg q \\
\downarrow & & \downarrow \\
p, \neg p, \neg q & & q, \neg p, \neg q \\
\times & & \times
\end{array}$$

Which one is better?

Heuristic: *Process conjunctions before disjunctions*

α-formulas

α	α_1	α_2
$\neg\neg A_1$	A_1	
$A_1 \wedge A_2$	A_1	A_2
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \rightarrow A_2)$	A_1	$\neg A_2$
$A_1 \leftrightarrow A_2$	$A_1 \rightarrow A_2$	$A_2 \rightarrow A_1$

β-formulas

β	β_1	β_2
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \vee B_2$	B_1	B_2
$B_1 \rightarrow B_2$	$\neg B_1$	B_2
$\neg(B_1 \leftrightarrow B_2)$	$\neg(B_1 \rightarrow B_2)$	$\neg(B_2 \rightarrow B_1)$

Termination of Tableau Construction

Completed Tableau

- A tableau whose construction has terminated is a *completed tableau*.
- A completed tableau is *closed* is all leaves are closed;
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Theorem

The construction of a tableau for any formula ψ terminates. When the construction terminates, all the leaves are marked $\times$ or $\odot$

Termination of Tableau Construction

How to do proofs of termination?

Termination of Tableau Construction

How to do proofs of termination?

Ranking Function

Come up with a *ranking function* such it:

- The function has a *lower boun*d (say it is always non-negative)
- The function *decreases* with each step of the algorithm

Termination of Tableau Construction

Proof

- For the set of (sub)formulas in the pending list, let
  - b(l): total number of binary operators
  - n(l): total number of negations
- Let us use a ranking function over the subformulas in list l in a node: $W(l) = 3.b(l) + n(l)$
- For each α-formula, argue that $W(l)$ decreases
- For each β-formula, argue that each child has a lower $W(l)$ value than the parent

Analytic tableaux (Smullyan, 1968)

Two alternations for efficiency:

- Subformulas not copied from parent to child
- Subformulas for decomposition selected from all nodes from root to the current node (not only from the parent node)
- Check for contradiction on the path (not only on the node)

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$$(p \vee q) \wedge (\neg p \wedge \neg q)$$

$$\downarrow$$

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$$\checkmark$$

$$\downarrow$$

$$p \vee q, \neg p \wedge \neg q$$

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$$(p \lor q) \land (\neg p \land \neg q)$$
$$\checkmark$$
$$\downarrow$$
$$p \lor q, \neg p \land \neg q$$

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$$\swarrow \qquad \searrow$$

$$p \qquad\qquad q$$

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$$\checkmark$$

$$\downarrow$$

$$p \vee q, \neg p \wedge \neg q$$

$$\checkmark$$

$$\downarrow$$

$$\neg p, \neg q$$

$$\checkmark$$

$$\swarrow \qquad \searrow$$

$$p \qquad\qquad q$$

Correctness

We need to connect the *syntactic transformations* by the algorithm to *semantic meaning (models)* of the formula

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Soundness and Completeness

A completed tableau, $\mathscr{T}$, for a formula A is *unsatisfiable* if and only if $\mathscr{T}$ is *closed*.

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A completed tableau, $\mathscr{T}$, for a formula A is *unsatisfiable* if and only if $\mathscr{T}$ is *closed*.

Corollaries

- $A$is satisfiable if and only if $\mathscr{T}$ is open.
- A is valid if and only if the tableau for $\neg A$ closes.
- The method of semantic tableaux is a decision procedure for validity in propositional logic.
  - Semantic tableau for $\neg A$ terminates and closes

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If $\mathscr{T}$ is *closed* the respective formula A is *unsatisfiable.*

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Soundness

if $\mathscr{T}_n$ *, the subtree rooted at node n of* $\mathscr{T}$ *, closes then the set of formulas* $U(n)$ *labeling n is unsatisfiable.*

Proof

$$
\begin{array}{c}
n : \{A_1 \wedge A_2\} \cup U_0 \\
| \\
n' : \{A_1, A_2\} \cup U_0
\end{array}
$$

Proof

$$n : \{A_1 \wedge A_2\} \cup U_0$$

$$n' : \{A_1, A_2\} \cup U_0$$

$$n : \{B_1 \vee B_2\} \cup U_0$$

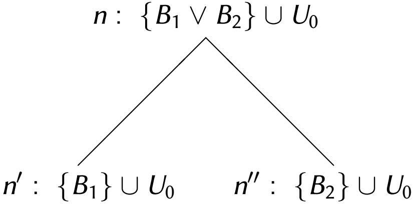
$$n' : \{B_1\} \cup U_0 \qquad n'' : \{B_2\} \cup U_0$$

Proof

- Proof by induction on h_n, height of node n in $\mathscr{T}_n$
- Base case:
  - $h_n = 0$ and $\mathscr{T}_n$ closes
  - so $\mathscr{T}_n$ must have only literals (it is a leaf), and at least one pair of contradictory literals (it closes)

Inductive step: Case I (Node n has an α-formula)

- We only show for $A_1 \wedge A_2$
- $\mathscr{T}_{n'}$ is unsatisfiable (by induction hypothesis)
- Two possibilities of why an arbitrary interpretation I is false on n' (child):
  - U_0 is unsatisfiable: $v_I(A_0) = F$, but the same U_0 is carried to the parent, so it must be unsatisfiable as well
  - One of A_1 or A_2 is unsatisfiable in n': by semantics of $\wedge$, if any of A_1 or A_2 is unsat, $A_1 \wedge A_2$ is unsat



Inductive step: Case II (Node n has an β-formula)

- We only show for $A_1 \vee A_2$
- $\mathscr{T}_{n'}$ and $\mathscr{T}_{n''}$ are <u>both</u> unsatisfiable (by induction hypothesis)
- Two possibilities of why an arbitrary interpretation I if false on n' and n'' (children):
  - U_0 is unsatisfiable: $v_I(B_0) = F$, but the same U_0 is carried to the parent, so it must be unsatisfiable as well
  - Both B_1 and B_2 is unsatisfiable in n' and n'': by semantics of $\vee$, if both B_1 and B_2 is unsat, $B_1 \vee B_2$ is unsat

$$n : \ \{A_1 \wedge A_2\} \cup U_0$$

$$n' : \ \{A_1, A_2\} \cup U_0$$

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Completeness (contrapositive)

If some tableau for A is open, then A is satisfiable.

Gentzen System $\mathcal{G}$

α	α_1	α_2
$\neg\neg A$	A	
$\neg(A_1 \wedge A_2)$	$\neg A_1$	$\neg A_2$
$A_1 \vee A_2$	A_1	A_2
$A_1 \rightarrow A_2$	$\neg A_1$	A_2
$\neg(A_1 \leftrightarrow A_2)$	$\neg(A_1 \rightarrow A_2)$	$\neg(A_2 \rightarrow A_1)$

β	β_1	β_2
$B_1 \wedge B_2$	B_1	B_2
$\neg(B_1 \vee B_2)$	$\neg B_1$	$\neg B_2$
$\neg(B_1 \rightarrow B_2)$	B_1	$\neg B_2$
$B_1 \leftrightarrow B_2$	$B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$

$$\frac{}{\{q, p_1, p_2, \ldots, p_n, \neg q\}}$$

$$\frac{\vdash U_1' \cup \{\alpha_1, \alpha_2\}}{\vdash U_1' \cup \{\alpha\}}$$

$$\frac{\vdash U_1' \cup \{\beta_1\} \qquad \vdash U_2' \cup \{\beta_2\}}{\vdash U_1' \cup U_2' \cup \{\beta\}}$$

Proofs in $\mathscr{G}$ and the semantic tableau

$$
\begin{tikzcd}[column sep=small]
\neg p, q, p \arrow[dr] & & \neg q, q, p \arrow[dl] \\
& \neg(p \lor q), q, p \arrow[d] & \\
& \neg(p \lor q), (q \lor p) \arrow[d] & \\
& (p \lor q) \to (q \lor p) &
\end{tikzcd}
$$

Proofs in $\mathscr{G}$ and the semantic tableau

$$\begin{tikzcd}[column sep=small]
\neg p, q, p \arrow[dr] & & \neg q, q, p \arrow[dl] \\
& \neg(p \vee q), q, p \arrow[d] & \\
& \neg(p \vee q), (q \vee p) \arrow[d] & \\
& (p \vee q) \rightarrow (q \vee p) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& \neg[(p \vee q) \rightarrow (q \vee p)] \arrow[d] & \\
& p \vee q, \neg(q \vee p) \arrow[d] & \\
& p \vee q, \neg q, \neg p \arrow[dl] \arrow[dr] & \\
p, \neg q, \neg p & & q, \neg q, \neg p \\
\times & & \times
\end{tikzcd}$$

Hilbert System $\mathcal{H}$

Axiom 1 $\vdash (A \to (B \to A))$,

Axiom 2 $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

Axiom 3 $\vdash (\neg B \to \neg A) \to (A \to B)$.

Modus ponens $$\frac{\vdash A \qquad\qquad \vdash A \to B}{\vdash B}$$

Hilbert System $\mathcal{H}$

$$
\begin{aligned}
&\textbf{Axiom 1} \quad \vdash (A \rightarrow (B \rightarrow A)),\\
&\textbf{Axiom 2} \quad \vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)),\\
&\textbf{Axiom 3} \quad \vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B).
\end{aligned}
$$

$$
\textbf{Modus ponens} \qquad \frac{\vdash A \qquad\qquad \vdash A \rightarrow B}{\vdash B}
$$

$A \wedge B \equiv \neg(A \rightarrow \neg B)$

$A \vee B \equiv \neg A \rightarrow B$

Tricks to argue on

- Semantics: argue on an arbitrary interpretation $\mathscr{I}$.
- Proofs: argue on an arbitrary proof or arbitrary path on the proof tree.
- Semantic tableaux: argue on an arbitrary path in a completed tableaux.

Soundness

We have looked at a proof system (natural deduction), a decision procedure (semantic tableaux) and propositional semantics (as truth tables).

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Soundness of ND (w.r.t. propositional semantics)

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ holds.

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Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas. If the tableaux for $\phi_1, \phi_2, \ldots, \phi_n \rightarrow \neg\psi$ closes, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ holds (ψ is valid) .

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Proof by induction on length of proof.

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Proof by induction on height of semantic tableaux.

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We prove the contrapositive: If the tableaux for $\phi_1, \phi_2, \ldots, \phi_n \to \neg\psi$ is open, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ is falsifiable (is not valid)

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Refining it: If the tableaux for $\phi_1, \phi_2, \ldots, \phi_n \rightarrow \psi$ is open, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ is satisfiable

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Refining it: If the tableaux for $\phi_1, \phi_2, \ldots, \phi_n \rightarrow \psi$ is open, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ is satisfiable
Four steps:

- Define a sets of formulas S (Hintikka's set)
- Connect this set S to the semantic tableaux (show that the union of formulas along an open path in a completed tableaux forms a Hintikka's set)
- Prove a property ϕ for this defined set S (prove that all Hintikka's sets

Setp 1: Defining a special set

Hintikka's set

Let U be a set of formulae s.t.

1. if $p \in \mathscr{P}$, either $p \in U$ or $p \notin U$ (contraditions not allowed)
2. if $A \in U$ is an α-formula, both $\{A_1, A_2\} \in U$
3. If $B \in U$ is a β-formula, one of $B_1 \in U$ or $B_2 \in U$

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Hintikka's sets are **downward saturated**: if $F \in U$ and if U can be extended to some U' by using the α/β rules on F as given above, then $U = U'$.

Step 2: Connect special set to semantic tableaux

Set of formulas on an open path in ST has a Hintikka's set

We pick an arbitrary open path:

1. No rule to decompose literals p or $\neg p$ (simply get copied); open node $\Longrightarrow$ no complemantary set can appear interpretation, so either $p \in U$ or $\neg p \in U$
2. $A_1 \wedge A_2$ (α-formula): Tablueax is completed, so $A_1, A_2 \in U$
3. $B_1 \vee B_2$ (β-formula): Tablueax is completed, so one of $B_1, B_2 \in U$

Step 3: Special set has a special property

Hintikka's sets are SAT

- Induction on size of formulae: Hintikka's sets with formulae less then size k are satisfiable (Hintikka's sets with larger formulae can be constructed by joining elements from this set)
- Let us "design" an interpretation:
  - $\mathscr{I}(p) = T$ if $p \in U$
  - $\mathscr{I}(p) = F$ if $p \in U$
  - $\mathscr{I}(p) = T$ if $p, \neg p \notin U$
- For any element $F \in U$
  - F is a literal: trivially satisfied by the construction of the set
  - Inductive step: for an arbitrary interpretation $\mathscr{I}$
    - F is α-formula: $v(A_1) = v(A_2) = T$ by structural induction; $v(A_1 \wedge A_2) = T$ (by definition of sementics on this $\mathscr{I}$)
    - F is β-formula: $v(B_1) = T$ or $v(B_2) = T$ by structural induction; $v(B_1 \vee B_2) = T$ (by definition of sementics on this $\mathscr{I}$)

Step 4: Final Inference

Root formula is SAT

- Root formula is in U
- If $\mathscr{I}$ is a model for a set of formula U, it is a model for all formulas $F \in U$ (by definition of satisfiability over set of formulae).
- So $\mathscr{I}$ is a model of root formula.

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Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas. If $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds.

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Steps:

- We prove $\vDash \psi \implies \vdash \psi$ (if ψ is a tautology, then ψ is a theorem)
- We prove $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi \implies \phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

if ψ is a tautology, then ψ is a theorem

To prove: whenever valuation over all interpretations of ψ is true, there exists some (at least one) proof for ψ

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We need to come up with a mechanical way of constructing such a proof for an arbitrary ψ

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Our strategy

- Encode each row in the truth table into a sequent
- Construct proofs of these 2^n sequents
- Combine these proofs into a proof of ψ

Encode each row in the truth table into a sequent

Lemma

For a formula ψ with propositions $p_1, p_2 \dots p_n$. For a line l in the truth table of ψ:

- Let $\hat{p}_i = \begin{cases} p_i & p_i(l) = T \\ \neg p_i & p_i(l) = F \end{cases}$

Then,

1. $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \psi$ has a proof if $\psi(l) = T$
2. $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\psi$ has a proof if $\psi(l) = F$

Construct proofs of these 2^n sequents

Proof by structural induction of ψ

Construct proofs of these 2^n sequents

Proof by structural induction of ψ

1. If ψ is a proposition: $p \vdash p$ and $\neg p \vdash \neg p$ are one liners
2. If $\psi = \neg\phi$; cases:
   1. $\psi(l)$ is T; then $\phi(l) = F$
      1. using induction hypothesis, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\phi$
      2. same as $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \psi$
   2. $\psi(l)$ is F; then $\phi(l) = T$
      1. using induction hypothesis, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi$
      2. Using $\neg\neg i$, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\neg\phi$
      3. same as $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\psi$

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Note: $\frac{\Gamma_1 \vdash \phi}{\Gamma_1, \Gamma_2 \vdash \phi}$ (can always strengthen premises)

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1. If ψ is $\phi_1 \wedge \phi_2$:
   1. Case 1: $\phi_1(l) = T$ and $\phi_2(l) = T$
      1. using IH, we have $\hat{q_1}, \hat{q_2}, \ldots, \hat{q_{n_1}} \vdash \phi_1$ and $\hat{r_1}, \hat{r_2}, \ldots, \hat{r_{n_2}} \vdash \phi_2$
      2. same as $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} \vdash \phi_1$ and $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} \vdash \phi_2$, where $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} = \{q_1, \hat{q_2}, \ldots, \hat{q_{n_1}}\} \cup \{\hat{r_1}, \hat{r_2}, \ldots, \hat{r_{n_2}}\}$ (using the above rule)
      3. So, $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} \vdash \phi_1 \wedge \phi_2$ ($\wedge i$)
   2. Case 2: $\phi_1(l) = F$ and $\phi_2(l) = T$
      1. using IH, we have $\hat{q_1}, \hat{q_2}, \ldots, \hat{q_{n_1}} \vdash \neg\phi_1$ and $\hat{r_1}, \hat{r_2}, \ldots, \hat{r_{n_2}} \vdash \phi_2$
      2. same as $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} \vdash \neg\phi_1$ and $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} \vdash \phi_2$, where $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} = \{q_1, \hat{q_2}, \ldots, \hat{q_{n_1}}\} \cup \{\hat{r_1}, \hat{r_2}, \ldots, \hat{r_{n_2}}\}$ (using the above rule)
      3. So, $\hat{p_1}, \hat{p_2}, \ldots, \hat{p_n} \vdash \neg\phi_1 \wedge \phi_2$ ($\wedge i$)
      4. To prove: $\neg\phi_1 \wedge \phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$
   3. Case 3: $\phi_1(l) = T$ and $\phi_2(l) = F$
   4. Case 4: $\phi_1(l) = F$ and $\phi_2(l) = F$
2. If ψ is $\phi_1 \vee \phi_2$: ...
3. If ψ is $\phi_1 \rightarrow \phi_2$: ...

Mini Lemma

1. To prove: $\neg\phi_1 \wedge \phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$
    1. $\neg\phi_1 \wedge \phi_1$
    2. $\neg\phi_1$
    3.
        1. $\phi_1 \wedge \phi_2$ (assumption)
        2. ϕ_1 $(\wedge e)$
        3. $\perp$
2. $\neg(\phi_1 \wedge \phi_2)$

Combine the sequent proofs

(law of excluded middle): $\frac{}{\phi \vee \neg \phi}$

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Illustration for an example: $p \wedge q \to p$

1. Proof the sequents for each line of the truth table (using theorem above)
   1. $p, q \vdash p \wedge q \to p$
   2. $\neg p, q \vdash p \wedge q \to p$
   3. $p, \neg q \vdash p \wedge q \to p$
   4. $\neg p, \neg q \vdash p \wedge q \to p$

Combine the sequent proofs

1	$p \lor \neg p$							LEM
2	p			ass	$\neg p$			ass
3	$q \lor \neg q$			LEM	$q \lor \neg q$			LEM
4	q	ass	$\neg q$	ass	q	ass	$\neg q$	ass
5	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
6								
7	$p \land q \to p$		$p \land q \to p$		$p \land q \to p$		$p \land q \to p$	
8	$p \land q \to p$			$\lor$e	$p \land q \to p$			$\lor$e
9	$p \land q \to p$							$\lor$e

(credits: Huth and Ryan. Modelling and Reasoning about Systems)

Wrapping up the general case...

Prove $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi \implies \phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

1. Prove: $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ has a logical consequence $\vDash (\phi_1 \rightarrow (\phi_2 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi))))$ (argue over semantics)
2. Prove: $\vDash (\phi_1 \rightarrow (\phi_2 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi)))$ implies $\vdash (\phi_1 \rightarrow (\phi_2 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi))))$ (directly from the above proof $\vDash \gamma \implies \vdash \gamma$)
3. Prove: $\vdash (\phi_1 \rightarrow (\phi_2 \rightarrow \cdots \rightarrow (\phi_n \rightarrow \psi))))$ can prove $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ (argue over proof system)

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Compactness (corollary)

Let S be a countably infinite set of formulas. If S is unsatisfiable, there must be at least one finite subset of S that is unsatisfiable.

Compactness (Proof)

Semantic tableaux for infinite formulae

- Let $S = A_1, A_2, ...$
- Set the label of the root to $\{A_1\}$.
- Whenever a rule is applied to a leaf of depth n, $A_n + 1$ will be added to the label(s) of its child(ren) in addition to the α_i or β_i .

Compactness (Proof)

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- Proof by contradiction: assume that every finite subset is satisfiable but the formula is unsatisfiable
- formula unsatisfiable, implies, semantic tableaux is closed
- Then there must be a finite number of formula labelling each leaf node, each providing a finite subset of formula in S that is unsatisfiable
- contradicts the assumption that every finite subset is satisfiable

What cannot be done in propositional logic

"All students registered in CS202 aboslutely love it!"

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How to encode on propositional logic:

- p_A: A is a student
- $q_{A,B}$: A is registered in course B
- $r_{A,B}$: A loves course B

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$$(p_{Aniket} \wedge q_{Aniket,CS202} \rightarrow r_{Aniket,CS202}) \wedge (p_{Shashwat} \wedge q_{Shashwat,CS202} \rightarrow r_{Shashwat,CS202}) \wedge \cdots \wedge p_{Ayush} \wedge q_{Ayush,CS202} \rightarrow r_{Ayush,CS202}$$

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Need for more powerful logics:

- Support for predicates (eg. Loves(Student,Course))
- Support for quantifiers

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$$\forall X, Y.\ Student(X) \wedge Registered(X, CS202) \rightarrow Loves(X, CS202)$$