CS203B : Mathematics for Computer Science - III
CSE, IIT Kanpur

Practice sheet 7

1. **Exponential random variable**
   If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is an exponential random variable with parameter λ/c.
   Hint: Start with the probability distribution function of cX, express it in terms of that of X, and then differentiate it.

2. **Exponential random variable**
   The time (in hours) required to repair a machine is an exponential random variable with parameter $\lambda = 1/2$. What is

   (a) the probability that a repair time exceeds 2 hours;

   (b) the probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours ?

   Answers: $e^{-1}, e^{-1/2}$.

3. **Memorylessness of exponential distribution**
   Consider a post office that is staffed by two clerks. Suppose that when Mr. Smith enters the post office, he discovers that Ms. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Jones or Brown leaves. If the amount of time that a clerk spends with a customer is exponentially distributed with parameter λ, what is the probability that, of the 3 customers, Mr. Smith is the last to leave the post office ?
   Hint: 1/2.

4. **Probability density function of a random variable**
   Let X be a random variable with probability density function f_X. Calculate the probability density function of random variable Y, defined by $Y = aX + b$.
   Hint: Start with the probability distribution function of Y, express it in terms of that of X, and then differentiate it.

5. **Probability density function of a random variable**
   If X is exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.
   Hint: Start with the probability distribution function of Y, express it in terms of that of X, and then differentiate it.

6. **Monochromatic cliques**
   First, let us define some terminologies for undirected graphs.

   - K_r denotes a complete graph on r vertices, that is, there is an edge between each pair of vertices.
   - Coloring of edges of a graph means assigning colors to its edges.
   - Once we have assigned colors to all the edges of a graph, a subgraph is said to be a monochromatic subgraph if each edge in the subgraph has the same color.

Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.
Hint: Proceed along similar lines as that of the solution of the "tournament problem" in Lecture 20.

7. **Line intersecting a circle**
   There are several circles of total circumference 10 inside a square of side length 1. Prove that there is a line that intersects at least 4 of the circles.
   Hint : Choose any one side of the square. Select a random point on this side and draw a line perpendicular to the side. What will be expected number of circles it will intersect ?

8. **Large cut**
   In the lecture class, we showed that a graph having m edges has a cut of size at least $m/2$. In fact, this bound can be further improved slightly: If G has $2n+1$ vertices and m edges, then it has a cut of size at least $m(n+1)/(2n+1)$.
   Hint : Use another simple randomized algorithm to partition the vertices *as evenly as possible.*

9. **Generating a random permutation using very few random bits**
   Let n be a prime number and let $S = \{1, 2, ..., n-1\}$. We are given an array A storing a permutation of S. We wish to permute A randomly such that the following condition is satisfied.

   $$\mathbf{P}(A[i] = j) = \frac{1}{n-1}$$

   How will you do it using $O(\log n)$ random bits only ?
   Hint: Refer to the early part of the solution of the problem "sum-free subset of size at least $n/3$" in Lecture 20.

10. **sum-free subset**
    Give the complete details of the proof we gave for the following theorem:
    Each set B of n positive integers has a sum-free subset of size at least $n/3$.
    Hint: it is already in the slides of Lecture 20. Go through it slowly and rigorously.

Note: There are a few questions in this sheet which were asked during the lectures.