CS203B : Mathematics for Computer Science - III
CSE, IIT Kanpur

Practice sheet 5

1. Let X and Y be independent random variables, both being equally likely to be any of the values $1, 2, ..., m$. Show that

$$E[|X-Y|] = \frac{(m-1)(m+1)}{3m}$$

   Hint: Derive the formula in terms of any given value of X, and then use conditional expectation.

2. Suppose X is a Binomial random variable with parameter n and p. Let Y be a random variable defined as $Y = 1/(1+X)$. Suppose you wish to calculate $\mathbf{E}[Y]$. You will, as first instinct, start writing the formula required to calculate it, and start using your mathematical skills to solve it. But wait... The following problem gives an indirect and elegant way of finding $\mathbf{E}[Y]$...

   There are $n+1$ participants in a game. Each person, independently, is a winner with probability p. The winners share a total prize of 1 unit. For instance, if 4 persons win, then each of them receives $1/4$, whereas if there are no winners, then none of the participants receive anything. Let A denote a specified one of the players, and let X denote the amount that is received by A.

   (a) Compute the expected total prize shared by the players.

   (b) Argue that

$$\mathbf{E}[X] = \frac{1-(1-p)^{n+1}}{n+1}$$

   (c) Compute $\mathbf{E}[X]$ by conditioning on whether A is a winner, and conclude that

$$\mathbf{E}[(1+B)^{-1}] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

   where B is a Binomial random variable with parameters n and p.

   Hint: All steps are already given.

3. This problem shows that Markov's inequality is as tight as it could possibly be. Given a positive integer k, describe a random variable X that assumes only nonnegative values such that

$$\mathbf{P}(X \geq k\mathbf{E}[X]) = \frac{1}{k}$$

   Hint: Define a random variable that takes value k with probability $1/k$ and takes value 0 with probability $1-1/k$.

4. Let X be a number chosen randomly uniformly from $[-k, k]$. Find $\mathbf{E}[X]$. Find $\mathbf{Var}[X]$.
   Hint: Just follow the definition of $\mathbf{E}[X]$ and $\mathbf{Var}[X]$.

5. Let X be a Poisson random variable with parameter λ. What is $\mathbf{Var}[X]$?
   Hint: λ.

6. Suppose that we roll a standard fair dice 100 times. Let X be the sum of the numbers that appear over the 100 rolls. Use Chebyshev's inequality to bound $\mathbf{P}(|X - 350| \geq 50)$.
   Hint: First calculate $\mathbf{E}[X^2]$, then $\mathbf{Var}[X]$, and then apply Chebyshev's inequality appropriately.

7. Suppose $X_1, \ldots, X_n$ be n random variables defined over a probability space. Moreover, $\mathbf{E}[X_i \cdot X_j] = \mathbf{E}[X_i] \cdot \mathbf{E}[X_j]$. Prove that
$$\mathbf{Var}[\sum_i X_i] = \sum_i \mathbf{Var}[X_i].$$
   Hint: Unfold the expression of $\mathbf{Var}[X]$, use $\mathbf{E}[X_i \cdot X_j] = \mathbf{E}[X_i] \cdot \mathbf{E}[X_j]$, and use cancellation wherever possible.

8. Suppose X and Y are random variables defined over a probability space. Moreover, they are independent. Prove that $\mathbf{Var}[X - Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$.
   Hint: Unfold the left hand side. The negative term will be nullified due to independence.

9. Find an example of a random variable with finite expectation and unbounded variance.
   Hint: X takes value n with probability $1/n$ and takes value 0 with probability $1 - 1/n$.

10. Let Y be a nonnegative integer-valued random variable with positive expectation. Prove that
$$\mathbf{E}[Y] \geq \mathbf{P}[Y \neq 0] \geq \frac{(\mathbf{E}[Y])^2}{\mathbf{E}[Y^2]}$$
    Hint: The first inequality follows obviously. For the 2nd inequality, proceed as follows. Use the following equalities to derive an upper bound on $\mathbf{P}[Y = 0]$ using Chebyshev's inequality.
$$\mathbf{P}[Y = 0] = \mathbf{P}[Y - \mathbf{E}[Y] = -\mathbf{E}[Y]] \leq \mathbf{P}[|Y - \mathbf{E}[Y]| = \mathbf{E}[Y]] \leq \mathbf{P}[|Y - \mathbf{E}[Y]| \geq \mathbf{E}[Y]]$$

11. Recall the ball-bin problem discussed many times in the course. Suppose the number of balls = number of bins = n. Let X be the random variable for the number of empty bins. Calculate $\mathbf{Var}[X]$.
    Hint: Follow the formula for variance and express X as sum of X_i's. Note that the random variables X_i's are not independent.

12. Alice and Bob play checkers often. Alice is a better player, so the probability that she wins any game is 0.6, independent of all other games. they decide to play a tournament of n games. Give the best possible bound on the probability that Alice loses the tournament.
    Hint: Just apply Chernoff bound.

13. We have a standard six-sided dice. Let X denote the number of times that a 6 occurs over n throws of the dice. Let p be the probability of the event $X \geq n/4$. Compare the best upper bounds on p using Markov's inequality, Chebyshev's inequality, and Chernoff's bound.
    Hint: Just apply the formulas.

14. Recall the proof of Chernoff bound that we discussed in the class. You are advised to go through the proof carefully and then try to answer the following questions. Some of these questions might not make sense if you have fully internalized the proof of Chernoff bound.
    (a) In which step (or steps), the independence of random variables was used ?
    (b) Where did we use the fact that e^x is an increasing function of x ?
    (c) Did we ever use the fact that e^x is a 1-1 function ?
    (d) Chernoff bound is based on Markov Inequality. Then how is it possible to achieve better bound using Chernoff bound ?
    (e) What if we had picked some other function that e^{tx} ? For example, what if we had picked $e^{e^{tx}}$? Would that have led to a better bound ?

    Hint: Just go through the slides of the proof of Chernoff bound carefully.

15. We toss a fair coin n times. Prove that the length of the the longest sequence of consecutive heads will not be more than $2 \log n$ with probability at least $1 - 1/n$.
Hint: Show that starting from any position i, the probability of getting sequence of $2 \log n$ consecutive heads is $1/n^2$. Now use Union theorem.

Note: There are a few questions in this sheet which were asked during the lectures.