First Order Logic

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Where are we?

First-Order Formula

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$\Phi := p(t_1, t_2, \ldots, t_n) \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \exists x_1, x_2, \ldots x_n.\Phi \mid \forall x_1, x_2, \ldots x_n.\Phi$

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Generally we always add a predicate for $=$(equality)

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The encoding is much nicer with relations and function than with raw predicates.

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Then, (R, F, C) is a **first order language** where $r \in R$, $f \in F$ and $c \in C$.

Universe (Domain) of Discourse

- Set of elements are we are discussing about
- The quantifiers run over this set
- **Intentional** versus **Extensional** description of the domain:
  - Intentional: what properties that this domain holds (necessary and sufficient conditions)
  - Extensional: listing of all the elements in the set or a enumerative description that shows how the set can be constructed

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- Andy and Paul have the same paternal grandfather.
  - $\forall x \forall y \forall u \forall v.\ (F(x, y) \wedge F(y, a) \wedge F(u, v) \wedge F(v, p) \rightarrow x = u)$ [Note the relation F(x,y) has no knowledge that y is the only father of x; so the following formulation says that "all grandfathers of Andy and Paul are same; using $\exists$ would say that at least one grandfather of Andy and Paul are same: both formulations are correct with the implicit knowledge of a unique father.)

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  - $f(f(a)) = f(f(p))$ (with functions, captures the knowledge that one has only one father)
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  - Ambigiuous! two possibilities:
  - $\exists x.\ (B(x,m) \wedge L(a,x))$ (Mary likes one of several brothers)

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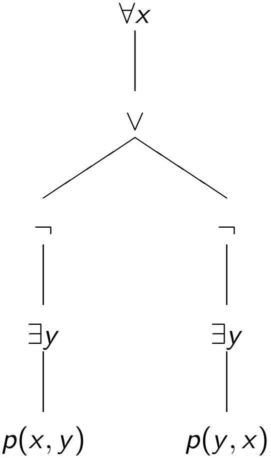
- $\forall x \exists y.\ op(x, y) = \epsilon$

Precedence and associativity

- $\neg$, $\forall y$ and $\exists y$ bind most tightly;
- then $\bigvee$ and $\bigwedge$;
- then $\rightarrow$, which is right-associative.

Syntax Tree for $\forall x(\neg\exists y p(x, y) \vee \neg\exists y p(y, x))$



Free and Bound variables

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Let ϕ be a formula in predicate logic.

- occurrence of x in ϕ is **free** (in ϕ) if
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- For $\forall$x ϕ, or $\exists$x ϕ, we say that ϕ — minus any of ψ's subformulas $\exists$x ψ, or $\forall$x ψ — is the **scope** of $\forall$x, respectively $\exists$x.

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$$(\forall x(P(x) \bigwedge Q(x))) \rightarrow (\neg P(x) \bigvee Q(y))$$

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- Incorrect failure can lead to (binding) capture.

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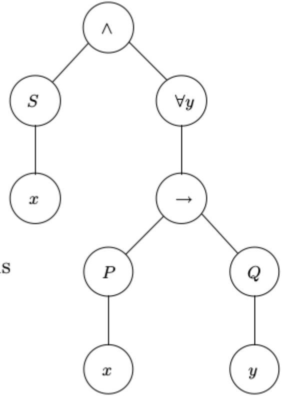

the term $f(y, y)$ is not free for x in this formula

Natural Deduction

Equality

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Here equality does not mean syntactic, or intensional, equality, but equality in terms of computation results

Natural Deduction

Universal Quantification

$$\frac{\forall x.\ \phi}{\phi[t/x]}\,\forall xe \qquad \frac{\boxed{\begin{array}{c} x_0 \text{ fresh} \\ \ldots \\ \phi[x_0/x] \end{array}}}{\forall x.\ \phi}\,\forall i$$

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Note the *scope* (shown by the box): the scope contains *local assumptions* and fresh variables, and hence the derived formulas and fresh variables are not valid/occur outside the proof

Natural Deduction

Existential Quantification

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- Proof system
  - Easy to prove something is valid (show a proof for it)
  - Difficult to prove that something is not valid (maybe you are just not able to construct a proof
  - For satisfiability, easy to prove contradictions (give proof of $\neg\phi$)
- Semantics
  - Easy to prove something is not valid (show a falsifiable interpretation)
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So, both are important!

Semantics

Interpretation

Given a set of formulas U with a set of relations R, set of functions F and a set of constants C, an interpretation $\mathscr{I}$ is a four tuple $(D, r_1^{\mathscr{I}}, r_2^{\mathscr{I}}, \ldots, r_n^{\mathscr{I}} \in R^{\mathscr{I}}, f_1^{\mathscr{I}}, f_2^{\mathscr{I}}, \ldots, f_m^{\mathscr{I}} \in F^{\mathscr{I}}, c_1^{\mathscr{I}}, c_2^{\mathscr{I}}, \ldots, c_l^{\mathscr{I}} \in C^{\mathscr{I}})$, where

- D is the domain of discourse
- For each relation symbol, $r_i \in R$, there exists a concrete relation $r_i^{\mathscr{I}} \in R^{\mathscr{I}}$ (of same arity)
- For each function symbol, $f_i \in F$, there exists a concrete function $f_i^{\mathscr{I}} \in F^{\mathscr{I}}$ (of same arity)
- For each function symbol, $c_i \in C$, there exists a concrete constant $c_i^{\mathscr{I}} \in C^{\mathscr{I}}$

Valuations

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  - $\mathscr{I} \vDash_I \phi$, $I \in \Gamma$ can now be questioned.
- For simplicity, we will only restrict overselves to closed formulas

Example: State Transition System

Given $\mathscr{I} =$
$(\{a, b, c\}, \{\{(a, a), (a, b), (a, c), (b, c), (c, c)\}, \{b, c\}\}, \{(a, b), (b, c), (c, a)\}, \{$

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- $D = \{a, b, c\}$ (states)
- There are two relations, $R^{\mathscr{I}} = \{r_1^{\mathscr{I}}, r_2^{\mathscr{I}}\}$:
  - $r_1^{\mathscr{I}} = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$ (transition relation)
  - $r_2^{\mathscr{I}} = \{b, c\}$ (a uninary relation, i.e. a set) (final states)
- There is one function, $F^{\mathscr{I}} = \{f_1^{\mathscr{I}}\}$ (additional "failure" function)
  - $f_1^{\mathscr{I}}(a) = b; f_1^{\mathscr{I}}(b) = c; f_1^{\mathscr{I}}(c) = a;$
- There are two constants: $C^{\mathscr{I}} = \{a, b\}$ (initial states)

Check on $\mathscr{I}$

- $\exists y.\ R(i, y)$
- $\neg F(i)$
- $\forall x, y, z\ (R(x, y) \rightarrow R(x, z) \rightarrow y = z)$
- $\forall x \exists y.\ R(x, y)$

Ground Terms

- A ground term is a term which does not contain any variables.
- A ground atomic formula is an atomic formula, all of whose terms are ground.
- A ground literal is a ground atomic formula or the negation of one.
- A ground formula is a quantifier-free formula, all of whose atomic formula are ground.
- A is a ground instance of a quantifier free formula A' iff it can be obtained from A' by substituting ground terms for the (free) variables in A'.

Theorem: The set of ground terms is countable.

Proving Semantic Consequence

Harder than propositional logic (building truth tables)—needs argument on sets (1-ary relations), relations and functions

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$\forall x(P(x) \rightarrow Q(x)) \vDash \forall x P(x) \rightarrow \forall x Q(x)$

- Let $\mathcal{M}$ be a model of $\forall x(P(x) \rightarrow Q(x))$; need to show it is a model of $\forall x P(x) \rightarrow \forall x Q(x)$
- Case I: all $x \in P$
  - then, all $x \in Q$ (from $\forall x(P(x) \rightarrow Q(x))$)
- Case II: let some $x_0 \notin P$
  - then, $\forall x P(x)$ is false, so $\mathcal{M}$ satisfies.

Proving Semantic Consequence

- $\forall x P(x) \rightarrow \forall x Q(x) \vDash \forall x (P(x) \rightarrow Q(x))$
  - Let $\mathcal{M}$ be a model of $\forall x P(x) \rightarrow \forall x Q(x)$; need to show it is a model of $\forall x (P(x) \rightarrow Q(x))$
  - Case I: if all $x_0 \in P$, then for all $x_0 \in Q$. Consequent holds.
  - Case II: if there is some $x \notin P$, the premise is vacously true asseting no constraints, allowing sets P and Q to be arbitrarily. Let is create counterexample: $(\{a, b\}, \{\{a\}, \{b\}\}, \{\}, \{a, b\})$.

Soundness and Completeness

Natural Deduction is sound and complete with respect to first order semantics as described above.

Compactness

Compactness Theorem

Let Γ be a set of sentences in predicate logic. If all finite subsets of Γ are satisfiable, then so is Γ.

Proof

- proof by contradiction
- let all finite subsets of Γ are satisfiable and Γ is not satisfiable
- then, $\Gamma \vDash \bot$ (Γ can have infinite premises)
- by completeness, $\Gamma \vdash \bot$
- so there exists a proof for $\Gamma \vdash \bot$
- a proof implies that it can use only a finite premises, say some Δ (Δ is finite)
- so, $\Delta \vdash \bot$
- by soundness, $\Delta \vDash \bot$

Application of Compactness

Reachability

Reachability is not expressibe in predicate logic

Proof

- Let there be such a first order formula ψ
- $\Phi_0 \stackrel{def}{=} c = c'$; $\phi_1 = R(c, c')$;
  $\Phi_n \stackrel{def}{=} \exists x_1, x_2, \ldots x_{n-1}(R(c, x_1) \wedge R(x_1, x_2) \wedge \cdots \wedge R(x_{n-1}, c')$
- here, interpretations are graphs
- Φ_n : there exists a path of length n
- Let $\Delta = \{\neg\Phi_i | i \geq 0\} \cup \{\psi[c/u][c'/v]\}$
- Δ: there does not exist a path of length 1, 2, ... but a finite path from c to c'
- But, every finite subset is satisfiable
- Δ is unsatisfiable

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Normal Forms

PCNF (prenex conjunctive normal form)

$Q_1 x_1 Q_2 x_2 \dots Q_n x_n M$

- $Q_1 x_1 Q_2 x_2 \dots Q_n x_n$ is prefix
- M is the matrix

Clausal form

$Q_1 x_1 Q_2 x_2 \dots Q_n x_n M$

- If all Q_i in prefix is universal quantification,
- M is written as CNF

then formula can we written as a list of clauses.

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Skolemization

For a closed formula $\exists x A(x, y)$, $\exists x A = A[f(y)/x]$

- $\forall x \exists y p(x, y)$: for all x, produce a y such that p(x,y) holds
- f(y): produces one such value of x (for a given y) for which p(x,y) holds
- so, equisat (not all interpretations are retained) but not equivalent

Skolemization Algorithm

$\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$

- Rename bound variables
  $\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall y p(y) \rightarrow \forall z q(z))$
- Transform to only $\vee$ and $\wedge$
  $\neg\forall x(\neg p(x) \vee q(x)) \vee (\neg\forall y p(y) \vee \forall z q(z))$
- Push negations inside
  $\exists x(p(x) \wedge \neg q(x)) \vee \exists y \neg p(y) \vee \forall z q(z)$
- Extract quantifiers from matrix (out to in)
  $\exists y \exists x \forall z(p(x) \wedge \neg q(x)) \vee \neg p(y) \vee q(z))$
- Skolemization (add functions with arguments for universal quantifier outside it)
  - $\exists y \exists x \forall z(p(x) \wedge \neg q(x)) \vee \neg p(y) \vee q(z))$
  - No, universal quantifier outside existentials:
    $\forall z(p(a) \wedge \neg q(a)) \vee \neg p(b) \vee q(z))$
  - $\forall z \exists y \exists x(p(x) \wedge \neg q(x)) \vee \neg p(y) \vee q(z))$
  - universal quantifier outside existentials is z:
    $\forall z(p(f(z)) \wedge \neg q(f(z))) \vee \neg p(g(z)) \vee q(z))$

Herbrand Models

- cannonical interpretations for set of models
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Herbrand Base

set of ground atomic formulae that can be formed from predicate symbols in S and terms in H_s

A relation over Herbrand universe is simply a subset of Herbrand base.

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Gives a semi-decision procedure to solve first order satisfiability/validity.