CS203B : Mathematics for Computer Science - III
CSE, IIT Kanpur

Practice sheet 6

1. Let X and Y be random variables defined over a probability space (Ω, P). Moreover, it is also given that X and Y are independent. Let f and g be any two real valued functions defined for all real numbers. Let $Z_1 = f(X)$ and $Z_2 = g(Y)$. Prove that Z_1 and Z_2 are independent.
   Hint: Start with $\mathbf{P}(Z_1 = a_1 | Z_2 = a_2)$, use the formula for conditional probability, express it in terms of probability of events of the form "X taking some values" and "Y taking some values", do cancellations wherever possible, and arrive at $\mathbf{P}[Z_1 = a_1]$.

2. Let Z be a uniformly distributed random variable with corresponding interval $[a, b]$. In the class, we calculated $\mathbf{E}[Z^2]$. Find $\mathbf{Var}(Z)$.
   Hint: $(b-a)^2/12$

3. Let X be a random variable with the probability density function

$$f_X(x) = \begin{cases} c(1-x^2) & \text{for} \quad -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of c ?
   (b) What is the probability distribution function $F_X(x)$?

   Hint: For the first part, use the fact that the integration of f_X over $-1 < x < 1$ should be 1. For the second part, do integration of f_X from -1 to x.

4. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f_X(x) = xe^{-x} \qquad \text{for} \quad x \geq 0$$

   Compute the expected lifetime of such a tube.
   Hint: Integrate $x^2 e^{-x}$ over the range $x \geq 0$. You might like to use the formula for integration by parts. Do it properly since it may be used for variance of exponential random variable as well.

5. X is a continuous random variable with probability distribution function F_X and Y is another continuous random variable with probability distribution function F_Y. X and Y are independent. Let $Z = \min(X, Y)$. What is probability distribution function F_Z for Z ?
   Hint: Try to express $Z \leq x$ as union of two events, use the formula for union of events and exploit independence of events, to conclude that $F_Z(x)$ is $F_X(x) + F_Y(x) - F_X(x) \cdot F_Y(x)$.

6. A square frame is placed with only one of its sides touching the ground. The frame subtends an angle with the ground which is uniformly distributed in the interval $[0, \pi]$. What is the expected area of the shadow of the frame ?
   Hint: Due to symmetry, we need to solve the problem for $[0, \pi/2]$. The answer is $2/\pi$.

7. A stick of length 1 has a point p marked on it. The point is located at distance ℓ from one of the endpoints of the stick. The stick is split at a point U that is distributed randomly uniformly over $(0, 1)$. Determine the expected length of the piece that contains the point p.
   Hint: $1/2 + p(1-p)$.

8. A man and a woman agree to meet at a certain location about 12:30 PM. If the man arrives at a time uniformly distributed between 12:15 and 12:45 and if the woman independently arrives at a time uniformly distributed between 12:00 and 1:00 PM, find the probability that the first to arrive waits no longer than 5 minutes.
   Hint: Solve the problem based on the time when women arrives. You need to work for the following intervals :12:10 to 12:15, 12:15 to 12:20, 12:20 to 12:40, 12:40 to 12:45, 12:45 to 12:50. You might like to use symmetry to reduce the number of intervals to be analysed.

9. An ambulance travels back and forth, at a constant speed, along a road of length L. At a certain moment of time an accident occurs at a point uniformly distributed on the road. [That is, its distance from one of the fixed end of the road is uniformly distributed over $(0, L)$.]. Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, compute, assuming independence, the expected distance that the ambulance will have to travel.
   Hint: $L/3$.

10. Three points X_1, X_2, X_3 are selected at random on a line L. What is the probability that X_2 lies between X_1 and X_3 ?
    Hint: 1/3. You might like to use the solution of the previous problem. You may also use symmetry.

Note: There are a few questions in this sheet which were asked during the lectures.