CS203B: Mathematics for Computer Science - III
Final Exam
Max Marks=60

Name:

Roll Number:

Important Instructions:

1. This is the question paper as well as the answer sheet.
2. For each question which has choices, you **must** attempt **at most one choice**. If you attempt both, you will not get any marks for that question.
3. For each question, please provide answer only in the space that follows it.
4. You are provided **rough sheets** separately. Do all rough work and calculations there only and not on the answer sheet.
5. Please keep all the rough sheets and the answer sheet always on your desk. You must not keep it on the neighbouring seat.
6. There is a plenty of time. So please make decision about the choices for each question judiciously. Think carefully before writing your answer. **No extra answer sheet will be provided to you**.
7. The last 2 pages of the answer sheet are blank and may be used in case you attempted one particular choice of a question but later changed your mind. Make sure to cross the solution you wrote for the earlier choice.

This space is only for grading purpose.

Question	1	2	3	4	5	6	7	8
Marks								

1. For each question on this page, just write the answers in the space provided. There is no need to provide any explanation.

   (a) (*marks = 4*)
   There is a square field $ABCD$ with each side of length 1000 meters. See Figure 1. F is the mid point of side CD and O is the center of the square field. There is a dog standing at each of the points A, B, F, and O. Although dogs at A,B,O are free to move, the dog at F is tied to F with a rope of radius 250 meters. So the dog standing at F can not go farther than 250 meters from F. A piece of meat is dropped at a uniformly random location in the square field. Seeing it, each dog starts running towards it at its maximum speed. Maximum speed at which a dog can run is the same for each of these 4 dogs. The dog that grabs the piece of meat is the one that reaches it first. Let P_A, P_B, P_O, P_F be respectively the probability that dog at location A, B, O, F grabs the piece of meat. What are P_A, P_B, P_O, and P_F ?

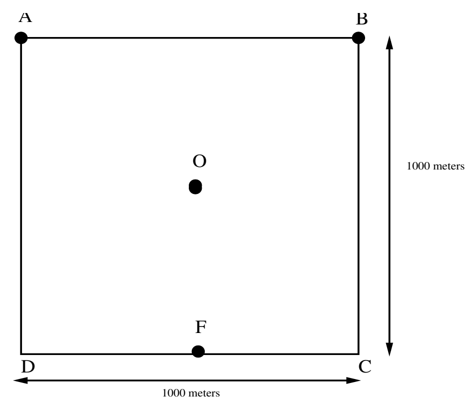

Figure 1: Square Field.

	P_A	P_B	P_O	P_F
Answer:	1/8	1/8	$3/4 - \pi/32$	$\pi/32$

   (b) (*marks = 1,2*)
   Ram has a coin that gives head with probability p independently in each toss. He keeps tossing the coin till he gets 10 heads. For any $1 \leq i \leq 10$, let X_i be the random variable for the toss number that results in the ith head.

   i. What is $\mathbf{E}[X_{10}]$?
   Answer: $10/p$.

   ii. What is $\mathbf{E}[X_9 | X_{10} = 51]$?
   Answer: This conditional expectation is independent of p. This is because given that 10th head appears at 51th toss, the first 9 toss may appear anywhere uniformly in the first 50 tosses. Now using the discrete analogue of "sampling n points uniformly on a line" in Lecture 18, the 10 intervals defined by the 9 heads have identical probability distribution. The total length of the interval is 51 (think over it). Hence the conditional expectation of X_9 is the expected sum of the first 9 intervals, which is $51 \cdot \frac{9}{10} = 45.9$.

   (c) (*marks = 2*)
   If Y is uniformly distributed in the interval $(0, 5)$, what is the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real ?
   Answer: $3/5 = 0.6$

2. Attempt exactly one of the following problems.

   (a) (*marks = 4*)
   A man possesses four coins, one of which is double-headed, one is double-tailed and two are normal. He shuts his eyes, picks a coin at random, and tosses it. He opens his eyes and sees that the coin is showing head; what is the probability that the lower face is head? He shuts his eyes again, and tosses the coin again. He opens his eyes and sees that the coin is showing head; what is the probability that the lower face is a head? He discards this coin, and picks another at random, and tosses it, what is the probability that it shows head? **Answer: 1/2, 2/3, 7/18**

   (b) (*marks = 8*)
   A bag contains 203 red balls and 1 blue ball. Balls are selected randomly uniformly from the bag and discarded until the first time that a ball, say B, is removed having a different colour from its predecessor. The ball B is now replaced into the bag and the procedure is restarted. This process continues until only one ball is left. What is the probability that this ball is a red ball ? You must give formal reason to justify your claim. No marks for incomplete or informal justification even if the answer is correct.

Answer:
Let $\mathcal{E}_n$ be the event that the random experiment beginning with n red balls and 1 blue balls ends with a red colored ball. Let $p_n = \mathbf{P}[\mathcal{E}_n]$. It is easy to see that $p_0 = 0$ and $p_1 = 1/2$. So in the following discussion, we assume $n > 1$.

A generic form of the procedure undertaken in the experiment is the following: *we take out a sequence of zero or more red balls followed by the blue ball.* So we can partition the sample space based on the location of the blue ball in this procedure. Using uniformity in sampling, the blue ball may appear at each of the places from 1 to $n+1$ with equal probability. You may refer to the exercise of *star ball* that we did in the class while discussing partition theorem. Hence

$$p_n = \mathbf{P}[\mathcal{E}_n] = \sum_{1 \le i \le n+1} \frac{1}{n+1} \cdot \mathbf{P}[\mathcal{E}_n | \text{blue ball appears at } i\text{th place}] \tag{1}$$

If the blue ball appears as the first place, this ball is going to be discarded and hence the last ball is surely going to be red. So $\mathbf{P}[\mathcal{E}_n|\text{blue ball appears at 1st place}] = 1$. If blue ball appears at ith place with $i > 1$, the procedure restarts with $n - i + 1$ red balls and blue ball. So for $i > 1$, $\mathbf{P}[\mathcal{E}_n|\text{blue ball appears at ith place}] = p_{n-i+1}$. So Equation 1 can be rewritten as a recurrence as follows.

$$p_n = \frac{1}{n+1} + \frac{1}{n+1} \sum_{i=2}^{i=n+1} p_{n-i+1}$$

Multiplying both sides by $(n+1)$ and changing the labels of the sum suitably, we get

$$(n+1)p_n = 1 + \sum_{0 \le j \le n-1} p_j$$

Rewriting this equation by substituting $n-1$ for n, we get

$$np_{n-1} = 1 + \sum_{0 \le j \le n-2} p_j$$

Subtracting this equation from the one above it, we get

$$(n+1)p_n - np_{n-1} = p_{n-1}$$

Hence $p_n = p_{n-1}$ for each $n > 1$. Since $p_1 = 1/2$, so we get $p_n = 1/2$ for all $n > 0$.

3. Attempt exactly one of the following problems.

   (a) (*marks = 3*)
   A fair coin is tossed n times. What is the expected number of times the pattern HTH appears ? Note that the patterns may be overlapping as well. For example, in the sequence $HHTHTHTHTHTT$, the pattern HTH appears 4 times. **Answer:** $(n-2)/8$.

   (b) (*marks = 5*)
   There are n line segments in a plane. Total number of points of intersections of these segments is m. Moreover, no three segments intersect at the same point. We pick a uniformly random sample S of i segments. Each possible subset of i segments is equally likely to be picked as S. Calculate the expected number of points of intersections among the segments of set S.

Answer: $\frac{i(i-1)}{n(n-1)}m$.
Let the set of points of intersections be $A = \{p_1, \ldots, p_m\}$. Let X be the random variable for the number of points from A that appear in the sample S of segments. For each $1 \leq j \leq m$, let X_j be the random variable that takes value 1 if p_j appears in the sample. From construction, it follows that $X = \sum_j X_j$. So by linearity of expectation,

$$\mathbf{E}[X] = \sum_j \mathbf{E}[X_j] = m\mathbf{P}[X_j = 1]$$

For X_j to be 1, p_j has to appear in the sample. For p_j to appear in the sample, the two segments that define it must be selected. Once we have fixed 2 segments, the remaining $i-2$ segments can be picked from $n-2$ segments in $\binom{n-2}{i-2}$ ways only. The total number of ways of selecting i segments out of n segments is $\binom{n}{i}$. So the probability that a specific pair of segments is selected in the sample is

$$\binom{n-2}{i-2} \Big/ \binom{n}{i} = \frac{i(i-1)}{n(n-1)}$$

Hence, $\mathbf{E}[X] = m\frac{i(i-1)}{n(n-1)}$.

4. (*marks = 5*)
The following result is well known in geometric probability:
Result: *If 3 points P, Q, and R are picked randomly uniformly and independently inside a unit square, the expected area of triangle PQR is* $11/144$.

We pick 4 points uniformly randomly and independently inside a unit square. Using the result mentioned above, or otherwise, calculate the probability that these four points form a convex quadrilateral.
Answer: $25/36$.
Let the four points to be picked be denoted as P, Q, R, S. Define event $\mathcal{E}_P$ as the event that point P lies inside the $\triangle QRS$. The events $\mathcal{E}_Q, \mathcal{E}_R, \mathcal{E}_S$ are defined accordingly. Since points are selected uniformly, the probability S lies inside $\triangle PQR$ is equal to expected area of $\triangle PQR = 11/144$. So $\mathbf{P}[\mathcal{E}_S] = 11/144$. Since each point is selected independent of others, the order in which the points are picked does not matter. So $11/144$ is also the probabilty of each of the remaining 3 events, namely $\mathcal{E}_P, \mathcal{E}_Q, \mathcal{E}_R$.

Let $\mathcal{E}$ be the event that the quadrilateral $PQRS$ is not convex. It follows that $\mathcal{E} = \mathcal{E}_P \cup \mathcal{E}_Q \cup \mathcal{E}_R \cup \mathcal{E}_S$. If a point, say Q, lies inside the triangle PRS, then surely P (likewise R, S) lies outside the $\triangle QRS$ (likewise $\triangle PQS$, $\triangle PQR$). Hence, each of the events $\mathcal{E}_P, \mathcal{E}_Q, \mathcal{E}_R, \mathcal{E}_S$ are mutually exclusive. So $\mathbf{P}[\mathcal{E}] = \mathbf{P}[\mathcal{E}_P] + \mathbf{P}[\mathcal{E}_Q] + \mathbf{P}[\mathcal{E}_R] + \mathbf{P}[\mathcal{E}_S] = 4\frac{11}{144} = 44/144$. Hence probability that PQRS is convex $= 1 - \mathbf{P}[\mathcal{E}]$ $= 100/144 = 25/36$.

5. Attempt exactly one of the following problems.

   (a) (*marks = 4*)
   Prove that an undirected graph on n vertices and m edges has a cut of size at least $m/2$.

   (b) (*marks = 8*)
   A random graph $G(n, p)$ is constructed on n vertices as follows :

   add an edge between each pair of vertices with probability p independently.

   Length of a path between two vertices is the number of edges lying on the path. A path connecting two vertices is said to be a shortest path if there is no other path between them with a shorter length. Distance between two vertices is the length of the shortest path connecting them. Diameter of a graph is the maximum distance between any pair of vertices in the graph. If $p = \sqrt{\frac{4\log_e n}{n}}$, show that $G(n, p)$ has a diameter larger than 2 with probability $< 1/n$ for asymptotically large value of n.

Answer:
Let $\mathcal{E}$ be the event that the diameter of the graph is more than 2. Consider any pair of vertices $u, v \in V$. Define event $\mathcal{E}_{u,v}$ as the event that Distance(u, v) is strictly greater than 2. It follows that $\mathcal{E} = \cup_{u,v\in V}\mathcal{E}_{u,v}$. Using Union Theorem, in order to show that $\mathbf{P}[\mathcal{E}] < 1/n$, it suffices to show that $\mathbf{P}[\mathcal{E}_{u,v}] < 1/n^3$.

Distance(u, v) will be 1 iff $(u, v) \in E$. Distance(u, v) will be 2 if there is any common neighbour of u and v. Hence Distance(u, v) will be more than 2 iff there is no edge joining u and v and there is no common neighbour for u and v. So the probability of event $\mathcal{E}_{u,v}$ can be bounded from above as follows.

$$
\begin{aligned}
\mathbf{P}[\mathcal{E}_{u,v}] &= (1-p)(1-p^2)^{n-2} \\
&< (1-p^2)^{n-2} \\
&\leq e^{-p^2(n-2)} \quad \text{(since } 1 + x \leq e^x \text{ for all } x\text{)} \\
&= e^{-4\frac{n-2}{n}\log_e n} \quad \text{(plugging in the value of } p.\text{)} \\
&\leq e^{-3\log_e n} \quad \text{(for } n > 8\text{)} \\
&= 1/n^3.
\end{aligned}
$$

6. Attempt exactly one of the following problems.

   (a) (*marks = 3*)
   Let X be a random variable distributed randomly uniformly in $[0, \pi/2]$. Let $Y = \sin(X)$. Calculate the probability density function of Y.
   Answer: $2/(\pi\sqrt{(1-x^2)})$.

   (b) (*marks = 7*)
   Let $X_1, X_2, ...X_n$ be n independent exponential random variables with same parameter λ. Let $Z = \min(X_1, \ldots, X_n)$. Calculate the probability density function of Z. What is $\mathbf{E}[Z]$?

Answer:
Since X_i is an exponential random variable with parameter λ, so $\mathbf{P}[X_i \leq x] = 1 - e^{-\lambda x}$. Hence $\mathbf{P}[X_i > x] = e^{-\lambda x}$.

$$\begin{aligned}
\mathbf{P}[Z \leq x] = 1 - \mathbf{P}[Z > x] &= 1 - \mathbf{P}[\min(X_1, X_2, \ldots, X_n) > x] \\
&= 1 - \mathbf{P}[X_1 > x \cap X_2 > x \cap \ldots \cap X_n > x] \\
&= 1 - \Pi_{i=1}^{i=n} \mathbf{P}[X_i > x] \quad \text{since } X_i\text{'s are independent} \\
&= 1 - \Pi_{i=1}^{i=n} e^{-\lambda x} \\
&= 1 - e^{-n\lambda x}.
\end{aligned}$$

In other words Z is an exponential random variable with parameter $n\lambda$. Hence, as discussed in the class, $\mathbf{E}[Z] = \frac{1}{n\lambda}$.

7. Attempt exactly one of the following problems.

   (a) (*marks = 4*)
   X is a random variable distributed uniformly in the interval $[0, 1]$. Let $Y = \min(X, 1 - X)$. Calculate the expected value of Y.
   Answer: $1/4$.

   (b) (*marks = 8*)
   Two points A and B are picked independently and uniformly at random inside a circle C of unit radius. Calculate the probability that the circle with A as center and $|AB|$ as radius lies inside C.
   Answer: $1/6$.
   An exercise involving continuous probability and integration. It was surprising to see that not many could solve it though they did similar exercise in their 12th class.
   First pick A and then pick B.
   As mentioned in the class, picking a point uniformly from a 2-D space of Area R (inside circle) means the following: The point is picked from a region of area a with probability a/R. In the current problem, $R = \pi$ since we the 2-D space is circle of unit radius. Splitting the circle into concentric strips of very small width, the probability A lies in a strip of radius r and width Δr is $\approx (2\pi r \Delta r)/\pi$. Once A is picked, B must lie in a circle centered at A and having radius $1 - r$. The calculations follow easily...

8. Recall the algorithm for generating random permutation of $\{1, 2, \ldots, n\}$ that we discussed in the class.

(a) (*marks = 3*)
Fill in the blanks of the following code for generating random permutation of $\{1, 2, \ldots, n\}$. A is an array with indices from 1 to n. $A[i]$ is initialized to 0 for each i. You may invoke function **Rand**(n) in this algorithm which returns an integer picked randomly uniformly and independently from 1 to n. S is a string that stores the permutation at the end of the algorithm.

```
S ← Empty-String;
for i = 1 to n do
    repeat
        x ← ······rand(n)·········
    until A[x] = ······0·········;;
    S ← ······S :: x·········;
    A[x] ← 1;
end
return S;
```

(b) (*marks = 7*)
Let X be the random variable for the number of calls to **Rand** that the algorithm makes. In the class, we showed that $\mathbf{E}[X] < n \log_2 n$. Calculate the tightest possible bound on $\mathbf{P}[X - \mathbf{E}[X] > 3\mathbf{E}[X]]$ that you can.

Answer. Chernoff bound won't work. Chebyshev's bound will give very weak bound. Union theorem rocks again !
In the class, we proved that $\mathbf{E}[X] \approx n \log_e n < n \log_2 n$. Let $\mathcal{E}$ be the event that the number of calls to **rand** is more than $4\mathbf{E}[X]$. If $\mathcal{E}$ happens, then surely there is at least one number $i \leq n$ which has not been generated even once in first $4\mathbf{E}[X]$ calls to the random number generator **rand**. For each $i \leq n$, let $\mathcal{E}_i$ be the event that number i is not generated even once in $4\mathbf{E}[X]$ calls to **rand**. Clearly $\mathcal{E} = \cup_i \mathcal{E}_i$. Hence, by Union Theorem, it follows that $\mathbf{P}[\mathcal{E}] \leq n\mathbf{P}[\mathcal{E}_i]$. A call to **rand** will generate i with probability $1/n$. Outcome of each call to **rand** is independent of other calls. Hence, probability of event $\mathcal{E}_i$ can be bounded as follows.

$$\begin{aligned}
\mathbf{P}[\mathcal{E}_i] &= (1 - 1/n)^{4\mathbf{E}[X]} \\
&= (1 - 1/n)^{4n \log_e n} \\
&\leq e^{\frac{-1}{n} 4n \log_e n} \qquad \text{since } 1 + x \leq e^x \text{ for all } x \\
&= e^{-4 \log_e n} \\
&= \frac{1}{n^4}
\end{aligned}$$

Hence $\mathbf{P}[\mathcal{E}] \leq n\mathbf{P}[\mathcal{E}_i] \leq 1/n^3$.