ESO207: Data Structures and Algorithms	*26 August*
After Class Questions: 1	

he following questions were among those asked at the end of class by some students. They appear informative enough to merit a discussion.

Problem 1.

a. Can we always argue that $T(n+c)$ is asymptotically $O(T(n))$ for any constant $c \ge 1$ and use it in our arguments?

b. Suppose $f(n)$ and $g(n)$ are two non-negative, monotonically increasing functions. Is it true that one of $f(n) = O(g(n))$ or $g(n) = O(f(n))$ holds.

For super-exponential functions, that is, functions that grow faster than exponential functions, $T(n+c)$ may not be $O(T(n))$. Simple example is $T(n) = n!$. Then, $T(n+1) = (n+1)T(n)$ and is not $O(T(n))$. But for polynomial functions, that is, $T(n) = n^{O(1)}$, this would be true. Let $T(n) = O(n^d)$, so that $T(n) \le an^d$, asymptotically. So there exists a constant β such that

$$(n+c)^d = n^d(1+c/n)^d \le n^d(1+\beta/n) = n^d + \beta n^{d-1} \le 2n^d$$

for sufficiently large n. Hence, $T(n+c) = O(T(n))$.

The solution to part [b.] is more complicated. Let us design two functions $f(n)$ and $g(n)$ that are monotonically non-decreasing functions such that neither $f(n) = O(g(n))$ nor $g(n) = O(f(n))$. Extending this to increasing functions requires just a small modification.

Let f and g be defined to be 1 for $n = 0, 1, \ldots, 5$. We will construct these functions in the range $n = 7k, 7k+1, \ldots, 7k+5, 7k+6$, as $k = 1, 2, \ldots$. Informally, the construction proceeds as follows. Let $k = \lfloor n/7 \rfloor$. At $n = 7k$, define $f(n) = g(n) = k^k$.

Fix k, and consider the 6 numbers $7k+1, \ldots, 7k+6$. Let

$$f(7k+j) = f(7k) \times k^{\lfloor j/2 \rfloor} \ .$$

Then, $f(7k+1) = f(7k)$, $f(7k+2) = f(7k)\cdot k = f(7k+3)$ and $f(7k+4) = f(7k)\cdot k^2 = f(7k+5)$ and $f(7k+6) = f(7k)\cdot k^3$. We want $g(7k+j)$ to be significantly (i.e., super-constant) larger than $f(7k+j)$ for some values of j and be significantly smaller than $f(7k+j)$ at a few other values of j (and may equal $f(7k+j)$ at some other values as well). So define

$$g(7k+j) = g(7k)\cdot k^{1.5\lfloor j/3 \rfloor}$$

Then, $g(7k+1) = g(7k+2) = g(7k)$, $g(7k+3) = g(k)\cdot k^{1.5} = g(7k+4) = g(7k+5)$ and $g(7k+6) = g(k)\cdot k^3 = f(7k+6)$.

The points of interest are n of the form $7k+2$ and $7k+3$. Note that $f(7k+2) = k^{k+1}$ and $g(7k+2) = k^k$. Also, $f(7k+3) = k^{k+1}$ and $g(7k+3) = k^{k+1.5}$. Hence, in every span of the multiple of 7, there is an n_1 of the form $7k+2$ such that $f(n_1)/g(n_1)$ is super-constant and, there is an n_2 of the form $7k+3$ such that $g(n_2)/f(n_2)$ is super-constant.

Hence, $f(n) = O(g(n))$ does not hold and $g(n) = O(f(n))$ also does not hold.