CS203B : Mathematics for Computer Science - III
CSE, IIT Kanpur

Practice sheet 1: **Elementary Probability Theory**

1. Let A and B be events with probabilities $P(A) = 3/4$ and $P(B) = 1/3$. Show that $1/12 \leq P(A \cap B) \leq 1/3$, and give examples to show that both extremes are possible. Find corresponding bounds for $P(AUB)$.

2. You are given that at least one of the event A_r, $1 \leq r \leq n$, is certain to occur, but certainly no more than two occur. If $P(A_r) = p$ for each $1 \leq r \leq n$, $P(A_r \cap A_s) = q$, $r \neq s$, show that $p \geq 1/n$ and $q \leq 2/n$.

3. A man possesses five coins, two of which are double-headed, one is double-tailed and two are normal. He shuts his eyes, picks a coin at random, and tossed it. What is probability that the lower face of coin is head? He opens his eyes and sees that the coin is showing head; what is the probability that the lower face is head? He shuts his eyes again, and tossed the coin again. What is the probability that the lower face is a head? He opens his eyes and sees that the coin is showing head; what is the probability that the lower face is a head? He discards this coin, and picks another at random, and tosses it, what is the probability that it shows heads?

   Solution: Already discussed in the class.

4. There are n urns of which the rth contains $r - 1$ red balls and $n - r$ blue balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that:
   a) the second ball is blue;
   b) the second ball is blue, given that the first is blue.

   Solution: 1/2,2/3. Try for 3 urns first, ...

5. We roll a dice n times. Let A_{ij} be the event that the ith and jth rolls produce the same number. Show that the events $\{A_{ij} : 1 \leq i < j \leq n\}$ are pairwise independent but not independent.
   (Hint for 2nd part: Consider any three distinct i, j, k. Analyse the corresponding events: A_{ij}, A_{jk}, A_{ik}.)

6. Two fair dice are rolled. Show that the event that their sum is 7 is independent of score shown by the first die.

7. There are two roads from A to B and two roads from B to C. Each of the four roads is blocked by snow with probability p, independent of the others.
   a) Find the probability that there is an open road from A to B.
   b) Find the probability that there is an open road from A to B given that there is no open route from A to C.
   c) If, in addition, there is a direct road from A to C, this road being blocked with probability p independently of others, Find the required conditional probability mentioned in part (b) above.

   Solution: $1 - p^2$, $\frac{(1-p^2)p^2}{1-(1-p^2)^2}$, $\frac{(1-p^2)p^2}{1-(1-p^2)^2}$ (same as part (b)).

8. A pack contains m cards, labelled $1, 2, , m$. The cards are dealt out in a random orders, one by one. Given that the label of kth card dealt is the largest of the first k cards dealt, what is the probability that it is also the largest in the pack? **Solution:** k/m.

9. A bowl contains twenty cherries, exactly fifteen of which have had their stones removed. A greedy pig eats five whole cherries, picked at random, without remarking on the presence or absence of stones. Subsequently, a cherry is picked randomly from remaining fifteen.
a) What is the probability that this cherry contains a stones?
b) Given that this cherry contains a stone, what is probability that the pig consumed at least one stone?

Solution: 5/20, $1 - \frac{15}{19} \cdot \frac{14}{18} \cdot \frac{13}{17} \cdot \frac{12}{16} \cdot \frac{11}{15}$.

10. The menages problem poses the following question. Some consider it to be desirable that men and women alternates when seated at the circular table. If n couples are seated randomly according to this rule, what is the probability that nobody sits next to his or her partner ?

Solution: Apply the theorem for union of events carefully, and try for some more days...

11. You choose r of the first n positive integers, and a lottery chooses a random subset L of the same size. What is the probability that:
a) L includes no consecutive integers?
b) L includes exactly one pair of consecutive integers?
c) the number in L are drawn in increasing order?
d) your choice of numbers is same as L?
e) there are exactly k of your numbers matching members of L?

Solution:
a) $\binom{n-r+1}{r} / \binom{n}{r}$.
b) $(r-1)\binom{n-r+1}{r-1} / \binom{n}{r}$.
c) $\frac{1}{r}$.
d) $\frac{1}{\binom{n}{r}}$.
e) $\binom{r}{k}\binom{n-r}{r-k} / \binom{n}{r}$.

12. You are lost in National Park of Bandrika. Tourists comprise two-thirds of the visitors to the park, and give a correct answer to requests for directions with probability 3/4. (Answer to repeated question are independent, even if the question and the person are the same). If you ask a Bandrikan for direction, the answer is always false.
a)You ask a passer-by whether the exit from park is east or west. The answer is East. What is the probability this is correct?
b)You ask the same person again, and received the same reply. Show the probability that it is correct is 1/2.
c) You ask the same person again, and received the same reply. What is the probability that it is correct?
d) You ask for the forth time, and receive the answer East. Show that the probability it is correct is 27/70.
e) Show that, had the forth answer been West instead, the probability that East is nevertheless correct is 9/10.

Balls into Bins

The experiment: There are m balls and n bins. Each ball selects a bin randomly uniformly and independent of other balls and falls into it.

Make sincere attempts to solve the following problems.

1. What is the probability that no bin is empty (assume $m > n$) ?
2. What is the probability that there are exactly k empty bins (assume $m > n$) ?
3. **Conditional probability**
For the following questions, assume $m = n$ and n is even.

(a) Find the conditional probability that bin 1 has one ball given that exactly one ball fell into the first three bins.

(b) Find the conditional probability that the number of balls in bin 1 is 2 under the condition that bin 2 received $n/2$ balls.

(c) What is the conditional probability that nth bin is empty given that the bins numbered 1 to $n/2$ are empty ?

Note: There are a few questions in this sheet which were asked during the lectures.