cs229 notes

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Table of contents

1 Matrix Derivatives

Good link: https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf

1.1 Some useful identities

1.

$$\frac{\partial}{\partial X} \log |X| = X^{-1}$$

Proof:

$$\frac{\partial}{\partial X} \log |X| = \frac{1}{|X|} \frac{\partial |X|}{\partial X}$$

We know that

$$(\frac{\partial |X|}{\partial X})_{ij} = \frac{\partial}{\partial X_{ij}} * det(X)$$

and

$$det(X) = X_{i1}C_{i1} + X_{i2}C_{i2} + \cdots + X_{in}C_{in}$$

where C_{ij} is the cofactor of X_{ij}. So,

$$\frac{\partial}{\partial X_{ij}} * det(X) = C_{ij}$$

$$\frac{\partial |X|}{\partial X} = C = adj(X)^T$$

where C is the cofactor matrix of X. $adj(X)$ is the adjugate matrix of X and $X^{-1} = \frac{adjX}{|X|}$.

so we get

$$\frac{\partial}{\partial X} \log |X| = \frac{1}{|X|} \frac{\partial |X|}{\partial X} = \frac{1}{|X|} adj(X)^T = (X^{-1})^T$$

Reference: kamper matrix calculus

2. $\frac{\partial}{\partial X}(z^T X^{-1} z) = -(X^{-1}) z z^T (X^{-1})$

Proof:

$$\frac{\partial}{\partial X}(z^T X^{-1} z)$$

Lets first compute the derivative of $z^T X^{-1} z$ with respect to X_{ij}

$$\frac{\partial}{\partial X_{ij}}(z^T X^{-1} z)$$

Lets first derive $\frac{\partial X^{-1}}{\partial X_{ij}}$

$$\frac{\partial X^{-1}}{\partial X_{ij}}$$

Using $X * X^{-1} = I$ we get

$$X^{-1} \frac{\partial X}{\partial X_{ij}} + \frac{\partial X^{-1}}{\partial X_{ij}} X = 0$$

i.e.

$$\frac{\partial X^{-1}}{\partial X_{ij}} = -X^{-1} \frac{\partial X}{\partial X_{ij}} X^{-1}$$

where $\frac{\partial X}{\partial X_{ij}}$ is the matrix of partial derivatives of X with respect to X_{ij} and it's elements are 0 except for the element at i, j which is 1.

So lets say $H = \frac{\partial \ tr(z^T X^{-1} z)}{\partial X}$

$$H_{ij} = \frac{\partial}{\partial X_{ij}} tr(z^T X^{-1} z)$$

Using cyclic property of trace we get

$$H_{ij} = \frac{\partial}{\partial X_{ij}} tr(z^T X^{-1} z) = \frac{\partial}{\partial X_{ij}} tr(zz^T (X^{-1}))$$

We know that

$$\partial(Tr(A)) = Tr(\partial(A))$$

because trace is linear. so

$$H_{ij} = tr(zz^T \frac{\partial}{\partial X_{ij}} (X^{-1})) = tr(zz^T (-X^{-1} \frac{\partial X}{\partial X_{ij}} X^{-1}))$$

Using cyclic property of trace we get

$$H_{ij} = tr(X^{-1} zz^T X^{-1} \frac{\partial X}{\partial X_{ij}})$$

Now suppose that

$$F = X^{-1} zz^T X^{-1}$$

then

$$tr(F \frac{\partial X}{\partial X_{ij}}) = F_{ji} = F_{ij}$$

since F is symmetric. Hint: You can think of the fact only the jth row of F is multiplied by the jth column, and only ith column of jth row of F is multiplied by the ith row of jth column of F leading to element at F_{jj} contributing and the rest being zero.

Hence: $H = -X^{-1} zz^T X^{-1}$

2 Prove that if $z^T H z \geq 0$ then H is positive semi-definite and cost function J is convex. H is Hessian matrix of J.

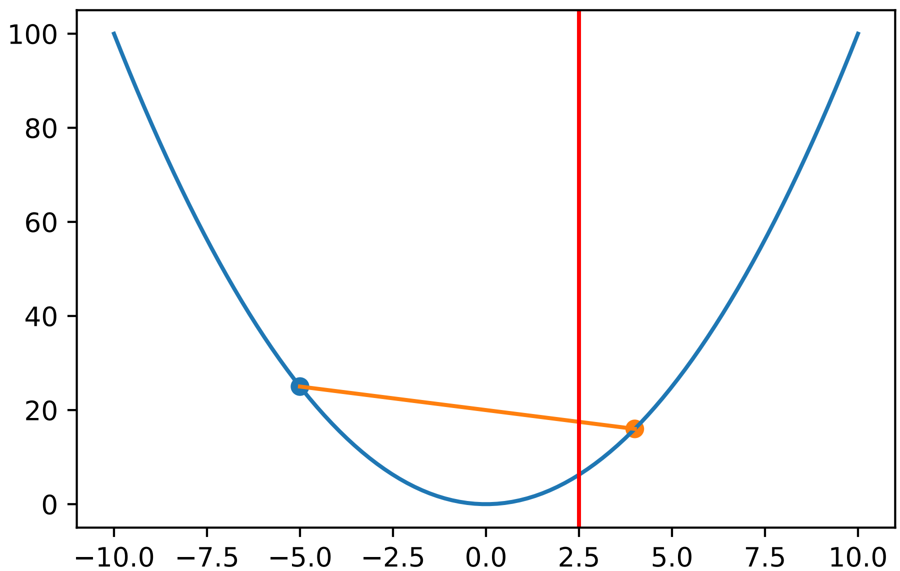
Some definitions first:

2.1 Convex function

A function f is convex if for any $x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$ we have

$$f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y)$$

This basically means that any line segment between two points on the graph of the function lies above the graph of the function.



2.2 Convex functions properties

Property 1: If f is convex then $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for any $\alpha \in [0,1]$ and $x, y \in \mathbb{R}^n$.

Proof: Let $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$. Since any point between x and y on the line segment $[x, y]$ is given by $\alpha x + (1-\alpha)y$, we have the following from the definition of convexity:

The value of function as point $z = \alpha x + (1-\alpha)y$ is $f(\alpha x + (1-\alpha)y)$. Now the equation of line is:

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Plugging values for z and x_1, x_2, y_1, y_2 we get:

$$\begin{aligned} y &= f(x) + \frac{f(y) - f(x)}{y - x}(\alpha.x + (1-\alpha)y - x) \\ &= f(x) + \frac{f(y) - f(x)}{y - x}(y - x)(1-\alpha) \\ &= f(x)(1 - 1 + \alpha) + f(y)(1-\alpha) \\ &= f(x)(\alpha) + f(y)(1-\alpha) \end{aligned}$$

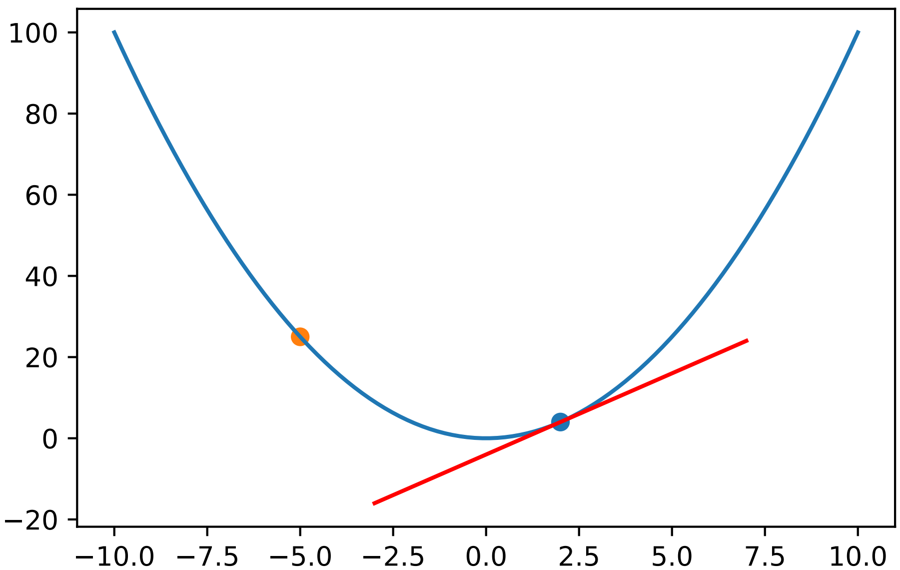
So according to the definition of convexity we have:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

Property 2: Suppose $f : \mathcal{R}^n \to \mathcal{R}$. Then

1. $f(y) \geq f(x) + \nabla f(x)^T(y - x)$ for all $x, y \in \mathcal{R}^n$ if and only if f is convex
2. $\nabla^2 \succeq 0$ if and only if f is convex

Proof:



1. Using the definition of convexity we have:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$
$$f(y) - f(x) \geq \frac{f(x + \alpha(y - x)) - f(x)}{\alpha}$$
$$\text{if } \alpha \to 0$$
$$f(y) - f(x) \geq \nabla f(x)^T(y - x)$$

Now we also need to prove the other direction. So suppose $f(y) \geq f(x) + \nabla f(x)^T(y - x)$ for all $x, y \in \mathcal{R}^n$. We need to prove that $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$. Let's consider $z = \alpha x + (1-\alpha)y$. Then we have:

$$
\begin{aligned}
f(x) &\geq f(z) + \nabla f(z)^T (x - z) \\
f(y) &\geq f(z) + \nabla f(z)^T (y - z)
\end{aligned}
$$

Multiply first with α and other by $1 - \alpha$ and add

$$
\begin{aligned}
\alpha f(x) + (1-\alpha) f(y) &\geq \alpha f(z) + (1-\alpha) f(z) + \\
\alpha \nabla f(z)^T (x - z) + (1-\alpha) \nabla f(z)^T (y - z) & \\
&\geq f(z) + \nabla f(z)^T (\alpha x + (1-\alpha) y - z) \\
\text{Since } \alpha x + (1-\alpha) y = z & \\
&\geq f(z) + \nabla f(z)^T (0) \\
&\geq f(z)
\end{aligned}
$$

2. Let's prove the second part. Suppose $\nabla^2 \succeq 0$ then we have:

Let us first prove it for $f : \mathbb{R} \to \mathbb{R}$. Let $x, y \in dom(f)$ and $x \leq y$ then we have:

$$
\begin{aligned}
f(y) - f(x) &\geq f'(x)(y - x) \\
f(x) - f(y) &\geq f'(y)(x - y) \\
\implies \frac{f'(x) - f'(y)}{x - y} &\geq 0 \\
\text{if } x \to y \text{ then} & \\
\frac{f'(x) - f'(y)}{x - y} \rightarrowtail f''(x) &\geq 0
\end{aligned}
$$

We can prove the other direction using the mean value version of the Taylor's theorem. Suppose $f''(x) \geq 0$ then there exists a point $z \in (x, y)$ such that:

$$
\begin{aligned}
f(y) = f(x) + f'(x)(y - x) + f''(z) \frac{(y - x)^2}{2} \\
\text{Since } f''(z) \geq 0 \\
f(y) \geq f(x) + f'(x)(y - x)
\end{aligned}
$$

Now we need to prove the same for $f : \mathbb{R}^n \to \mathbb{R}$. Remember that a convex function is convex along all lines. i.e. if $f : \mathbb{R}^n \to \mathbb{R}$ is convex then $g(\alpha) = f(x + \alpha(v))$ is convex for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

$$
g''(\alpha) = v^T \nabla^2 f(x + \alpha v) v
$$