Unit 1: Linear Relationships and Equations

Topic 5: Slope and Rate of Change

Concept Summary

The **slope** of a line describes how steep the line is — it measures how much y changes when x increases by 1. In other words, slope represents the **rate of change** between two quantities.

$$\text{Slope } (m) = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

If you imagine moving along a line: - The numerator $(y_2 - y_1)$ represents the “rise” (vertical change). - The denominator $(x_2 - x_1)$ represents the “run” (horizontal change).

Slope as Rate: On the SAT, slope often represents a real-world rate such as:

miles per hour,   dollars per item,   points per game,   etc.

Types of Slope

$$\begin{aligned} m > 0 &\Rightarrow \text{line rises left to right (positive slope)} \\ m < 0 &\Rightarrow \text{line falls left to right (negative slope)} \\ m = 0 &\Rightarrow \text{horizontal line} \\ \text{Undefined slope} &\Rightarrow \text{vertical line} \end{aligned}$$

Slope Relationships

- **Parallel lines** have the same slope.
- **Perpendicular lines** have slopes that are negative reciprocals:

$$m_1 \cdot m_2 = -1$$

Core Skills

- Find slope from two points, a graph, or an equation.
- Interpret slope as a rate of change in context.

- Recognize slope patterns for parallel and perpendicular lines.
- Identify slope units in real-world models.

Example 1: Finding Slope from Two Points

Find the slope of the line passing through the points $(2, 5)$ and $(6, 13)$.

Step 1: Label the points. $(x_1, y_1) = (2, 5)$, $(x_2, y_2) = (6, 13)$

Step 2: Use the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13 - 5}{6 - 2} = \frac{8}{4} = 2$$

Final Answer: $\boxed{m = 2}$

Interpretation: For every 1 unit increase in x, y increases by 2 units.

Example 2: Slope as a Rate of Change

A car travels 120 miles in 3 hours. What is its average rate of change in miles per hour?

Step 1: Identify the change in distance and change in time.

$$\text{Change in distance} = 120 \text{ miles}, \quad \text{Change in time} = 3 \text{ hours}$$

Step 2: Compute the rate.

$$m = \frac{\text{change in distance}}{\text{change in time}} = \frac{120}{3} = 40$$

Final Answer: $\boxed{\text{40 miles per hour}}$

Interpretation: The slope of the line on a distance–time graph would be 40, showing a steady rate of travel.

Key Takeaways

- Slope measures the rate of change between two variables.
- Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find slope from points.
- Parallel lines have equal slopes; perpendicular lines have negative reciprocal slopes.

- In word problems, slope often represents a real-world rate such as “cost per item” or “distance per time.”

Practice Questions: Slope and Rate of Change

Part A: Finding Slope from Points

1. Find the slope of the line through $(2, 3)$ and $(6, 11)$.
2. Find the slope of the line through $(5, 7)$ and $(9, 15)$.
3. Find the slope of the line through $(-3, 4)$ and $(5, 0)$.
4. Find the slope of the line through $(0, 8)$ and $(3, -4)$.
5. Find the slope of the line through $(-2, -3)$ and $(2, 5)$.

Part B: Identifying Slope from an Equation

6. Find the slope of $y = 3x + 7$.
7. Find the slope of $y = -2x + 4$.
8. Find the slope of $2x + 5y = 10$.
9. Find the slope of $4x - y = 12$.
10. Find the slope of $3y + 6x = 9$.

Part C: Parallel and Perpendicular Lines

11. Find the slope of a line parallel to $y = \frac{1}{2}x - 3$.
12. Find the slope of a line perpendicular to $y = -\frac{2}{3}x + 4$.
13. Determine whether the lines $y = 2x + 5$ and $y = -\frac{1}{2}x - 3$ are perpendicular.
14. Determine whether the lines $3x - 2y = 6$ and $y = \frac{3}{2}x + 1$ are parallel, perpendicular, or neither.
15. The line through $(1, 4)$ and $(3, 8)$ — find the slope of any line perpendicular to it.

Part D: Rate of Change in Context

16. A runner covers 400 meters in 50 seconds. Find the rate of change (meters per second).
17. The price of gas increased from \$3.20 to \$4.00 over 4 months. Find the average rate of change per month.
18. The temperature dropped from 68°F to 50°F over 3 hours. Find the rate of change per hour.
19. A business earns \$500 for 20 units sold and \$650 for 30 units sold. Find the rate of change in dollars per unit.
20. A car travels 150 miles in 3 hours. What is its average speed?

Part E: SAT-Style Applications

21. The equation $y = 25x + 100$ models the total cost y (in dollars) for renting a car for x days. What does the number 25 represent in this context?
22. The line passing through $(0, 50)$ and $(10, 80)$ represents a company's revenue over time. Find the rate of change and interpret its meaning.
23. A student's test score increased from 70 to 85 over 3 exams. Find the rate of change per exam.
24. The graph of a line has slope -4. What does this tell you about the relationship between x and y?
25. Two lines have slopes $m_1 = 3$ and $m_2 = -\frac{1}{3}$. Are the lines parallel, perpendicular, or neither?

Answer Key and Solutions: Slope and Rate of Change

Part A Solutions: Finding Slope from Points

1. $(2, 3)$ to $(6, 11)$: $m = \dfrac{11 - 3}{6 - 2} = \dfrac{8}{4} = \boxed{2}$
2. $(5, 7)$ to $(9, 15)$: $m = \dfrac{15 - 7}{9 - 5} = \dfrac{8}{4} = \boxed{2}$
3. $(-3, 4)$ to $(5, 0)$: $m = \dfrac{0 - 4}{5 - (-3)} = \dfrac{-4}{8} = \boxed{-\frac{1}{2}}$
4. $(0, 8)$ to $(3, -4)$: $m = \dfrac{-4 - 8}{3 - 0} = \dfrac{-12}{3} = \boxed{-4}$
5. $(-2, -3)$ to $(2, 5)$: $m = \dfrac{5 - (-3)}{2 - (-2)} = \dfrac{8}{4} = \boxed{2}$

Part B Solutions: Identifying Slope from an Equation

6. $y = 3x + 7 \Rightarrow m = \boxed{3}$
7. $y = -2x + 4 \Rightarrow m = \boxed{-2}$
8. $2x + 5y = 10 \Rightarrow 5y = -2x + 10 \Rightarrow y = -\frac{2}{5}x + 2 \Rightarrow m = \boxed{-\frac{2}{5}}$
9. $4x - y = 12 \Rightarrow -y = -4x + 12 \Rightarrow y = 4x - 12 \Rightarrow m = \boxed{4}$
10. $3y + 6x = 9 \Rightarrow 3y = -6x + 9 \Rightarrow y = -2x + 3 \Rightarrow m = \boxed{-2}$

Part C Solutions: Parallel and Perpendicular Lines

11. Parallel line to $y = \frac{1}{2}x - 3$: $m = \boxed{\frac{1}{2}}$
12. Perpendicular to $y = -\frac{2}{3}x + 4$: $m = \boxed{\frac{3}{2}}$ (negative reciprocal)
13. Slopes 2 and $-\frac{1}{2}$: $2 \cdot (-\frac{1}{2}) = -1 \Rightarrow \boxed{\textit{Perpendicular}}$
14. $3x - 2y = 6 \Rightarrow y = \frac{3}{2}x - 3$ and $y = \frac{3}{2}x + 1$: same slope $\frac{3}{2} \Rightarrow \boxed{\textit{Parallel}}$
15. Slope through $(1, 4)$ and $(3, 8)$: $m = \dfrac{8 - 4}{3 - 1} = \dfrac{4}{2} = 2$. Perpendicular slope $= \boxed{-\frac{1}{2}}$.

Part D Solutions: Rate of Change in Context

16. $\dfrac{400 \text{ m}}{50 \text{ s}} = \boxed{8 \text{ m/s}}$

17. $\dfrac{4.00 - 3.20}{4} = \dfrac{0.80}{4} = \boxed{\$0.20 \text{ per month}}$

18. $\dfrac{50 - 68}{3} = \dfrac{-18}{3} = \boxed{-6^\circ\text{F per hour}}$

19. $\dfrac{650 - 500}{30 - 20} = \dfrac{150}{10} = \boxed{\$15 \text{ per unit}}$

20. $\dfrac{150 \text{ miles}}{3 \text{ h}} = \boxed{50 \text{ mph}}$

Part E Solutions: SAT-Style Applications

21. In $y = 25x + 100$, the slope 25 is the $\boxed{\text{cost per day}}$.

22. Slope $\dfrac{80 - 50}{10 - 0} = \dfrac{30}{10} = \boxed{3}$. Interpretation: revenue increases by \$3 per unit of time.

23. $\dfrac{85 - 70}{3} = \dfrac{15}{3} = \boxed{5 \text{ points per exam}}$

24. Slope -4: as x increases by 1, y $\boxed{decreases by 4}$. Negative linear relationship.

25. $m_1 = 3,\ m_2 = -\frac{1}{3} \Rightarrow 3 \cdot (-\frac{1}{3}) = -1 \Rightarrow \boxed{\text{Perpendicular}}$