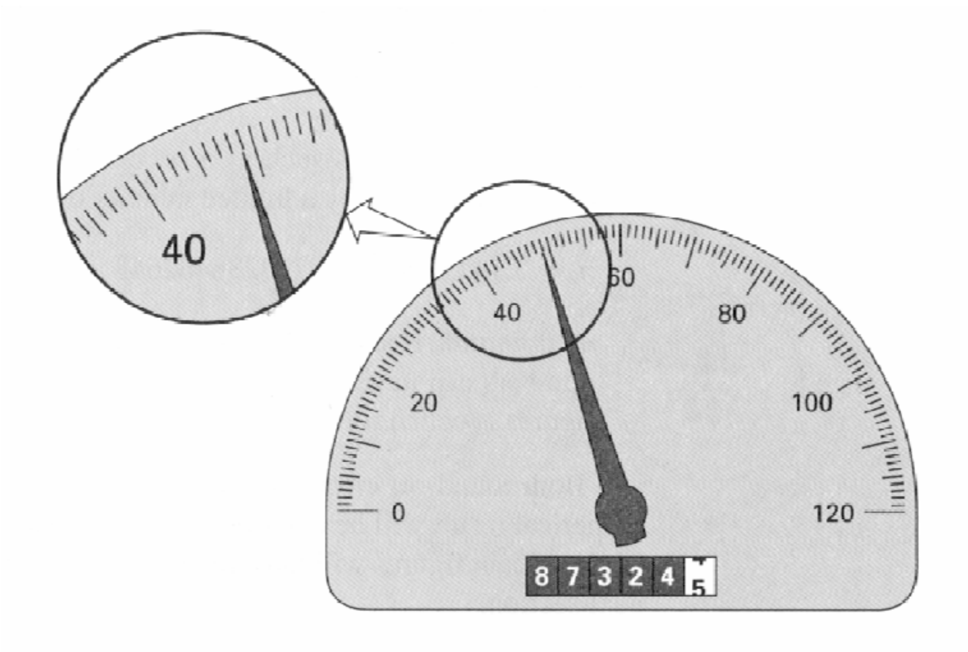
Lecture 03: Round off and truncation errors

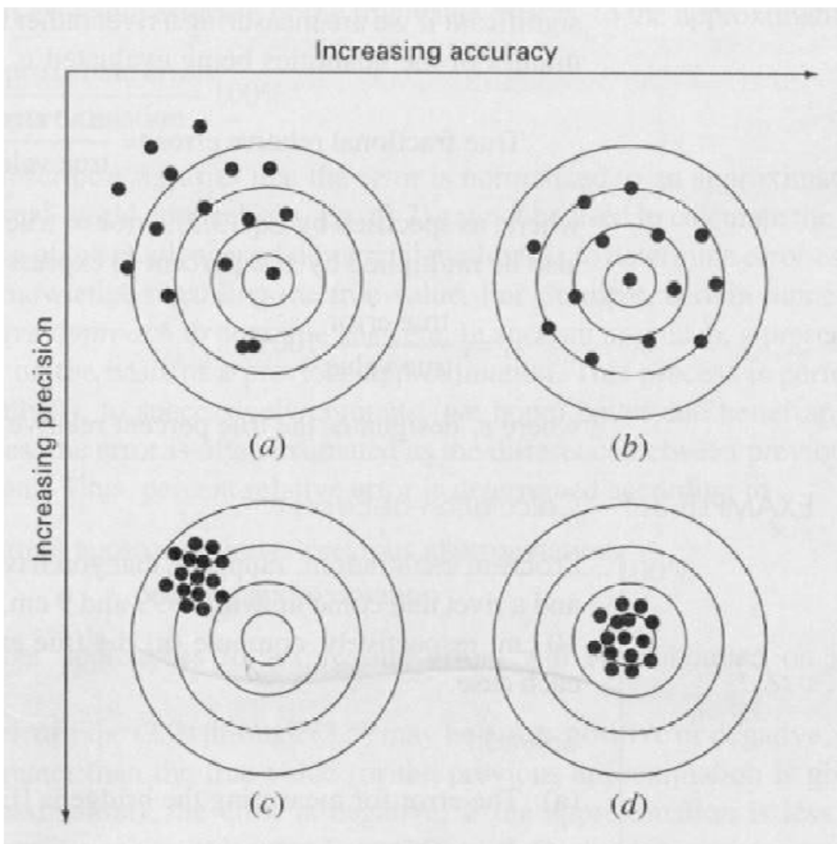
Readings: Chapters 3 and 4

SIGNIFICANT FIGURES



The number of certain digits + 1 estimated digit

ACCURACY vs. PRECISION



ERROR DEFINITIONS

True value = Approximation + True Error

True error = E_t = True value - Approximation

$$\varepsilon_t = \frac{\text{True error}}{\text{True value}} 100\%$$

True value = Approximation + Error

$$\varepsilon_t = \frac{1}{10{,}000} 100\% = 0.01\% \qquad \varepsilon_t = \frac{1}{10} 100\% = 10\%$$

Bridge **Rivet**

APPROXIMATE ERROR

Approximate error

$$\varepsilon_a = \frac{\textbf{approximate error}}{\textbf{approximation}} \textbf{100\%}$$

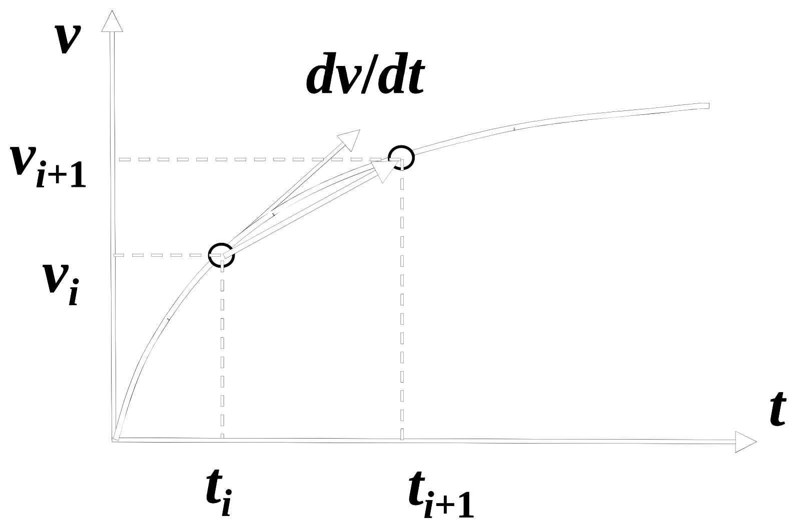
Estimated approximate error

$$\varepsilon_a = \frac{\textbf{current approximation – previous approximation}}{\textbf{current approximation}} \textbf{100\%}$$

Stopping criterion

$$|\varepsilon_a| < \varepsilon_s$$

TRUNCATION ERRORS



$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

TAYLOR'S THEOREM

If a function f and its first $n + 1$ derivatives are continuous on an interval containing x_i and $x_{i+1} = x_i + h$, then the value of the function at x_{i+1} is given by

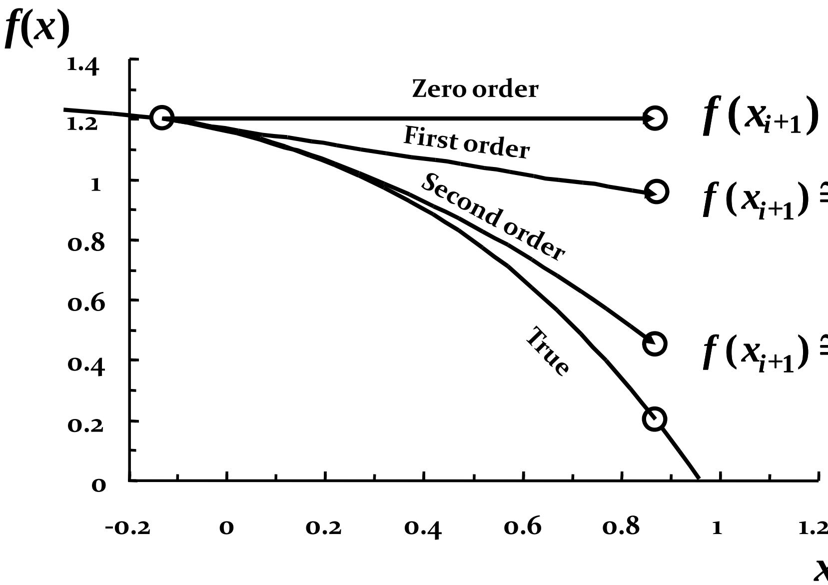
$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n$$

where

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

TAYLOR SERIES OF A POLYNOMIAL

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

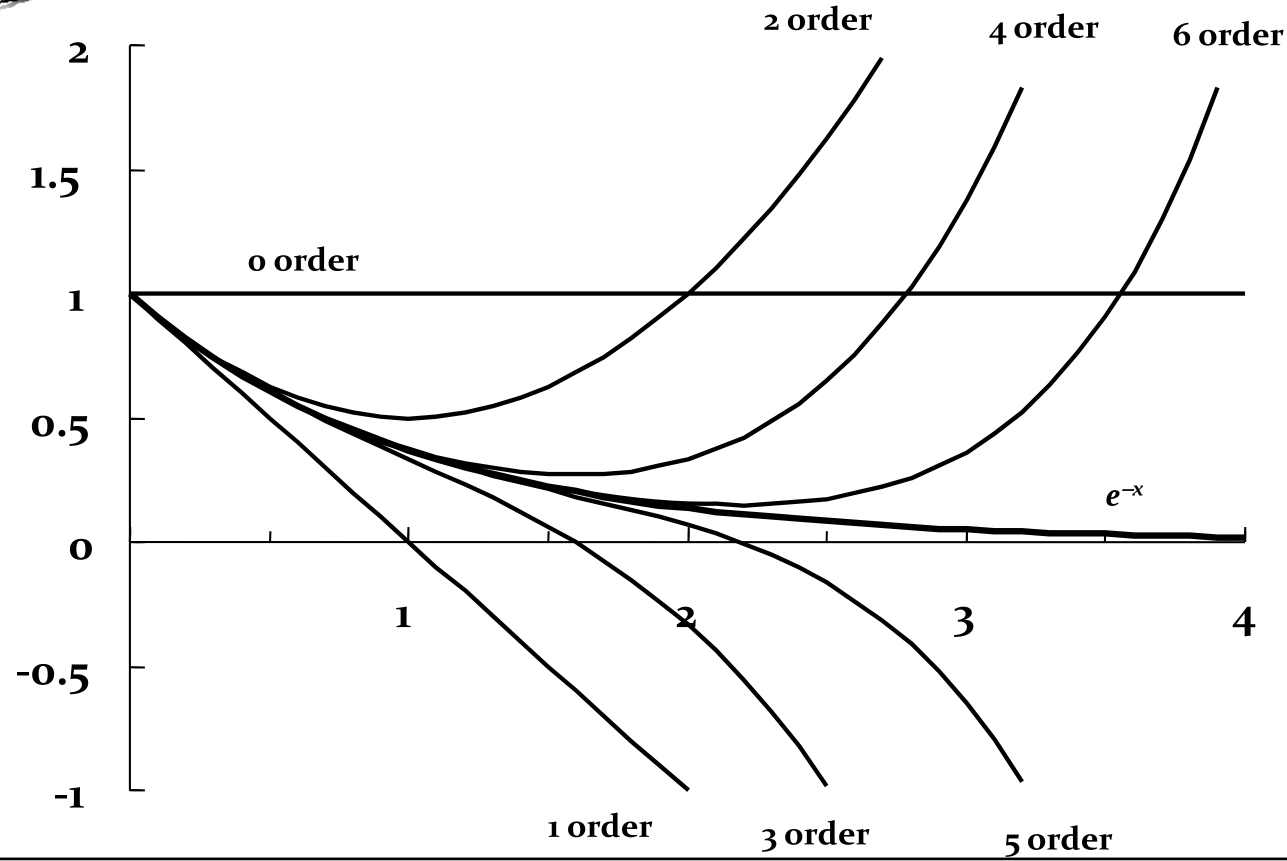


$$f(x_{i+1}) \cong f(x_i)$$

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

$$f(x) = e^{-x} = 1 - x + x^2/2! - x^3/3! + \cdots$$



TAYLOR SERIES

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2$$

$$+ \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n$$

where

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

$$R_n = O(h^{n+1})$$

ANY SMOOTH FUNCTION CAN BE APPROXIMATED AS A POLYNOMIAL!!!

ESTIMATING TRUNCATION ERRORS

$$v(t_{i+1}) = v(t_i) + v'(t_i)\,(t_{i+1} - t_i) + \frac{v''(t_i)}{2!}(t_{i+1} - t_i)^2 + \cdots + R^n$$

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1$$

$$v'(t_i) = \underbrace{\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}}_{\text{First-order Approximation}} - \underbrace{\frac{R_1}{t_{i+1} - t_i}}_{\text{Truncation error}}$$

$$\frac{R_1}{t_{i+1} - t_i} = \frac{v''(\xi)}{2!}(t_{i+1} - t_i) = O(t_{i+1} - t_i)$$

NUMERICAL DIFFERENTIATION

First forward divided difference

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

First backward divided difference

$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$

CENTERED DIFFERENCE

First centered divided difference

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \cdots$$

$$-\left[f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \cdots \right]$$

$$f(x_{i+1}) = f(x_{i-1}) + 2f'(x_i)h + \frac{2f^{(3)}(x_i)}{3!}h^3 + \cdots$$

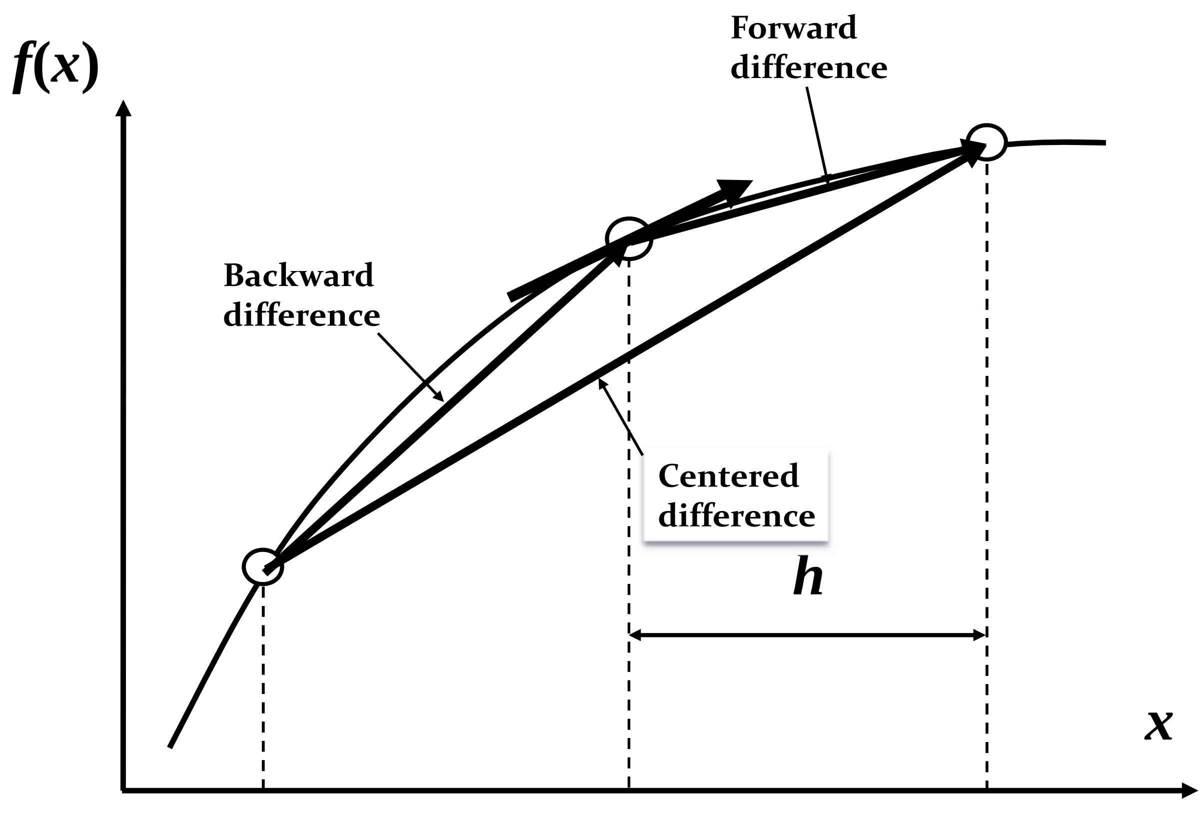
CENTERED DIFFERENCE

Solve for

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - \frac{f^{(3)}(x_i)}{6} h^2 + \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - O(h^2)$$

GRAPHICAL REPRESENTATION

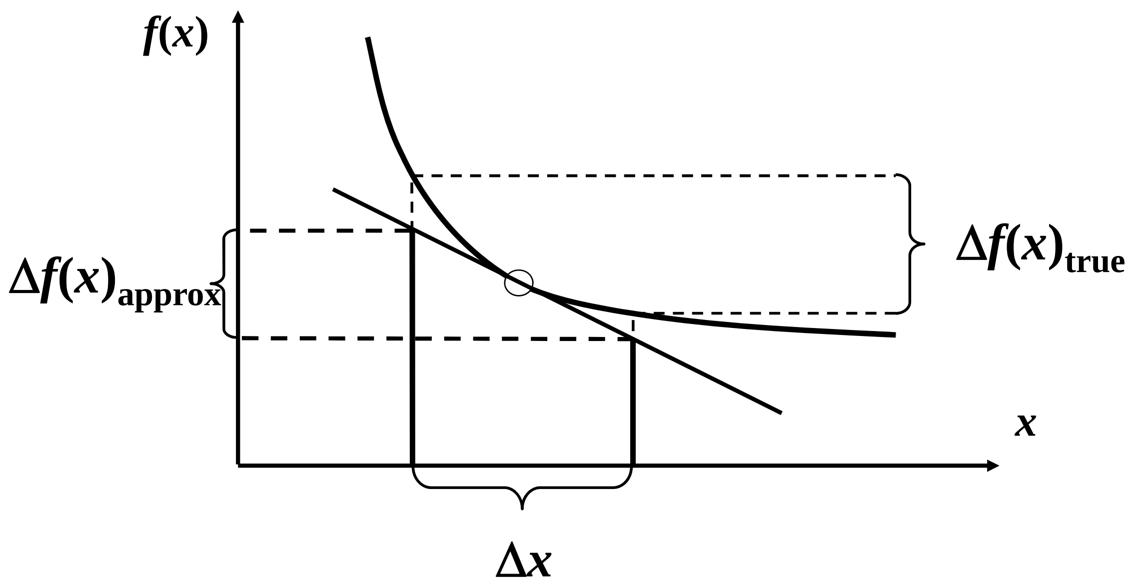


CENTERED SECOND DIVIDED DIFFERENCE

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)$$

$$f''(x_i) \cong \frac{\dfrac{f(x_{i+1}) - f(x_i)}{h} - \dfrac{f(x_i) - f(x_{i-1})}{h}}{h}$$

ERROR PROPAGATION



$$\Delta f(x) = |f'(x)|\Delta x$$ ⇒ **FIRST-ORDER ERROR PROPAGATION**

SECOND-ORDER TS

$$f(x_{i+1}, y_{i+1}) = f(x_i, y_i)$$

$$+\frac{\partial f}{\partial x}(x_{i+1} - x_i) + \frac{\partial f}{\partial y}(y_{i+1} - y_i)$$

$$+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x_{i+1} - x_i)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_{i+1} - x_i)(y_{i+1} - y_i) + \frac{\partial^2 f}{\partial y^2}(y_{i+1} - y_i)^2\right]$$

$$+ \circ \circ \circ$$

MULTI-DIMENSIONAL, FIRST-ORDER ERROR PROPAGATION

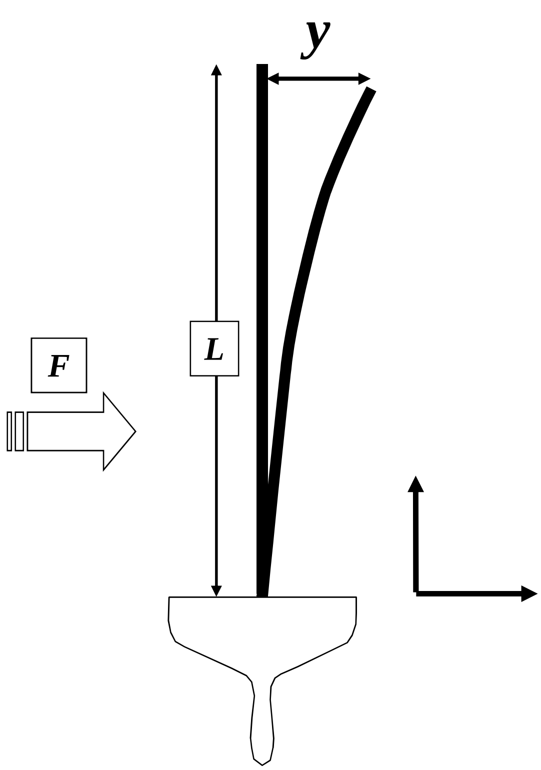
$$f(x_{i+1}, y_{i+1}) = f(x_i, y_i) + \frac{\partial f}{\partial x}(x_{i+1} - x_i) + \frac{\partial f}{\partial y}(y_{i+1} - y_i)$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x_{i+1} - x_i)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_{i+1} - x_i)(y_{i+1} - y_i) + \frac{\partial^2 f}{\partial y^2}(y_{i+1} - y_i)^2\right] + \bullet\bullet\bullet$$

$$\Delta f(x,y) = \left|\frac{\partial f}{\partial x}\right|\Delta x + \left|\frac{\partial f}{\partial y}\right|\Delta y$$

$$\Delta f(x_1, x_2, \ldots, x_n) \cong \left|\frac{\partial f}{\partial x_1}\right|\Delta x_1 + \left|\frac{\partial f}{\partial x_2}\right|\Delta x_2 + \cdots + \left|\frac{\partial f}{\partial x_n}\right|\Delta x_n$$

DEFLECTION OF A SAILBOAT MAST



Wind Force

Length

$$y = \frac{FL^4}{8\,EI}$$

Modulus Of Elasticity

Moment Of Inertia

EXAMPLE 4.6 Error Propagation in a Multivariable Function

Problem Statement. The deflection y of the top of a sailboat mast is

$$y = \frac{FL^4}{8EI}$$

where F = a uniform side loading (lb/ft), L = height (ft), E = the modulus of elasticity (lb/ft^2), and I = the moment of inertia (ft^4). Estimate the error in y given the following data:

$\tilde{F} = 50$ lb/ft $\qquad \Delta\tilde{F} = 2$ lb/ft

$\tilde{L} = 30$ ft $\qquad \Delta\tilde{L} = 0.1$ ft

$\tilde{E} = 1.5 \times 10^8$ lb/ft^2 $\qquad \Delta\tilde{E} = 0.01 \times 10^8$ lb/ft^2

$\tilde{I} = 0.06$ ft^4 $\qquad \Delta\tilde{I} = 0.0006$ ft^4

Solution. Employing Eq. (4.27) gives

$$\Delta y(\tilde{F}, \tilde{L}, \tilde{E}, \tilde{I}) = \left|\frac{\partial y}{\partial F}\right| \Delta\tilde{F} + \left|\frac{\partial y}{\partial L}\right| \Delta\tilde{L} + \left|\frac{\partial y}{\partial E}\right| \Delta\tilde{E} + \left|\frac{\partial y}{\partial I}\right| \Delta\tilde{I}$$

or

$$\Delta y(\tilde{F}, \tilde{L}, \tilde{E}, \tilde{I}) \cong \frac{\tilde{L}^4}{8\tilde{E}\tilde{I}}\Delta\tilde{F} + \frac{\tilde{F}\tilde{L}^3}{2\tilde{E}\tilde{I}}\Delta\tilde{L} + \frac{\tilde{F}\tilde{L}^4}{8\tilde{E}^2\tilde{I}}\Delta\tilde{E} + \frac{\tilde{F}\tilde{L}^4}{8\tilde{E}\tilde{I}^2}\Delta\tilde{I}$$

$$\Delta y = 0.0225 + 0.0075 + 0.00375 + 0.005625 = 0.039375$$

$$y = 0.5625 \pm 0.039375 \text{ ft} \quad (\pm 7\%)$$