Experiment Spring Pendulum

Tasks

1) Record the spring characteristic.
2) Determine the spring constant of a parallel and a serial connection.
3) Determine the spring constant by measuring the oscillation time.

1 Theory

1.1 Hooke's Law

Pulling with a force F on a spring elongates the spring by an elongation Δl. The elongation Δl is proportional to the applied force F. This relation is expressed by Hooke's law:

$$\boxed{F = k\Delta l. \tag{1}}$$

The constant of proportionality k is the spring constant, that can be determined by measuring the elongations for given forces. The linear behaviour of the spring described by Hooke's law is only valid in certain limits, if the spring is for example overstretched the relation between force and elongation is no longer linear.

Several springs with spring constants $k_1, k_2, \ldots, k_n$ can be combined in two elementary ways (see figure 1). On the one hand side it is possible to combine the springs in a parallel connection. Then the force F that pulls on all springs is distributed on the single springs and the elongation Δl of the springs is the same for all springs, i.e.

$$F = F_1 + F_2 + \cdots + F_n \tag{2}$$

$$k_{\text{tot}}\Delta l = k_1\Delta l + k_2\Delta l + \cdots + k_n\Delta l. \tag{3}$$

The sping constant of the parallel connection is then calculated with (3) as:

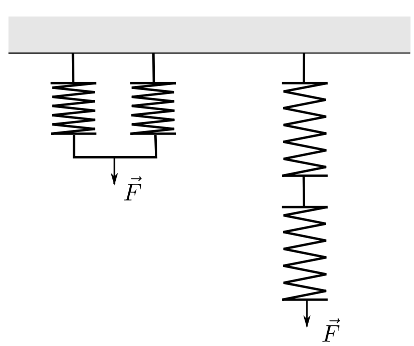

Figure 1: Parallel and serial connection of springs.

$$k_{\text{tot}} = k_1 + k_2 + \cdots + k_n \qquad \text{(parallel connection).} \tag{4}$$

The second elementary way to combine springs is a serial connection. The elongation of the serial connection is now the sum of the single elongations, while the force F pulling at the serial connection is pulling at each single spring (neglecting the weight of the single springs):

$$\Delta l_{\text{tot}} = \Delta l_1 + \Delta l_2 + \cdots + \Delta l_n \tag{5}$$

$$\frac{F}{k_{\text{tot}}} = \frac{F}{k_1} + \frac{F}{k_2} + \cdots + \frac{F}{k_n}. \tag{6}$$

Then follows with (6) for the spring constant of a serial connection:

$$\frac{1}{k_{\text{tot}}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} \qquad \text{(serial connection).} \tag{7}$$

1.2 Linear oscillations

If a mass m is attached to a spring with spring constant k, the spring is elongated by the weight force $F_{\text{G}} = mg$. The elongation is calcualted with (1) as $\Delta l = {}^{mg}/_{k}$ and the corresponding position of the mass is the equilibrium position of the spring-mass system. This position is chosen as the zero position of the y-axis (see figure 2).

Deflecting the mass in y-direction and releasing it, leads to an oscillation of the mass around its equilibrium position. In order to calculate the oscillation time of this oscillation, one considers the forces acting on the mass. Since it is a one-dimensional system, it suffices to consider the y-components of the force vectors.

In the equilibrium position $y = 0$ the weight force pointing downwards and of the spring force pointing upwards, just compensate, i.e. there is no resulting force in this position: $F_{\text{s}} + F_{\text{G}} = 0$. Lifting the mass on a position $y > 0$ reduces the tension of the spring and the spring force is reduced by ky. As a consequence the weight force is no longer compensated completely, so that there is a resulting force $F_{\text{s}} + F_{\text{G}} = -ky$ pointing downwards. Pulling the mass downwards on a negative position $y < 0$ leads on the other hand side to an increase of the spring force by $-ky$ (negative sign, because $y < 0$, but force pointing upwards). Again weight force and spring force are not completely compensated and one has a resulting force $F_{\text{s}} + F_{\text{G}} = -ky$ pointing upwards (see figure 2). So in all three cases $y = 0$, $y > 0$ and $y < 0$ the mass in position y is affected by a force

$$F_{\text{s}} + F_{\text{G}} = -ky. \tag{8}$$

This force is according to Newton's second law equal to the inertial force ma, where a is the acceleration of the mass in y-direction, i.e. $a(t) = \ddot{y}(t) = \frac{d^2}{dt^2}y(t)$. Equating the force acting on the mass and the inertial force leads to the following differential equation

$$-ky = ma \tag{9}$$

$$\Leftrightarrow \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0. \tag{10}$$

This differential equation is solved by the ansatz

$$y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{11}$$

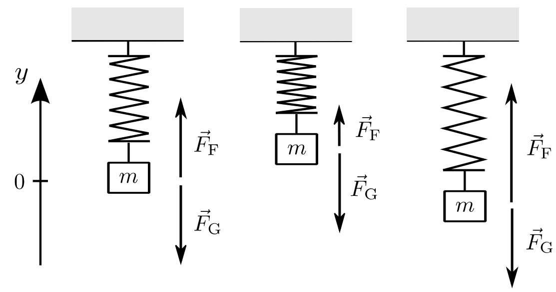

Figure 2: Forces acting on the mass during an oscillation.

where A and B are parameters. Inserting the ansatz in the differential equation gives $-\omega_0^2 y(t) + \frac{k}{m} y(t) = 0$, so that one obtains for the angular frequency $\omega_0 = \sqrt{\frac{k}{m}}$ and for the oscillation time $T_0 = \frac{2\pi}{\omega_0}$ follows:

$$\boxed{T_0 = 2\pi \sqrt{\frac{m}{k}}.} \tag{12}$$

The two parameters A and B in (11) can be calculated using the initial conditions. In the case where the mass is first lifted to an initial position y_0 and then released, the initial conditions are given by

$$y(t=0) = y_0 \qquad \text{and} \qquad \dot{y}(t=0) = 0. \tag{13}$$

Inserting (11) in the two initial conditions leads to two equations that allow to determine the parameters as $A = y_0$ and $B = 0$, so that the oscillation is in this case described by

$$y(t) = y_0 \cos(\omega_0 t). \tag{14}$$

In the above calculation the mass of the spring itself was neglected, because the mass of the spring is usually much smaller than the mass attached to the spring. If the attached mass and the mass of the spring are of comparable size, the mass of the spring m_{s} can also contribute to the resulting oscillation time.

The non-neglectable mass of the spring is taken into account by considering the movement of the spring during the oscillation. But the single sections of the spring move in different ways: The upper section of the spring, that is directly at the suspension of the spring does not move at all, while the lowest section of the spring is moving just as the attached mass up and down. For this reason a spring of length l and mass m_{s} is divided into infinitesimal sections of mass dm_{s}. Such a section with distance s to the suspension has a velocity

$$v_{\mathrm{s}}(t) = \frac{s}{l} \dot{y}(t), \tag{15}$$

where $\dot{y}(t)$ is the velocity of the mass attached to the spring. The spring section at the upper end of the spring with $s = 0$ has always the velocity zero, while the spring section at the lower end of the spring with $s = l$ has a velocity equal to the velocity of the attached mass.

The mass dm_{s} of a spring section can be described by introducing a linear mass density $\rho_l = {}^{m_{\mathrm{s}}}/_{l}$, so that

$$dm_{\mathrm{s}} = \frac{m_{\mathrm{s}}}{l} ds, \tag{16}$$

where ds is the length of a spring section. The kinetic energy of the spring section can then be calculated with (15) and (16) as

$$dE_{\mathrm{kin,s}} \;=\; \frac{1}{2} dm_{\mathrm{s}} v_{\mathrm{s}}^2 \;=\; \frac{1}{2} m_{\mathrm{s}} \dot{y}^2 \frac{1}{l^3}\, s^2\, ds \tag{17}$$

and the total kinetic energy of the spring is obtained by integrating over the length of the spring:

$$E_{\mathrm{kin,s}} \;=\; \frac{1}{2} m_{\mathrm{s}} \dot{y}^2 \frac{1}{l^3} \int\limits_0^l s^2\, ds \;=\; \frac{1}{2} \frac{m_{\mathrm{s}}}{3} \dot{y}^2. \tag{18}$$

The total energy of the spring pendulum is the sum of the kinetic energy of the spring $E_{\mathrm{kin,s}}$, the kinetic energy of the mass $E_{\mathrm{kin},m} = \frac{1}{2} m \dot{y}^2$ and its potential energy, that is given by

$$E_{\mathrm{pot},m} = -\int\limits_0^y F(y')\, dy' = -\int\limits_0^y (-ky')\, dy' = \frac{1}{2} k y^2, \tag{19}$$

where (8) was used for the force. The total energy of the spring pendulum is then

$$E_{\mathrm{tot}} \;=\; E_{\mathrm{kin},m} + E_{\mathrm{kin,s}} + E_{\mathrm{pot},m} \;=\; \frac{1}{2}\left(m + \frac{m_{\mathrm{s}}}{3}\right) \dot{y}^2 + \frac{1}{2} k y^2. \tag{20}$$

Conservation of energy means

$$0 \;=\; \frac{d}{dt} E_{\mathrm{tot}} \;=\; \frac{1}{2}\left(m + \frac{m_{\mathrm{s}}}{3}\right) 2\dot{y}\ddot{y} + \frac{1}{2} k 2 y \dot{y}, \tag{21}$$

which leads to the following differential equation

$$\frac{d^2 y}{dt^2} + \frac{k}{m + \frac{m_{\mathrm{s}}}{3}}\, y = 0. \tag{22}$$

Comparing this result with (10) shows that m was substituted by $m + \frac{m_{\mathrm{s}}}{3}$. The solution of the differential equation leads then in analogy to the calculation done before to the following oscillation time

$$\boxed{T = 2\pi \sqrt{\frac{m + \frac{m_{\mathrm{s}}}{3}}{k}}.} \tag{23}$$

2 Measurement Procedure

1) Determination of the spring constant

- ☐ Hang the first spring to the hook and position the upper index at the height of the lower end of the spring. Measure the height of the lower end of the spring.
- ☐ Put four 10 g weights on the weight holder and attach it to the first spring. Measure then the height of the lower end of the spring. Increase the the mass in steps of 50 g and measure the corresponding heights.
- ☐ Repeat these measurements with the other springs.

Evaluation:

- ☐ Calculate for all masses the corresponding weight force and elongation Δl.
- ☐ Create a coordinate system with the elongation on the horizontal axis and the weight force on the vertical axis. Plot the data points including the measurement uncertainties as error bars into the coordinate system. Use for the data points of the different springs different symbols and different colors.
- ☐ Use MATLAB to calculate for the data points of each spring a regression line. Plot the three regression lines into the coordinate system with the data points.

Answer the following question:

- What are the spring constants of the different springs (including their measurement uncertainties)?

2) Determination of the spring constant for a serial and a parallel connection

- ☐ Choose two springs out of the three springs you measured in 1). Hang the two springs in a serial connection to the hook and do the same measurement as in 1) now for the serial connection of the springs.
- ☐ Now choose one of the three springs measured in 1) and hang two of these springs in a parallel connection to the hook and do the same measurement as in 1) now for the parallel connection of the springs.

Evaluation:

- ☐ Repeat the evaluation of 1) for the serial and parallel connections.

Answer the following question:

- Do the measured spring constants of the serial and the parallel connection agree with the values one obtains using (4) and (7) with the values of the spring constants measured in 1)?

3) Measurement of the oscillation time for different amplitudes

- ☐ Choose the weakest of the three springs you measured in 1) and attach 200 g to the spring.
- ☐ Set the lower index to the lower end of the mass in the equilibrium position and position the upper index 2 cm above the lower index. Lift the mass then by 2 cm up to the upper index, release it and measure the time of ten oscillations.
- ☐ Position then the upper index 4 cm above the lower index, lift then the weight by 4 cm up to the upper index and release it. Measure again ten oscillations.
- ☐ Repeat this measurement for 6 cm up to 12 cm in steps of 2 cm.

Evaluation:

- ☐ Calculate for each triple of measurements the mean and the measurement uncertainty of the oscillation time.
- ☐ Create a coordinate system with the amplitude on the horizontal axis and the oscillation time on the vertical axis. Plot the data points with their measurement uncertainties as error bars into the coordinate system.

☐ Use MATLAB to calculate for the data points of each spring a regression line.

Answer the following question:

- Does the oscillation time depend on the amplitude?

4) Determination of the spring constant using the oscillation time

☐ Hang the first spring you measured in 1) to the hook. Put four 10 g weights and on the weight holder and attach the weight holder to the spring.

☐ Measure three times the oscillation time of ten oscillations.

☐ Increase the mass in steps of 50 g and measure for each weight three times the time of ten oscillations.

☐ Repeat the measurement for the two other springs.

Evaluation:

☐ Calculate for each triple of measurements the mean and the measurement uncertainty of the oscillation time.

☐ Create a coordinate system with the square root of the mass on the horizontal axis and the oscillation time on the vertical axis. Plot the data points with their measurement uncertainties as error bars into the coordinate system. Use for the data points of the different springs different symbols and different colors.

☐ Use MATLAB to calculate for the data points of each spring a regression line. Plot the three regression lines into the coordinate system with the data points.

Answer the following question:

- What are the spring constants you obtain now? Compare them with the spring constants measured in 1) and explain any differences.

5) Measurement of the oscillation time for a non-neglectable mass of the spring

☐ Measure the mass of the soft spring ($k = 3$ N/m). Measure three times the oscillation time of ten oscillations for a attached mass of 20 g and 30 g correspondingly.

Evaluation:

☐ Calculate the mean value of the oscillation times and then the ratio $T_{20\,\mathrm{g}}/T_{30\,\mathrm{g}}$ of the two mean values.

☐ Calculate $T_{20\,\mathrm{g}}/T_{30\,\mathrm{g}}$ with the right side of (23). What value do you get for $m_s = 0$?

Answer the following question:

- Does equation (12) or equation (23) give a better agreement with the measured values?

3 Preparatory Exercises

a) Insert ansatz (11) in differential equation (10) and show that (12) follows as the result for the oscillation time.

b) Calculate A and B in (11) for the initial conditions (13).

c) Calculate the measurement uncertainty $u(T_0)$ of the oscillation time (12), if the spring constant k has a measurement uncertainty of $u(k)$.

Setup No.	Names:	Date:
		Sign. of Supervisor:

1) Determination of the spring constant

Measurement uncertainty of the elongation $u(\Delta l)$=___________________

Spring 1 (l =______ cm, colour:______)							
mass	0 g	50 g	100 g	150 g	200 g	250 g	300 g
height in cm							

Spring 2 (l =______ cm, colour:______)							
mass	0 g	50 g	100 g	150 g	200 g	250 g	300 g
height in cm							

Spring 3 (l =______, colour:______ cm)							
mass	0 g	50 g	100 g	150 g	200 g	250 g	300 g
height in cm							

2) Determination of the spring constant for a serial and a parallel connection

Serial connection with spring ___ (l =___ cm, colour:_____) and spring ___ (l =____ cm, colour:_____)							
mass	0 g	50 g	100 g	150 g	200 g	250 g	300 g
height in cm							

Parallel connection with spring ___ (l =______ cm, colour:______)							
mass	0 g	50 g	100 g	150 g	200 g	250 g	300 g
height in cm							

sketch of the plots for 1) and 2):

slope =

formula for the plot:

3) Measurement of the oscillation time for different amplitudes

Spring ___ (l =______ cm, colour:______) with 200 g mass							
amplitude		2 cm	4 cm	6 cm	8 cm	10 cm	12 cm
$10T$ in s	1.						
	2.						
	3.						

4) Determination of the spring constant using the oscillation time

Spring 1 (l =______ cm, colour:______)							
mass		50 g	100 g	150 g	200 g	250 g	300 g
$10T$ in s	1.						
	2.						
	3.						

Spring 2 (l =______ cm, colour:______)							
mass		50 g	100 g	150 g	200 g	250 g	300 g
$10T$ in s	1.						
	2.						
	3.						

Spring 3 (l =______ cm, colour:______)							
mass		50 g	100 g	150 g	200 g	250 g	300 g
$10T$ in s	1.						
	2.						
	3.						

sketch of the plot:

slope =

formula for the plot:

5) Measurement of the oscillation time for a non-neglectable mass of the spring

Soft spring with mass m_s =________				
20 g mass	$10T$ in s			
30 g mass	$10T$ in s			

Results Experiment Spring Pendulum

1) Determination of the spring constant

- Regression lines for the three springs:

$$F_1 = (\mathtt{a}_1 \pm \mathtt{sa}_1)\Delta l + (\mathtt{b}_1 \pm \mathtt{sb}_1) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\Delta l + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$
$$F_2 = (\mathtt{a}_2 \pm \mathtt{sa}_2)\Delta l + (\mathtt{b}_2 \pm \mathtt{sb}_2) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\Delta l + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$
$$F_3 = (\mathtt{a}_3 \pm \mathtt{sa}_3)\Delta l + (\mathtt{b}_3 \pm \mathtt{sb}_3) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\Delta l + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$

- Spring constants of the three springs:

$$(k_1 \pm u(k_1)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_\_, \quad \text{colour} : \_\_\_$$
$$(k_2 \pm u(k_2)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_\_, \quad \text{colour} : \_\_\_$$
$$(k_3 \pm u(k_3)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_\_, \quad \text{colour} : \_\_\_$$

2) Determination of the spring constant for a serial and a parallel connection

- Regression line for serial and parallel connection:

$$F_s = (\mathtt{a}_s \pm \mathtt{sa}_s)\Delta l + (\mathtt{b}_s \pm \mathtt{sb}_s) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\Delta l + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$
$$F_p = (\mathtt{a}_p \pm \mathtt{sa}_p)\Delta l + (\mathtt{b}_p \pm \mathtt{sb}_p) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\Delta l + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$

- Spring constants of the serial and the parallel connection

	calculated	measured
serial connection	$k_s = \_\_\_\_\_\_ \ \_\_$	$(k_s \pm u(k_s)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_$
parallel connection	$k_p = \_\_\_\_\_\_ \ \_\_$	$(k_p \pm u(k_p)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_$

3) Measurement of the oscillation time for different amplitudes

- Regression line:

$$T(A) = (\mathtt{a} \pm \mathtt{sa})A + (\mathtt{b} \pm \mathtt{sb}) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)A + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$

4) Determination of the spring constant using the oscillation time

- Regression lines for the three springs:

$$T_1 = (\mathtt{a}_1 \pm \mathtt{sa}_1)\sqrt{m} + (\mathtt{b}_1 \pm \mathtt{sb}_1) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\sqrt{m} + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$
$$T_2 = (\mathtt{a}_2 \pm \mathtt{sa}_2)\sqrt{m} + (\mathtt{b}_2 \pm \mathtt{sb}_2) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\sqrt{m} + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$
$$T_3 = (\mathtt{a}_3 \pm \mathtt{sa}_3)\sqrt{m} + (\mathtt{b}_3 \pm \mathtt{sb}_3) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\sqrt{m} + (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)$$

- Spring constants of the three springs:

$$(k_1 \pm u(k_1)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_\_, \quad \text{colour} : \_\_\_$$
$$(k_2 \pm u(k_2)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_\_, \quad \text{colour} : \_\_\_$$
$$(k_3 \pm u(k_3)) = (\_\_\_\_\_\_ \pm \_\_\_\_\_\_)\_\_\_, \quad \text{colour} : \_\_\_$$

5) Measurement of the oscillation time for a non-neglectable mass of the spring

$$\frac{T_{m_1}}{T_{m_2}} = \_\_\_\_\_\_, \qquad \sqrt{\frac{m_1 + {}^{m_s}\!/_3}{m_2 + {}^{m_s}\!/_3}} = \_\_\_\_\_\_, \qquad \sqrt{\frac{m_1}{m_2}} = \_\_\_\_\_\_$$