Namespace Anudeep ;)

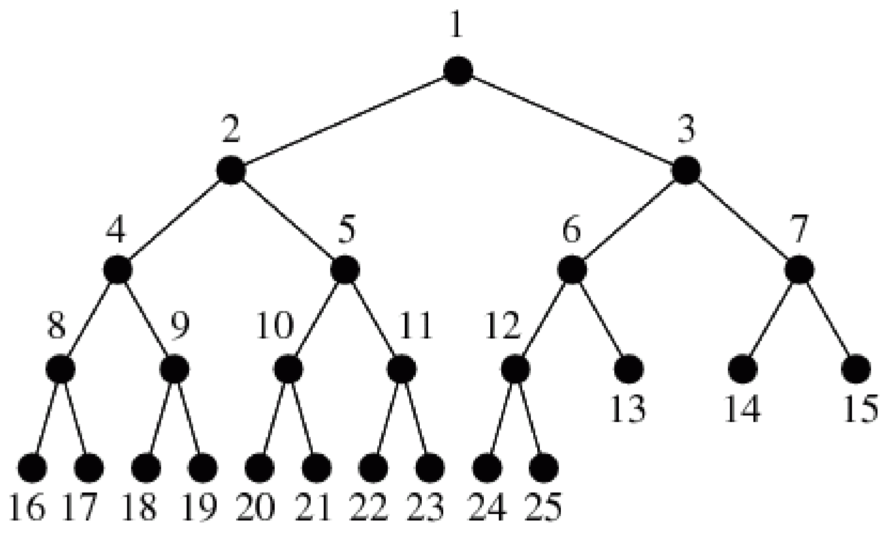
Soft copy of my thoughts and imaginations..

Heavy Light Decomposition

April 11, 2014 | anudeep2011 ⊕ Go to comments ∇ Leave a comment(38)

Long post with lot of explanation, targeting novice. If you are aware of how HLD is useful, skip to "Basic Idea".

Why a Balanced Binary Tree is good?



Balanced Binary Tree

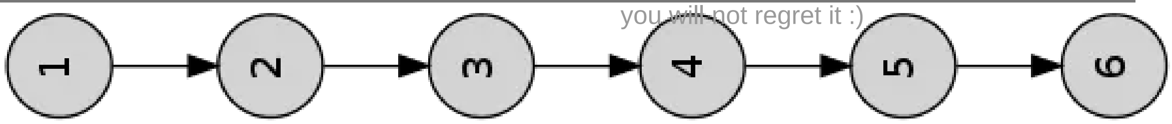
A balanced binary tree with **N** nodes has a height of **log N**. This gives us the following properties:

- You need to visit at most **log N** nodes to reach **root** node from any other node
- You need to visit at most **2 * log N** nodes to reach from any node to any other node in the tree

The **log** factor is always good in the world of competitive programming

Now, if a balanced binary tree with N nodes is given, then many queries can be done with **O(log N)** complexity. Distance of a path, Maximum/Minimum in a path, Maximum contiguous sum etc etc.

you will not regret it :)



A chain is a set of nodes connected one after another. It can be viewed as a simple array of nodes/numbers. We can do many operations on array of elements with **O(log N)** complexity using **segment tree / BIT /** other data structures. You can read more about segment trees here – A tutorial by Utkarsh .

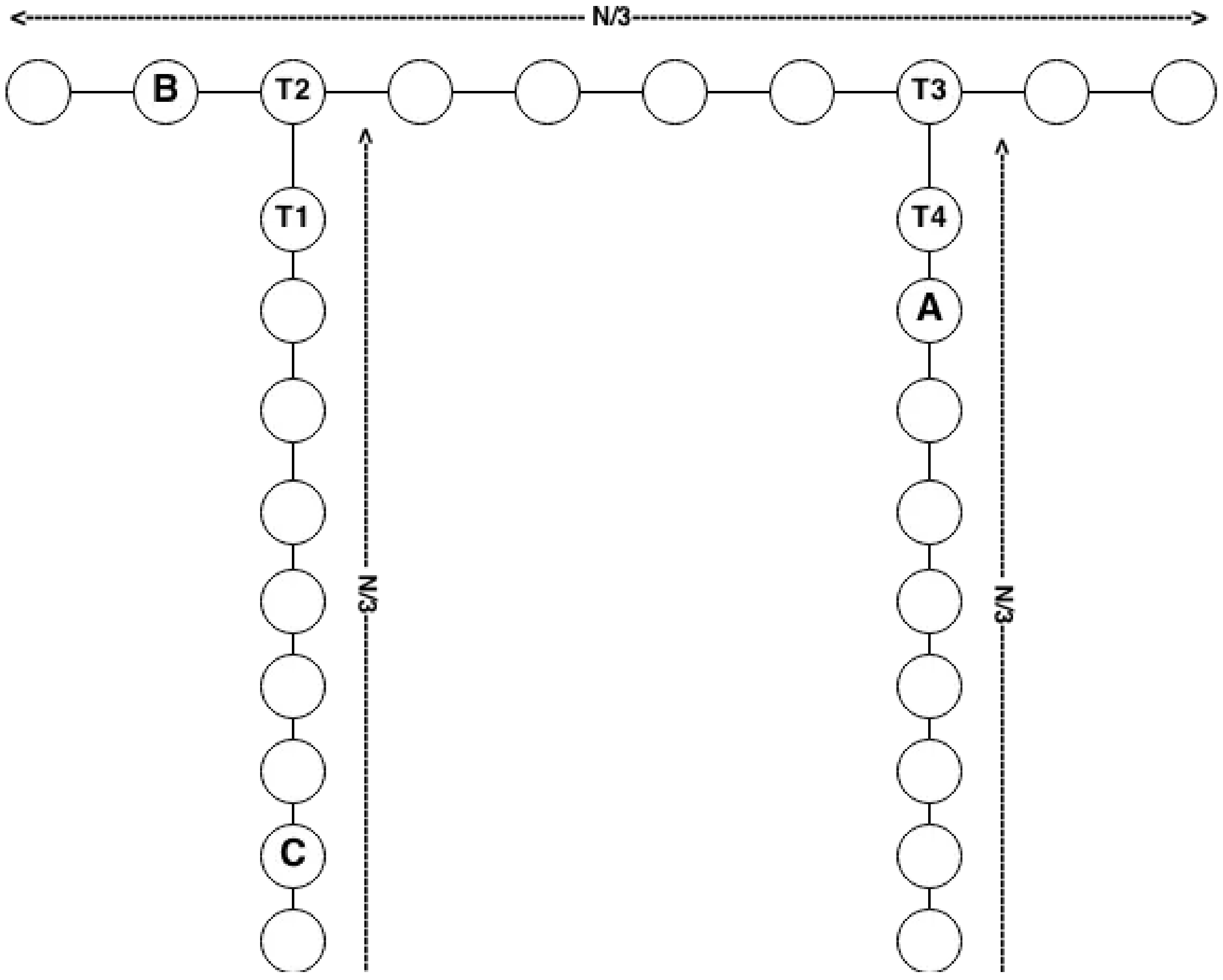
Now, we know that Balanced Binary Trees and arrays are good for computation. We can do a lot of operations with **O(log N)** complexity on both the data structures.

Why an Unbalanced Tree is bad?

Height of unbalanced tree can be arbitrary. In the worst case, we have to visit **O(N)** nodes to move from one node to another node. So Unbalanced trees are not computation friendly. We shall see how we can deal with unbalanced trees.

Consider this example..

Consider the following question: Given **A** and **B**, calculate the sum of all node values on the path from **A** to **B**.

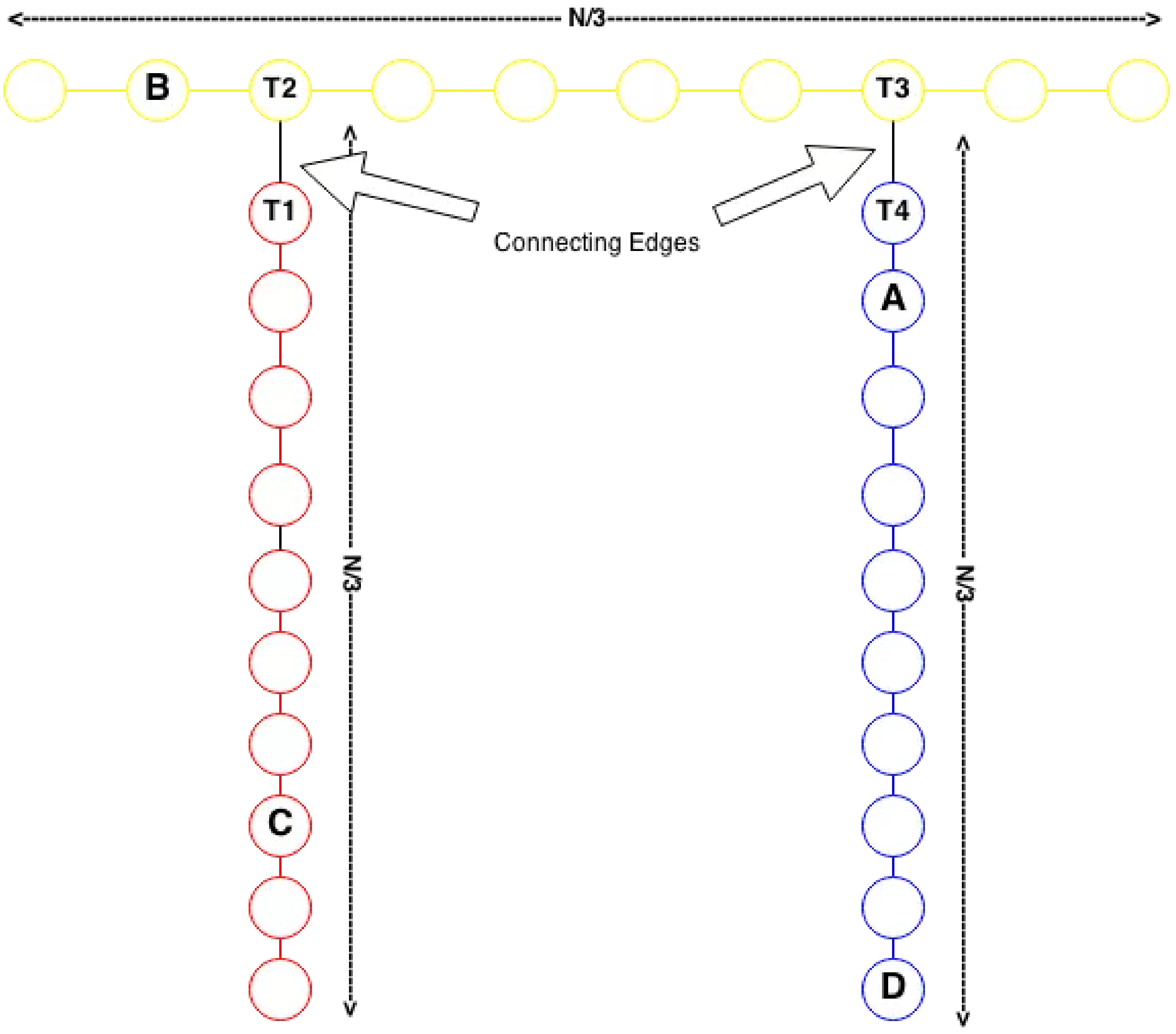


Here are details about the given images

1. The tree in the image has **N** nodes.
2. We need to visit **N/3** nodes to travel from **A** to **D**.
3. We need to visit **>N/3** nodes to travel from **B** to **D**.
4. We need to visit **>N/2** nodes to travel from **C** to **D**.

It is clear that moving from one node to another can be up to **O(N)** complexity.

This is important: What if we broke the tree in to 3 chains as shown in image below. Then we consider each chain as an independent problem. We are dealing with chains so we can use Segment Trees/other data structures that work well on linear list of data.



Here are the details after the trick

1. The tree still has **N** nodes, but it is **DECOMPOSED** into 3 chains each of size **N/3**. See 3 different colors, each one is a chain.
2. **A** and **D** belong to the same chain. We can get the required sum of node values of path from **A** to **D** in **O(log N)** time using segment tree data structure.
3. **B** belongs to yellow chain, and **D** belongs to blue chain. Path from **B** to **D** can be broken as **B** to **T3** and **T4** to **D**. Now we are dealing with 2 cases which are similar to the above case. We can calculate required sum in **O(log N)** time for **B** to **T3** and **O(log N)** time for **T4** to **D**. Great, we reduced this to **O(log N)**.
4. **C** belongs to red chain, and **D** belongs to blue chain. Path from **C** to **D** can be broken as **C** to **T1**, **T2** to **T3** and **T4** to **D**. Again we are dealing with 3 cases similar to 2nd case. So we can again do it in **O(log N)**.

Awesome!! We used concepts of **Decomposition** and **Segment Tree DS**, reduced the query complexity from **O(N)** to **O(log N)**. As I said before, competitive programmers always love the **log** factors 😀 😀

But wait the tree in the example is special, only 2 nodes had degree greater than 2. We did a simple decomposition and achieved better complexity, but in a general tree we need to do some thing little more complex to get better complexity. And that little more complex decomposition is called **Heavy Light Decomposition**.

Basic Idea

We will divide the tree into **vertex-disjoint chains** (Meaning no two chains has a node in common) in such a way that to move from **any node** in the tree to the **root** node, we will have to **change at most log N** chains. To put it in another words, the path from **any node** to **root** can be broken into pieces such that the each piece belongs to only one chain, then we will have no more than **log N** pieces.

Let us assume that the above is done, So what?. Now the path from any node **A** to any node **B** can be broken into two paths: **A** to **LCA(A, B)** and **B** to **LCA(A, B)**. Details about LCA - Click Here or Here. So at this point we need to only worry about paths of the following format: **Start at some node and go up the tree** because **A** to **LCA(A, B)** and **B** to **LCA(A, B)** are both such paths.

What are we up to till now?

- We assumed that we can break tree into chains such that we will have to **change at most log N** chains to move from any node up the tree to any other node.
- Any path can be broken into two paths such both paths start at some node and move up the tree
- We already know that queries in each chain can be answered with **O(log N)** complexity and there are at most **log N** chains we need to consider per path. So on the whole we have **O(log^2 N)** complexity solution. Great!!

Till now I have explained how **HLD** can be used to reduce complexity. Now we shall see details about how **HLD** actually decomposes the tree.

Note : My version of HLD is little different from the standard one, but still everything said above holds.

Terminology

Common tree related terminology can be found here.

Special Child : Among all child nodes of a node, the one with maximum sub-tree size is considered as **Special child**. Each non leaf node has exactly one **Special child**.

Special Edge : For each non-leaf node, the edge connecting the node with its **Special child** is considered as **Special Edge**.

Read the next 3 paras until you clearly understand every line of it, every line makes sense (Hope!). Read it 2 times 3 times 10 times 2 power 10 times .. , until you understand!!

What happens if you go to each node, find the special child and special edge and mark all special edges with green color and other edges are still black? Well, what happens is **HLD**. What would the graph look like then? Colorful yes. Not just colorful. Green edges actually forms vertex disjoint chains and black edges will be the connectors between chains. Let us explore one chain, start at root, move to the special child of root (there is only one special child, so easy pick), then to its special child and so on until you reach a leaf node, what you just traveled is a chain which starts at root node. Let us assume that root node has **m** child nodes. Note that all **m-1** normal child nodes are starting nodes of some other chains.

What happens if you move from a node to a normal child node of it. This is the **most important part**. When you move from a node to any of its normal child, the sub-tree size is at most half the sub-tree size of current node. Consider a node **X** whose sub-tree size is **s** and has **m** child nodes. If **m=1**, then the only child is special child (So there is no case of moving to normal child). For **m>=2**, sub-tree size of any normal child is **<=s/2**. To prove that, let us talk about the sub-tree size of special child. What is the least sub-tree size possible for special child? Answer is **ceil(s/m)** (what is ceil? click here). To prove it, let us assume it is less than **ceil(s/m)**. As this child is the one with maximum sub-tree size, all other normal child nodes will be at most as large as special child, **m** child nodes with each less than **ceil(s/m)** will not sum up to **s**, so with this counter-intuition. We have the following: The mininum sub-tree size possible for special child is **ceil(s/m)**. This being said, the maximum size for normal child is **s/2. So when ever you move from a node to a normal child, the sub-tree size is at most half the sub-tree size of parent node.**

We stated early that to move from **root** to any node (or viceversa) we need to change at most **log N** chains. Here is the proof; Changing a chain means we are moving for a node to a normal child, so each time we change chain we are at least halving the sub-tree size. For a tree with size **N**, we can halve it at most **log N** times (Why? Well, take a number and keep halving, let me know if it takes more than **ceil(log N**) steps).

At this point, we know what **HLD** is, we know why one has to use **HLD**, basic idea of **HLD**, terminology and proof. We shall now see implementation details of **HLD** and few related problems.

Implementation

Algorithm

```
HLD(curNode, Chain):
    Add curNode to curChain
    If curNode is LeafNode: return                    //Nothing left to do
    sc := child node with maximum sub-tree size       //sc is the special child
    HLD(sc, Chain)                                    //Extend current chain to special
    for each child node cn of curNode:                //For normal childs
        if cn != sc: HLD(cn, newChain)                //As told above, for each normal
```

Above algorithm correctly does **HLD**. But we will need bit more information when solving **HLD** related problems. We should be able to answer the following questions:

1. Given a node, to which chain does that node belong to.
2. Given a node, what is the position of that node in its chain.
3. Given a chain, what is the head of the chain
4. Given a chain, what is the length of the chain

So let us see a **C++** implementation which covers all of the above

```
int chainNo=0,chainHead[N],chainPos[N],chainInd[N],chainSize[N];
void hld(int cur) {
    if(chainHead[chainNo] == -1) chainHead[chainNo]=cur;
    chainInd[cur] = chainNo;
    chainPos[cur] = chainSize[chainNo];
    chainSize[chainNo]++;

    int ind = -1,mai = -1;
    for(int i = 0; i < adj[cur].sz; i++) {         if(subsize[ adj[cur][i] ] > mai) {
            mai = subsize[ adj[cur][i] ];
            ind = i;
        }
    }

    if(ind >= 0) hld( adj[cur][ind] );

    for(int i = 0; i < adj[cur].sz; i++) {
        if(i != ind) {
            chainNo++;
            hld( adj[cur][i] );
        }
    }
}
```

Initially all entries of chainHead[] are set to -1. So in line 3 when ever a new chain is started, chain head is correctly assigned. As we add a new node to chain, we will note its position in the chain and increment the chain length. In the first for loop (lines 9-14) we find the child node which has maximum sub-tree size. The following if condition (line 16) is failed for leaf nodes. When the if condition passes, we expand the chain to special child. In the second for loop (lines 18-23) we recursively call the function on all normal nodes. chainNo++ ensures that we are creating a new chain for each normal child.

Example

Problem : SPOJ – QTREE

Solution : Each edge has a number associated with it. Given 2 nodes **A** and **B**, we need to find the edge on path from **A** to **B** with maximum value. Clearly we can break the path into **A** to **LCA(A, B)** and **B** to **LCA(A, B)**, calculate answer for each of them and take the maximum of both. As mentioned above as the tree need not be balanced, it may take upto **O(N)** to travel from **A** to **LCA(A, B)** and find the maximum. Let us use **HLD** as detailed above to solve the problem.

Solution Link : Github – Qtree.cpp (well commented solution)
I will not explain all the functions of the solution. I will explain how query works in detail

main() : Scans the tree, calls all required functions in order.
dfs() : Helper function. Sets up depth, subsize, parent of each node.
LCA() : Returns Lowest Common Ancestor of two node
make_tree() : Segment tree construction
update_tree() : Segment tree update. Point Update
query_tree() : Segment tree query. Range Update
HLD() : Does HL-Decomposition, similar to one explained above
change() : Performs change operation as given in problem statemenm

query() : We shall see in detail about the query function.

```
int query(int u, int v) {
    int lca = LCA(u, v);
    return max( query_up(u, lca), query_up(v, lca) );
}
```

we calculate LCA(u, v). we call query_up function twice once for the path u to lca and again for the path v to lca. we take the maximum of those both as the answer.

query_up() : This is **important.** This function takes 2 nodes u, v such that v is ancestor of u. That means the path from u to v is like going up the tree. We shall see how it works.

```
int query_up(int u, int v) {
    int uchain, vchain = chainInd[v], ans = -1;

    while(1) {
        if(uchain == vchain) {
            int cur = query_tree(1, 0, ptr, posInBase[v]+1, posInBase[u]+1);
            if( cur > ans ) ans = cur;
            break;
        }
        int cur = query_tree(1, 0, ptr, posInBase[chainHead[uchain]], posInBase[u]+1);
        if( cur > ans ) ans = cur;
        u = chainHead[uchain];
        u = parent(u);
    }
    return ans;
}
```

uchain and **vchain** are chain numbers in which **u** and **v** are present respectively. We have a while loop which goes on until we move up from **u** till **v**. We have 2 cases, one case is when both **u** and **v** belong to the same chain, in this case we can query for the range between **u** and **v**. We can stop our query at this point because we reached **v**.
Second case is when **u** and **v** are in different chains. Clearly **v** is above in the tree than **u**. So we need to completely move up the chain of **u** and go to next chain above **u**. We query from **chainHead[u]** to **u**, update the answer. Now we need to change chain. Next node after current chain is the parent of **chainHead[u]**.

Following image is the same tree we used above. I explained how query works on a path from **u** to **v**.

Problems

SPOJ – QTREE - http://www.spoj.com/problems/QTREE/

SPOJ – QTREE2 - http://www.spoj.com/problems/QTREE2/

SPOJ – QTREE3 - http://www.spoj.com/problems/QTREE3/

SPOJ – QTREE4 - http://www.spoj.com/problems/QTREE4/

SPOJ – QTREE5 - http://www.spoj.com/problems/QTREE5/

SPOJ – QTREE6 - http://www.spoj.com/problems/QTREE6/

SPOJ – QTREE7 - http://www.spoj.com/problems/QTREE7/

SPOJ – COT - http://www.spoj.com/problems/COT/

SPOJ – COT2 - http://www.spoj.com/problems/COT2/

SPOJ – COT3 - http://www.spoj.com/problems/COT3/

SPOJ – GOT - http://www.spoj.com/problems/GOT/

SPOJ – GRASSPLA - http://www.spoj.com/problems/GRASSPLA/

SPOJ – GSS7 - http://www.spoj.com/problems/GSS7/

CODECHEF – GERALD2 - http://www.codechef.com/problems/GERALD2

CODECHEF – RRTREE - http://www.codechef.com/problems/RRTREE

CODECHEF – QUERY - http://www.codechef.com/problems/QUERY

CODECHEF – QTREE - http://www.codechef.com/problems/QTREE

CODECHEF – DGCD - http://www.codechef.com/problems/DGCD

CODECHEF – MONOPLOY - http://www.codechef.com/problems/MONOPLOY

All codechef problems have Editorials.

QTREE : Explained above as example

QTREE2 : Segment trees this time should support sum operation over range. There is no update operation. Note that you can also use **BIT.** For finding the Kth element on a path, we can use **LCA** style jumping, it can be done in **O(log N)**.

QTREE3 : Simple. Segment tree should answer the following: index of left most black node in a range.

QTREE4 : You need the maximum distance. Clearly we can take the left most white node and right most white node. Distance between them will be the required answer. So our segment tree should be able to answer left most white node index as well as range sum.

QTREE5 : Bit hard. Read editorial of **QTREE6** to get idea. (Correct me if I am wrong, I did not solve it yet)

QTREE6 : Editorial – Click Here

QTREE7 : Hard. Read editorial of **MONOPOLY** and **GERALD2**.

MONOPOLY : Editorial – Click Here

GERALD2 : Editorial – Click Here

The End!

Finallyyy!!! Long one!! When I was learning various algorithms and doing programming, the topic that bugged me the most was HLD, I did not understand it for a long time, also there was not any good tutorial. So I decided to start with HLD, I hope it will help a lot 🙂 This is my first attempt in writing a post/tutorial. Do share your views on it. Also I took a lot of time setting up this blog on EC2 (3 days, well I am new to this stuff), you can use contact me to write anything related to this blog, about cautions or bugs or will-do-good changes.

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38 Comments.

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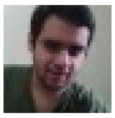

rtheman August 27, 2015 at 2:07 PM

Excellent explanation! Loved it!!

Reply

gaurav chandel April 6, 2015 at 12:01 PM

Brilliant explanation…You have done a lot of hardwork….Its easy and its fun…..Thank you very much……and please , dont stop writing such tutorials …

Reply

shubham March 22, 2015 at 11:09 AM

nice tutorial

Reply

Parshant garg March 21, 2015 at 10:57 PM

awesome tutorial. I need more tutorial on some topics trie, suffix array.

Reply

Saurabh Verma March 13, 2015 at 3:47 AM

You made it too easy for me

Reply

zuizui123 February 9, 2015 at 8:58 PM

I want more super-easy-to-understand posts like this. Thank you so much!

Reply

Nikhil Gupta January 20, 2015 at 2:43 PM

Fantastic tutorial!

Reply

Shubham January 7, 2015 at 11:57 AM

Hi, In query_up function,why in this line “query_tree(1, 0, ptr, posInBase[v]+1, posInBase[u]+1);’” posInBase[v]+1 & posInBase[u]+1 is used ?
Shouldn’t be posInBase[v] & posInBase[u]?
Thnx. for the great tutorial !!!

Reply

Aditya December 25, 2014 at 12:59 PM

Anudeep, in your explanation og QTREE4 can you what you mean by the left most and right most white nodes? In the question it is stated that the distance between nodes can be negative.

Reply

> **anudeep2011** December 28, 2014 at 8:38 AM
>
> Note that the solution is not trivial.
> We need a, b such that a and b are white and distance(a, b) is maximum.
> What ever the a, b be, for node LCA(a, b) one will be in its left subtree, other will be in its right subtree.
> For each node x, if you have the left farthest white node and right farthest white node, you can get the answer.
>
> Reply

Vladimir December 22, 2014 at 12:18 AM

“Each non leaf node has exactly one Special child.” And what if:
4-2-1
|
2-1
Who is special here?

Reply

> **anudeep2011** December 26, 2014 at 9:25 PM
>
> It can be either of them.
>
> Reply

Vivek Rai December 13, 2014 at 11:44 AM

Thats really great 🙂 =D !!!!!! Thanks !!

Reply

Tom November 29, 2014 at 3:24 AM

“The mininum sub-tree size possible for special child is ceil(s/m). This being said, the maximum size for normal child is ceil(s/m) and m>2.” – I think the first sentence is true, but the second one is not.

Reply

> **anudeep2011** December 26, 2014 at 9:38 PM
>
> Corrected 🙂
>
> Reply

Mohammad Samiul Islam November 26, 2014 at 11:58 PM

I liked the way you chained a node with random node when there was no heavy edge connected to it. Nice optimization.

Reply

James Porcelli November 23, 2014 at 8:59 AM

Thank you very much!

Reply

Sharad November 2, 2014 at 6:23 PM

Hi, anudeep i think there is some problem with the 2nd para in which you said that the sub-tree size at least becomes half and on the other hand you are saying that it’s size should be <=s/2 which indicates that instead of at-least it should be at-most which is ambiguous in your description .

Reply

> **anudeep2011** December 26, 2014 at 9:32 PM
>
> Made it clear. Thank you!
>
> Reply

Aravind Srinivas November 2, 2014 at 1:29 AM

Nice tutorial.

Reply

Ahmed Morst October 21, 2014 at 1:34 AM

your post is really helpful thnx alot

Reply

---

sunil October 15, 2014 at 12:04 AM

anudeep can u please add a driver main for which this fuction hld will run
thankyou

Reply

---

greendragons October 9, 2014 at 6:47 AM

Really beautiful technique.. and explanation made it so easy to grab, which otherwise would have been tough to get at once… 😀
If you have time, please write about O(n) suffix array and LCP construction.. there are lots of O(nlg^2n) material on web, but for O(n) either mosly links are redirected to research papers or explained in very cryptic style..

Reply

> **anudeep2011** December 26, 2014 at 2:32 PM
>
> Thank you.
> About the O(N) suffix array,
> 1) I never tried to learn it, for competitive programming it was not worth it.
> 2) Almost always a O(N * log N) is good enough.
> I will anyway add that to my TODO list (To learn it) and then decide about writing a blog or not 😀
>
> Reply

Niloy D Rocker September 23, 2014 at 11:19 AM

Great tutorial man, you make great hard work to understand this, but make a way for us to learn the thing only in 10 mins! 😀 Hats off to you buddy! Carry on! 😀

Reply

Jules August 30, 2014 at 12:39 AM

“This being said, the maximum size for normal child is ceil(s/m) and m>2.”

This is false, right?
You can only say it's s/2 maximum, but for example if m=3, there can be one child with 5, another with 5 and the third one with 2 ; and ceil(s/m) = 4.
(What you actually proved is that not all normal child can be greater than ceil(s/m))
That being said, thank you for the tutorial ! 😃

Reply

anudeep2011 December 26, 2014 at 9:35 PM

Corrected! Thank you

Reply

Pratik kumar August 18, 2014 at 5:20 PM

Hey Anudeep nice tutorial…..easy to understand and to the point.
Please look my code for errors:
http://ideone.com/foVgD4
I used the same concept but kept edge cost at the child side of the edge so as to avoid ambiguity as you did and hence queried using the child sided node…..but still it returns sigsegv error on submission on spoj.

Reply

anudeep2011 August 19, 2014 at 2:14 AM

Looking into code and finding bug is just very hard. I would not want to do that.

Reply

Sikander August 12, 2014 at 12:40 AM

Had been postponing learning HLD since months because of it being a hard topic, learned it in less than an hour thanks to your amazing explanation.

Reply

---

virat sharma August 3, 2014 at 2:40 PM

Can u explain How can we solve http://www.spoj.com/problems/COT/ using HLD bcz i didn't figure out.

Reply

Saumye July 24, 2014 at 6:22 PM

Wow! Just landed up on your blog and it's very informative.
You explain stuff very well. Should leave Google and start conducting workshops in India.
PS. If you can please do a blog article on different uncommon data structures(not the common ones). 😃

Reply

anudeep2011 July 27, 2014 at 1:08 AM

Thank you! Yes, I always try to blog about stuff that has least resources online, or the ones that are not common with sub-continent programmers!

Reply

---

rahul kumar July 23, 2014 at 1:23 PM

hi anudeep ,thank u for this greate tutorial ..finally i learnt how the hld works…
i implemented QTREE but it gives TLE …i used similar code for HLD as of yours
here is the link https://github.com/princerk/code/blob/master/.gitignore
i uesd segment tree discusssed in utkarsh's blog and calculated LCA using HLD
(im sure it wont take more than 5 min of urs)

Reply

---

rohit verma July 14, 2014 at 2:32 AM

nice tutorial 😛

Reply

Mriganka Basu Roy Chowdhury July 13, 2014 at 8:12 PM

This is great. Yes, totally agreed, HLD has very less number of tutorials; so thanks a lot. Could you please refer me / write one (this is even better :D) tutorial on 2D segtrees? This year I went to IOITC, and there was a problem on 2D BIT; in IOI 2013 there was one on 2D seg trees, so I thought knowing 2D segtrees would help.

Reply

anudeep2011 July 13, 2014 at 8:54 PM

Thank you. I will surely consider that topic.

Reply

---

parijat July 4, 2015 at 6:35 PM

got it thanks.
amazing tutorial. please continue this good work

Reply

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