Fixed Parameter Algorithms

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Classical complexity

A brief review:

- We usually aim for **polynomial-time** algorithms: the running time is $O(n^c)$, where n is the input size.
- Classical polynomial-time algorithms: shortest path, mathching, minimum spanning tree, 2SAT, convext hull, planar drawing, linear programming, etc.
- It is unlikely that polynomial-time algorithms exist for **NP-hard** problems.
- Unfortunately, many problems of interest are NP-hard: Hamiltonian cycle, 3-coloring, 3SAT, etc.
- We expect that these problems can be solved only in exponential time (i.e., c^n).

Can we say anything nontrivial about NP-hard problems?

Parameterized complexity

Main idea: Instead of expressing the running time as a function $T(n)$ of n, we express it as a function $T(n, k)$ of the input size n and some parameter k of the input.

In other words: we do not want to be efficient on all inputs of size n, only for those where k is small.

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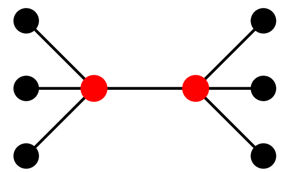
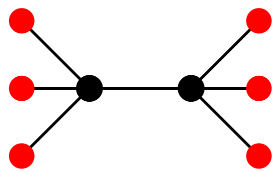
In other words: we do not want to be efficient on all inputs of size n, only for those where k is small.

What can be the parameter k?

- The size k of the solution we are looking for.
- The maximum degree of the input graph.
- The diameter of the input graph.
- The length of clauses in the input Boolean formula.
- ...

Parameterized complexity

Problem:	MINIMUM VERTEX COVER	MAXIMUM INDEPENDENT SET
Input:	Graph G, integer k	Graph G, integer k
Question:	Is it possible to cover the edges with k vertices? 	Is it possible to find k independent vertices? 
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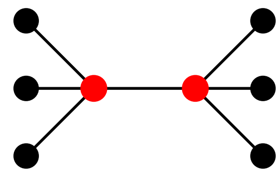
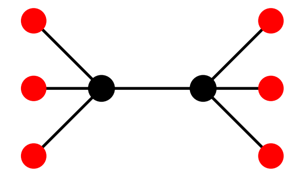
Question:	Is it possible to cover the edges with k vertices? 	Is it possible to find k independent vertices? 
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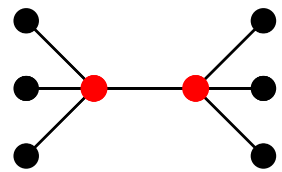
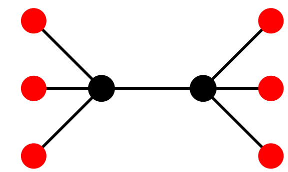
Question:	Is it possible to cover the edges with k vertices? 	Is it possible to find k independent vertices? 
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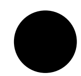
Complete enumeration:	$O(n^k)$ possibilities	$O(n^k)$ possibilities
	$O(2^k n^2)$ algorithm exists 	No $n^{o(k)}$ algorithm known 

Bounded search tree method

Algorithm for MINIMUM VERTEX COVER:

$e_1 = x_1 y_1$



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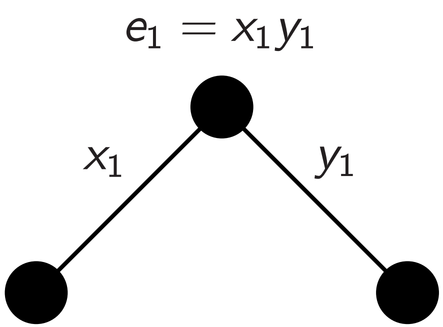

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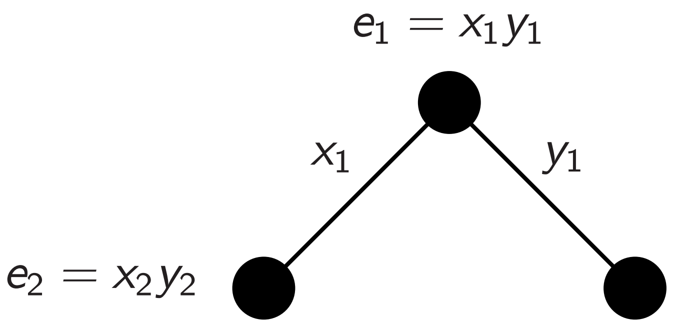

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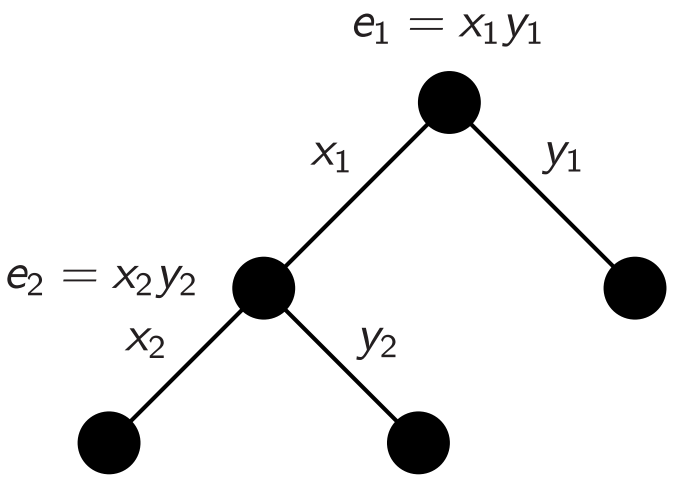

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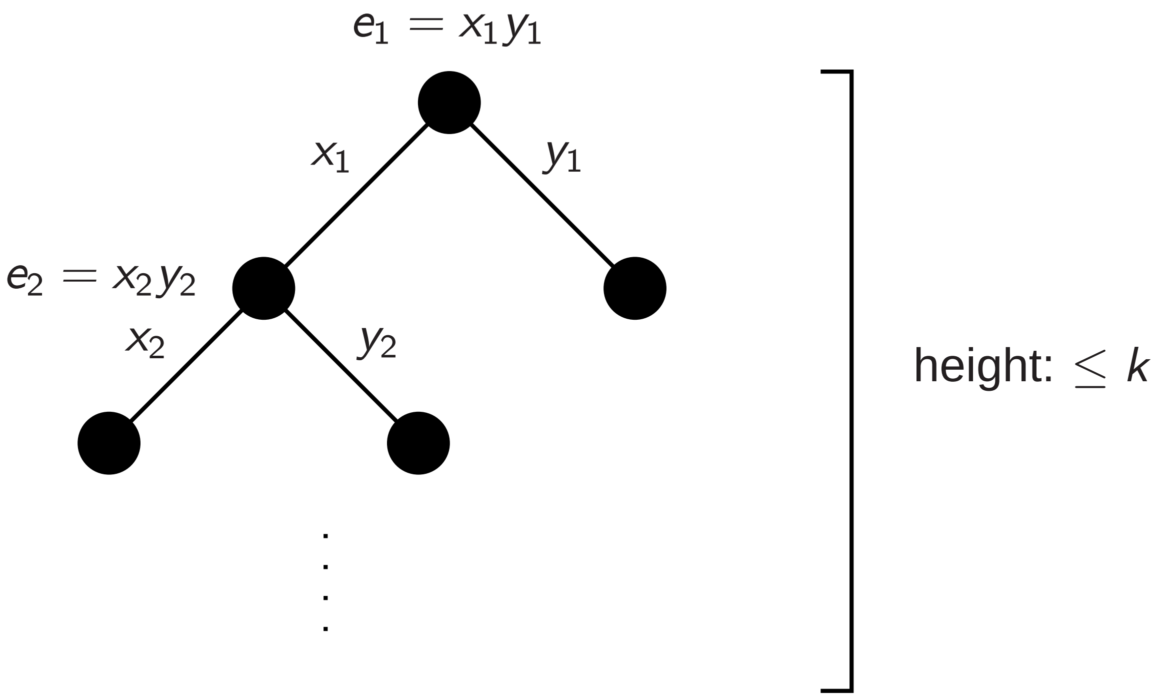

Height of the search tree is $\leq k \Rightarrow$ number of leaves is $\leq 2^k \Rightarrow$ complete search requires $2^k \cdot$ poly steps.

Fixed-parameter tractability

Definition: A **parameterization** of a decision problem is a function that assigns an integer parameter k to each input instance x.

The parameter can be

- explicit in the input (for example, if the parameter is the integer k appearing in the input (G, k) of VERTEX COVER), or
- implicit in the input (for example, if the parameter is the diameter d of the input graph G).

Main definition:

> A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

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Example: MINIMUM VERTEX COVER parameterized by the required size k is FPT: we have seen that it can be solved in time $O(2^k + n^2)$.

Better algorithms are known: e.g, $O(1.2832^k k + k|V|)$.

Main goal of parameterized complexity: to find FPT problems.

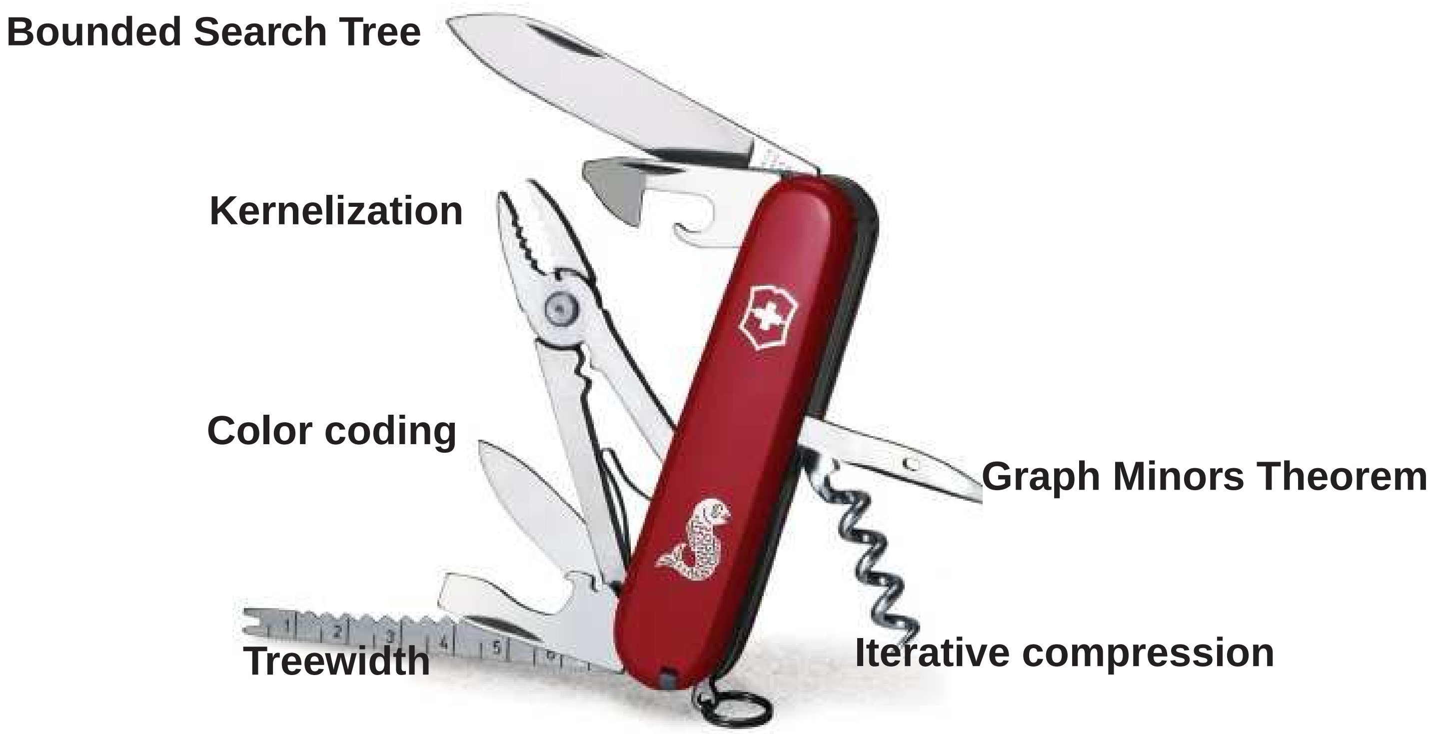
FPT problems

Examples of NP-hard problems that are FPT:

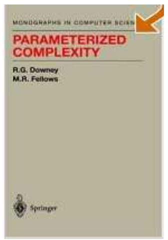
- Finding a vertex cover of size k.
- Finding a path of length k.
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossings.
- Finding disjoint paths that connect k pairs of points.
- …

FPT algorithmic techniques

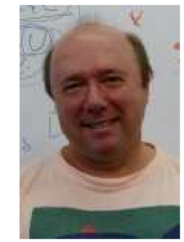
- Significant advances in the past 20 years or so (especially in recent years).
- Powerful toolbox for designing FPT algorithms:

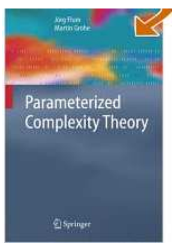


Books

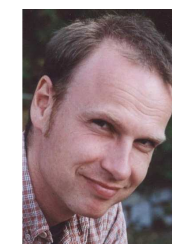


Downey-Fellows: Parameterized Complexity, Springer, 1999





Flum-Grohe: Parameterized Complexity Theory, Springer, 2006



Niedermeier: Invitation to Fixed-Parameter Algorithms, Oxford University Press, 2006.

Goals of the course

- Demonstrate techniques that were successfully used in the analysis of parameterized problems.
  - Determine quickly if a problem is FPT.
  - Design fast algorithms (improve the function $f(k)$).
- Introduce the basics of parameterized hardness theory (W[1]-hardness).

Notes

- **Warning:** The results presented for particular problems are not necessarily the best known results or the most useful approaches for these problems.
- Conventions:
  - Unless noted otherwise, k is the parameter.
  - O^* notation: $O^*(f(k))$ means $O(f(k) \cdot n^c)$ for some constant c.
  - Citations are mostly omitted (only for classical results).
  - We gloss over the difference between decision and search problems.

Kernelization

Kernelization

Definition: **Kernelization** is a polynomial-time transformation that maps an instance (I, k) to an instance (I', k') such that

- (I, k) is a yes-instance if and only if (I', k') is a yes-instance,
- $k' \leq k$, and
- $|I'| \leq f(k)$ for some function $f(k)$.

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Simple fact: If a problem has a kernelization algorithm, then it is FPT.
Proof: Solve the instance (I', k') by brute force.

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Proof: Solve the instance (I', k') by brute force.

Converse: Every FPT problem has a kernelization algorithm.
Proof: Suppose there is an $f(k)n^c$ algorithm for the problem.

- If $f(k) \leq n$, then solve the instance in time $f(k)n^c \leq n^{c+1}$, and output a trivial yes- or no-instance.
- If $n < f(k)$, then we are done: a kernel of size $f(k)$ is obtained.

Kernelization for VERTEX COVER

General strategy: We devise a list of reduction rules, and show that if none of the rules can be applied and the size of the instance is still larger than $f(k)$, then the answer is trivial.

Reduction rules for VERTEX COVER instance (G, k):

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$
Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k-1)$

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If neither Rule 1 nor Rule 2 can be applied:

- If $|V(G)| > k(k+1) \Rightarrow$ There is no solution (every vertex should be the neighbor of at least one vertex of the cover).
- Otherwise, $|V(G)| \leq k(k+1)$ and we have a $k(k+1)$ vertex kernel.

Kernelization for VERTEX COVER

Let us add a third rule:

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$

Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k-1)$

Rule 3: If $d(v) = 1$, then we can assume that its neighbor u is in the solution $\Rightarrow (G \setminus (u \cup v), k-1)$.

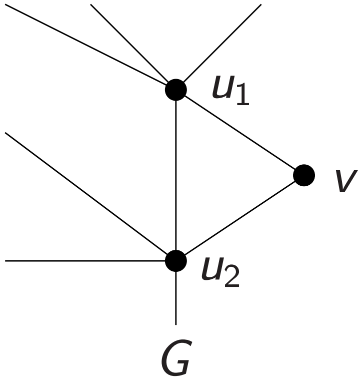
If none of the rules can be applied, then every vertex has degree at least 2.
$\Rightarrow |V(G)| \leq |E(G)|$

- If $|E(G)| > k^2 \Rightarrow$ There is no solution (each vertex of the solution can cover at most k edges).
- Otherwise, $|V(G)| \leq |E(G)| \leq k^2$ and we have a k^2 vertex kernel.

Kernelization for VERTEX COVER

Let us add a fourth rule:

Rule 4a: If v has degree 2, and its neighbors u_1 and u_2 are adjacent, then we can assume that u_1, u_2 are in the solution $\Rightarrow (G \setminus \{u_1, u_2, v\}, k-2)$.



Kernelization for VERTEX COVER

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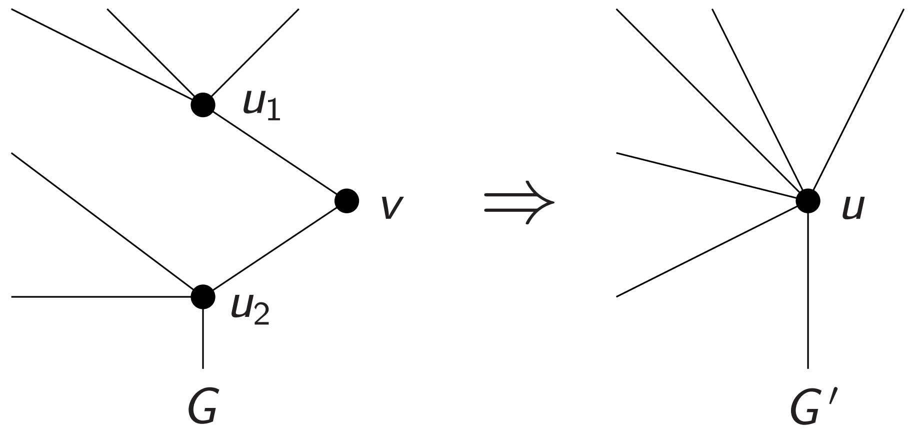

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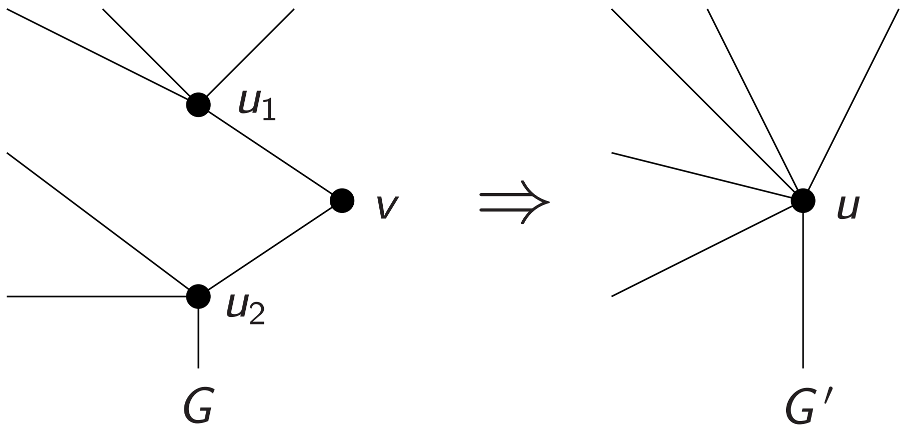

Correctness:

Let S' be a vertex cover of size $k-1$ for G'.

If $u \in S \Rightarrow (S' \setminus u) \cup \{u_1, u_2\}$ is a vertex cover of size k for G.

If $u \notin S \Rightarrow S' \cup v$ is a vertex cover of size k for G.

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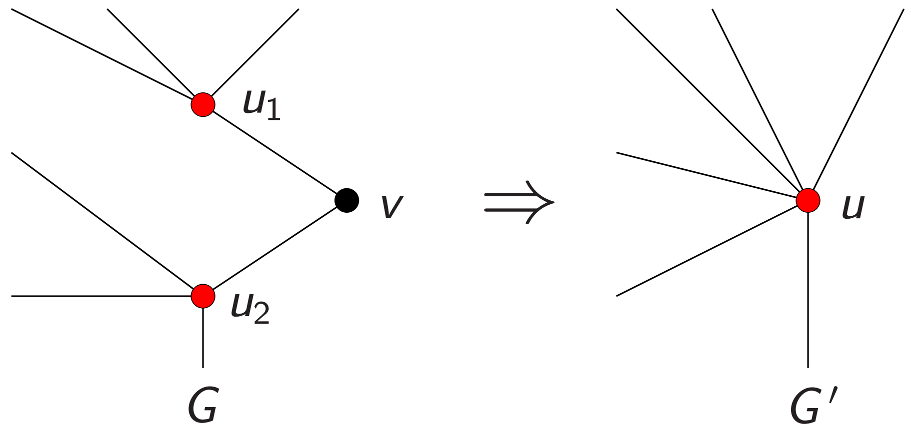

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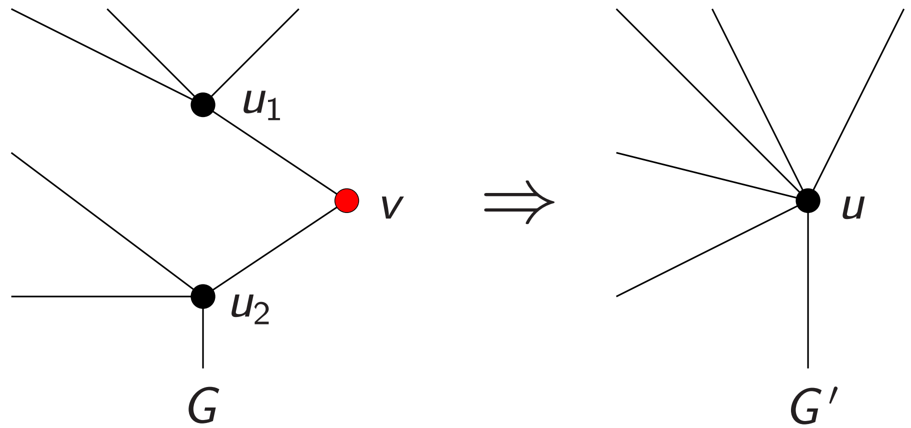

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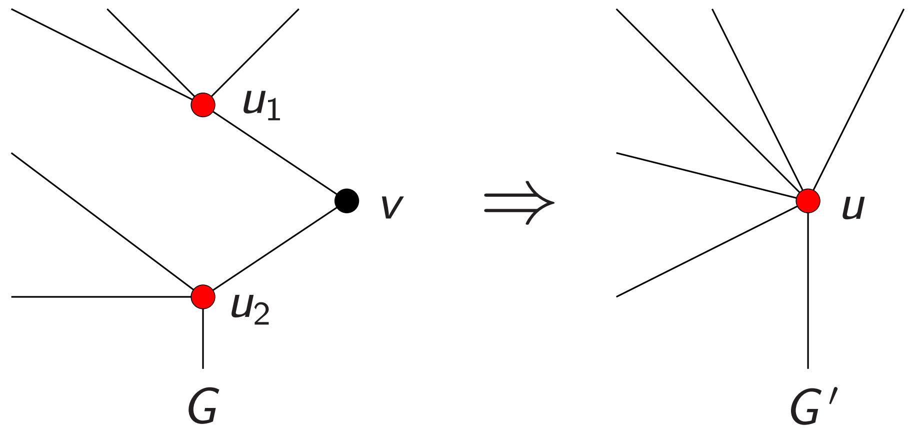

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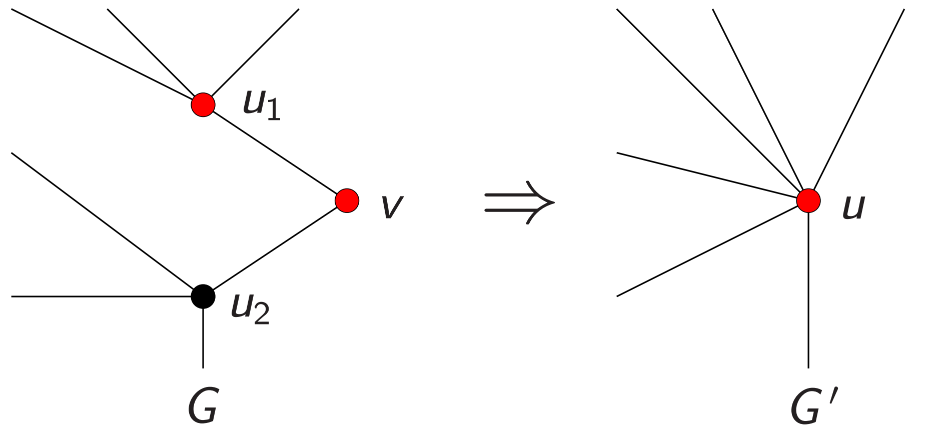

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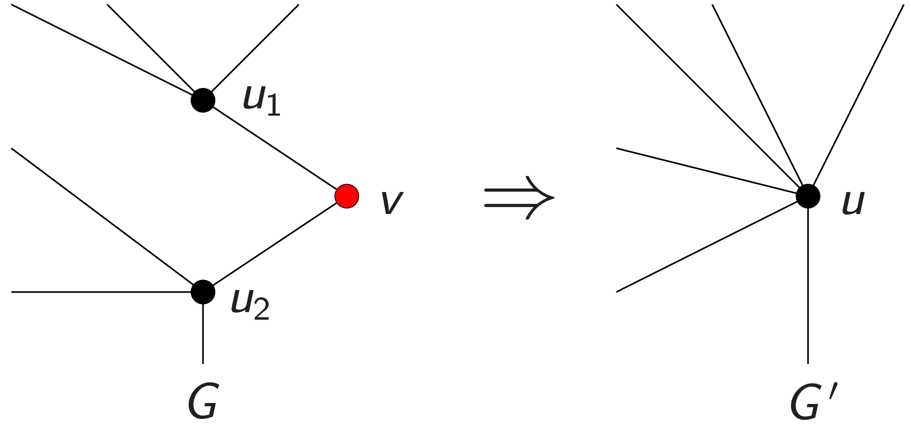

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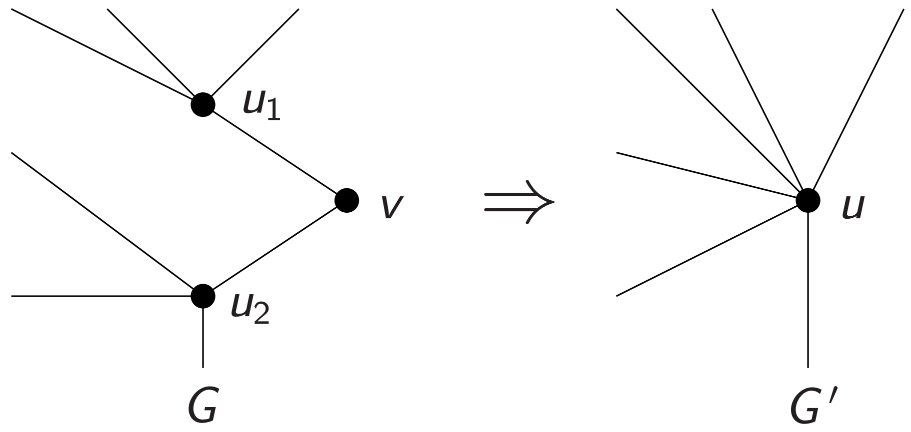
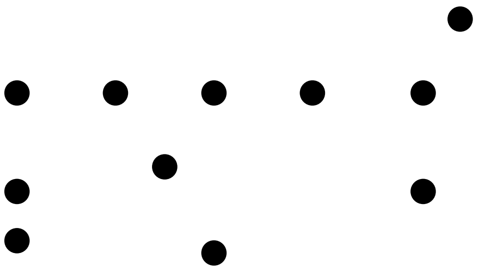


Kernel size:

- If $|E(G)| > k^2 \Rightarrow$ There is no solution (each vertex of the solution can cover at most k edges).
- Otherwise, $|V(G)| \leq 2|E(G)|/3 \leq \frac{2}{3}k^2$ and we have a $\frac{2}{3}k^2$ vertex kernel.

COVERING POINTS WITH LINES

Task: Given a set P of n points in the plane and an integer k, find k lines that cover all the points.



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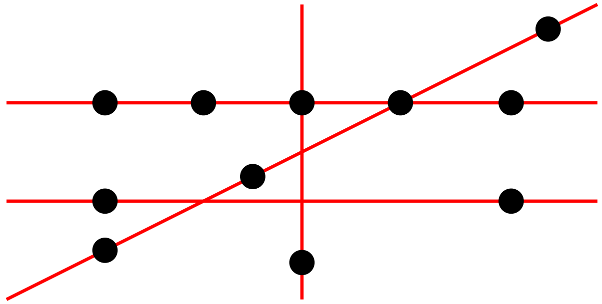

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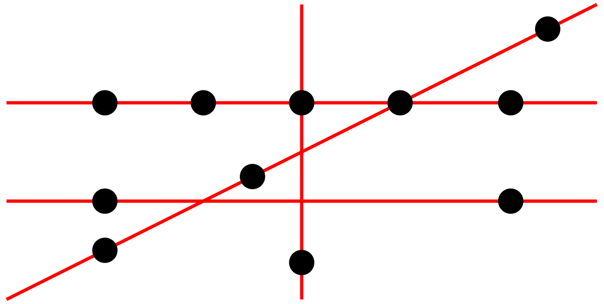

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Reduction Rule:

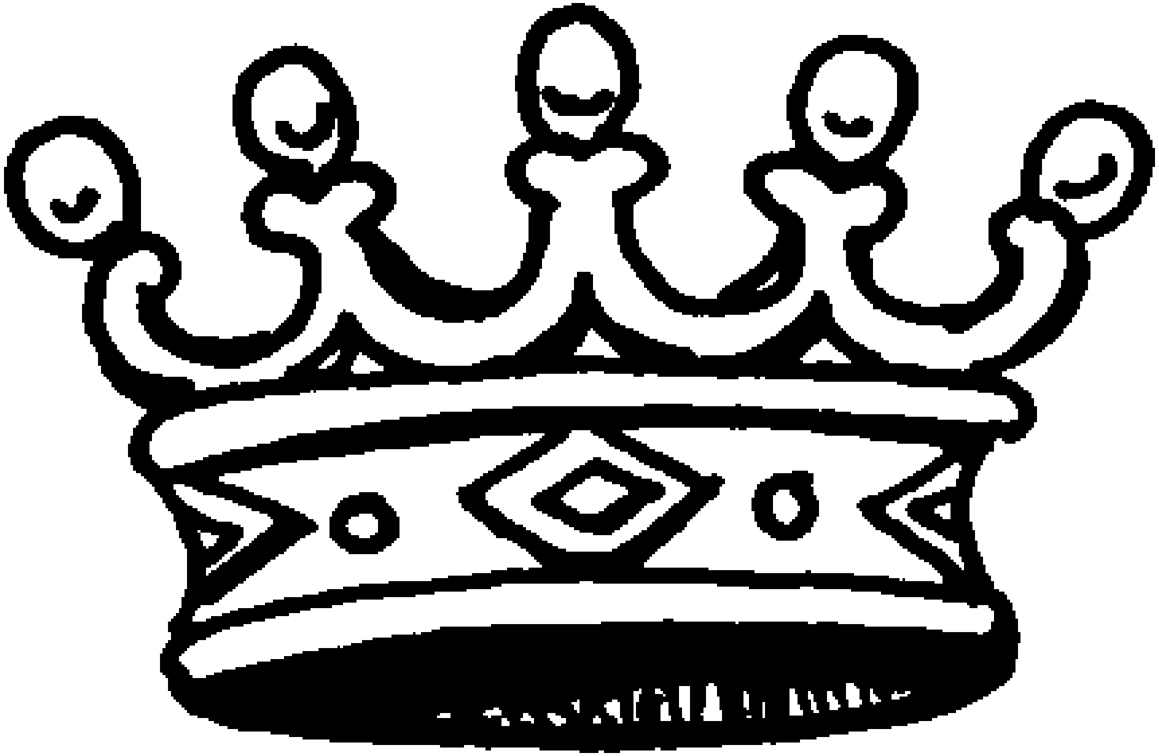
If a candidate line covers a set S of more than k points $\Rightarrow$ $(P \setminus S, k-1)$.

If this rule cannot be applied and there are still more than k^2 points, then there is no solution $\Rightarrow$ Kernel with at most k^2 points.

Kernelization

- Kernelization can be thought of as a polynomial-time preprocessing before attacking the problem with whatever method we have. “It does no harm” to try kernelization.
- Some kernelizations use lots of simple reduction rules and require a complicated analysis to bound the kernel size…
- … while other kernelizations are based on surprising nice tricks (Next: Crown Reduction and the Sunflower Lemma).
- Possibility to prove lower bounds.

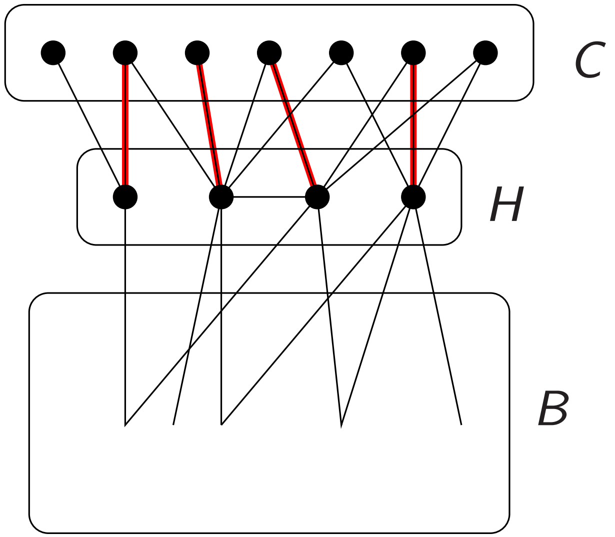
Crown Reduction



Crown Reduction

Definition: A **crown decomposition** is a partition $C \cup H \cup B$ of the vertices such that

- C is an independent set,
- there is no edge between C and B,
- there is a matching between C and H that covers H.



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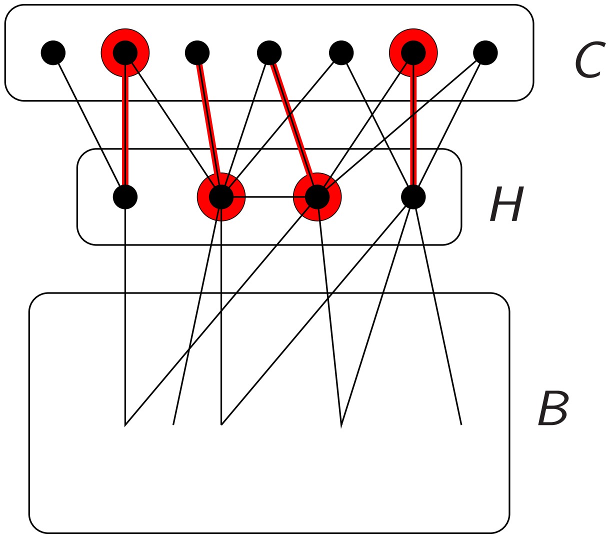

Crown rule for VERTEX COVER:

The matching needs to be covered and we can assume that it is covered by H (makes no sense to use vertices of C)

$\Rightarrow (G \setminus (H \cup C), k - |H|)$.

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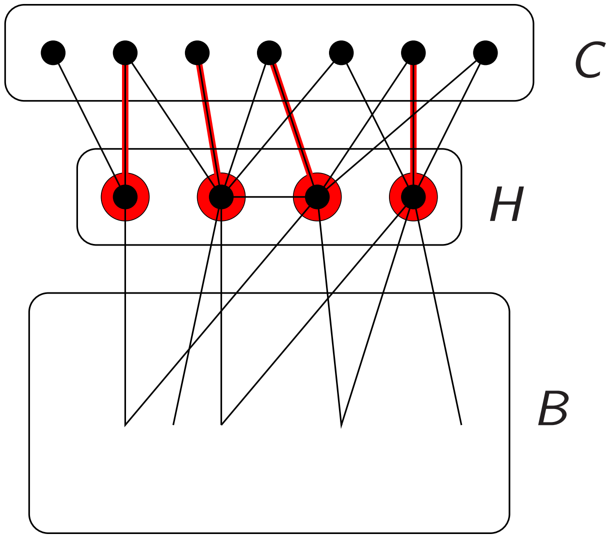

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Crown Reduction

Key lemma:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$,
- find a crown decomposition,
- or conclude that the graph has at most $3k$ vertices.

Crown Reduction

Key lemma:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$, $\Rightarrow$ No solution!
- find a crown decomposition, $\Rightarrow$ Reduce!
- or conclude that the graph has at most $3k$ vertices.

$\Rightarrow 3k$ vertex kernel!

This gives a 3k vertex kernel for VERTEX COVER.

Proof

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$,
- find a crown decomposition,
- or conclude that the graph has at most $3k$ vertices.

For the proof, we need the classical Kőnig's Theorem.

$\tau(G)$: size of the minimum vertex cover
$\nu(G)$: size of the maximum matching (independent set of edges)

Theorem: [Kőnig, 1931] If G is **bipartite,** then

$$\tau(G) = \nu(G)$$

Proof

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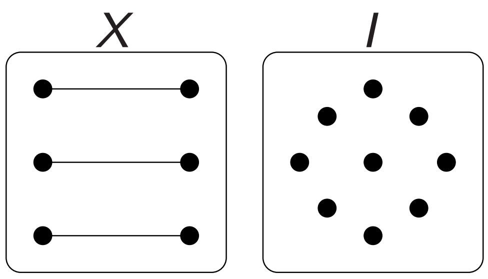
Proof: Find (greedily) a maximal matching; if its size is at least $k+1$, then we are done. The rest of the graph is an independent set I.



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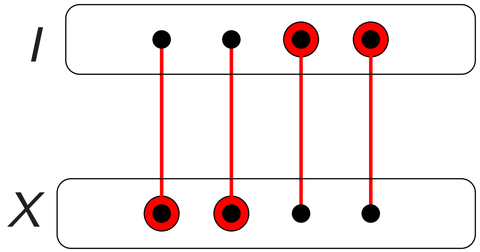

Find a maximum matching/minimum vertex cover in the bipartite graph between X and I.

Proof

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k + 1$,
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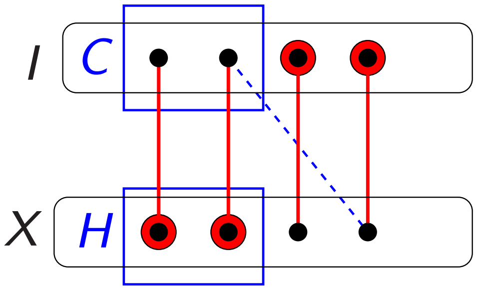
Proof:

Case 1: The minimum vertex cover contains at least one vertex of X

$\Rightarrow$ There is a crown decomposition.



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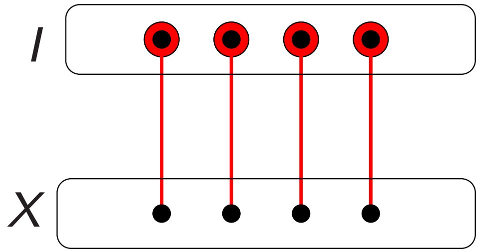

Case 2: The minimum vertex cover contains only vertices of I $\Rightarrow$ It contains every vertex of I

$\Rightarrow$ There are at most $2k + k$ vertices.

DUAL OF VERTEX COLORING

Parameteric dual of k-COLORING. Also known as SAVING k COLORS.

Task: Given a graph G and an integer k, find a vertex coloring with $|V(G)| - k$ colors.

Crown rule for DUAL OF VERTEX COLORING:

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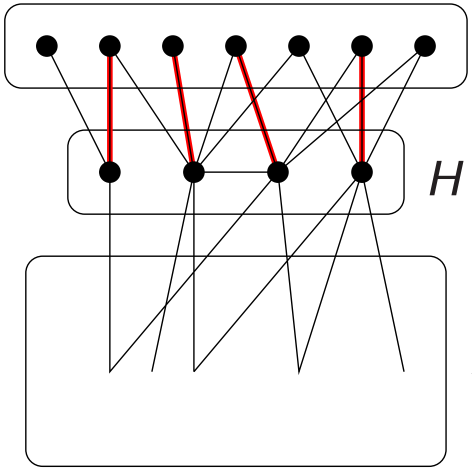
Suppose there is a crown decomposition for the **complement graph** $\overline{G}$.

- C is a clique in G: each vertex needs a distinct color.
- Because of the matching, it is possible to color H using only these $|C|$ colors.
- These colors cannot be used for B.
- $(G \setminus (H \cup C), k - |H|)$



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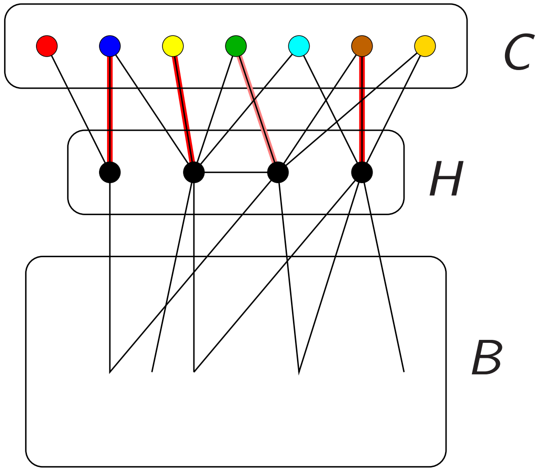

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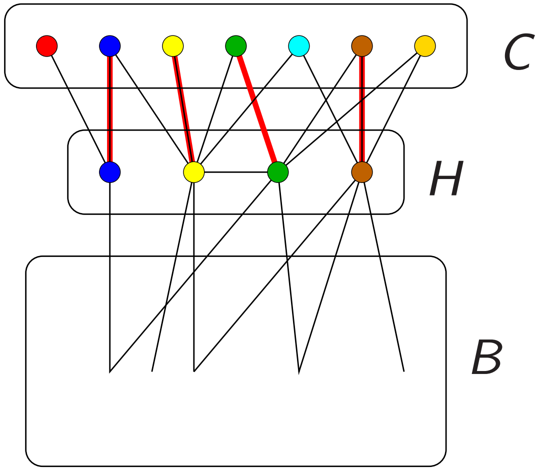


Crown Reduction for DUAL OF VERTEX COLORING

Use the key lemma for the complement $\overline{G}$ of G:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$, $\Rightarrow$ YES: we can save k colors!
- find a crown decomposition, $\Rightarrow$ Reduce!
- or conclude that the graph has at most $3k$ vertices.

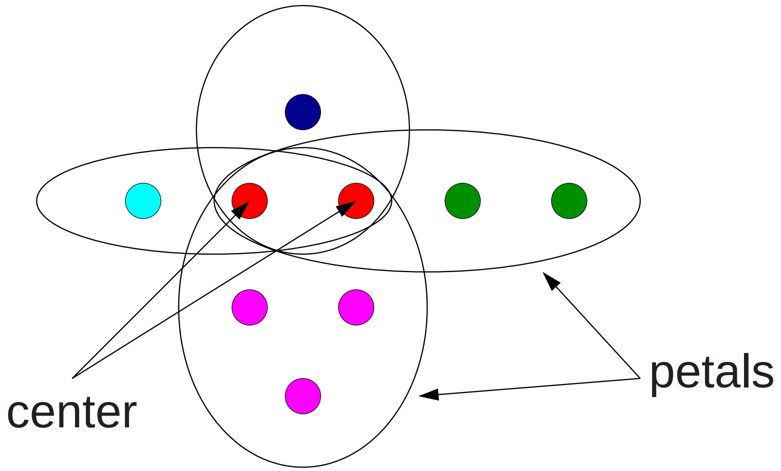
  $\Rightarrow$ $3k$ vertex kernel!

This gives a 3k vertex kernel for DUAL OF VERTEX COLORING.

Sunflower Lemma

Sunflower lemma

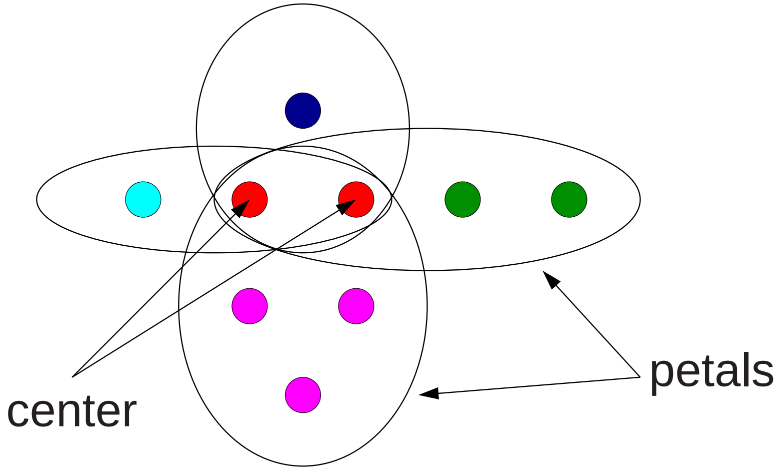
Definition: Sets S_1, S_2, ... , S_k form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \cdots \cap S_k)$ are disjoint.



Lemma: [Erdős and Rado, 1960] If the size of a set system is greater than $(p-1)^d \cdot d!$ and it contains only sets of size at most d, then the system contains a sunflower with p petals. Furthermore, in this case such a sunflower can be found in polynomial time.

Sunflowers and d-HITTING SET

d-HITTING SET: Given a collection $\mathcal{S}$ of sets of size at most d and an integer k, find a set S of k elements that intersects every set of $\mathcal{S}$.



Reduction Rule: If $k+1$ sets form a sunflower, then remove these sets from $\mathcal{S}$ and add the center C to $\mathcal{S}$ (S does not hit one of the petals, thus it has to hit the center).

Note: if the center is empty (the sets are disjoint), then there is no solution.

If the rule cannot be applied, then there are at most $O(k^d)$ sets.

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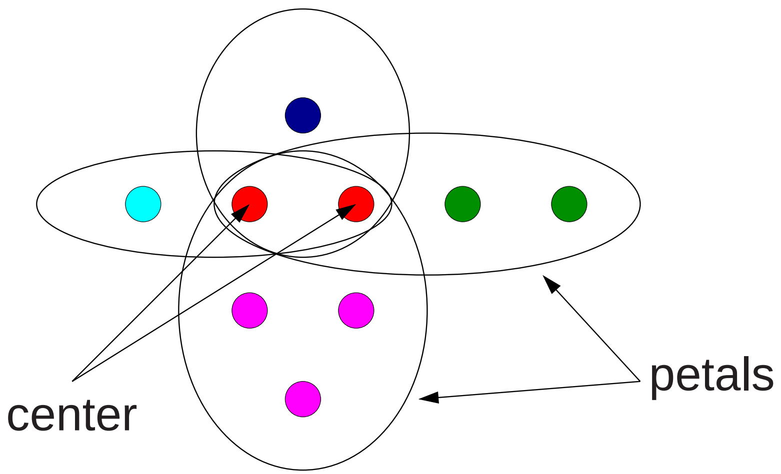


Reduction Rule (variant): Suppose more than $k+1$ sets form a sunflower.

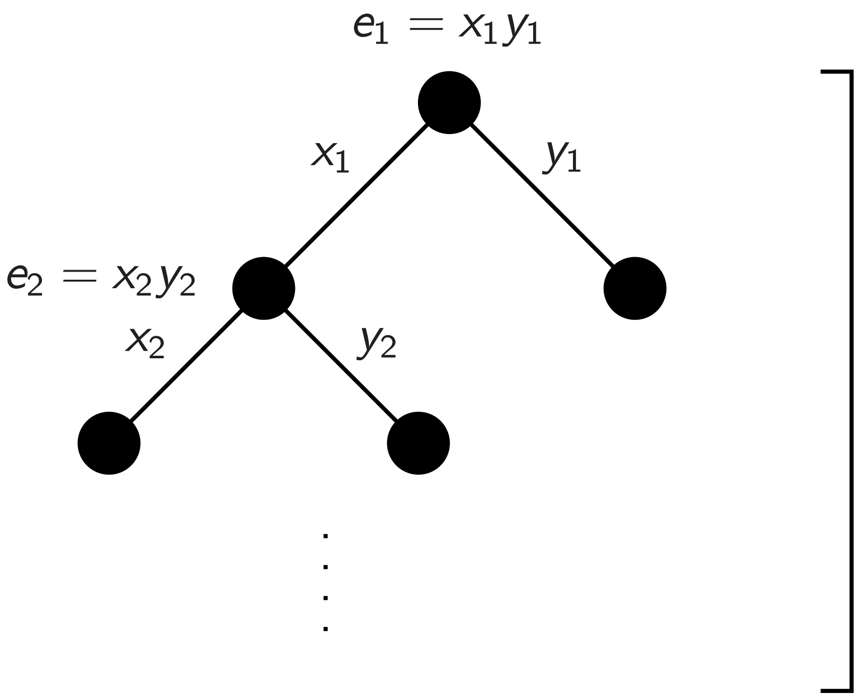
- If the sets are disjoint $\Rightarrow$ No solution.
- Otherwise, keep only $k+1$ of the sets.

If the rule cannot be applied, then there are at most $O(k^d)$ sets.

Branching and bounded search trees

Bounded search tree method

Recall how we solved MINIMUM VERTEX COVER:



height: $\leq k$

Bounded search tree method

We solve the problem by one or more **branching rules.**

Each rule makes a “guess” in such a way that at least one guess will lead to a correct solution.

If we have branching rules such that

- each rule branches into at most $b(k)$ directions, and
- applying a rule decreases the parameter,

then the problem can be solved in time $O^*(b(k)^k)$.

In many cases, this crude upper bound can be improved by better analysis.

VERTEX COVER

Improved algorithm for VERTEX COVER.

- If every vertex has degree ≤ 2, then the problem can be solved in polynomial time.
- **Branching rule:** If there is a vertex v with at least 3 neighbors, then
  - either v is in the solution,
  - or every neighbor of v is in the solution.

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

VERTEX COVER

Improved algorithm for VERTEX COVER.

- If every vertex has degree ≤ 2, then the problem can be solved in polynomial time.
- **Branching rule:** If there is a vertex v with at least 3 neighbors, then
  - either v is in the solution, $\Rightarrow k$ decreases by 1
  - or every neighbor of v is in the solution. $\Rightarrow k$ decreases by at least 3

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

But it is somewhat better than that, since in the second branch, the parameter decreases by at least 3.

Better analysis

Let $t(k)$ be the maximum number of leaves of the search tree if the parameter is at most k (let $t(k) = 1$ for $k \leq 0$).

$$t(k) \leq t(k-1) + t(k-3)$$

There is a standard technique for bounding such functions asymptotically.

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$$t(k) \leq t(k-1) + t(k-3)$$

There is a standard technique for bounding such functions asymptotically.

We prove by induction that $t(k) \leq c^k$ for some $c > 1$ as small as possible.

What values of c are good? We need:

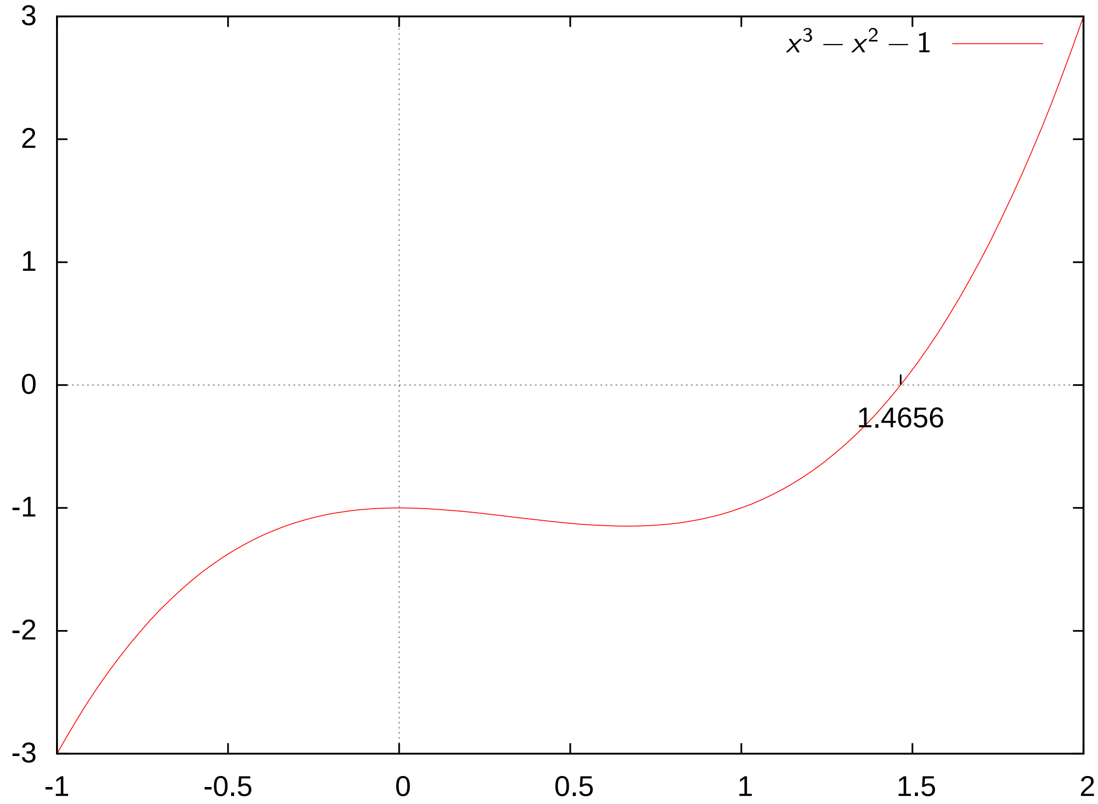
$$c^k \geq c^{k-1} + c^{k-3}$$

$$c^3 - c^2 - 1 \geq 0$$

We need to find the roots of the **characteristic equation** $c^3 - c^2 - 1 = 0$.

Note: it is always true that such an equation has a unique positive root.

Better analysis



$c = 1.4656$ is a good value $\Rightarrow$ $t(k) \leq 1.4656^k$

$\Rightarrow$ We have a $O^*(1.4656^k)$ algorithm for VERTEX COVER.

Better analysis

We showed that if $t(k) \leq t(k-1) + t(k-3)$, then $t(k) \leq 1.4656^k$ holds.

Is this bound tight? There are two questions:

- Can the function $t(k)$ be that large?
  Yes (ignoring rounding problems).
- Can the search tree of the VERTEX COVER algorithm be that large?
  Difficult question, hard to answer in general.

Branching vectors

The **branching vector** of our $O^*(1.4656^k)$ VERTEX COVER algoritm was $(1, 3)$.

Example: Let us bound the search tree for the branching vector $(2, 5, 6, 6, 7, 7)$.
(2 out of the 6 branches decrease the parameter by 7, etc.).

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The **branching vector** of our $O^*(1.4656^k)$ VERTEX COVER algoritm was $(1, 3)$.

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The value $c > 1$ has to satisfy:

$$c^k \geq c^{k-2} + c^{k-5} + 2c^{k-6} + 2c^{k-7}$$
$$c^7 - c^5 - c^2 - 2c - 2 \geq 0$$

Unique positive root of the characteristic equation: $1.4483 \Rightarrow t(k) \leq 1.4483^k$.

It is hard to compare branching vectors intuitively.

Branching vectors

Example: The roots for branching vector (i, j) (1 $\leq i, j \leq 6)$.

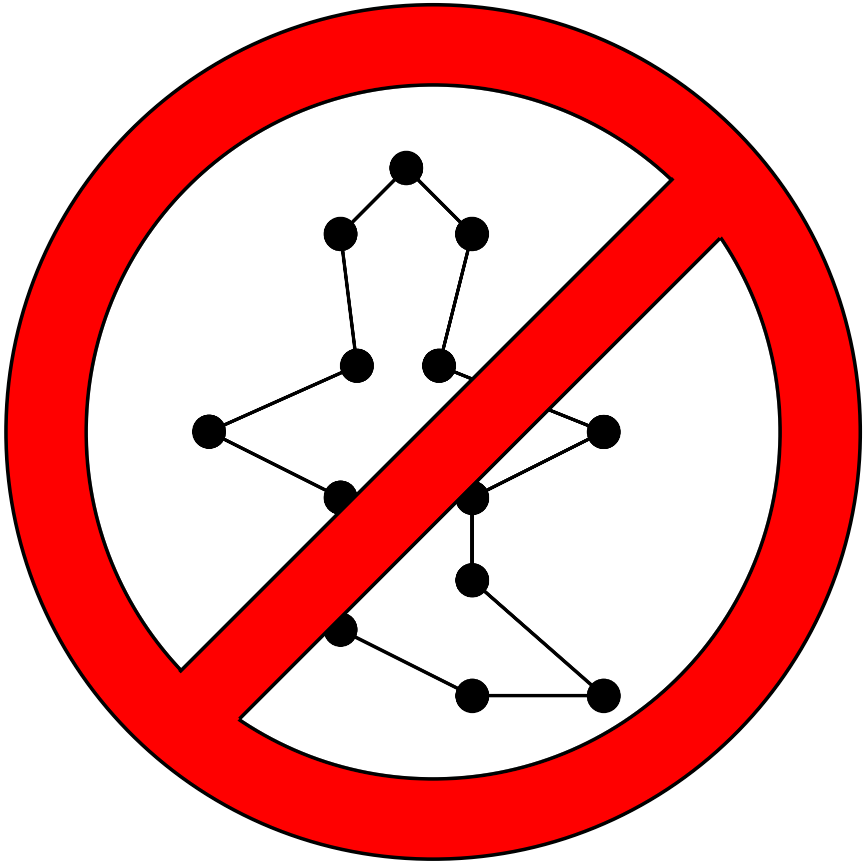
$$t(k) \leq t(k-i) + t(k-j) \Rightarrow c^k \geq c^{k-i} + c^{k-j}$$

$$c^j - c^{j-i} - 1 \geq 0$$

We compute the unique positive root.

	1	2	3	4	5	6
1	2.0000	1.6181	1.4656	1.3803	1.3248	1.2852
2	1.6181	1.4143	1.3248	1.2721	1.2366	1.2107
3	1.4656	1.3248	1.2560	1.2208	1.1939	1.1740
4	1.3803	1.2721	1.2208	1.1893	1.1674	1.1510
5	1.3248	1.2366	1.1939	1.1674	1.1487	1.1348
6	1.2852	1.2107	1.1740	1.1510	1.1348	1.1225

Forbidden subgraphs



Forbidden subgraphs

General problem class: Given a graph G and an integer k, transform G with at most k modifications (add/remove vertices/edges) into a graph having property $\mathcal{P}$.

Example:

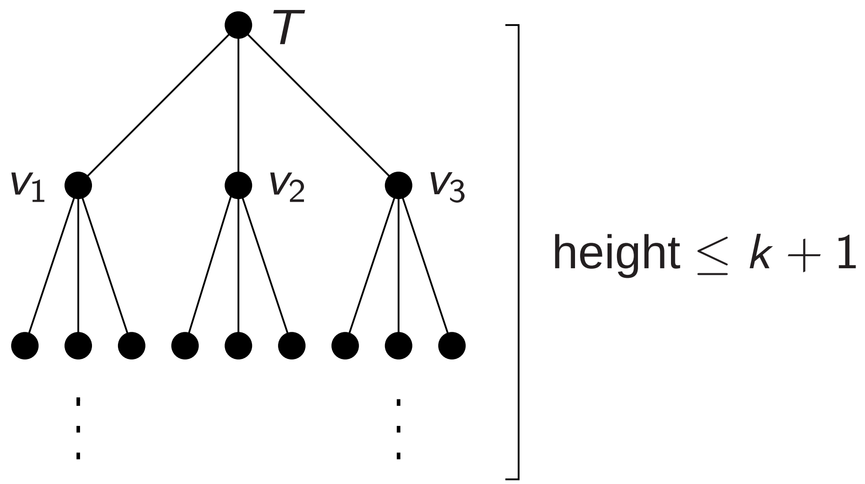
TRIANGLE DELETION: make the graph triangle-free by deleting at most k vertices.

Branching algorithm:

- If the graph is triangle-free, then we are done.
- **Branching rule:** If there is a triangle $v_1 v_2 v_3$, then at least one of v_1, v_2, v_3 has to be deleted $\Rightarrow$ We branch into 3 directions.

TRIANGLE DELETION

Search tree:



The search tree has at most 3^k leaves and the work to be done is polynomial at each step $\Rightarrow O^*(3^k)$ time algorithm.

Note: If the answer is “NO”, then the search tree has **exactly** 3^k leaves.

Hereditary properties

Definition: A graph property $\mathcal{P}$ is **hereditary** if for every $G \in \mathcal{P}$ and induced subgraph G' of G, we have $G' \in \mathcal{P}$ as well.

Examples: triangle-free, bipartite, interval graph, planar

Observation: Every hereditary property $\mathcal{P}$ can be characterized by a (finite or infinite) set $\mathcal{F}$ of forbidden induced subgraphs:

$$G \in \mathcal{P} \iff \forall H \in \mathcal{F},\, H \not\subseteq_{\text{ind}} G$$

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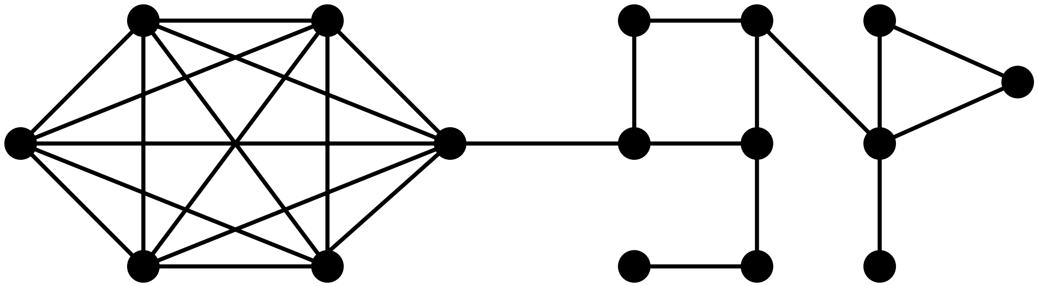
Proof:

- Suppose that every graph in $\mathcal{F}$ has at most r vertices. Using brute force, we can find in time $O(n^r)$ a forbidden subgraph (if exists).
- If a forbidden subgraph exists, then we have to delete one of the at most r vertices or add/delete one of the at most $\binom{r}{2}$ edges
  $\Rightarrow$ Branching factor is a constant c depending on $\mathcal{F}$.
- The search tree has at most c^k leaves and the work to be done at each node is $O(n^r)$.

CLUSTER EDITING

Task: Given a graph G and an integer k, add/remove at most k edges such that every component is a clique in the resulting graph.



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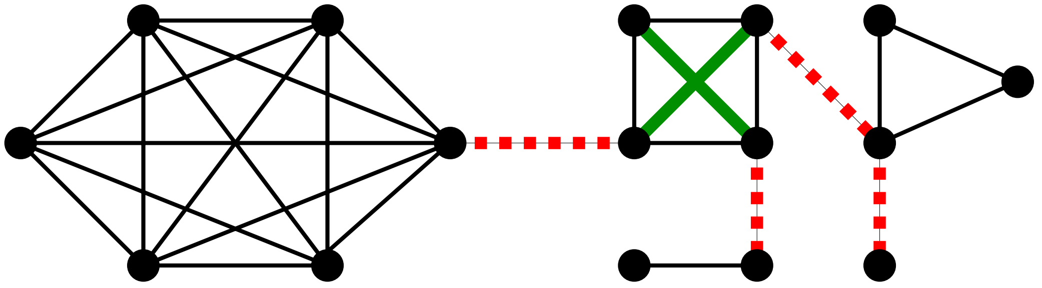

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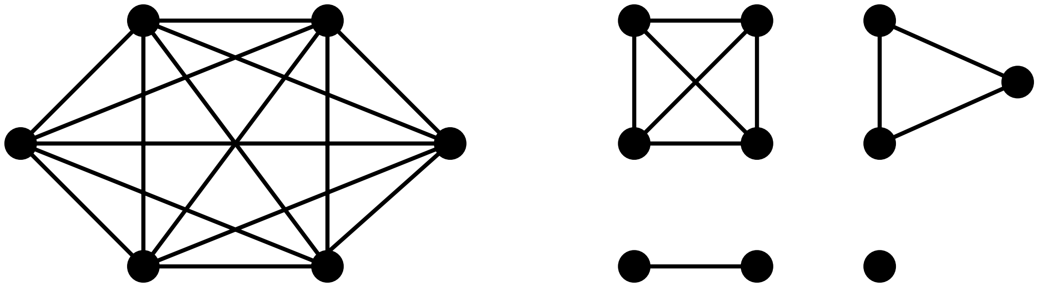

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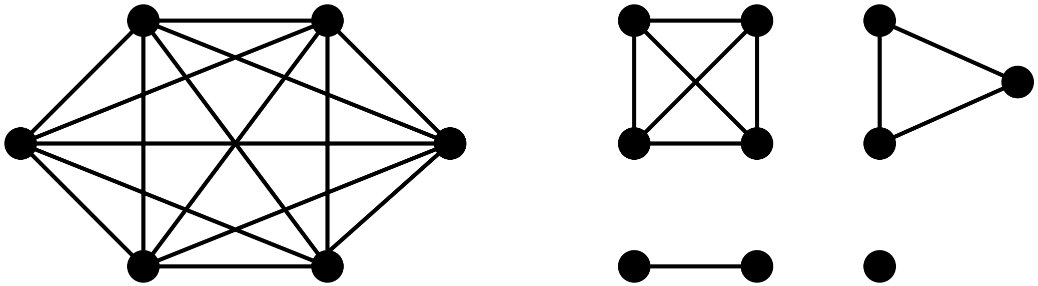

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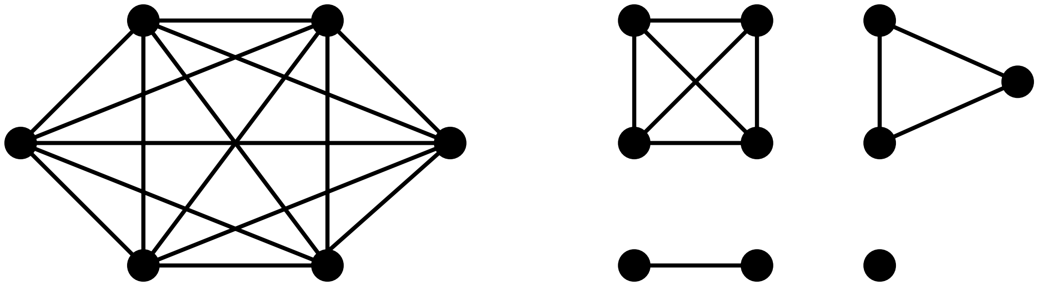

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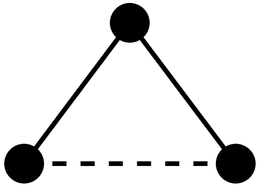
Forbidden induced subgraph:



$O^*(3^k)$ time algorithm.

CHORDAL COMPLETION

Definition: A graph is **chordal** if it does not contain an induced cycle of length greater than 3.

CHORDAL COMPLETION: Given a graph G and an integer k, add at most k edges to G to make it a chordal graph.

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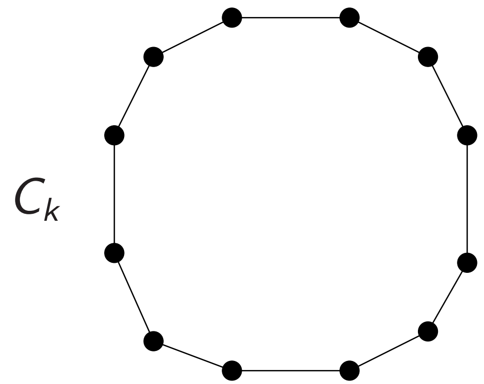
Lemma: At least $k-3$ edges are needed to make a k-cycle chordal.
Proof: By induction. $k=3$ is trivial.



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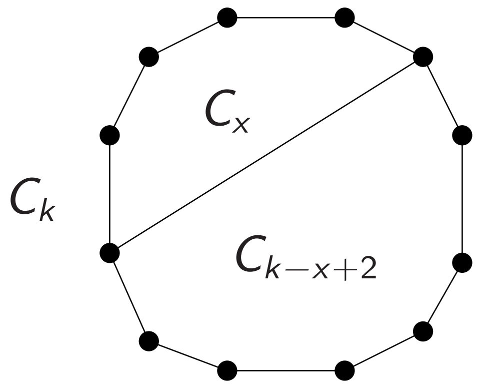


C_x: $x-3$ edges
C_{k-x+2}: $k-x-1$ edges
C_k: $(x-3)+(k-x-1)+1=k-3$ edges

Chordal Completion

Algorithm:

- Find an induced cycle C of length at least 4 (can be done in polynomial time).
- If no such cycle exists $\Rightarrow$ Done!
- If C has more than $k+3$ vertices $\Rightarrow$ No solution!
- Otherwise, one of the

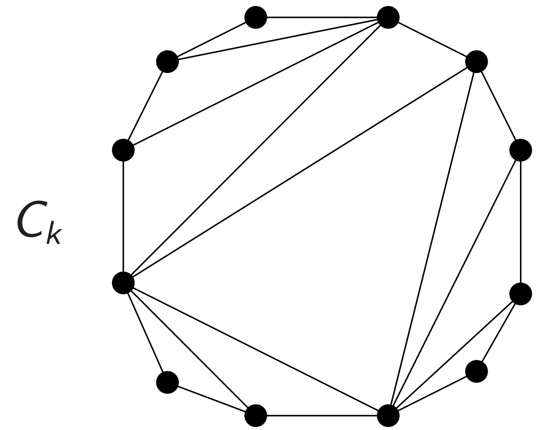
$$\binom{|C|}{2} - |C| \leq (k+3)(k+2)/2 - k = O(k^2)$$

  missing edges has to be added $\Rightarrow$ Branch!

Size of the search tree is $k^{O(k)}$.

Chordal Completion – *more efficiently*

Definition: Triangulation of a cycle.



Lemma: Every chordal supergraph of a cycle C contains a triangulation of the cycle C.

Lemma: The number of ways a cycle of length k can be triangulated is exactly the $(k-2)$-nd Catalan number

$$C_{k-2} = \frac{1}{k-1}\binom{2(k-2)}{k-2} \leq 4^{k-3}.$$

CHORDAL COMPLETION – *more efficiently*

Algorithm:

- Find an induced cycle C of length at least 4 (can be done in polynomial time).
- If no such cycle exists $\Rightarrow$ Done!
- If C has more than $k + 3$ vertices $\Rightarrow$ No solution!
- Otherwise, one of the $\leq 4^{|C|-3}$ triangulations has to be in the solution $\Rightarrow$ Branch!

Claim: Search tree has at most $T_k = 4^k$ leaves.

Proof: By induction. Number of leaves is at most

$$T_k \leq 4^{|C|-3} \cdot T_{k-(|C|-3)} \leq 4^{|C|-3} \cdot 4^{k-(|C|-3)} = 4^k.$$

Iterative compression

Iterative compression

- A surprising small, but very powerful trick.
- Most useful for deletion problems: delete k things to achieve some property.
- Demonstration: ODD CYCLE TRANSVERSAL aka BIPARTITE DELETION aka GRAPH BIPARTIZATION: Given a graph G and an integer k, delete k vertices to make the graph bipartite.
- Forbidden induced subgraphs: odd cycles. There is no bound on the size of odd cycles.

BIPARTITE DELETION

Solution based on iterative compression:

- **Step 1:**

  Solve the **annotated problem** for bipartite graphs:

  > Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.

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- **Step 2:**

  Solve the **compression problem** for general graphs:

  Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.

BIPARTITE DELETION

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- **Step 2:**

  Solve the **compression problem** for general graphs:

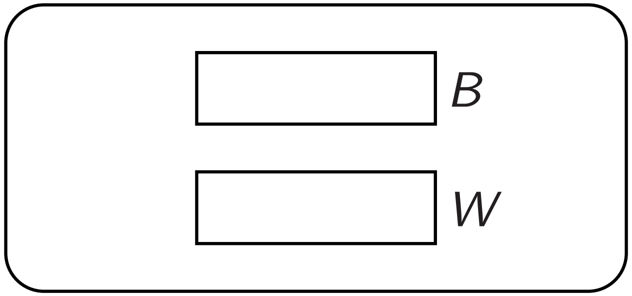
  > Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.

- **Step 3:**

  Apply the magic of iterative compression...

Step 1: The annotated problem

Given a bipartite graph G, two sets B, $W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.



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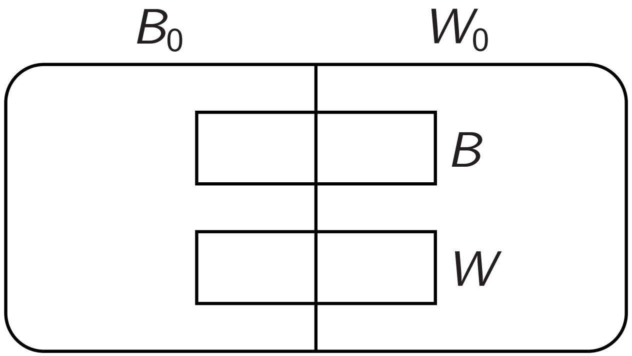

Find an arbitrary 2-coloring (B_0, W_0) of G.

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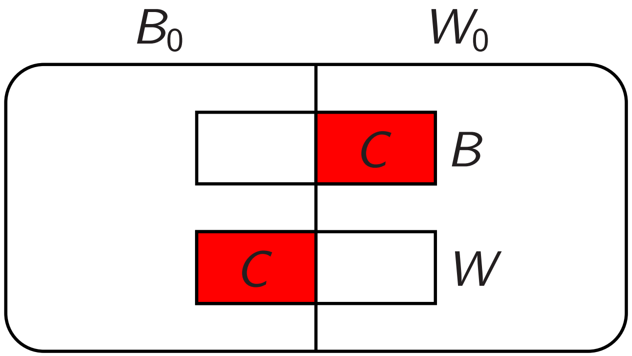

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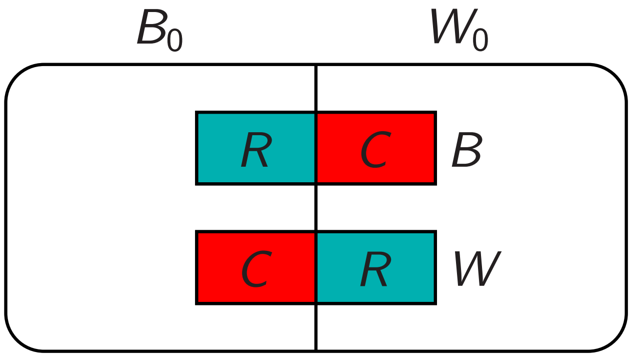

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Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Step 1: The annotated problem

Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Proof:

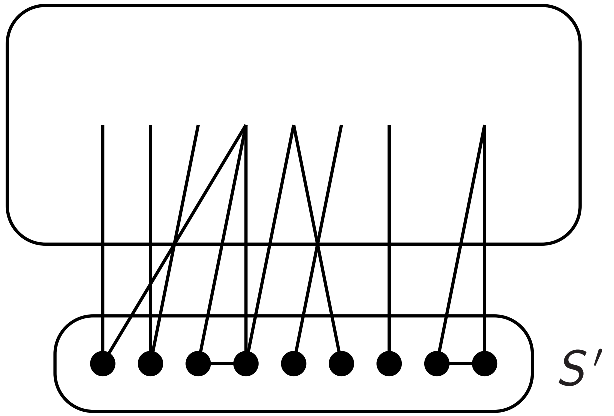
$\Rightarrow$ In a 2-coloring of $G \setminus S$, each vertex either remained the same color or changed color. Adjacent vertices do the same, thus every component either changed or remained.

$\Leftarrow$ Flip the coloring of those components of $G \setminus S$ that contain vertices from $C \setminus S$. No vertex of R is flipped.

Algorithm: Using max-flow min-cut techniques, we can check if there is a set S that separates C and R. It can be done in time $O(k|E(G)|)$ using k iterations of the Ford-Fulkerson algorithm.

Step 2: The compression problem

Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.



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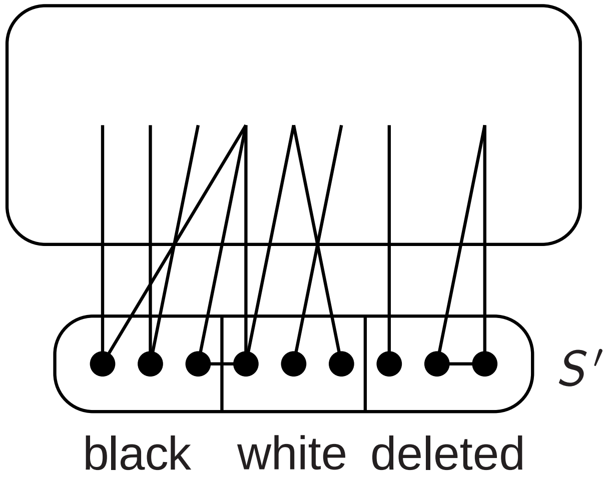

Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.

Trivial check: no edge between two black or two white vertices.

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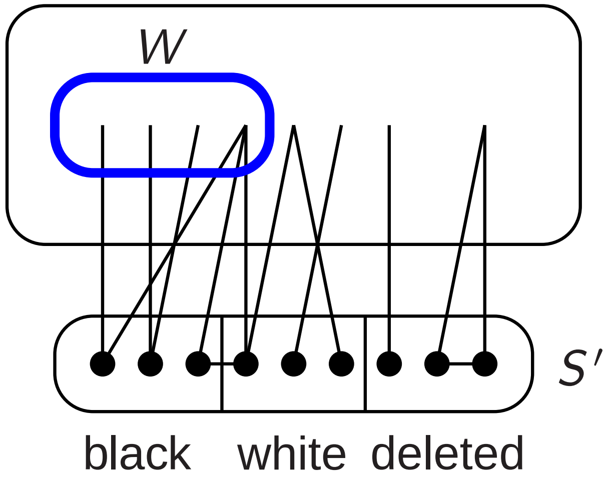

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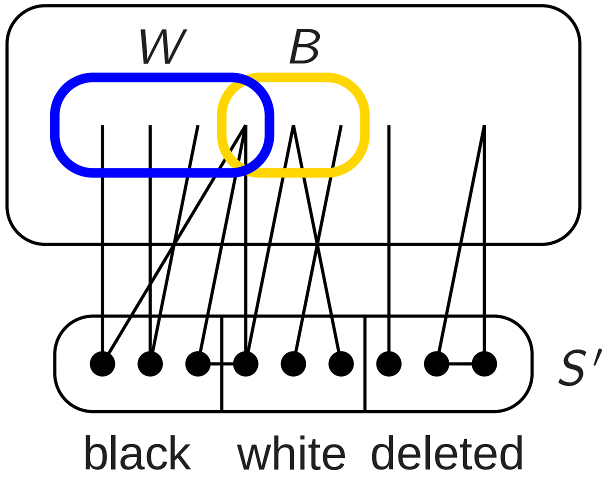

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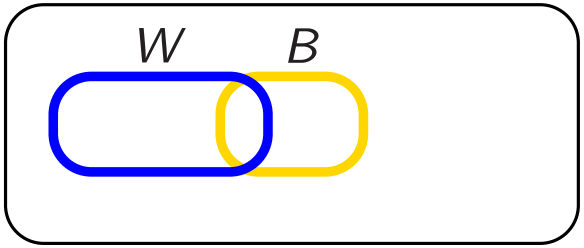

The vertices of S' can be disregarded. Thus we need to solve the annotated problem on the bipartite graph $G \setminus S'$.

Running time: $O(3^k \cdot k|E(G)|)$ time.

Step 3: Iterative compression

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We get it for free!

Let $V(G) = \{v_1, \ldots, v_n\}$ and let G_i be the graph induced by $\{v_1, \ldots, v_i\}$.

For every i, we find a set S_i of size k such that $G_i \setminus S_i$ is bipartite.

- For G_k, the set $S_k = \{v_1, \ldots, v_k\}$ is a trivial solution.
- If S_{i-1} is known, then $S_{i-1} \cup \{v_i\}$ is a set of size $k+1$ whose deletion makes G_i bipartite $\Rightarrow$ We can use the compression algorithm to find a suitable S_i in time $O(3^k \cdot k|E(G_i)|)$.

Step 3: Iterative Compression

Bipartite-Deletion(G, k)

1. $S_k = \{v_1, \ldots, v_k\}$
2. for $i := k + 1$ to n
3. Invariant: $G_{i-1} \setminus S_{i-1}$ is bipartite.
4. Call Compression$(G_i, S_{i-1} \cup \{v_i\})$
5. If the answer is “NO” $\Rightarrow$ return “NO”
6. If the answer is a set $X \Rightarrow S_i := X$
7. Return the set S_n

Running time: the compression algorithm is called n times and everything else can be done in linear time

$\Rightarrow O(3^k \cdot k|V(G)| \cdot |E(G)|)$ time algorithm.

Graph Minors

Neil Robertson

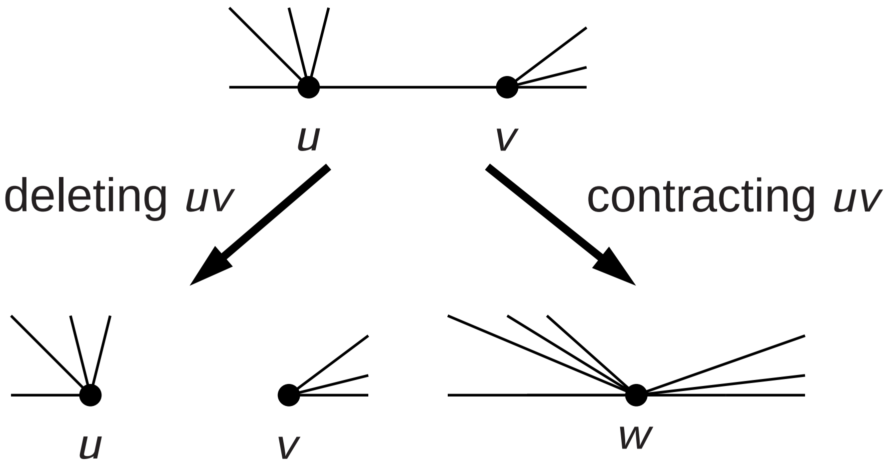
Paul Seymour

Graph Minors

- Some consequences of the Graph Minors Theorem give a quick way of showing that certain problems are FPT.
- However, the function $f(k)$ in the resulting FPT algorithms can be HUGE, completely impractical.
- History: motivation for FPT.
- Parts and ingredients of the theory are useful for algorithm design.
- New algorithmic results are still being developed.

Graph Minors

Definition: Graph H is a **minor** G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

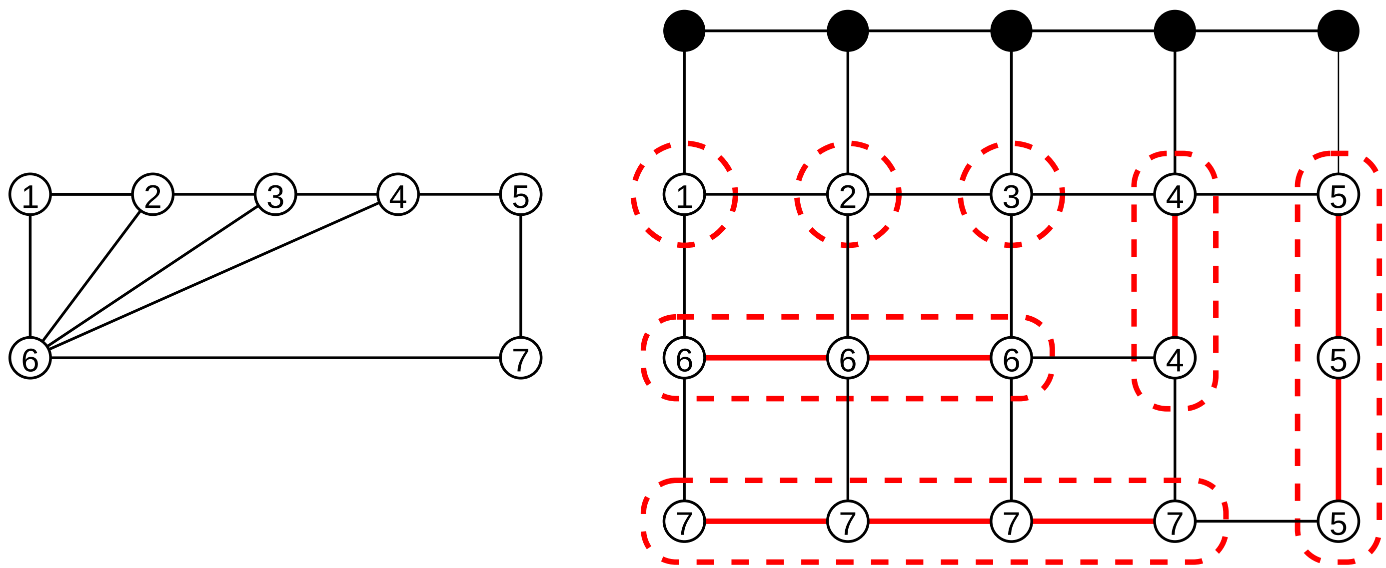


Example: A triangle is a minor of a graph G if and only if G has a cycle (i.e., it is not a forest).

Graph minors

Equivalent definition: Graph H is a **minor** of G if there is a mapping ϕ that maps each vertex of H to a connected subset of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.



Minor closed properties

Definition: A set $\mathcal{G}$ of graphs is **minor closed** if whenever $G \in \mathcal{G}$ and $H \leq G$, then $H \in \mathcal{G}$ as well.

Examples of minor closed properties:

planar graphs

acyclic graphs (forests)

graphs having no cycle longer than k

empty graphs

Examples of not minor closed properties:

complete graphs

regular graphs

bipartite graphs

Forbidden minors

Let $\mathcal{G}$ be a minor closed set and let $\mathcal{F}$ be the set of "minimal bad graphs": $H \in \mathcal{F}$ if $H \notin \mathcal{G}$, but every proper minor of H is in $\mathcal{G}$.

Characterization by forbidden minors:

$$G \in \mathcal{G} \iff \forall H \in \mathcal{F}, H \not\leq G$$

The set $\mathcal{F}$ is the **obstruction set** of property $\mathcal{G}$.

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The set $\mathcal{F}$ is the **obstruction set** of property $\mathcal{G}$.

Theorem: [Wagner] A graph is planar if and only if it does not have a K_5 or $K_{3,3}$ minor.

In other words: the obstruction set of planarity is $\mathcal{F} = \{K_5, K_{3,3}\}$.

Does every minor closed property have such a finite characterization?

Graph Minors Theorem

Theorem: [Robertson and Seymour] Every minor closed property $\mathcal{G}$ has a finite obstruction set.

Note: The proof is contained in the paper series “Graph Minors I–XX”.

Note: The size of the obstruction set can be astronomical even for simple properties.

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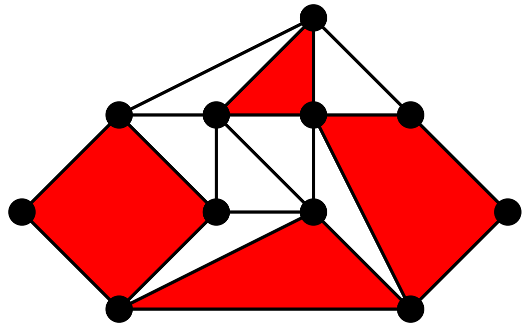
Note: The size of the obstruction set can be astronomical even for simple properties.

Theorem: [Robertson and Seymour] For every fixed graph H, there is an $O(n^3)$ time algorithm for testing whether H is a minor of the given graph G.

Corollary: For every minor closed property $\mathcal{G}$, there is an $O(n^3)$ time algorithm for testing whether a given graph G is in $\mathcal{G}$.

Applications

PLANAR FACE COVER: Given a graph G and an integer k, find an embedding of planar graph G such that there are k faces that cover all the vertices.



One line argument:

For every fixed k, the class $\mathcal{G}_k$ of graphs of yes-instances is minor closed.

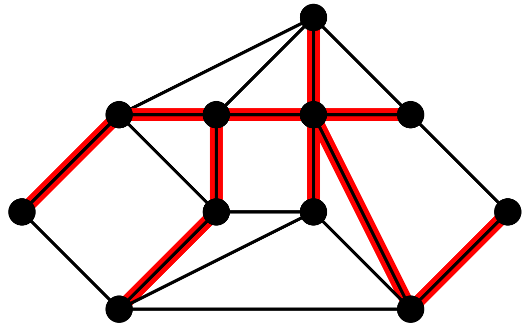
$$\Downarrow$$

For every fixed k, there is a $O(n^3)$ time algorithm for PLANAR FACE COVER.

Note: non-uniform FPT.

Applications

k-LEAF SPANNING TREE: Given a graph G and an integer k, find a spanning tree with **at least** k leaves.



Technical modification: Is there such a spanning tree for at least one component of G?

One line argument:

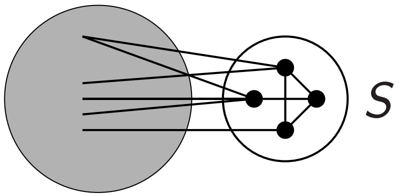
For every fixed k, the class $\mathcal{G}_k$ of no-instances is minor closed.

$$\Downarrow$$

For every fixed k, k-LEAF SPANNING TREE can be solved in time $O(n^3)$.

$\mathcal{G} + k$ vertices

Let $\mathcal{G}$ be a graph property, and let $\mathcal{G} + kv$ contain graph G if there is a set $S \subseteq V(G)$ of k vertices such that $G \setminus S \in \mathcal{G}$.



Lemma: If $\mathcal{G}$ is minor closed, then $\mathcal{G} + kv$ is minor closed for every fixed k.

$\Rightarrow$ It is (nonuniform) FPT to decide if G can be transformed into a member of $\mathcal{G}$ by deleting k vertices.

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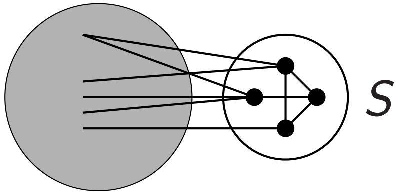

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- If $\mathcal{G} =$ forests $\Rightarrow \mathcal{G} + kv =$ graphs that can be made acyclic by the deletion of k vertices $\Rightarrow$ FEEDBACK VERTEX SET is FPT.
- If $\mathcal{G} =$ planar graphs $\Rightarrow \mathcal{G} + kv =$ graphs that can be made planar by the deletion of k vertices (k-apex graphs) $\Rightarrow$ k-APEX GRAPH is FPT.
- If $\mathcal{G} =$ empty graphs $\Rightarrow \mathcal{G} + kv =$ graphs with vertex cover number at most k $\Rightarrow$ VERTEX COVER is FPT.

Color coding

Color coding

- Works best when we need to ensure that a small number of “things” are disjoint.
- We demonstrate it on the problem of finding an s-t path of length **exactly** k.
- Randomized algorithm, but can be derandomized using a standard technique.
- Very robust technique, we can use it as an “opening step” when investigating a new problem.

k-PATH

Task: Given a graph G, an integer k, two vertices s, t, find a **simple** s-t path with exactly k internal vertices.

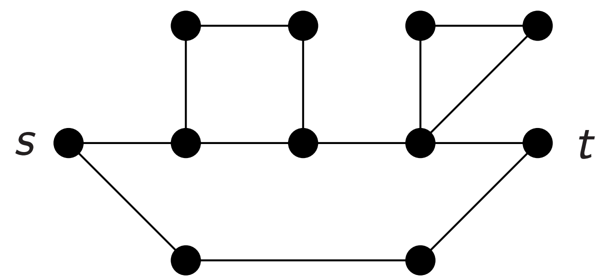
Note: Finding such a **walk** can be done easily in polynomial time.

Note: The problem is clearly NP-hard, as it contains the s-t HAMILTONIAN PATH problem.

The k-PATH algorithm can be used to check if there is a cycle of length exactly k in the graph.

k-PATH

- Assign colors from $[k]$ to vertices $V(G) \setminus \{s, t\}$ uniformly and independently at random.



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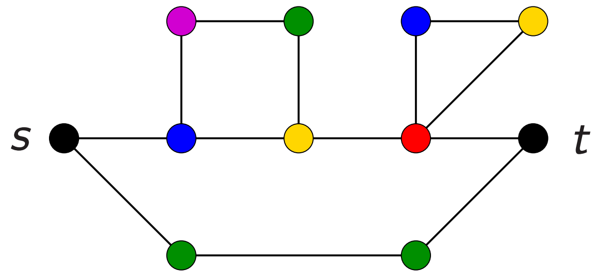

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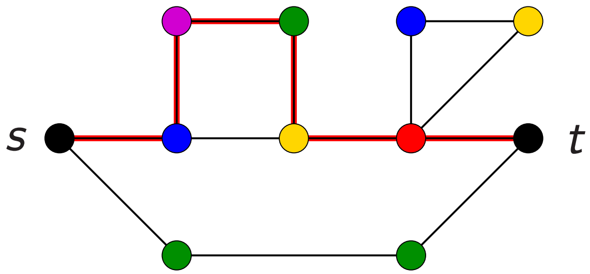

- Check if there is a **colorful** s-t path: a path where each color appears exactly once on the internal vertices; output "YES" or "NO".

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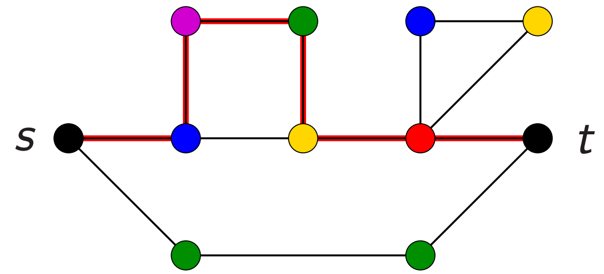

- Check if there is a **colorful** s-t path: a path where each color appears exactly once on the internal vertices; output “YES” or “NO”.
  - If there is no s-t k-path: no such colorful path exists $\Rightarrow$ “NO”.
  - If there is an s-t k-path: the probability that such a path is colorful is

$$\frac{k!}{k^k} > \frac{\left(\frac{k}{e}\right)^k}{k^k} = e^{-k},$$

thus the algorithm outputs “YES” with at least that probability.

Error probability

- **Useful fact:** If the probability of success is at least p, then the probability that the algorithm **does not** say “YES” after $1/p$ repetitions is at most

$$(1-p)^{1/p} < (e^{-p})^{1/p} = 1/e \approx 0.38$$

- Thus if $p > e^{-k}$, then error probability is at most $1/e$ after e^k repetitions.
- Repeating the whole algorithm a constant number of times can make the error probability an arbitrary small constant.
- For example, by trying $100 \cdot e^k$ random colorings, the probability of a wrong answer is at most $1/e^{100}$.

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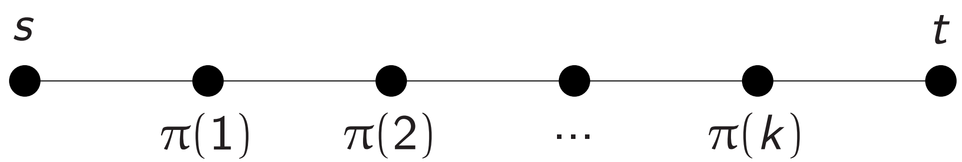
It remains to see how a colorful s-t path can be found.

Method 1: Trying all permutations.
Method 2: Dynamic programming.

Method 1: Trying all permutations

The colors encountered on a colorful s-t path form a permutation π of $\{1, 2, \ldots, k\}$:



We try all possible $k!$ permutations. For a fixed π, it is easy to check if there is a path with this order of colors.

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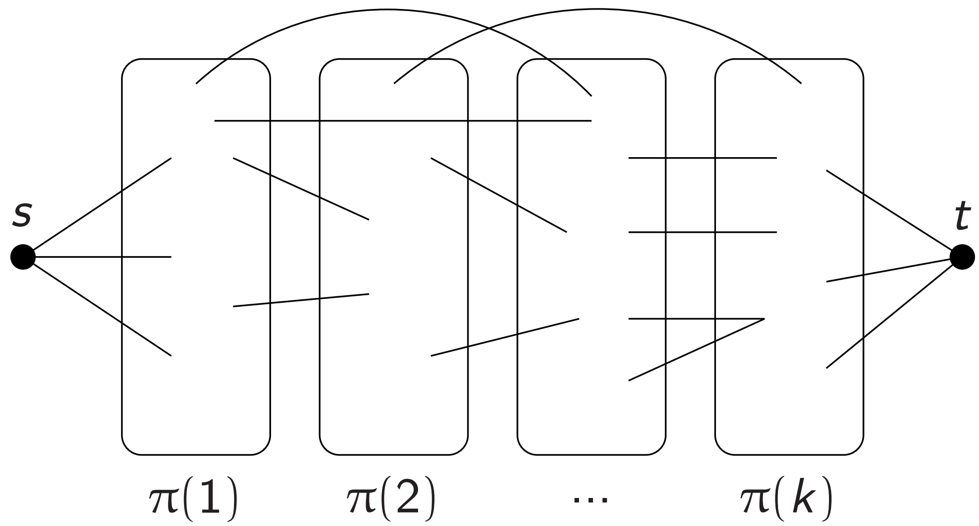

- Edges connecting nonadjacent color classes are removed.
- The remaining edges are directed.
- All we need to check if there is a directed s-t path.
- Running time is $O(k! \cdot |E(G)|)$.

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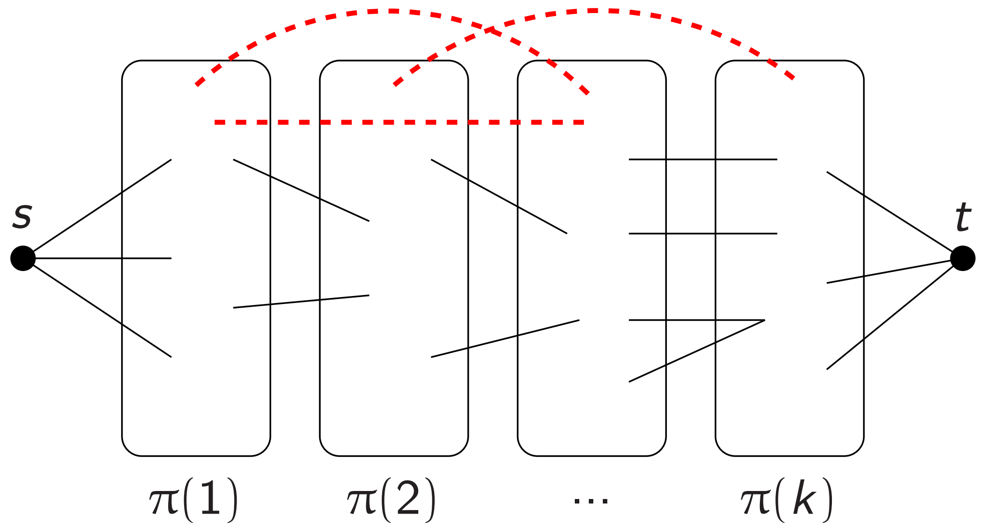

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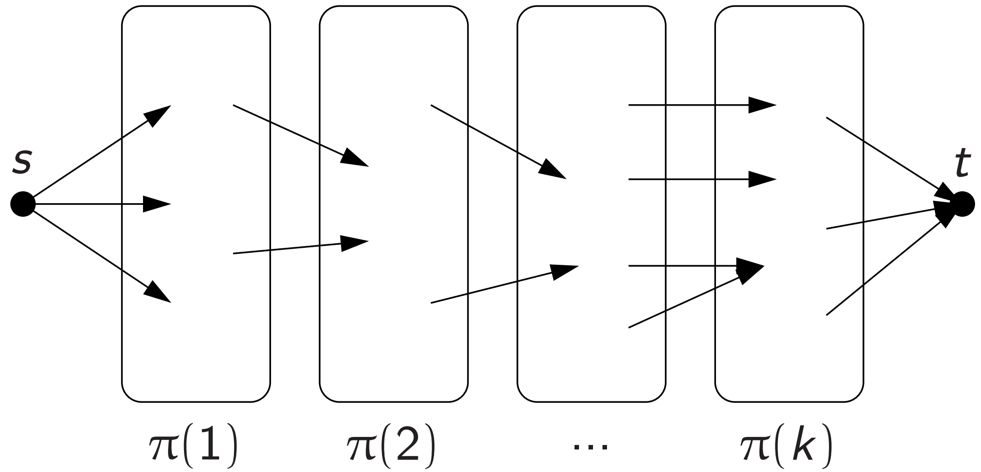

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Method 2: Dynamic Programming

We introduce $2^k \cdot |V(G)|$ Boolean variables:

$$\boxed{\begin{array}{c} x(v, C) = \text{TRUE for some } v \in V(G) \text{ and } C \subseteq [k] \\ \Updownarrow \\ \text{There is an } s\text{-}v \text{ path where each color in } C \text{ appears exactly once} \\ \text{and no other color appears.} \end{array}}$$

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Clearly, $x(s, \emptyset) = \text{TRUE}$. Recurrence for vertex v with color r:

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If we know every $x(v, C)$ with $|C| = i$, then we can determine every $x(v, C)$ with $|C| = i + 1$ $\Rightarrow$ All the values can be determined in time $O(2^k \cdot |E(G)|)$.

There is a colorful s-t path $\iff$ $x(v, [k]) = \text{TRUE}$ for some neighbor of t.

Derandomization

Using Method 2, we obtain a $O^*((2e)^k)$ time algorithm with constant error probability. How to make it deterministic?

Definition: A family $\mathcal{H}$ of functions $[n] \to [k]$ is a k**-perfect** family of hash functions if for every $S \subseteq [n]$ with $|S| = k$, there is a $h \in \mathcal{H}$ such that $h(x) \neq h(y)$ for any $x, y \in S$, $x \neq y$.

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Instead of trying $O(e^k)$ random colorings, we go through a k-perfect family $\mathcal{H}$ of functions $V(G) \to [k]$. If there is a solution $\Rightarrow$ The internal vertices S are colorful for at least one $h \in \mathcal{H}$ $\Rightarrow$ Algorithm outputs “YES”.

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Using Method 2, we obtain a $O^*((2e)^k)$ time algorithm with constant error probability. How to make it deterministic?

Definition: A family $\mathcal{H}$ of functions $[n] \to [k]$ is a k-**perfect** family of hash functions if for every $S \subseteq [n]$ with $|S| = k$, there is a $h \in \mathcal{H}$ such that $h(x) \neq h(y)$ for any $x, y \in S$, $x \neq y$.

Instead of trying $O(e^k)$ random colorings, we go through a k-perfect family $\mathcal{H}$ of functions $V(G) \to [k]$. If there is a solution $\Rightarrow$ The internal vertices S are colorful for at least one $h \in \mathcal{H}$ $\Rightarrow$ Algorithm outputs "YES".

Theorem: There is a k-perfect family of functions $[n] \to [k]$ having size $2^{O(k)} \log n$ (and can be constructed in time polynomial in the size of the family).

$\Rightarrow$ There is a **deterministic** $2^{O(k)} \cdot n^{O(1)}$ time algorithm for the k-PATH problem.

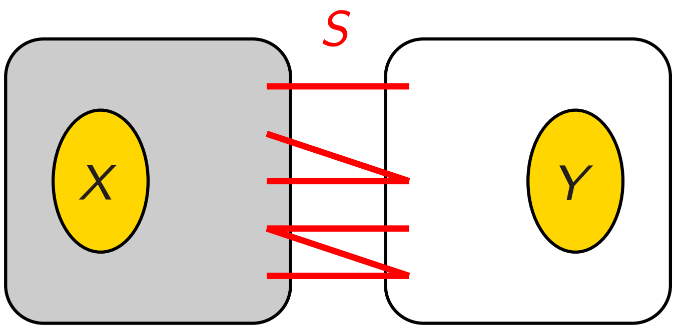
Cut problems

MULTIWAY CUT

Task: Given a graph G, a set T of vertices, and an integer k, find a set S of at most k edges that separates T (each component of $G \setminus S$ contains at most one vertex of T).

Polynomial for $|T| = 2$, but NP-hard for $|T| = 3$.

Theorem: MULTIWAY CUT is FPT parameterized by k.



$\delta(R)$: set of edges leaving R
$\lambda(X,Y)$: minimum number of edges in an (X,Y)-separator

Submodularity

Fact: The function δ is **submodular:** for arbitrary sets A, B,

$$|\delta(A)| \quad + \quad |\delta(B)| \quad \geq \quad |\delta(A \cap B)| \quad + \quad |\delta(A \cup B)|$$

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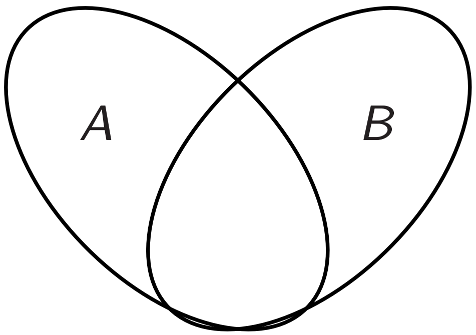
Proof: Determine separately the contribution of the different types of edges.



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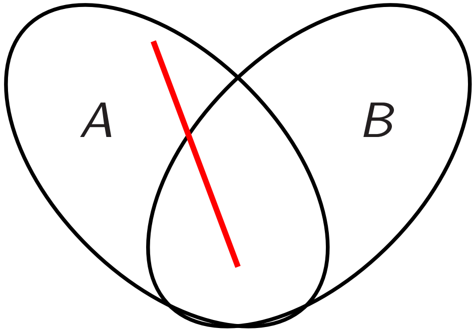

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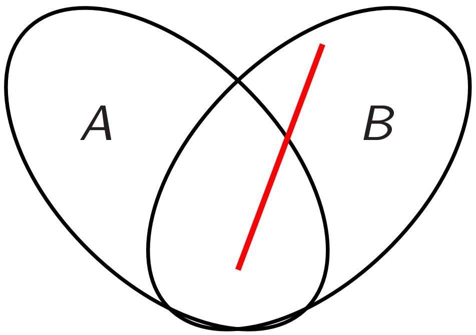

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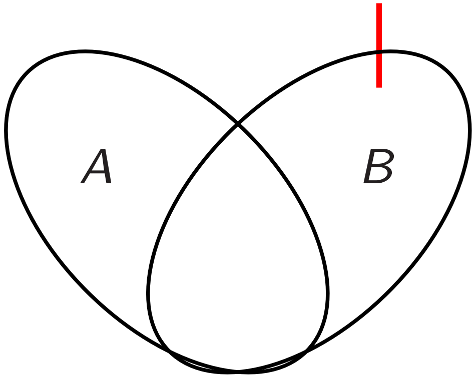

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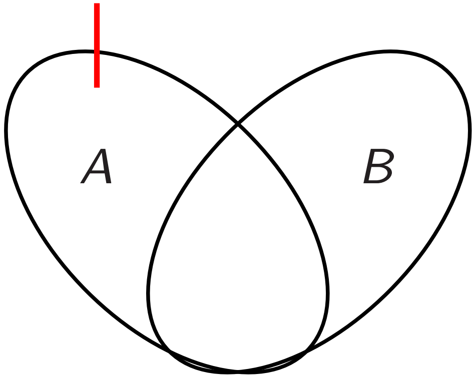

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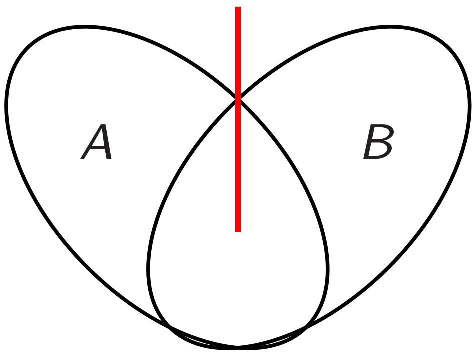

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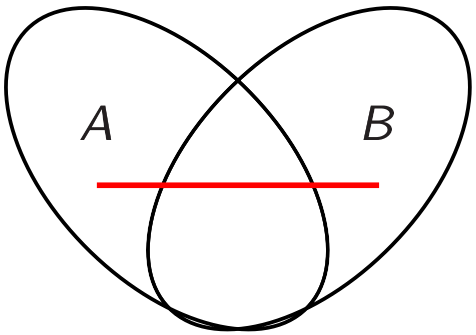
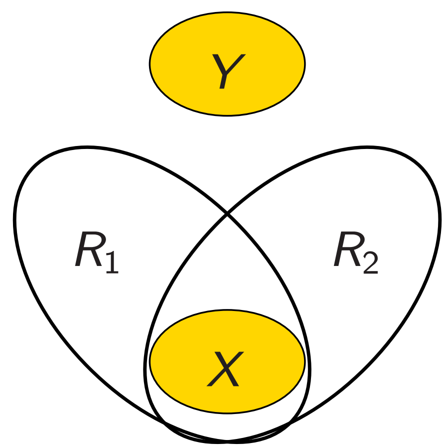


Submodularity

Consequence: There is a unique maximal $R_{\max} \supseteq X$ such that $\delta(R_{\max})$ is an (X, Y)-separator of size $\lambda(X, Y)$.

Proof: Let $R_1, R_2 \supseteq X$ be two sets such that $\delta(R_1), \delta(R_2)$ are (X, Y)-separators of size $\lambda := \lambda(X, Y)$.



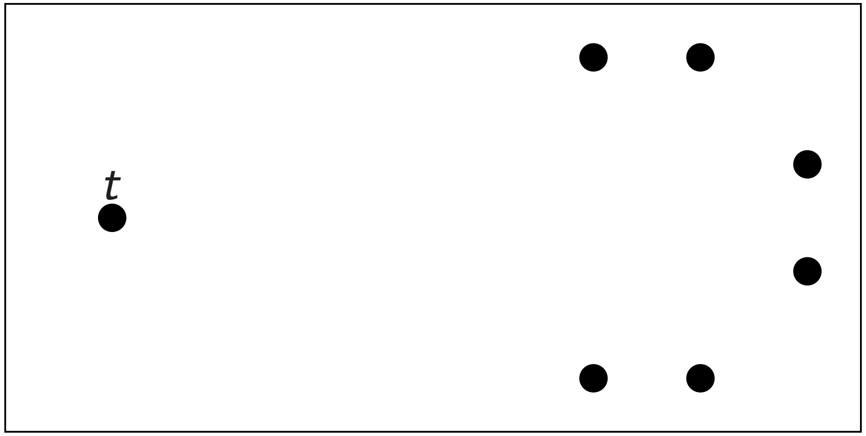
$$\begin{array}{ccccccc} |\delta(R_1)| & + & |\delta(R_2)| & \geq & |\delta(R_1 \cap R_2)| & + & |\delta(R_1 \cup R_2)| \\ \lambda & & \lambda & & \geq \lambda & & \end{array}$$

$\Rightarrow |\delta(R_1 \cup R_2)| \leq \lambda$

Note: Analogous result holds for a unique minimal $R_{\min}$.

MULTIWAY CUT

Intuition: Consider a $t \in T$. A subset of the solution separates t and $T \setminus t$.



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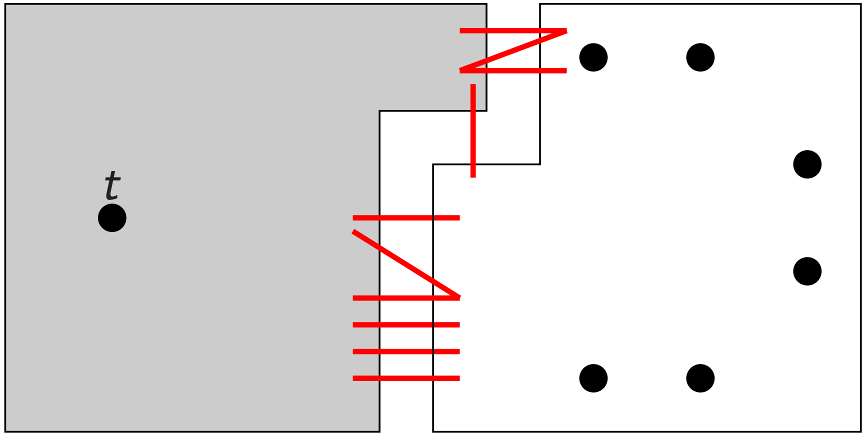

There are many such separators.

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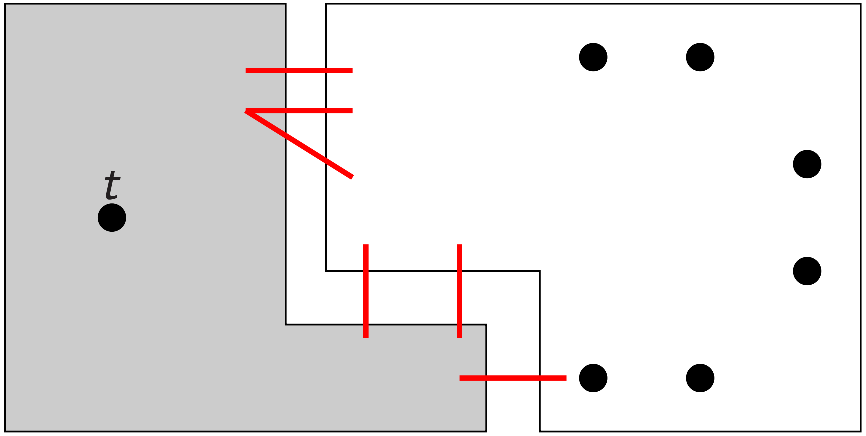

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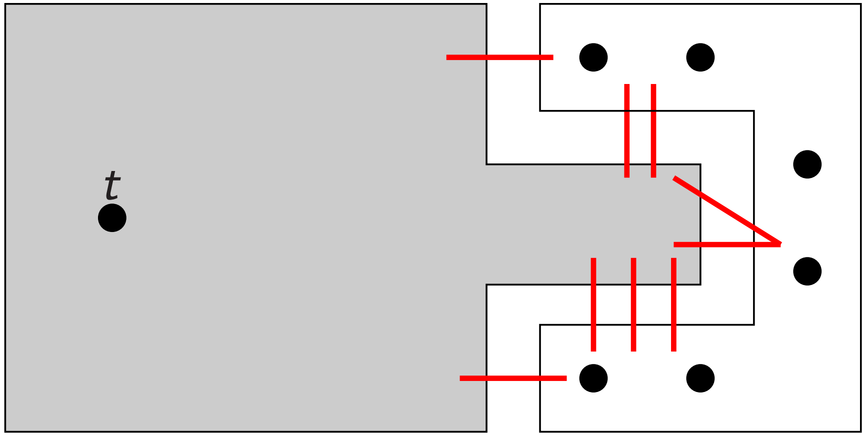
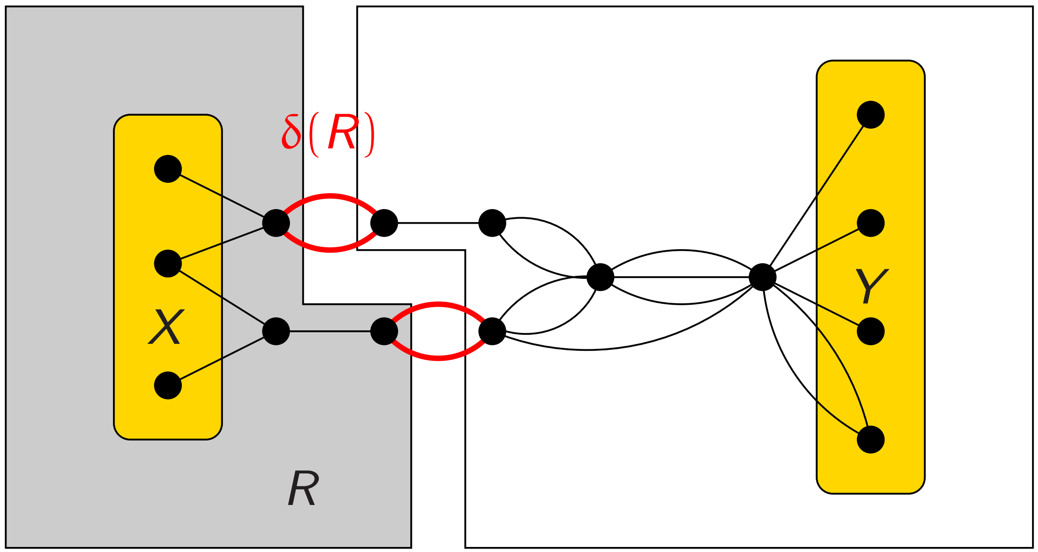


There are many such separators.

But a separator farther from t and closer to $T \setminus t$ seems to be more useful.

Important separators

Definition: An (X, Y)-separator $\delta(R)$ is **important** if there is no (X, Y)-separator $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$.



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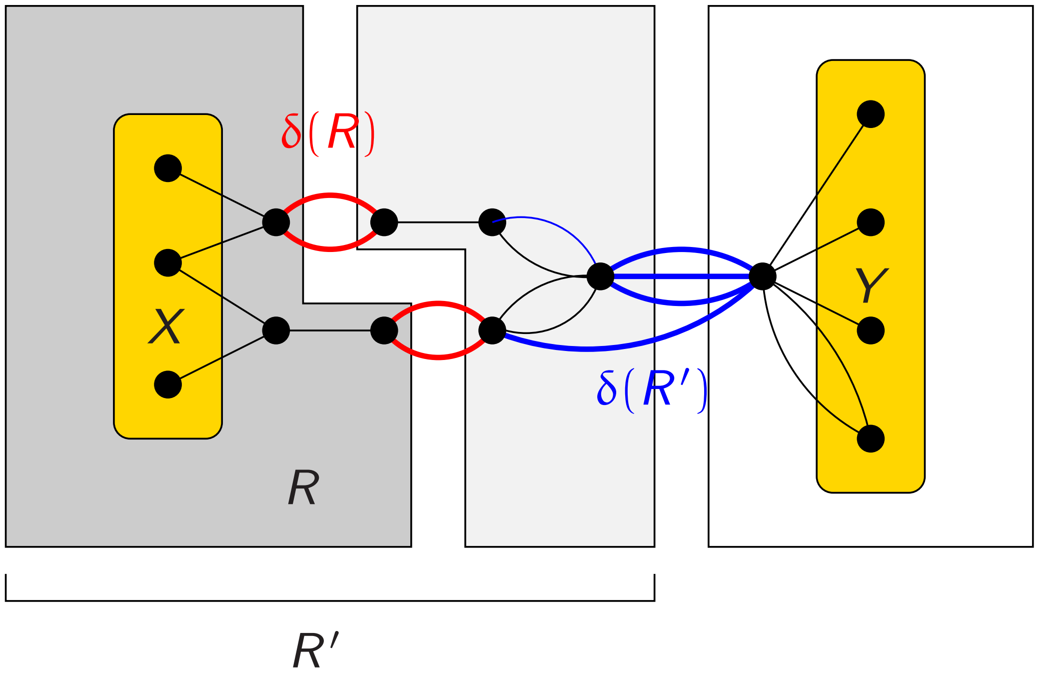

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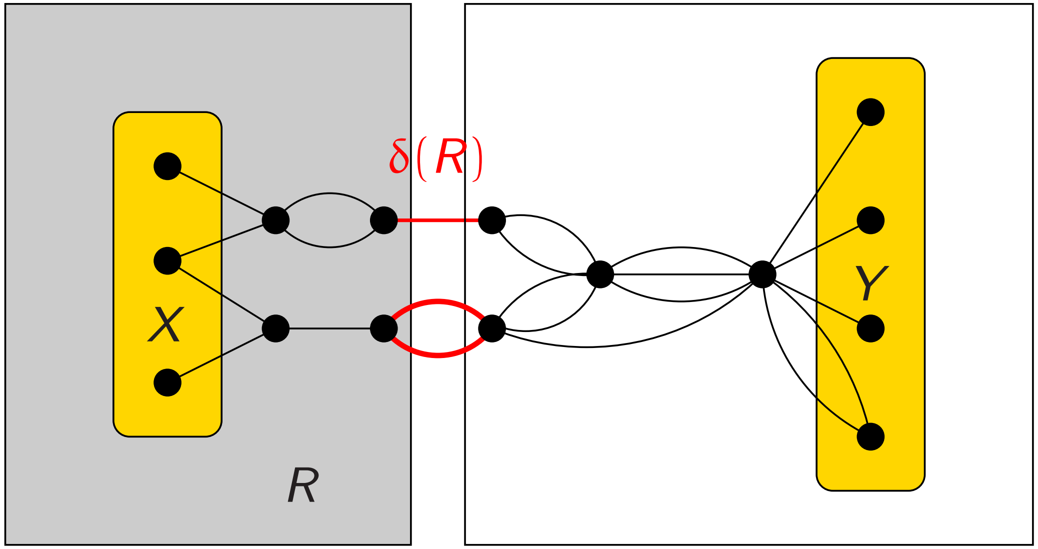

MULTIWAY CUT *and important separators*

Lemma: Let $t \in T$. The MULTIWAY CUT problem has a solution S that contains an important $(t, T \setminus t)$-separator.

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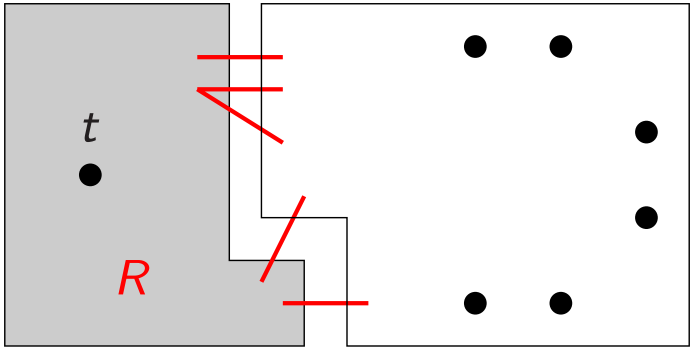

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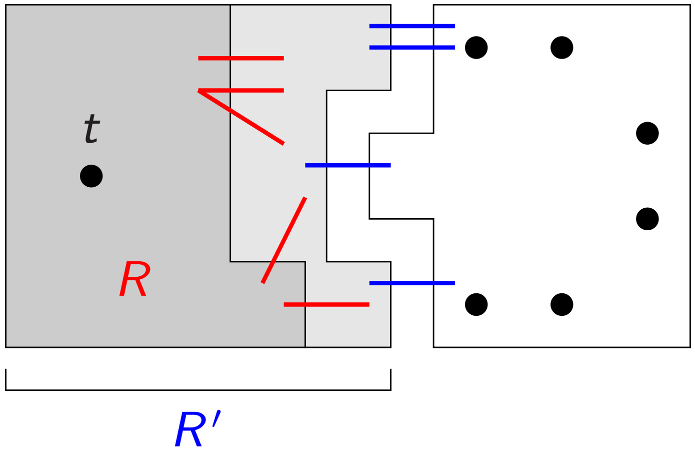

If $\delta(R)$ is not important, then there is an important separator $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$. Replace S with $S' := (S \setminus \delta(R)) \cup \delta(R') \Rightarrow |S'| \leq |S|$

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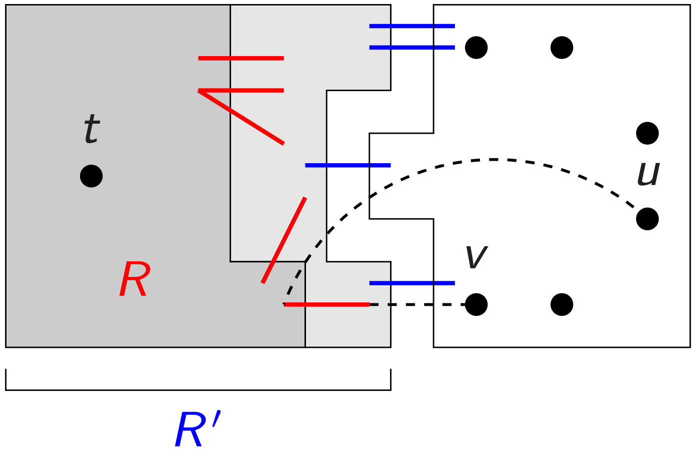

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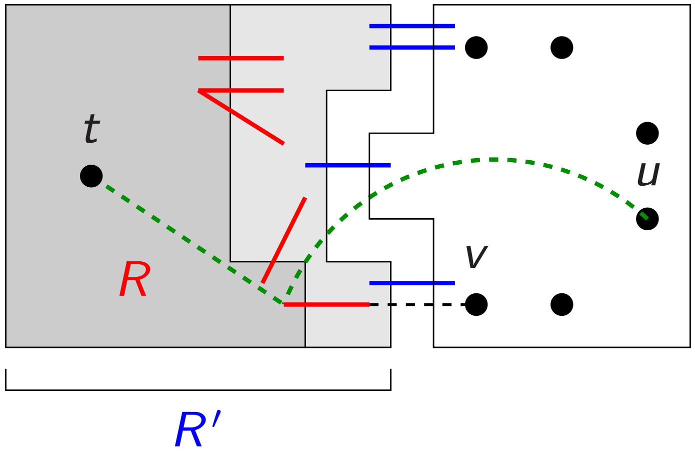

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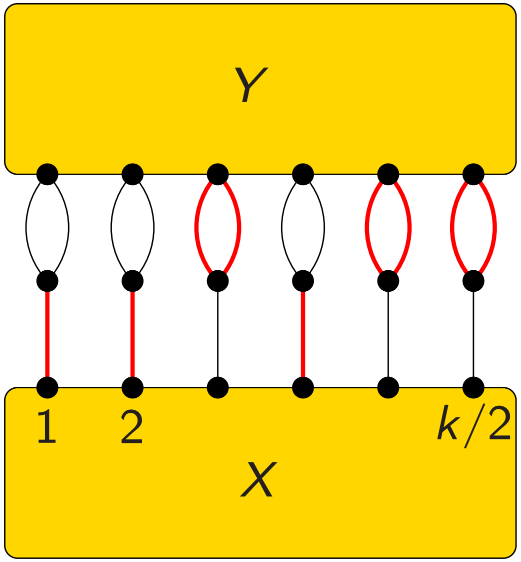
Important separators

Lemma: There are at most 4^k important (X, Y)-separators of size at most k.

Example:



There are exactly $2^{k/2}$ important (X, Y)-separators of size at most k in this graph.

Important separators

Lemma: There are at most 4^k important (X, Y)-separators of size at most k.

Proof: First we show that $R_{\max} \subseteq R$ for every important separator $\delta(R)$.

$$\underbrace{|\delta(R_{\max})|}_{\lambda} + |\delta(R)| \geq \underbrace{|\delta(R_{\max} \cap R)|}_{\geq \lambda} + |\delta(R_{\max} \cup R)|$$

$$\Downarrow$$

$$|\delta(R_{\max} \cup R)| \leq |\delta(R)|$$

$$\Downarrow$$

If $R \neq R_{\max} \cup R$, then $\delta(R)$ is not important.

Thus the important (X, Y)- and $(R_{\max}, Y)$-separators are the same.

$\Rightarrow$ We can assume $X = R_{\max}$.

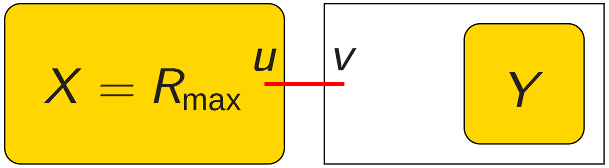
Important separators

Lemma: There are at most 4^k important (X, Y)-separators of size at most k.

Search tree algorithm for finding all these separators:

An (arbitrary) edge uv leaving $X = R_{\max}$ is either in the separator or not.

Branch 1: If $uv \in S$, then $S \setminus uv$ is an important (X, Y)-separator of size at most $k - 1$ in $G \setminus uv$.



Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$-separator of size at most k in G.

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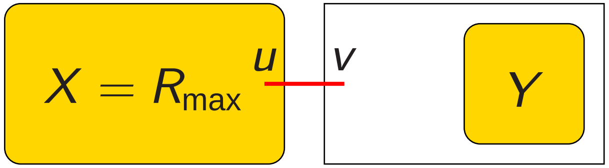
$\Rightarrow$ k decreases by one, λ decreases by at most 1.



Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$-separator of size at most k in G.

$\Rightarrow$ k remains the same, λ increases by 1.

The measure $2k - \lambda$ decreases in each step.

$\Rightarrow$ Height of the search tree $\leq 2k \Rightarrow \leq 2^{2k} = 4^k$ important separators.

Algorithm for MULTIWAY CUT

1. If every vertex of T is in a different component, then we are done.
2. Let $t \in T$ be a vertex with that is not separated from every $T \setminus t$.
3. Branch on a choice of an important $(t, T \setminus t)$ separator S of size at most k.
4. Set $G := G \setminus S$ and $k := k - |S|$.
5. Go to step 1.

We branch into at most 4^k directions at most k times.

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We branch into at most 4^k directions at most k times.
Better estimate of the search tree size:

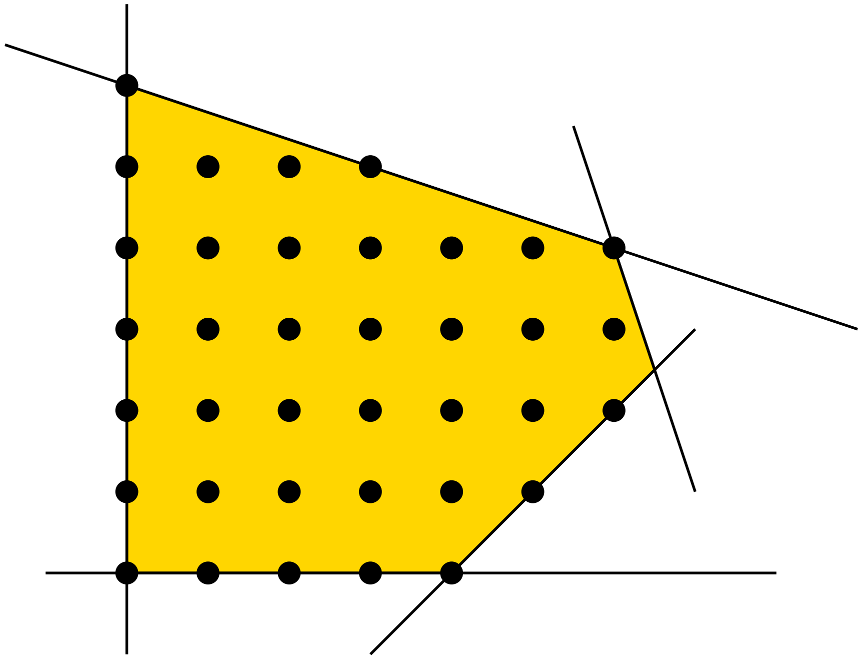
- When choosing the important separator, $2k - \lambda$ decreases at each branching, until λ reaches 0.
- When choosing the next vertex t, λ changes from 0 to positive, thus $2k - \lambda$ does not increase.

Size of the search tree is at most $2^{2k} = 4^k$.

Other separation problems

- Some other variants:
  - $|T|$ as a parameter
  - MULTITERMINAL CUT: pairs $(s_1, t_1), \ldots, (s_\ell, t_\ell)$ have to be separated.
  - Directed graphs
  - Planar graphs
- Useful for deletion-type problems such as DIRECTED FEEDBACK VERTEX SET (via iterative compression).
- Important separators: is it relevant for a given problem?

Integer Linear Programming



Integer Linear Programming

Linear Programming (LP): important tool in (continuous) combinatorial optimization. Sometimes very useful for discrete problems as well.

$$\max c_1x_1 + c_2x_2 + c_3x_3$$

$$\text{s.t.}$$

$$x_1 + 5x_2 - x_3 \leq 8$$

$$2x_1 - x_3 \leq 0$$

$$3x_2 + 10x_3 \leq 10$$

$$x_1, x_2, x_3 \in \mathbb{R}$$

Fact: It can be decided if there is a solution (feasibility) and an optimum solution can be found in polynomial time.

Integer Linear Programming

Integer Linear Programming (ILP): Same as LP, but we require that every x_i is integer.

Very powerful, able to model many NP-hard problems. (Of course, no polynomial-time algorithm is known.)

Theorem: ILP with p variables can be solved in time $p^{O(p)} \cdot n^{O(1)}$.

Closest String

Task: Given strings $s_1, \dots, s_k$ of length L over alphabet Σ, and an integer d, find a string s (of length L) such that $d(s, s_i) \leq d$ for every $1 \leq i \leq k$.

Note: $d(s, s_i)$ is the Hamming distance.

Theorem: Closest String parameterized by k is FPT.

Theorem: Closest String parameterized by d is FPT.

Theorem: Closest String parameterized by L is FPT.

Theorem: Closest String is NP-hard for $\Sigma = \{0, 1\}$.

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Closest String

An instance with $k = 5$ and a solution for $d = 4$:

s_1	CBDCCACBB
s_2	ABDBCABDB
s_3	CDDBACCBD
s_4	DDABACCBD
s_5	ACDBDDCBC
	ADDBCACBD

Each column can be described by a partition $\mathcal{P}$ of $[k]$.

The instance can be described by an integer $c_{\mathcal{P}}$ for each partition $\mathcal{P}$: the number of columns with this type.

Closest String

An instance with $k = 5$ and a solution for $d = 4$:

s_1	**CB**D**C**CACB**B**
s_2	A**B**DBCA**BDB**
s_3	**C**DDB**AC**CBD
s_4	**D**D**A**B**AC**CBD
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CLOSEST STRING

Each column can be described by a partition $\mathcal{P}$ of $[k]$.
The instance can be described by an integer $c_{\mathcal{P}}$ for each partition $\mathcal{P}$: the number of columns with this type.

Describing a solution: If C is a class of $\mathcal{P}$, let $x_{\mathcal{P},C}$ be the number of type $\mathcal{P}$ columns where the solution agrees with class C.

There is a solution iff the following ILP has a feasible solution:

$$\sum_{C \in \mathcal{P}} x_{\mathcal{P},C} \leq c_{\mathcal{P}} \qquad \forall \text{partition } \mathcal{P}$$

$$\sum_{i \notin C, C \in \mathcal{P}} x_{\mathcal{P},C} \leq d \qquad \forall 1 \leq i \leq k$$

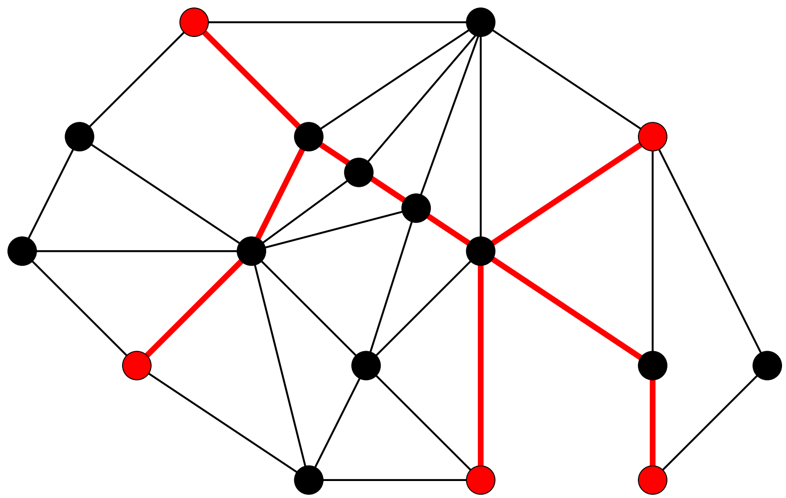
$$x_{\mathcal{P},C} \geq 0 \qquad \forall \mathcal{P}, C$$

Number of variables is $\leq B(k) \cdot k$, where $B(k)$ is the no. of partitions of $[k]$
$\Rightarrow$ The ILP algorithm solves the problem in time $f(k) \cdot n^{O(1)}$.

STEINER TREE

STEINER TREE

Task: Given a graph G with weighted edges and a set S of k vertices, find a tree T of minimum weight that contains S.



Known to be NP-hard. For fixed k, we can solve it in polynomial time: we can guess the Steiner points and the way they are connected.

Theorem: STEINER TREE is FPT parameterized by $k = |S|$.

STEINER TREE

Solution by dynamic programming. For $v \in V(G)$ and $X \subseteq S$,

$c(v, X) :=$ minimum cost of a Steiner tree of X that contains v

$d(u, v) :=$ distance of u and v

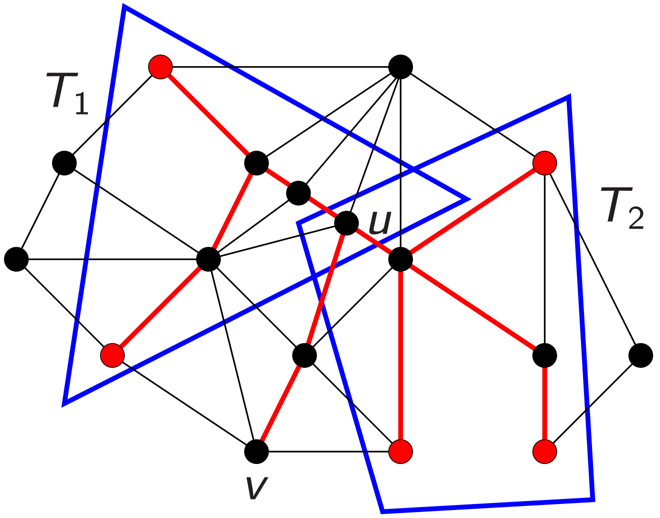
Recurrence relation:

$$c(v, X) = \min_{\substack{u \in V(G) \\ \emptyset \subset X' \subset X}} c(u, X' \setminus u) + c(u, (X \setminus X') \setminus u) + d(u, v)$$

Recurrence relation:

$$c(v, X) = \min_{\substack{u \in V(G) \\ \emptyset \subset X' \subset X}} c(u, X' \setminus u) + c(u, (X \setminus X') \setminus u) + d(u, v)$$

- $\leq$: A tree T_1 realizing $c(u, X' \setminus u)$, a tree T_2 realizing $c(u, (X \setminus X') \setminus u)$, and the path uv gives a (superset of a) Steiner tree of X containing v.

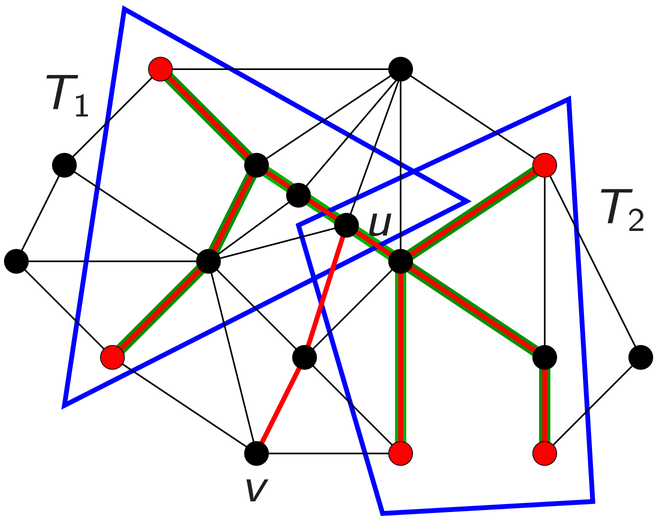


STEINER TREE

Recurrence relation:

$$c(v, X) = \min_{\substack{u \in V(G) \\ \emptyset \subset X' \subset X}} c(u, X' \setminus u) + c(u, (X \setminus X') \setminus u) + d(u, v)$$

- $\geq$: Suppose T realizes $c(v, X)$, let T' be the minimum subtree containing X. Let u be a vertex of T' closest to v. If $|X| > 1$, then there is a component C of $T \setminus u$ that contains a subset $\emptyset \subset X' \subset X$ of terminals. Thus T is the disjoint union of a tree containing $X' \setminus u$ and u, a tree containing $(X \setminus X') \setminus u$ and u, and the path uv.



STEINER TREE

Recurrence relation:

$$c(v,X) = \min_{\substack{u\in V(G)\\ \emptyset\subset X'\subset X}} c(u,X'\setminus u) + c(u,(X\setminus u)\setminus X') + d(u,v)$$

Running time:

$2^k|V(G)|$ variables $c(v,X)$, determine them in increasing order of $|X|$. Variable $c(v,X)$ can be determined by considering $2^{|X|}$ cases. Total number of cases to consider:

$$\sum_{X\subseteq T} 2^{|X|} = \sum_{i=1}^{k} \binom{k}{i} 2^i \le (1+2)^k = 3^k.$$

Running time is $O^*(3^k)$.

Note: Running time can be reduced to $O^*(2^k)$ with clever techniques.

Conclusions

- Many nice techniques invented so far — and probably many more to come.
- A single technique might provide the key for several problems.
- How to find new techniques? By attacking the open problems!
- Next (January):
  - Treewidth
  - Hardness theory