SPS6

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11:16 AM

PHY 356F
Problem set 6: Solutions

1. We have

$$\hat{\mathcal{P}}_- = \int_{-L/2}^{0} dx\, |x\rangle \langle x|,$$
$$\hat{\mathcal{P}}_+ = \int_{0}^{L/2} dx\, |x\rangle \langle x|.$$

(a) We want to show $\hat{\mathcal{P}}_-^2 = \hat{\mathcal{P}}_-$, and $\hat{\mathcal{P}}_+^2 = \hat{\mathcal{P}}_+$. We have

$$\begin{aligned}
\hat{\mathcal{P}}_-^2 &= \left(\int_{-L/2}^{0} dx\, |x\rangle \langle x|\right)\left(\int_{-L/2}^{0} dx'\, |x'\rangle \langle x'|\right) \\
&= \int_{-L/2}^{0}\int_{-L/2}^{0} dx dx'\, |x\rangle \langle x|x'\rangle \langle x'| \\
&= \int_{-L/2}^{0}\int_{-L/2}^{0} dx dx'\, |x\rangle \delta(x-x') \langle x'| \\
&= \int_{-L/2}^{0} dx\, |x\rangle \langle x| = \hat{\mathcal{P}}_-,
\end{aligned}$$

and similarly

$$\begin{aligned}
\hat{\mathcal{P}}_+^2 &= \left(\int_{0}^{L/2} dx\, |x\rangle \langle x|\right)\left(\int_{0}^{L/2} dx'\, |x'\rangle \langle x'|\right) \\
&= \int_{0}^{L/2}\int_{0}^{L/2} dx dx'\, |x\rangle \langle x|x'\rangle \langle x'| \\
&= \int_{0}^{L/2}\int_{0}^{L/2} dx dx'\, |x\rangle \delta(x-x') \langle x'| \\
&= \int_{0}^{L/2} dx\, |x\rangle \langle x| = \hat{\mathcal{P}}_+.
\end{aligned}$$

(b) The probability that outcome (-1) is the result is

$$\begin{aligned}
\left\langle \psi|\hat{\mathcal{P}}_-|\psi \right\rangle &= \left\langle 1|\hat{\mathcal{P}}_-|1 \right\rangle \\
&= \int_{-L/2}^{0} dx\, \langle 1|x\rangle \langle x|1\rangle \\
&= \int_{-L/2}^{0} dx\, |\psi_1(x)|^2 \\
&= \frac{2}{L}\int_{-L/2}^{0} dx \cos^2\left(\frac{\pi x}{L}\right) \\
&= \frac{2}{L}\left(\frac{1}{2}x + \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right)_{-L/2}^{0} \\
&= \frac{2}{L}\left(0 - \left(-\frac{L}{4}\right)\right) \\
&= \frac{1}{2},
\end{aligned}$$

as might be expected from symmetry.

(c) If the result is (-1), we take

$$|\psi\rangle \to \frac{\hat{\mathcal{P}}_- |\psi\rangle}{\left\langle \psi|\hat{\mathcal{P}}_-|\psi\right\rangle^{1/2}},$$

so after the measurement the ket is

$$\begin{aligned} |\psi\rangle &= \frac{\hat{\mathcal{P}}_- |1\rangle}{\left\langle 1|\hat{\mathcal{P}}_-|1\right\rangle^{1/2}} \\ &= \sqrt{2}\hat{\mathcal{P}}_- |1\rangle \\ &= \sqrt{2}\int_{-L/2}^{0} dx\, |x\rangle \langle x|1\rangle \\ &= \sqrt{2}\int_{-L/2}^{0} dx\, |x\rangle\, \psi_1(x) \\ &= \frac{2}{\sqrt{L}}\int_{-L/2}^{0} dx\, |x\rangle \cos\left(\frac{\pi x}{L}\right). \end{aligned} \tag{1}$$

The probability that E_2 will then be the result in an energy measurement is $|\langle 2|\psi\rangle|^2$. We have

$$\begin{aligned} \langle 2|\psi\rangle &= \frac{2}{\sqrt{L}}\int_{-L/2}^{0} dx\, \langle 2|x\rangle \cos\left(\frac{\pi x}{L}\right) \\ &= \frac{2}{\sqrt{L}}\int_{-L/2}^{0} dx \psi_2^*(x) \cos\left(\frac{\pi x}{L}\right) \\ &= \frac{2\sqrt{2}}{L}\int_{-L/2}^{0} dx \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) \\ &= \frac{2\sqrt{2}}{L}\left(-\frac{\cos\left(\frac{\pi x}{L}\right)}{\frac{2\pi}{L}} - \frac{\cos\left(\frac{3\pi x}{L}\right)}{\frac{6\pi}{L}}\right)_{-L/2}^{0} \\ &= \frac{2\sqrt{2}}{L}\left(\left(-\frac{1}{\frac{2\pi}{L}} - \frac{1}{\frac{6\pi}{L}}\right) - \left(-\frac{\cos\left(\frac{\pi}{2}\right)}{\frac{2\pi}{L}} - \frac{\cos\left(\frac{3\pi}{2}\right)}{\frac{6\pi}{L}}\right)\right) \\ &= \frac{2\sqrt{2}}{L}\left(-\frac{1}{\frac{2\pi}{L}} - \frac{1}{\frac{6\pi}{L}}\right) \\ &= -\frac{2\sqrt{2}}{L}\left(\frac{L}{2\pi} + \frac{L}{6\pi}\right) \\ &= -\frac{2\sqrt{2}}{L}\frac{2L}{3\pi} = -\frac{4\sqrt{2}}{3\pi}. \end{aligned}$$

So the probability of finding E_2 is

$$|\langle 2|\psi\rangle|^2 = \frac{32}{9\pi^2}\,(\approx 0.36)\,.$$

Notes to the interested student:

This problem illustrates two interesting points. The first is that projector value measures (PVM's) should be considered, at least operationally, as more fundamental than Hermitian operators. For it is really the PVM $\left\{\hat{\mathcal{P}}_-, \hat{\mathcal{P}}_+\right\}$ that characterizes the measurement. We chose to associate (-1) and $(+1)$ with the results, but we could – for example – have chosen to associate (-2) and $(+2)$ with the results. Then the Hermitian operator would have been

$$\hat{A}' = (-2)\hat{\mathcal{P}}_- + (+2)\hat{\mathcal{P}}_+.$$

But this would have just been a "relabelling" of the names we give to the outcomes, a sort of change in language but nothing else. In general, the eigenvalues that appear in the spectral representation of a Hermitian operator are just a particular "labelling" of the outcomes. It is really the projectors that appear in the spectral representation – the set of which define the PVM – that are more fundamental.

The second interesting point is that, for position measurements, the kind of PVM $\left\{\hat{\mathcal{P}}_-, \hat{\mathcal{P}}_+\right\}$ that we introduced here is somewhat unrealistic, at least if later measurements are to be considered. For the PVM produces, after measurement, a ket such as (1) with a coordinate representation that is discontinuous:

$$\begin{aligned}\psi(x) &= \langle x|\psi\rangle = \frac{2}{\sqrt{L}}\int_{-L/2}^{0} dx' \langle x|x'\rangle \cos\left(\frac{\pi x'}{L}\right) \\ &= \frac{2}{\sqrt{L}}\int_{-L/2}^{0} dx' \delta(x-x') \cos\left(\frac{\pi x'}{L}\right),\end{aligned}$$

so

$$\begin{aligned}\psi(x) &= \frac{2}{\sqrt{L}}\cos\left(\frac{\pi x}{L}\right) \text{ for } -L/2 < x < 0, \\ &= 0 \text{ otherwise.}\end{aligned} \tag{2}$$

This wave function is discontinuous at the origin. And for such a ket the expectation value of the energy is infinite,

$$\left\langle \psi|\hat{H}|\psi\right\rangle \to \infty.$$

(To see this, consider the expectation value of the energy for a smooth but rapid variation in $\psi(x)$ near the origin, and then consider the limit to (2)). So if $\left\{\hat{\mathcal{P}}_-, \hat{\mathcal{P}}_+\right\}$ really characterized the position measurement, then the average energy found in a succeeding energy measurement would be infinite! For more proper description of actual position measurements, one must move to "positive operator valued measures" (POVMs) which are generalizations of PVMs that we will not have time to consider in this course.

2. (a) We have $\left[\hat{A}, \hat{B}\right] = i\hbar\hat{I}$, showing that the result is true for $n = 1$ (we have used $\hat{A}^0 = \hat{I}$). Now consider the case $n+1$:

$$\begin{aligned}\left[\hat{A}^{n+1}, \hat{B}\right] &= \hat{A}\left[\hat{A}^n, \hat{B}\right] + \left[\hat{A}, \hat{B}\right]\hat{A}^n \\ &= \hat{A}\left(i\hbar n\hat{A}^{n-1}\right) + (i\hbar)\hat{A}^n \\ &= i\hbar(n+1)\hat{A}^n\end{aligned}$$

where we have used the assumed formula $\left[\hat{A}^n, \hat{B}\right] = i\hbar n\hat{A}^{n-1}$ in the second step. The fact that the formula is true for $n = 1$, and that the assumption of its validity for n implies that it is valid for $n+1$ means that it is true for all integers n.

(b) Note that, if the function $f(a)$ can be written as a Taylor series, i.e. $f(a) = \sum_{n=0}^{\infty} f^{(n)}(0)a^n/n!$, this result implies that

$$\left[f(\hat{A}), \hat{B}\right] = i\hbar f'(\hat{A}).$$

Thus the result follows immediately.

(c)

$$\begin{aligned}\hat{H}\exp(i\omega\hat{T})|E\rangle &= \left(\hat{H}\exp(i\omega\hat{T}) - \exp(i\omega\hat{T})\hat{H} + \exp(i\omega\hat{T})\hat{H}\right)|E\rangle \\ &= \left(\left[\hat{H}, \exp(i\omega\hat{T})\right] + \exp(i\omega\hat{T})E\right)|E\rangle \\ &= \left(\hbar\omega\exp(i\omega\hat{T}) + \exp(i\omega\hat{T})E\right)|E\rangle \\ &= (E+\hbar\omega)\exp(i\omega\hat{T})|E\rangle\end{aligned}$$

Thus, we have $\exp(i\omega\hat{T})|E\rangle = c|E + \hbar\omega\rangle$, where c is a constant. Renormalizing, and using the fact that $\hat{T}$ is Hermitian, (and hence $\exp(i\omega\hat{T})$ is unitary), we find that $|c| = 1$. The phase can be set to zero without loss of generality.