Ph205 Notes

November 13, 2012

Contents

1 Intro

QFT is about the fields and how particles affect those fields

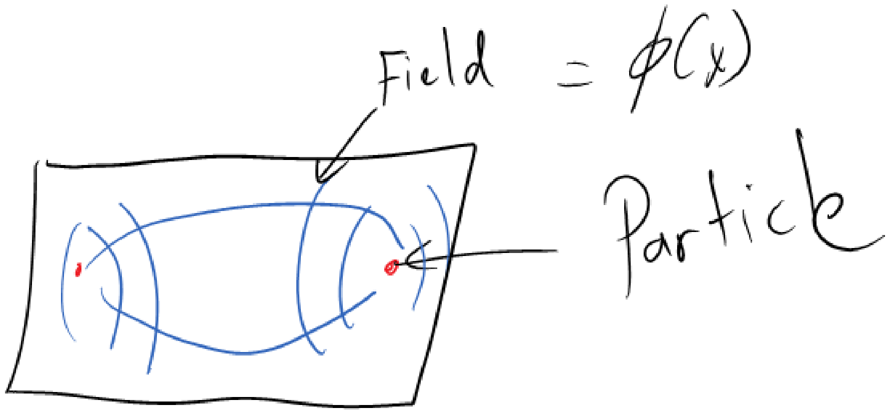

Figure 1.0.1: Particles and fields.

1.1 Key concepts

Key players in the theory of QFT are:

1. The function "field" $\phi(x)$
2. The action $S[\phi]$.

2 Classical Field Theory/Terminology that will be seen in QFT

Intro/Stuff that will be useful in QFT:

1. Wave physics and many waves equations
2. Theory of solitons. in 1d, solitons are kinks. 2d are vortices. 3d are monopoles and 4d instantons. Solitons are tightly related to symmetry breaking (e.g. 2.1.9).

2.1 The action

The action (which is a functional) is defined by

$$S[\phi] = \int_{M^n} \mathcal{L}[\phi(x)]\, d^n x \tag{2.1.1}$$

where $\mathcal{L}$ is the lagrangian that describes the system,

$$\mathcal{L} = T - V$$

(T is K.E. and V is P.E.), $\phi(x)$ is the "field" and M^n (Minkowsky space of dimension n) is the spactime and combines space dimensions (1,2,3) with time dimension (and x is a coordinate of that spacetime).

The action and the Lagrangian completely characterize the system.

2.1.1 EOMS

How to reformulate functional minimization problems (like action) into PDEs:

$$\partial_\phi S = \partial_\phi \mathcal{L} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{2.1.2}$$

where $\partial_\phi S$ kind of means $\phi(x) \to \phi(x) + \partial\phi(x)$.

2.1.2 Examples of actions

2.1.2.1 Klein-Gordon theory

Consider the following action

$$S = \int d^n x \left(\underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)}_{K.E.} - \underbrace{\frac{m}{2}\phi^2(x)}_{P.E.} \right), \tag{2.1.3}$$

where m is a constant/parameter. The EOMs (equation (2.1.2)) become

$$\left(\Box + m^2\right)\phi(x) = 0, \tag{2.1.4}$$

where $\Box = \partial_t^2 - \nabla^2$. To obtain the EOMs we differentiated a quadratic function and so we obtained a linear PDE (?).The solutions are of the form:

$$\phi = e^{ikx + i\omega t} \tag{2.1.5}$$

2.1.2.2 4th order theory

A 4th order (in ϕ) action :

$$S=\int d^{n}x\left(\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi-\underbrace{\frac{\lambda}{4}\left(\phi^{2}-a^{2}\right)^{2}}_{P.E.}\right) \tag{2.1.6}$$

(this is relevant to describing dynamics of Higgs-Boson). The potential is shown in the figure:

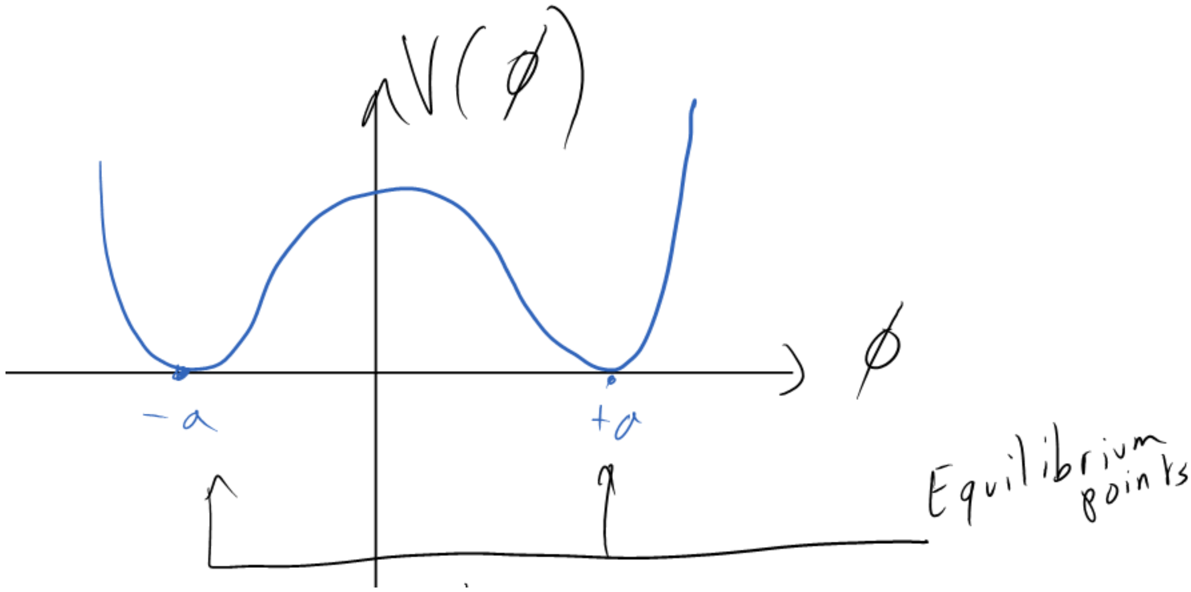

Figure 2.1.1: Klein-Gordon potential and equilibrium points

. Eoms (2.1.2) becomes that of Solitons with dimension $n=1$:

$$\Box\phi-\lambda\phi\left(\phi^{2}-a^{2}\right)=0 \tag{2.1.7}$$

(this is a cubic equation - called the Klein-Gordon equation - because we differentiated a quadratic Lagrangian and assume the K.E. part is obtained just like partial differentiating wrt to ϕ the K.E. term in the Lagrangian). If we assume $n=1$:

$$\nabla_{x}^{2}\phi-\lambda\phi\left(\phi^{2}-a^{2}\right)=0, \tag{2.1.8}$$

where $\nabla_{x}^{2}=d^{2}/dx^{2}$. Notice that this is a symmetric equation in $\phi\rightarrow-\phi$ but we will see that the soliton breaks the symmetry (it has to (??)). This equation turns out to have the solution

$$\phi\left(x\right)=\pm a\tanh\left(\sqrt{\frac{\lambda}{2}}ax\right) \tag{2.1.9}$$

and notice $\phi\left(x=\pm\infty\right)=\pm a$ and we say that ϕ migrates from one vacuum (which we define to be $\pm a$) to another vacuum. Such behavior defines a soliton. Also, the solution if not even like the EOM.

2.1.2.3 Sine-Gordon theory and Bogomolny Bound

Sine-Gordon theory:

$$S=\int d^{n}x\left(\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi-\left(1-\cos\phi\right)\right). \tag{2.1.10}$$

This gives the EOM:

$$\Box\phi+\sin\phi=0 \tag{2.1.11}$$

e.g. $\phi_{tt}-\phi_{xx}+\sin\phi=0$. This has soliton solutions that interpolate (alternate) between $\phi=2\pi n$ and $n\in\mathbb{Z}$ (these are the minima of the potential). The trick of solving such soliton equations is the Bogomolny Bound:

$$\begin{aligned}E &= T+V\\ &= \int_{-\infty}^{\infty}dx\left(\frac{1}{2}\left(\partial_{t}\phi\right)^{2}+\frac{1}{2}\left(\partial_{x}\phi\right)^{2}+\left(1-\cos\phi\right)\right)\end{aligned}$$

(where we consider that the K.E. is the part that contains derivatives) where each term is positive (adding contributions to E - trying to find minimum). Assume that we have a single kink that moves from $2\pi m$ to $2\pi(m+1)$. The potential will look like:

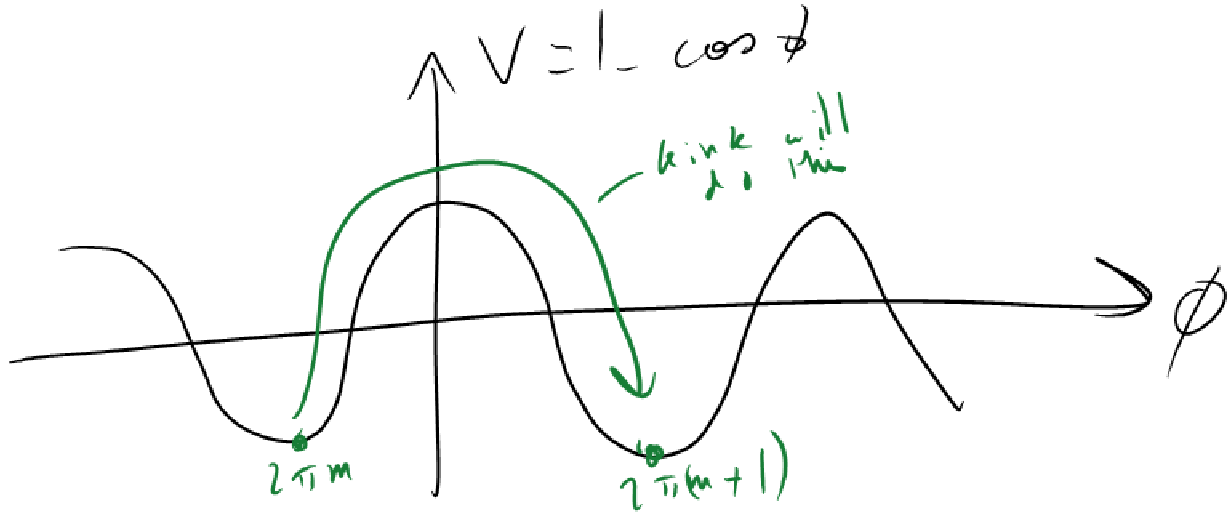

Figure 2.1.2: Sine-Gordon equation and assumed behavior of ϕ

$$
\begin{aligned}
E &\geq \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\phi_x^2 + (1-\cos\phi)\right) \\
&= \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\phi_x^2 + 2\sin^2\left(\frac{\phi}{2}\right)\right] dx \\
&= \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\left[\phi_x \pm 2\sin\left(\frac{\phi}{2}\right)\right]^2 \mp 2\sin\left(\frac{\phi}{2}\right)\underbrace{\phi_x}_{\frac{d\phi}{dx}}\right) dx \\
&= \left(\int_{-\infty}^{\infty} dx \frac{1}{2}\left[\phi_x \pm 2\sin\left(\frac{\phi}{2}\right)\right]^2\right) \pm 4\cos\left(\frac{\phi}{2}\right)\Big|_{-\infty}^{\infty} \\
&\geq 8
\end{aligned}
$$

In order to saturate the bound (i.e. to have $E = 8$) we have a 1st order equation (BPS equations):

$$\phi_x \pm 2\sin\frac{\phi}{2} = 0 \tag{2.1.12}$$

Assume ϕ increases then (direction that we move from one vacuum to another)

$$\phi_x - 2\sin\frac{\phi}{2} = 0 \tag{2.1.13}$$

We then obtin that

$$\int dx = \int \frac{d\phi}{2\sin\frac{\phi}{2}} = \log\tan\left(\frac{\phi}{4}\right) \tag{2.1.14}$$

and then we invert to obtain ϕ in terms of x:

$$\phi = 4\arctan e^{x-x_0} \tag{2.1.15}$$

This solution lends itself to interpreting ϕ as a particle: it is localized (arctan has an abrupt change - a kink- over a narrow window) and has a center of mass given by x_0 (this is the center of the region in which the kink is localized). With this solution it can be verified that $E = 8$.

2.1.2.4 Chern-Simons theory

Consider

$$S = \frac{k}{4\pi} \int_{M^3} Tr\left(AdA + A \wedge A \wedge A\right) \tag{2.1.16}$$

where M^3 is a 3d spacetime, A is the gauge "field" and d and $\wedge$ are differential geometry operators. EOMs will not be linear because the lagrangian is not quadratic.

$$\frac{\partial S}{\partial A} = 0 \tag{2.1.17}$$

implies that ('Euler Lagrange equations)

$$F_A \equiv dA + A \wedge A = 0 \tag{2.1.18}$$

For example, dA is the magnetic field and F_A is the field strength.

2.1.2.5 Yang-Mills theory

Consider

$$S\left[A\right] = Im\tau \int Tr\left(F_A \wedge F_A + iF_A \wedge *F_A\right), \tag{2.1.19}$$

where $Im\tau > 0$, $\tau = \theta/2\pi + 4\pi i/g^2$ with g is called the coupling constant.

3 Path from physics of space-time (special relativity) to quantization.

Space-time M^n →Function on M^n ("fields")→Action $S[\phi]$ and Lagrangian $L[\phi]$→"Quantization"

3.1 Space-time/Notation

3.1.1 Poincare and Lorentz symmetry

We will use space-time coordinates: x^μ, where greek letters (e.g. μ,ν) go from $0,...,n-1$ and label space and time. We have

$$\begin{aligned} x^\mu &= \left(x^0, x^i\right) && (3.1.1)\\ &\equiv \left(x^0, \vec{x}\right) \equiv \left(ct, x^i\right) && (3.1.2)\end{aligned}$$

where roman letters (e.g. i,j) go from $1,...,n-1$ and label space. Note that we often set fundamental constants such as c to be equal to 1 and so

$$x^0 = t. \quad \text{(units such as } c=1\text{)} \tag{3.1.3}$$

3.1.2 Metric to calculate distance

The metric is $g_{\mu\nu}$ and the *distance/interval* is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{3.1.4}$$

3.1.3 $n = 4$

There are many implications of $n = 4$:

$n=4$ Minkowski space	$n=4$ Euclidean space
$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
$x_\mu = g_{\mu\nu}x^\nu = \left(-x^0, x^i\right)$where $x^\mu = \left(x^0, x^i\right)$	$x_\mu = g_{\mu\nu}x^\nu = \left(x^0, x^i\right)$
$g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$ which is a Kronecker delta (=1 if $\mu=\rho$ and 0 otherwise)	$g^{\mu\nu}g_{\nu\rho} = \delta_{\mu\rho}$(??)

3.1.4 Antisymmetric tensor

Consider $\epsilon^{\mu_1\cdots\mu_n}$ and is a totally antisymmetric tensor .We define

$$\epsilon^{01..(n-1)} = 1 \tag{3.1.5}$$

and this ($+\epsilon$ is totally antisymmetric) completely defines ϵ

3.2 Transformations

Physics is the same in all frames:

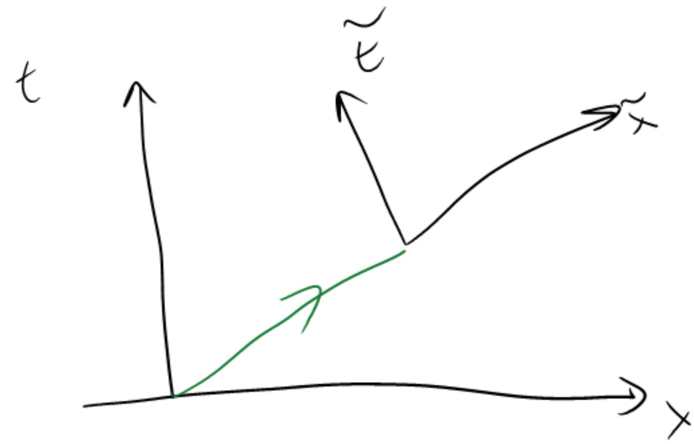

Figure 3.2.1: Change of frames

We have $x^\mu \to \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$. This transformation is called a Poincare transformation. and consists of two components:

1. $\Lambda^\mu_\nu x^\nu$ is a Lorentz transformation. The number of generators (the number of parameters needed to specify a Lorentz transformation) is given by $n\left(n-1\right)/2$ (this will be derived soon).

2. $a^\mu$is a translation. The number of generators for space time shifts it is n.

The transformations should preserve the intervals:

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu && (3.2.1)\\ &\to g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_d dx^\rho dx^d && (3.2.2)\\ &= g_{\mu\nu}dx^\mu dx^\nu && (3.2.3)\end{aligned}$$

(notice that translations have no effect as dx is a differential/difference so the translation a will cancel out). The last equality holds if

$$g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_d = g_{\rho d}, \tag{3.2.4}$$

which imposes some constraints on Λ.

For infinitesimal Lorentz transformations, $\Lambda^\mu_\nu = \delta^\mu_\nu + \delta\omega^\mu_\nu$, where δ^μ_ν is identity $\delta\omega$ describes a small rotation. Invariance (equation (3.2.4)) implies

$$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu} \tag{3.2.5}$$

and so there are $n\left(n-1\right)/2$ Lorentz generators because this is the number of parameters required to specify a $n\times n$ anti-symmetric matrix.

3.2.1 Remarks

1. Poincare (Lorentz) transformations form a group (if we compose two transformations, we will get a transformation of the same kind).

2. We found that in n space-time dimensions $\dim\left(\text{Lorentz}\right) = n\left(n-1\right)/2$ and that $\dim\left(\text{Poincare}\right) = n\left(n-1\right)/2 + n$.

Example 1. For $n=2$, Λ is a 2×2 matrix. Hence, $\dim\left(\text{Lorentz}\right) = n\left(n-1\right)/2 = 1$ and is parametrized by one variable θ:

$$\Lambda = \pm\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}. \tag{3.2.6}$$

3.2.2 Indefinite orthogonal group

In general, the group of linear transformations

$$x^\mu \to \Lambda^\mu_\nu x^\nu \tag{3.2.7}$$

that preserves a quadratic form $Q\left(x\right)$ of sign $\left(p,q\right)$ (it has p positive eigenvalues and q negative eigenvalues, and $p+q=n$)

$$Q\left(x\right) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2. \tag{3.2.8}$$

Such transformations are called *(indefinite) orthogonal group* $O\left(p,q\right)$. We have that $\dim O\left(p,q\right) = n\left(n-1\right)/2$.

3.2.3 The Lorentz group

The Lorentz group is

$$\text{Lorentz group} = \begin{cases} O(n,0) & \text{corresponds to Euclidean space-time} \\ O(n-1,1) & \text{corresponds to Mikowski space-time} \end{cases} \tag{3.2.9}$$

3.2.4 Proper/Improper Lorentz transformations

Since $\eta_{\mu\nu}$ (which refers to Mikowski or Euclidean space metric - although in many papers it refers to just Mikowski metric) satisfies

$$\eta_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{d} = \eta_{\rho d} \tag{3.2.10}$$

taking the determinant we have that

$$\det\Lambda = 1 \quad \text{"proper"} \tag{3.2.11}$$
$$\text{or} \tag{3.2.12}$$
$$\det\Lambda = -1 \quad \text{"improper"} \tag{3.2.13}$$

Proper Lorentz transformations form a group and is a subgroup of the group of Lorentz transformations. It is denoted $SO(n)$ for Euclidean space and $SO(n-1,1)$ for Minkowski space-time.

Improper Lorentz transformations do not form a group (because composing two improper transformations will give us a proper transformation). Members of improper Lorentz transformations include

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \text{(Parity transformtion)}$$
$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{(Time reversal)}$$

the latter is called time reversal because it flips the sign of time. We have $P^2 = T^2 = 1$.

3.2.5 Generators of Transformations

How do we implement transformations (such as Lorentz transformations) in a Lagrangian, etc ...

3.2.5.1 Translation generator

Let's shift by $a^\mu : x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu$. The corresponding operator is

$$P(a) = \exp\left(iP_\mu a^\mu/\hbar\right), \tag{3.2.14}$$

where P_μ is the generator of infinitesimal translations: $P_\mu = -i\hbar\delta_\mu$. Indeed

$$P(\delta a) = 1 - \frac{i}{\hbar}\delta a^\mu P_\mu. \tag{3.2.15}$$

We have that $P^0 = -P_0 \equiv H$ is the Hamiltonian (time translation is related to conserved quantities) - note we set $\hbar = 1$.

3.2.5.2 Lorentz generator

Lorentz transformations are realized by unitary operators $U(\Lambda)$ that obey

$$U\left(\Lambda'\Lambda\right) = U\left(\Lambda'\right)U(\Lambda) \tag{3.2.16}$$

(as Lorentz transformations form a group and this is a representation of the group).

Focusing on infinitesimal transformations by

$$\Lambda = 1 + \delta\omega \tag{3.2.17}$$

We have that (definition of M)

$$U(1+\delta\omega) \equiv 1 + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \tag{3.2.18}$$

(ω carries two indices so $\delta\omega_{\mu\nu}M^{\mu\nu}$ is a contraction that gives us a scalar). Note that such a choice satisfies equation (3.2.16) because $\delta\omega$ is antisymmetric. Furthermore, we chose to contract μ and ν because we wanted a form for U that is universal.

U should also obey

$$U(\Lambda)^{-1} U\left(\Lambda^{'}\right) U(\Lambda) = U\left(\Lambda^{-1}\Lambda^{'}\Lambda\right) \tag{3.2.19}$$

Writing Λ in infinitesimal form: $1+\delta\omega$, $\Lambda^{'} = 1+\delta\omega^{'}$ and expanding both sides, we get (a non-trivial solution that satisfies the above equation)

$$\left[M^{\mu\nu}, M^{\rho d}\right] = i\left(g^{\mu\rho}M^{\nu d} - g^{\nu\rho}M^{\mu d} - g^{\mu d}M^{\nu\rho} + g^{\nu d}M^{\nu\rho}\right) \tag{3.2.20}$$

This is called the Lie algebra of Lorentz transformation.

3.2.5.3 Generator of Poincare transformation

Following the same exercise, we can get for a full Poincare transformation that

$$\left[P^{\mu}, M^{\rho d}\right] = i\left(g^{\mu d}P^{\mu} - g^{\mu\rho}P^{d}\right) \tag{3.2.21}$$

$$\left[P^{\mu}, P^{\nu}\right] = 0 \tag{3.2.22}$$

This is called the Lie algebra of Poincare transformation.

In M^{n} space-time, there n P^{μ} parameters and the number of $M^{\mu\nu}$ is $n\left(n-1\right)/2$.

4 Quantum Field Theory (QFT)

In contrast with classical field theory, QFT uses functionals (such as the action) directly rather than EOMs.

4.1 The Partition function

Consider the particle-wave duality (that occurs like for example in the double slit experiment - http://en.wikipedia.org/wiki/File:Doubleslit3Dspectrum.gif - for electrons). The wave continuous part will help us calculate probabilities for the discrete entity: the particle.

The problem of calculating probabilities relies on the fundamental ingredient of the partition function:

$$Z = \int_{\text{possible fields}} \mathcal{D}\phi e^{-S[\phi]/\hbar}. \tag{4.1.1}$$

Since $\hbar$ is very small, we can often approximately do this integral in the saddle point approximation (we find configurations to satisfy $\partial_\phi S[\phi] = 0$).

4.2 Operators

A fundamental ingredient in the theory of QFT is the operator $\mathcal{O}$, which can relay to us information (analogy given: thermometer), e.g. $\mathcal{O}(x) = \phi(x)$ or $\phi^k(x)$, etc ... $\mathcal{O}$ are combinations of basic building blocks like the field ϕ. We can have also more complicated non-local operators such as $\mathcal{O} = Tr_R Holy(A(x))$, where R characterizes a knot (so $\mathcal{O}$ is a function of not just one point $\vec{x}$).

4.3 Analogy between stat Mechanics and QFT

Two types of randomness appear in physics:

1. Quantum, when certain parameters are in the order of $\hbar$. There will be a characteristic partition function: equation (4.1.1).
2. Thermal, with the characteristic parameter given by T = Temperature = β^{-1}. In statistical mechanics, $Z = \sum_{\text{all configurations}} e^{-\beta E}$, with $e^{-\beta E}$ being the statistical weight.

4.3.1 Ising Model

Consider

$$\sigma = \pm 1, \tag{4.3.1}$$

which characterizes spins on a lattice - to every point in space and time, a value is assigned. We can interpret σ as a field. We need to write down an action to describe the system. The analog of action in statistical mechanics is the energy:

$$E = -J \sum_{\langle i,j \rangle \text{ nearest neighbors}} \sigma_i \sigma_j - H \sum_i \sigma_i. \tag{4.3.2}$$

This is a good model for magnets if $J > 0$.

4.3.1.1 1D Ising Model

Assume a circular configuration of N spins with periodic conditions: $\sigma_i = \sigma_{i+N}$. as shown, for example, in the figure below

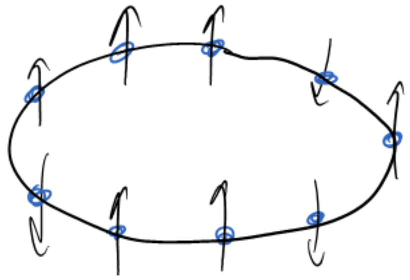

Figure 4.3.1: Example of a cicular Ising chain

We have that

$$Z = \sum_{\{\sigma\}} \exp\left(\beta J \sum_i \sigma_i \sigma_{j+1} + \beta H \sum_i \sigma_i\right).$$

This can be solved as we have a local interaction (i.e. the interaction the same wherever we look at it)

In the context of QFT, locality is a powerful concept and means that we have translational invariance.

To solve it, define the *transfer matrix*

$$\begin{aligned} T &= \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix} \\ &= \left\{ \left\langle \sigma \middle| T \middle| \sigma' \right\rangle \right\} \end{aligned}$$

and it specifies all possible values of the product $\left\langle \sigma \middle| T \middle| \sigma' \right\rangle$. Using T, we can rewrite the partition function to be

$$Z = Tr\left(T^N\right) \tag{4.3.3}$$

$$= \sum_{\sigma_1} \sum_{\sigma_2} \ldots \sum_{\sigma_N} T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} \ldots \tag{4.3.4}$$

The eigenvalues of T can be calculated to be

$$\lambda_{\pm} = e^{\beta J} \left[\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right] \tag{4.3.5}$$

Writing T in diagonal form, we have that

$$\begin{aligned} Z &= Tr\left(T^N\right) \\ &= \lambda_+^N + \lambda_-^N \approx \lambda_+^N \end{aligned}$$

for $N \gg 1$ and because $\lambda_+ > \lambda_-$.

4.3.1.2 Operators in the context of the Ising Model

One legitimate choice of an operator (and there is no unique choice) is

$$\mathcal{O} = \sigma_i. \tag{4.3.6}$$

We have that

$$\langle \mathcal{O} \rangle = \langle \sigma_i \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \sigma_i e^{-\beta E} \tag{4.3.7}$$

$$= \frac{1}{Z} Tr\left(T^N \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \tag{4.3.8}$$

$$= \frac{\sinh(\beta + 1)}{\sqrt{\sinh^2(\beta H) + e^{-4\beta J}}} \tag{4.3.9}$$

where the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflects the possible values of σ^*[1].

Exercise 1. Compute the correlation function $\langle \sigma_i \sigma_{i+r} \rangle$, which measures the (degree of) order of the system.
The answer is

$$\operatorname{cov}(\sigma_i, \sigma_{i+r}) = \frac{1}{1 + e^{4J\beta} \sinh^2(H\beta)} e^{r/\Delta l}, \tag{4.3.11}$$

where Δl is the correlation length

$$\Delta l \equiv \ln \left(\frac{\cosh(\beta H) + \sqrt{e^{-4\beta J} + \sinh^2(\beta H)}}{\cosh(\beta H) - \sqrt{e^{-4\beta J} + \sinh^2(\beta H)}} \right). \tag{4.3.12}$$

*

4.4 Perturbation theory (Feynmann diagrams)

In QFT, one can rarely compute the partition function exactly.
To do so, first decompose the action in the following way:

$$S = S_{free\text{(i..e quadratic)}} + S_{interaction}.$$

$S_{interaction}$ will often depend on small parameters of the theory, which allow us to carry out a perturbation calculation.

4.4.1 Examples

Example 2. As an example,

$$S[A] = \int \underbrace{AdA}_{S_{free}} + \underbrace{\frac{2}{3\sqrt{k}} A^3}_{S_{interaction}}. \tag{4.4.1}$$

Moreover, we can take $k \to \infty$, which will give us a small parameter and will allow us to do a nice perturbation. $k \to \infty$ is the 'nice' perturbative limit.

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[1] We have that

$$Tr\left(T^N\right) = \sum_{\sigma_1} \sum_{\sigma_2} \dots \sum_{\sigma_N} T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} \dots$$

so

$$? = \sum_{\sigma_1} \sum_{\sigma_2} \dots \sum_{\sigma_N} \sigma_2 T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} \dots$$

For simplicity, let $\sigma_i = \sigma_2$, then we claim that

$$\begin{aligned} \sum_{\sigma_2} \sigma_2 T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} &= \left[T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T \right]_{\sigma_1 \sigma_3}. \\ &\equiv (TGT)_{\sigma_1 \sigma_3}, \end{aligned}$$

where we defined

$$G \equiv \begin{pmatrix} 1 = \sigma_1 & 0 \\ 0 & -1 = \sigma_2 \end{pmatrix}$$

as

$$\begin{aligned} \left(T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T \right)_{ab} &= \sum_i T_{ai} (GT)_{ib} \\ &= \sum_{ij} T_{ai} G_{ij} T_{jb} \\ &= \sum_{ij} T_{ai} \sigma_i \delta_{ij} T_{jb} \\ &= \sum_i T_{ai} \sigma_i T_{ib} \end{aligned}$$

Hence,

$$\sum_{\sigma_1} \sum_{\sigma_2} \dots \sum_{\sigma_N} \sigma_l T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} \dots = Tr\left(T^l G T^k\right), \tag{4.3.10}$$

where l and k are some integers. Using the cyclic property of the trace $Tr\left(T^l G T^k\right) = Tr\left(G T^{k+l}\right) = Tr\left(G T^N\right)$.

Also,

Example 3. Take (and assume space consists of one point, i.e. $M^n = M^0$)

$$S = \underbrace{m^2\phi^2}_{S_{free}} + \underbrace{\delta m^2 \phi^2}_{S_{int}}, \tag{4.4.2}$$

where the second term is a small correction to the first. Since we are working in 0d, we have that $\int \mathcal{D}\phi = \int d\phi$ and so we have that ($\hbar = 1$)

$$Z = \int \mathcal{D}\phi e^{-S'[\phi] = -m^2\phi^2} = \frac{\sqrt{\pi}}{m}. \tag{4.4.3}$$

In our example,

$$\begin{aligned} Z &= \sqrt{\frac{\pi}{(m^2+\delta m^2)}} && (4.4.4)\\ &= \sqrt{\frac{\pi}{m}}\left(1+\frac{\delta m^2}{m^2}\right)^{-1/2} && (4.4.5)\\ &= \sqrt{\frac{\pi}{m}}\left(1-\frac{1}{2}\frac{\delta m^2}{m^2}+\frac{3}{8}\left(\frac{\delta m^2}{m^2}\right)+...\right). && (4.4.6) \end{aligned}$$

For the simple case of $S = m^2\phi^2$, using equation (4.4.9) $G = 1/m^2$. Since the free part can be written $S = \left(m^2 + \delta m^2\right)\phi^2$, the exact green function will be of the form $G = 1/\left(m^2 + \delta m^2\right)$

4.4.2 Analysis of the free part of the action

Consider

$$\Box G(x,y) = \delta(x-y), \tag{4.4.7}$$

where G is the green's function and will turn out to be the 2-point correlation function (in the example of the Ising model, it is equal to $\langle\sigma_i\sigma_{i+r}\rangle$). It is called the propagator

Use the Fourier transform:

$$f(p) = \int dx e^{ipx} f(x), \tag{4.4.8}$$

then $\Box G(x,y) = \delta(x-y)$ (Eq. equation (4.4.9)) becomes

$$\left(p^2+m^2\right)G(p) = 1$$

and so

$$G(p) = \frac{1}{p^2+m^2}. \tag{4.4.9}$$

4.4.3 Visualization of interactions

The effect of a simple propagator (i.e. one stemming from the free part of the action) can be visualized in the following fashion

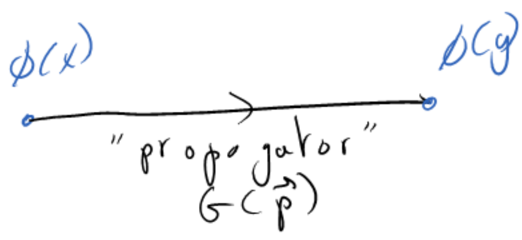


Figure 4.4.1: Visualization of a simple propagator.

For a general polynomial interaction term of the form $\lambda\phi^k(\vec{x})$, this can be visualized by

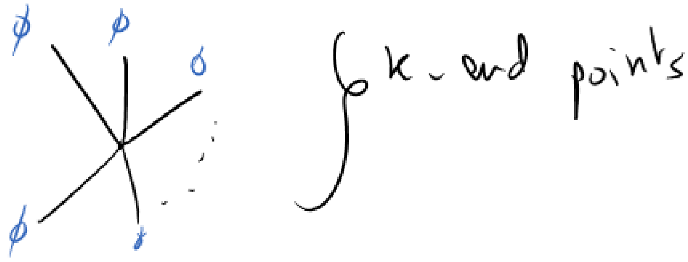


Figure 4.4.2: visualization of an interaction term of the form $\lambda\phi^k$.

4.5 Quantization of scalar fields

For us, $\phi(x)$ is a real scalar field. The procedure is not completely rigorous but it works. The ingredients are the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{4.5.1}$$

where $\partial_\mu\phi\partial^\mu\phi$ is the quadratic term, and m will be interpreted as the mass of a particle. We have also seen the EOM: the Euler-Lagrange equations:

$$\partial_\mu\partial^\mu\phi - m^2\phi = 0. \tag{4.5.2}$$

This is the Klein-Gordon equation and its general solution is

$$a \times e^{ikx} = ae^{i\vec{k}.\vec{x} - i\omega_{\vec{k}}.t}, \tag{4.5.3}$$

where k denotes 4-momentum $\left(\omega, \vec{k}\right)$ and x 4-point $(x^\mu = (t, \vec{x}))$ and

$$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}. \tag{4.5.4}$$

When this condition is satisfied, the solution/field/particle/etc is *"on the mass shell"*. We also have the 'dispersion relation' $k^2 = -m^2$.

Since ϕ is a real function,

$$\phi = ae^{ikx} + a^* e^{-ikx}. \tag{4.5.5}$$

The general solution (since the EOM are linear) is given by

$$\phi(x) = \int \frac{d^3\vec{k}}{f(k)} \left[a\left(\vec{k}\right)e^{ikx} + a^*\left(\vec{k}\right)e^{-ikx}\right], \tag{4.5.6}$$

where $f(k)$ depends only on $\left|\vec{k}\right|$ and is included only for convenience.

Let's assume that 3-dimensional space is compact: think of 3-d space as being a box of size $V = L^3$. Moreover, assume periodic boundary conditions. In particular, $\phi\left(t, x^1, x^2, x^3\right) = \phi\left(t, x^1 + L, x^2, x^3\right) = \phi\left(t, x^1, x^2 + L, x^3\right) = \phi\left(t, x^1, x^2, x^3 + L\right)$. What are the consequences of such a model?

Make the ansatz that

$$\phi(t, \vec{x}) = \sum_{k_1, k_2, k_3} \left(\frac{2\pi}{V\omega_{\vec{k}}}\right)^{1/2} \left(a\left(t, \vec{k}\right)e^{i\vec{k}.\vec{x}} + a^*\left(t, \vec{k}\right)e^{-i\vec{k}.\vec{x}}\right) \tag{4.5.7}$$

We have that

$$\vec{k} = (k_1, k_2, k_3) = \left(\frac{2\pi}{L}n_1, \frac{2\pi}{L}n_2, \frac{2\pi}{L}n_3\right). \tag{4.5.8}$$

Such an expression for ϕ is periodic in x^1, x^2 and x^3. Plugging this back into EOM:

$$\ddot{a}\left(t, \vec{k}\right) + \omega_{\vec{k}}^2 a\left(t, \vec{k}\right) = 0, \tag{4.5.9}$$

where

$$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} = \sqrt{m^2 + \frac{4\pi^2}{L^2}\left(n_1^2 + n_2^2 + n_3^2\right)}.$$

Such a solution is "on the mass shell" $k^2 = -m^2$ which means that $k^2 = k_\mu k^\mu = -m^2$.

How does the Ansatz depend on time? $a\left(t, \vec{k}\right)$ =amplitude of a harmonic oscillator labeled by $\vec{k}$ with frequency $\omega_{\vec{k}}$. This is equivalent to a system of harmonic oscillators labeled by $\vec{k}$.

"QFT is an infinite dimensional version of quantum mechanics" but this is not always the most optimal technique.

We have started with this formulation because it is close to traditional quantum mechanics.

4.5.1 Intermezzo: Harmonic oscillators

The usual harmonic oscillator is given by

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \tag{4.5.10}$$

Also

$$H\psi_n = E_n\psi_n \tag{4.5.11}$$

and

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{4.5.12}$$

and

$$\begin{aligned} a^\dagger\psi_n &= \sqrt{n+1}\psi_{n+1} \\ a\psi_n &= \sqrt{n}\psi_{n+1} \end{aligned}$$

and

$$\begin{aligned} a &= \frac{1}{\sqrt{2\omega}}\left(\omega\hat{q} + i\hat{p}\right) \\ a^\dagger &= \frac{1}{\sqrt{2\omega}}\left(\omega\hat{q} - i\hat{p}\right), \end{aligned}$$

where classically

$$\{p, q\} = 1, \tag{4.5.13}$$

which means that quantum mechanically

$$[\hat{p}, \hat{q}] = i\hbar. \tag{4.5.14}$$

As an example,

$$\hat{p} = -i\hbar\partial_q \tag{4.5.15}$$
$$\hat{q} = q \tag{4.5.16}$$

and

$$\hat{p} = p \tag{4.5.17}$$
$$\hat{q} = i\hbar\partial_p \tag{4.5.18}$$

Hence,

$$\left[a, a^\dagger\right] = 1 \tag{4.5.19}$$
$$[a, a] = 0 = \left[a^\dagger, a^\dagger\right]. \tag{4.5.20}$$

Moreover,

$$H = \frac{\omega}{2}\left(aa^\dagger + a^\dagger a\right). \tag{4.5.21}$$

Hence,

$$H\psi_n = \frac{\omega}{2}\left((n+1) + n\right)\psi_n \tag{4.5.22}$$
$$= \omega\left(n + \frac{1}{2}\right)\psi_n. \tag{4.5.23}$$

For the ground state, $a\psi_0 = 0$ so $\left(\omega q + \partial_q\right)\psi_0 = 0$ and so $\psi_0 \propto e^{-\omega q^2/2}$ and $\psi_n \left(a^\dagger\right)^n / \sqrt{n!}\psi_0$.

4.5.2 Many Quantum Harmonic oscillators

Let's go back to the notion that QFT is infinite dimensional QM. Take multiple oscillators

$$H = \sum_k H_k = \sum_k \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right). \tag{4.5.24}$$

Take

$$\left[a_k, a_l^\dagger \right] = \delta_{kl}.$$

Likewise,

$$[a_k, a_l] = 0. \tag{4.5.25}$$

Similarly,

$$\left[a_k^\dagger, a_l^\dagger \right] = 0 \tag{4.5.26}$$

The creation and annihilation operators are labeled by $\vec{k}$:

$$\begin{aligned} \left[a_{\vec{k}}, a_{\vec{k}'}^\dagger \right] &= \delta_{\vec{k}\vec{k}'} \\ \left[a_{\vec{k}}, a_{\vec{k}'} \right] &= \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger \right] = 0 \end{aligned}$$

The last step in this process is to take the continuum limit:

$$\left(\frac{2\pi}{L} \right)^3 \sum_{\vec{k}} \tag{4.5.27}$$

we replace this with

$$\int d^3\vec{k} \tag{4.5.28}$$

Moreover,

$$\left(\frac{L}{2\pi} \right)^3 \delta_{\vec{k}\vec{k}'} \to \delta\left(\vec{k} - \vec{k}' \right) \tag{4.5.29}$$

Moreover,

$$a_{\vec{k}} \to \left(\frac{2\pi}{L} \right)^{3/2} a\left(\vec{k} \right) \tag{4.5.30}$$

Under the continuum limit, such commutation relations will have the form

$$\left[a\left(\vec{k} \right), a^\dagger \left(\vec{k}' \right) \right] = \delta\left(\vec{k} - \vec{k}' \right) \tag{4.5.31}$$

$$\left[a\left(\vec{k} \right), a \left(\vec{k}' \right) \right] = 0 \tag{4.5.32}$$

$$\left[a^\dagger\left(\vec{k} \right), a^\dagger \left(\vec{k}' \right) \right] = 0. \tag{4.5.33}$$

4.5.2.1 Transition to QFT

Indeed,

$$\begin{aligned} q &\leftrightarrow \phi(x) \\ p &\leftrightarrow \Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \end{aligned} \tag{4.5.34}$$

where

$$\mathcal{L} = \pm \frac{1}{2}\dot{\phi}^2 \pm \left(\vec{\nabla}\phi^2 \right) + \text{mass term} \tag{4.5.35}$$

(not sure about sign) In our simple theory, $\Pi(x) = \dot{\phi}(x)$ and Hamiltonian is given by

$$H = \Pi\dot{\phi} - \mathcal{L} \tag{4.5.36}$$

$$= \frac{1}{2}\Pi^2 + \frac{1}{2}\left(\vec{\nabla}\phi \right)^2 + \frac{1}{2}m^2\phi^2 \tag{4.5.37}$$

$$= \frac{1}{2}\int \frac{d^3\vec{k}}{f(k)} \omega_{\vec{k}} \left(a^*\left(\vec{k} \right) a\left(\vec{k} \right) + a\left(\vec{k} \right) a^*\left(\vec{k} \right) \right). \tag{4.5.38}$$

The first two lines refer to the Hamiltonian density and the last line is the hamiltonian integrated over $d^3\vec{x}$ and we aplied a fourier transoform $\left(\phi(x) = \int \frac{dk}{f}\left(a(k)e + a^\dagger e^- ....\right)\right.$.

Quantization:

$$[\hat{p}, \hat{q}] = -i\hbar \leftrightarrow \left[\Pi(x), \phi\left(x'\right)\right] = -i\delta\left(x - x'\right). \tag{4.5.39}$$

This definition gives

$$\begin{aligned}
\left[a\left(\vec{k}\right), a^\dagger\left(\vec{k'}\right)\right] &= (2\pi)^3 2\omega\delta^3\left(\vec{k} - \vec{k'}\right) && (4.5.40)\\
\left[a\left(\vec{k}\right), a\left(\vec{k'}\right)\right] &= 0 && (4.5.41)\\
\left[a^\dagger\left(\vec{k}\right), a^\dagger\left(\vec{k'}\right)\right] &= 0. && (4.5.42)
\end{aligned}$$

Exercise 2. Derive the above commutation relations (note that this is question 3.1 of [1])

Our starting point is that

$$a\left(\vec{k}\right) = \int d^3x e^{-ikx}\left[i\partial_0\phi(x) + \omega\phi(x)\right]. \tag{4.5.43}$$

This comes from inverting the following formula:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left[a\left(\vec{k}\right)e^{ikx} + a^*\left(\vec{k}\right)e^{-ikx}\right]. \tag{4.5.44}$$

(see equation (4.5.6) with $f(k) = (2\pi)^3 2\omega$; this is choice needed to make $\phi(x)$ Lorentz invariant - see [1] chapter 3)

We will also use that

$$\Pi(x) = \dot{\phi}(x). \tag{4.5.45}$$

(see equation (4.5.34))

We have that

$$\begin{aligned}
\left[a\left(\vec{k}\right), a^\dagger\left(\vec{k'}\right)\right] &= \int d^3xd^3ye^{-ikx}e^{iky}\left[i\partial_0\phi(x) + \omega\phi(x), -i\partial_0\phi(y) + \omega\phi(y)\right] && (4.5.46)\\
&= \int d^3xd^3ye^{-ikx}e^{iky}\left(\left[i\partial_0\phi(x), \omega\phi(y)\right] + \left[\omega\phi(x), -i\partial_0\phi(y)\right]\right). && (4.5.47)
\end{aligned}$$

We have that $\partial_0 = \partial_{x^0} = \partial_t$ (for $c = 1$), so $\partial_0\phi(x) = \Pi$ and we can write

$$\begin{aligned}
\left[a\left(\vec{k}\right), a^\dagger\left(\vec{k'}\right)\right] &= \int d^3xd^3ye^{-ikx}e^{ik'y}\left(\left[i\Pi(x), \omega\phi(y)\right] + \left[\omega'\phi(x), -i\Pi(y)\right]\right) && (4.5.48)\\
&= \int d^3xd^3ye^{-ikx}e^{ik'y}\left(-i\omega i\left(\delta(x-y)\right) - i\omega' \times i\delta(x-y)\right) && (4.5.49)\\
&= \int d^3xd^3ye^{-ikx}e^{ik'y} \times \left(\omega + \omega'\right)\delta(x-y) && (4.5.50)\\
&= \int d^3x\left(\omega + \omega'\right)e^{ix\left(k'-k\right)} = \int d^3x\left(\omega + \omega'\right)e^{i\vec{x}\left(\vec{k'}-\vec{k}\right)}e^{-it\left(\omega'-\omega\right)} && (4.5.51)\\
&= (2\pi)^3 2\omega\delta^3\left(\vec{k'} - \vec{k}\right)e^{-it\times 0} && (4.5.52)\\
&= (2\pi)^3 2\omega\delta^3\left(\vec{k'} - \vec{k}\right) && (4.5.53)
\end{aligned}$$

where we used that $\left[\Pi(x), \phi\left(x'\right)\right] = -i\delta\left(x - x'\right)$ (equation (4.5.39))

A Klein-Gordon problem:

Exercise 3. (Problem 3.5 of [1]) "Consider a complex (that is, nonhermitian) scalar field ϕ with lagrangian density

$$\mathcal{L} = -\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi + \Omega_0. \tag{4.5.54}$$

a) Show that ϕ obeys the Klein-Gordon equation Take a small variation of the action $S = \int \mathcal{L}d^4x$:

$$\begin{aligned}
\delta S &= \int \left(\mathcal{L}(\phi + \delta\phi) - \mathcal{L}(\phi)\right)d^4x && (4.5.55)\\
&= \int \left(-\partial^\mu(\phi + \delta\phi)^\dagger\partial_\mu(\phi + \delta\phi) - m^2(\phi + \delta\phi)^\dagger(\phi + \delta\phi) + \partial^\mu\phi^\dagger\partial_\mu\phi + m^2\phi^\dagger\phi\right)d^4x && (4.5.56)
\end{aligned}$$

Hence

$$\delta S=\int\left(-\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi-m^{2}\phi^{\dagger}\phi-\partial^{\mu}\delta\phi^{\dagger}\partial_{\mu}\phi-m^{2}\delta\phi^{\dagger}\phi-\partial^{\mu}\phi^{\dagger}\partial_{\mu}\delta\phi-m^{2}\phi^{\dagger}\delta\phi-\partial^{\mu}\delta\phi^{\dagger}\partial_{\mu}\delta\phi-m^{2}\delta\phi^{\dagger}\delta\phi+\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi+m^{2}\phi^{\dagger}\phi\right)d^{4}xw \tag{4.5.57}$$

Ignore the second order terms in δS to obtain that

$$\begin{aligned}
\delta S &= \int\left(-\partial^{\mu}\delta\phi^{\dagger}\partial_{\mu}\phi-m^{2}\delta\phi^{\dagger}\phi-\partial^{\mu}\phi^{\dagger}\partial_{\mu}\delta\phi-m^{2}\phi^{\dagger}\delta\phi\right)d^{4}x && (4.5.58)\\
&= -\int\left(\partial^{\mu}\phi^{\dagger}\partial_{\mu}\delta\phi+m^{2}\phi^{\dagger}\delta\phi\right)d^{4}x+h.c. && (4.5.59)\\
&= -\int\left(\partial^{\mu}\phi^{\dagger}\left(\delta\phi\right)'+m^{2}\phi^{\dagger}\delta\phi\right)d^{4}x+h.c. && (4.5.60)\\
&= -\int\left(-\partial^{\mu}\partial_{\mu}\phi^{\dagger}\delta\phi+m^{2}\phi^{\dagger}\delta\phi\right)d^{4}x+\partial^{\mu}\phi^{\dagger}\delta\phi\big|_{-\infty}^{\infty}+h.c. && (4.5.61)
\end{aligned}$$

$\delta\phi$ vanishes at the boundaries, so we have that

$$\begin{aligned}
\delta S &= -\int\left(-\partial^{\mu}\partial_{\mu}\phi^{\dagger}\delta\phi+m^{2}\phi^{\dagger}\delta\phi\right)d^{4}x+h.c. && (4.5.62)\\
&= -\int\left(-\partial^{\mu}\partial_{\mu}\phi^{\dagger}+m^{2}\phi^{\dagger}\right)\delta\phi d^{4}x+h.c. && (4.5.63)\\
&= 0. && (4.5.64)
\end{aligned}$$

This equality has to hold for all $\delta\phi$, so we must have that

$$\begin{aligned}
\partial^{\mu}\partial_{\mu}\phi^{\dagger}-m^{2}\phi^{\dagger} &= 0 && (4.5.65)\\
\partial^{\mu}\partial_{\mu}\phi-m^{2}\phi &= 0 && (4.5.66)
\end{aligned}$$

This is the Klein-Gordon equation (equation (2.1.4)), as

$$\partial^{\mu}\partial_{\mu}\phi-m^{2}\phi=-\partial_{t}^{2}\phi+\triangle\phi-m^{2}\phi=0 \tag{4.5.67}$$

so $\left(\partial_{t}^{2}-\triangle\right)\phi+m^{2}\phi=\left(\Box+m^{2}\right)\phi=0$.

b) Treat ϕ and $\phi^{\dagger}$ as independent fields, and find the conjugate momentum for each. Compute the hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives). We have that the conjugate momentum is given by (equation (2.1.4))

$$\begin{aligned}
\Pi_{\phi} &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} && (4.5.68)\\
&= \partial_{\dot{\phi}}\left(-\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi-m^{2}\phi^{\dagger}\phi+\Omega_{0}\right) && (4.5.69)\\
&= \partial_{\dot{\phi}}\left(-\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi\right) && (4.5.70)\\
&= \partial_{\dot{\phi}}\left(-\left(-\partial_{t}\phi^{\dagger}\partial_{t}\phi+....\right)\right) && (4.5.71)\\
&= \partial_{0}\phi^{\dagger}. && (4.5.72)
\end{aligned}$$

Similarly $\Pi_{\phi^{\dagger}}=\partial_{0}\phi=\Pi_{\phi}^{\dagger}$.

The Hamiltonian is given by

$$\begin{aligned}
H &= \sum_{i}\dot{\phi}_{i}\Pi_{i}-\mathcal{L} && (4.5.73)\\
&= \Pi^{\dagger}\Pi+\Pi\Pi^{\dagger}+\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi+m^{2}\phi^{\dagger}\phi-\Omega_{0} && (4.5.74)\\
&= \Pi^{\dagger}\Pi+\Pi\Pi^{\dagger}+\left(\partial^{t}\phi^{\dagger}\partial_{t}\phi+\nabla\phi^{\dagger}.\nabla\phi\right)+m^{2}\phi^{\dagger}\phi-\Omega_{0} && (4.5.75)\\
&= \Pi^{\dagger}\Pi+\Pi\Pi^{\dagger}+\left(-\partial_{t}\phi^{\dagger}\partial_{t}\phi+\nabla\phi^{\dagger}.\nabla\phi\right)+m^{2}\phi^{\dagger}\phi-\Omega_{0} && (4.5.76)\\
&= \Pi^{\dagger}\Pi+\Pi\Pi^{\dagger}-\Pi\Pi^{\dagger}+\nabla\phi^{\dagger}.\nabla\phi+m^{2}\phi^{\dagger}\phi-\Omega_{0} && (4.5.77)\\
&= \Pi^{\dagger}\Pi+\nabla\phi^{\dagger}.\nabla\phi+m^{2}\phi^{\dagger}\phi-\Omega_{0}. && (4.5.78)
\end{aligned}$$

So

$$H=\Pi^{\dagger}\Pi+\nabla\phi^{\dagger}.\nabla\phi+m^{2}\phi^{\dagger}\phi-\Omega_{0} \tag{4.5.79}$$

and the Hamiltonian density is given by

$$\mathcal{H}=\int d^{3}xH. \tag{4.5.80}$$

c) Write the mode expansion of ϕ as

$$\phi(x) = \int \tilde{dk} \left[a\left(\vec{k}\right) e^{ikx} + b^{\dagger}\left(\vec{k}\right) e^{-ikx} \right], \tag{4.5.81}$$

where $\tilde{dk} = d^3k/(2\pi)^3 2\omega$. Express $a\left(\vec{k}\right)$ and $b\left(\vec{k}\right)$ in terms of ϕ and $\phi^{\dagger}$ and their time derivatives. We will be using that

$$e^{ikx} = e^{i\vec{k}.\vec{x} - i\omega t} \tag{4.5.82}$$

Apply:

$$\begin{aligned}
\int d^3x e^{-isx}\phi(x) &= \int d^3x e^{-isx} \int \tilde{dk} \left[a\left(\vec{k}\right) e^{ikx} + b^{\dagger}\left(\vec{k}\right) e^{-ikx} \right] && (4.5.83)\\
&= \int \tilde{dk} \left[a\left(\vec{k}\right) \int d^3x e^{-ix(s-k)} + b^{\dagger}\left(\vec{k}\right) \int d^3x e^{-ix(s+k)} \right] && (4.5.84)\\
&= \int \tilde{dk} \left[a\left(\vec{k}\right) e^{i\omega_s t} e^{-i\omega_k t} \int d^3x e^{-ix(s-k)} + b^{\dagger}\left(\vec{k}\right) e^{i\omega_s t} e^{i\omega_k t} \int d^3x e^{-ix(s+k)} \right] && (4.5.85)\\
&= \int \frac{d^3k}{(2\pi)^3 2\omega} \left[a\left(\vec{k}\right) e^{i\omega_s t} e^{-i\omega_k t} (2\pi)^3 \delta^3\left(\vec{s} - \vec{k}\right) + b^{\dagger}\left(\vec{k}\right) e^{i\omega_s t} e^{i\omega_k t} (2\pi)^3 \delta^3\left(\vec{s} + \vec{k}\right) \right] && (4.5.86)\\
\left(\text{because } \omega\left(\vec{k}\right) = \omega\left(-\vec{k}\right)\right) &= \frac{1}{2\omega} \left[a\left(\vec{k}\right) + e^{2i\omega t} b^{\dagger}\left(-\vec{k}\right) \right]. && (4.5.87)
\end{aligned}$$

Similarly, we can also show

$$\begin{aligned}
\int d^3x e^{-isx}\partial_0\phi(x) &= \int d^3x e^{-isx} \int \tilde{dk}\partial_0 \left[a\left(\vec{k}\right) e^{ikx} + b^{\dagger}\left(\vec{k}\right) e^{-ikx} \right] && (4.5.88)\\
&= \int d^3x e^{-isx} \int \tilde{dk}\partial_0 \left[a\left(\vec{k}\right) e^{i\vec{k}.\vec{x} - i\omega_k t} + b^{\dagger}\left(\vec{k}\right) e^{-i\vec{k}.\vec{x} + i\omega_k t} \right] && (4.5.89)\\
&= \int d^3x e^{-isx} \int \tilde{dk} \left[-i\omega_k a\left(\vec{k}\right) e^{i\vec{k}.\vec{x} - i\omega_k t} + i\omega_k b^{\dagger}\left(\vec{k}\right) e^{-i\vec{k}.\vec{x} + i\omega_k t} \right] && (4.5.90)\\
&= \int \tilde{dk} \left[-i\omega_k a\left(\vec{k}\right) e^{it(\omega_s - \omega_k)} \int d^3x e^{i\left(\vec{k} - \vec{s}\right).\vec{x}} + i\omega_k b^{\dagger}\left(\vec{k}\right) e^{it(\omega_s + \omega_k)} \int d^3x e^{-i\left(\vec{s} + \vec{k}\right).\vec{x}} \right] && (4.5.91)\\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[-i\omega_k a\left(\vec{k}\right) e^{it(\omega_s - \omega_k)} (2\pi)^3 \delta^3\left(\vec{s} - \vec{k}\right) + i\omega_k b^{\dagger}\left(\vec{k}\right) e^{it(\omega_s + \omega_k)} (2\pi)^3 \delta^3\left(\vec{s} + \vec{k}\right) \right] && (4.5.92)\\
&= \frac{1}{2\omega} \left[-i\omega a\left(\vec{k}\right) + i\omega b^{\dagger}\left(-\vec{k}\right) e^{2it\omega} \right] && (4.5.93)\\
&= \frac{-i}{2} a\left(\vec{k}\right) + \frac{i}{2} e^{2i\omega t} b^{\dagger}\left(-\vec{k}\right). && (4.5.94)
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\int d^3x e^{-ikx}\phi(x) &= \frac{1}{2\omega} \left[a\left(\vec{k}\right) + e^{2i\omega t} b^{\dagger}\left(-\vec{k}\right) \right] && (4.5.95)\\
\int d^3x e^{-ikx}\partial_0\phi(x) &= \frac{-i}{2} a\left(\vec{k}\right) + \frac{i}{2} e^{2i\omega t} b^{\dagger}\left(-\vec{k}\right). && (4.5.96)
\end{aligned}$$

Using mathematica* we obtain that

$$\begin{aligned}
a\left(\vec{k}\right) &= \int d^3x e^{-ikx} \left(\omega\phi(x) + i\partial_0\phi(x) \right) && (4.5.97)\\
b^{\dagger}\left(-\vec{k}\right) &= \int d^3x e^{-ikx} e^{-2i\omega t} \left(\omega\phi(x) - i\partial_0\phi(x) \right) && (4.5.98)\\
\implies b\left(-\vec{k}\right) &= \int d^3x e^{ikx} e^{2i\omega t} \left(\omega\phi^{\dagger}(x) + i\partial_0\phi^{\dagger}(x) \right) && (4.5.99)
\end{aligned}$$

and so

$$\begin{aligned}
b\left(\vec{k}\right) &= \int d^3x e^{i\left(-\vec{k}\right).\vec{x} - i\omega t} e^{2i\omega t} \left(\omega\phi^{\dagger}(x) + i\partial_0\phi^{\dagger}(x) \right) && (4.5.100)\\
&= \int d^3x e^{-i\vec{k}.\vec{x} + i\omega t} \left(\omega\phi^{\dagger}(x) + i\partial_0\phi^{\dagger}(x) \right) && (4.5.101)\\
&= \int d^3x e^{-ikx} \left(\omega\phi^{\dagger}(x) + i\partial_0\phi^{\dagger}(x) \right). && (4.5.102)
\end{aligned}$$

d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates. Assuming canonical commutation relations then

$$\left[\phi\left(x,t\right),\Pi\left(\vec{x}^{'},t\right)\right]=i\delta^{3}\left(\vec{x}-\vec{y}\right) \tag{4.5.103}$$

We obtain that

$$\begin{aligned}\left[a\left(\vec{k}\right),b^{\dagger}\left(\vec{s}\right)\right] &= \left[a\left(\vec{k}\right),a\left(\vec{s}\right)\right]=\left[b\left(\vec{k}\right),b\left(\vec{s}\right)\right]=0 &(4.5.104)\\ \left[a\left(\vec{k}\right),a^{\dagger}\left(\vec{s}\right)\right] &= \left[b\left(\vec{k}\right),b^{\dagger}\left(\vec{s}\right)\right]=2\left(2\pi\right)^{3}\omega_{k}\delta\left(\vec{k}-\vec{s}\right). &(4.5.105)\end{aligned}$$

*

e) Express the hamiltonian in terms of $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy? We have shown (equation (4.5.79)) that the hamiltonian density is given

$$\mathcal{H}=\Pi^{\dagger}\Pi+\nabla\phi^{\dagger}.\nabla\phi+m^{2}\phi^{\dagger}\phi-\Omega_{0}. \tag{4.5.106}$$

We also have that

$$\phi\left(x\right)=\int\tilde{dk}\left[a\left(\vec{k}\right)e^{ikx}+b^{\dagger}\left(\vec{k}\right)e^{-ikx}\right] \tag{4.5.107}$$

and

$$\Pi=\dot{\phi}^{\dagger}=\int\tilde{dk}\left[-i\omega a\left(\vec{k}\right)e^{ikx}+i\omega b^{\dagger}\left(\vec{k}\right)e^{-ikx}\right] \tag{4.5.108}$$

BLAH (follow page 27 of [1]) to get that ($H=\int d^{3}x\mathcal{H}$)

$$H=\int\tilde{dk}\omega\left[a^{\dagger}\left(\vec{k}\right)a\left(\vec{k}\right)+b^{\dagger}\left(\vec{k}\right)b\left(\vec{k}\right)\right]+\left(2\mathcal{E}_{0}-\Omega_{0}\right)V, \tag{4.5.109}$$

where V is the volume of the space and

$$\mathcal{E}_{0}=\frac{1}{2\left(2\pi\right)^{3}}\int d^{3}k\omega. \tag{4.5.110}$$

4.5.3 Note: Canonical quantization has its problems

How do we quantize q^2p^3? Is it

$$\hat{q}^{2}\hat{p}^{3}+2\hat{q}\hat{p}\hat{q}\hat{p}^{2}+... \tag{4.5.111}$$

Specifically, "operator ordering" is an issue.

4.5.4 Note: Variational derivatives

(Variational principle) Consider Lagrangians $\mathcal{L}$ or actions S. We often encounter something like

$$\frac{\partial}{\partial\phi}, \tag{4.5.112}$$

where ϕ is a field/function. In many ways, these behave like normal partial differentials like ∂_x. But, keep in mind that there is a distinction.

4.6 QFT in $n=0$ dimensions

Usually, $\phi\left(x\right)$ is a function of spacetime coordinate x. When $n=0$, ϕ is a scalar which we will denote by q.

Suppose $q\in V\sim\mathbb{R}^{k}$ (like having many particles at the same position and each has their own field; we do this to make the example more pedagogical and more relevant to higher n examples). Let $B\left(q,q\right)$ be a symmetric bilinear form on V. Then, perform "path integral"

$$\int_{V}\exp\left(-\frac{1}{2}B\left(q,q\right)\right)dq=1, \tag{4.6.1}$$

where we have chosen the normalization to be1.

In this "baby problem", we wish to compute

$$\int_{V}P\left(q\right)\exp\left(-\frac{1}{2}B\left(q,q\right)\right)dq=? \tag{4.6.2}$$

It is always a sum of terms of the form

$$\langle f_1 ... f_N \rangle_0 \equiv \int_V f_1(q) ... f_N(q) \exp\left(-\frac{1}{2}B(q,q)\right) dq, \tag{4.6.3}$$

and $f_1(q), ..., f_N$ are linear functions: $f = \alpha_i q^i$, where $i = 1, 2, ..., k$. We chose this form for f because the path integral is linear (?).

$$\langle f_1 ... f_N \rangle = 0 \tag{4.6.4}$$

if N is odd (remember B is symmetric - so $B(-q,-q) = B(q,q)$and it is even - and f is even). Hence, work with $N = 2m$ is an even number.

Wick's theorem:

$$\langle f_1 ... f_{2m} \rangle_0 = \sum_{s \in S_{2m}/\sim\text{"pairing"}} B^{-1}\left(f_{S(1)}, f_{S(2)}\right) ... B^{-1}\left(f_{S(n-1)}, f_{S(n)}\right), \tag{4.6.5}$$

where $B^{-1}(f,q) = \alpha_i B^{ij} \beta_j$ is the inverse form to B and S_{2m} is the symmetric group, and $s_1 \sim s_2$ (~meaning they are equivalent), where $s_1, s_2 \in S_{2m}$, if they define the same splitting of $\{1, 2, ..., 2m\}$ into m pairs (e.g. $\{(1,2), (3,4)\}$ is the same as $\{(2,1), (3,4)\}$). The total number of splitting is given by

$$\#\,(\text{splitting}) = \frac{(2m)!}{2^m m!}, \tag{4.6.6}$$

where $m!$ is because of the different ways we can permute the pairs and 2^m is because in each pair, the order does not matter.

Example 4. $m = 2$ and so we have

$$\frac{4!}{2^2 \times 2!} = 3 \tag{4.6.7}$$

possible pairings. Hence,

$$\langle f_1 f_2 f_3 f_4 \rangle = \underbrace{f_1 f_2}\,\underbrace{f_3 f_4} + \underbrace{f_1 f_3}\,\underbrace{f_2 f_4} + \underbrace{f_1 f_4}\,\underbrace{f_2 f_3}, \tag{4.6.8}$$

where $\underbrace{\quad}$ denotes a wick contraction.

4.7 0+1 dimensional QFT (Path Integrals in Quantum Mechanics)

We have 0 space dimensions and one time dimension:

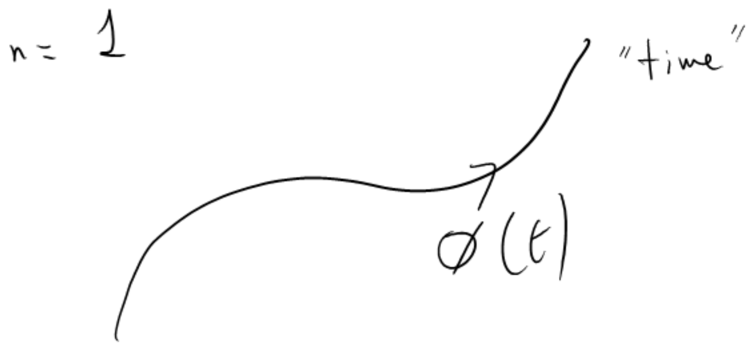


Figure 4.7.1: At each point in time, the particle has a different field.

. We have that

$$H(\hat{p}, \hat{q}) = \frac{1}{2}\hat{p}^2 + V(\hat{q}). \tag{4.7.1}$$

Position eigenstates are denoted with $|q\rangle$ such that $\hat{q}|q\rangle = q|q\rangle$.

The amplitude for a particle to go from q' at time t' to q'' at t'' is

$$\left\langle q'' \left| e^{-iHT} \right| q' \right\rangle = \left\langle q'' \left| e^{-iH\left(t''-t'\right)} \right| q' \right\rangle. \tag{4.7.2}$$

In the Heisenberg picture:

$$\hat{q}(t) \equiv e^{iHt} \hat{q} e^{-iHt}. \tag{4.7.3}$$

Moreover, define $|q,t\rangle$ by $\hat{q}(t)|q,t\rangle = q|q,t\rangle$. Explicitly,

$$|q,t\rangle \equiv e^{iHt}|q\rangle. \tag{4.7.4}$$

The amplitude is given by

$$\left\langle q^{''},t^{''} \middle| q^{'},t^{'}\right\rangle. \tag{4.7.5}$$

Divide the time interval $T = t^{''} - t^{'}$ into $N+1$ segments of size $\delta t = T/(N+1)$. In addition

$$\left\langle q^{''},t^{''} \middle| q^{'},t^{'}\right\rangle = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \prod_{j=1}^{N} dq_i \left\langle q^{''} \middle| e^{-iH\delta t} \middle| q_N \right\rangle \left\langle q_N \middle| e^{-iH\delta t} \middle| q_{N-1}\right\rangle ... \left\langle q_1 \middle| e^{-iH\delta t} \middle| q^{'}\right\rangle. \tag{4.7.6}$$

First, address each building block: use the Campbell-Baker-Hausdorf formula:

$$\exp(A+B) = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A,B] + \text{higher order commutators}\right). \tag{4.7.7}$$

This gives us (using that $\hat{H} = \hat{p}^2/2 + \hat{V}(\hat{q})$)

$$\exp(-iH\delta t) = \exp\left(-i\frac{\delta t}{2}\hat{p}^2\right)\exp\left(-i\frac{\delta t}{2}\hat{V}(\hat{q})\right)\exp\left(O\left(\delta t^2\right)\right). \tag{4.7.8}$$

We have that

$$\left\langle q_2 \middle| e^{-iH\delta t} \middle| q_1\right\rangle = \int dp_1 \left\langle q_2 \middle| e^{-i\delta t\hat{p}^2} \middle| p_1\right\rangle \left\langle p_1 \middle| e^{-i\delta t\hat{V}(q)} \middle| q_1\right\rangle, \tag{4.7.9}$$

where we used the completeness relation: $\int dp_1 |p_1\rangle\langle p_1| = I$. Continuing the calculation,

$$\left\langle q_2 \middle| e^{-iH\delta t} \middle| q_1\right\rangle = \int dp_1 e^{-i\delta t p_1^2/2} e^{-i\delta t V(q_1)} \langle q_2 | p_1\rangle \langle p_1 | q_1\rangle. \tag{4.7.10}$$

We have that

$$\langle q | p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipq}, \tag{4.7.11}$$

which when used, gives us that

$$\begin{aligned}
\left\langle q_2 \middle| e^{-iH\delta t} \middle| q_1\right\rangle &= \int \frac{dp_1}{2\pi} e^{-i\delta t p_1^2/2} e^{-i\delta t V(q_1)} e^{ip_1(q_2-q_1)} && (4.7.12)\\
&= \int \frac{dp_1}{2\pi} e^{-iH(p_1,q_1)\delta t} e^{ip_1(q_2-q_1)}. && (4.7.13)
\end{aligned}$$

Combining, we obtain that

$$\begin{aligned}
\left\langle q^{''},t^{''} \middle| q^{'},t^{'}\right\rangle &= \int ... \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(q_j,p_j)\delta t} && (4.7.14)\\
&= \int ... \int \prod_{k=1}^{N}\prod_{j=0}^{N} \left(dq_k \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(q_j,p_j)\delta t}\right) && (4.7.15)
\end{aligned}$$

In the limit that $\delta t \to 0$ (and so $N \to \infty$)

$$\left\langle q^{''},t^{''} \middle| q^{'},t^{'}\right\rangle = \int \mathcal{D}q(t)\mathcal{D}p(t)\exp\left[i\int_{t^{'}}^{t^{''}} dt\,(p(t)\dot{q}(t) - H(p(t),q(t)))\right], \tag{4.7.16}$$

where $\mathcal{D}q(t)$ denotes functional integration and

$$\dot{q} = \frac{q_{j+1}-q_j}{\delta t}$$

as $\delta t \to 0$. Hence

$$\left\langle q^{''},t^{''} \middle| q^{'},t^{'}\right\rangle = \int \mathcal{D}q\mathcal{D}p e^{iS}, \tag{4.7.17}$$

where

$$S = \int_{t^{'}}^{t^{''}} dt\,(p\dot{q} - H), \tag{4.7.18}$$

where $p\dot{q}-H$ is the Lagrangian density (in 0 space dimensions Lagrangian=Lagrangian density). S weights each possible path:

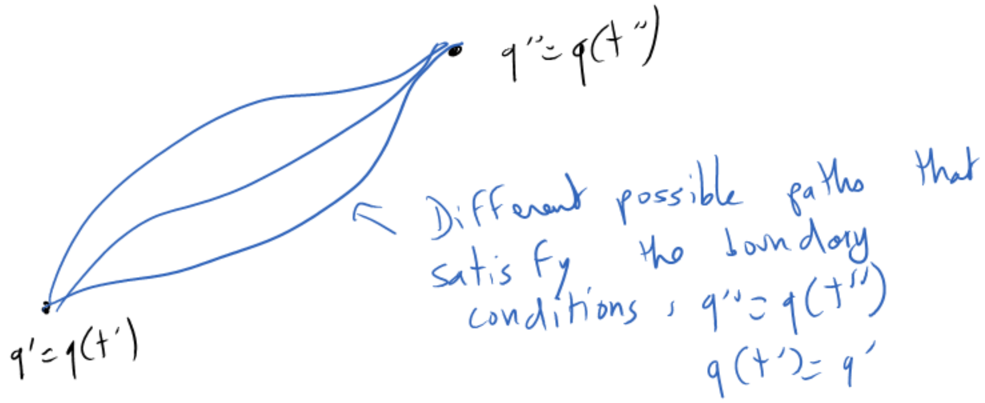


Figure 4.7.2: Visualization of Path integral

Path integral formulation (which is very infinite dimensional) might be useful for quantum mechanics, but it is the only tool available in QFT.

If $H(p,q)$ is quadratic in p, then $\int \mathcal{D}p\,(...)$ is Gaussian, and so it is dominated by its stationary point. What is it?

$$0=\partial_p\left(p\dot{q}-H(p,q)\right)=\dot{q}-\partial_pH(p,q) \tag{4.7.19}$$

gives $p(q,\dot{q})$ and $L(\dot{q},q)=p\dot{q}-H$.

4.7.1 Operators

Consider (we are working in the Heisenberg picture)

$$\begin{aligned}\left\langle q^{''},t^{''}\middle|\hat{q}(t_1)\middle|q^{'},t^{'}\right\rangle &= \left\langle q^{''}\middle|e^{-iH\left(t^{''}-t_1\right)}\hat{q}e^{-iH\left(t_1-t^{'}\right)}\middle|q^{'}\right\rangle &(4.7.20)\\ &= \int \mathcal{D}q\mathcal{D}p\,q(t_1)\,e^{iS}. &(4.7.21)\end{aligned}$$

We don't have to deal with non-commuting operators anymore! But, the trade-off is that now we are dealing with functional integrals.

In the other direction

$$\int \mathcal{D}q\mathcal{D}p\,q(t_1)\,q(t_2)\,e^{iS}=\left\langle q^{''},t^{''}\middle|T\hat{q}(t_1))\,\hat{q}(t_2)\middle|q^{'},t^{'}\right\rangle, \tag{4.7.22}$$

where T is called the *time ordering* of operators:

$$T\hat{q}(t_1)\,\hat{q}(t_2)=\begin{cases}\hat{q}(t_1)\,\hat{q}(t_2) & t_1>t_2\\ \hat{q}(t_2)\,\hat{q}(t_1) & t_1<t_2\end{cases} \tag{4.7.23}$$

4.7.2 Generalization to sources/correlation functions

Introduce the following generalization:

$$H(p,q)\to H(p,q)-f(t)\,q(t)-h(t)\,p(t), \tag{4.7.24}$$

where $f(t)$ and $h(t)$ are called *source functions.* Moreover, define

$$\left\langle q^{''},t^{''}\middle|q^{'},t^{'}\right\rangle_{f,h}=\int \mathcal{D}p\mathcal{D}q\exp\left[i\int_{t^{'}}^{t^{''}}dt\left(p\dot{q}-H+fq+hp\right)\right], \tag{4.7.25}$$

this is read in "the background of f and h" (vacuum background is $f=h=0$).

Consider the variational derivative wrt f, then

$$\frac{1}{i}\partial_{f(t_1)}\left\langle q^{''},t^{''}\middle|q^{'},t^{'}\right\rangle_{f,h}=\int\mathcal{D}p\mathcal{D}q\,q\left(t_1\right)\exp\left[i\int dt\left(p\dot{q}-H+fq+hp\right)\right]. \tag{4.7.26}$$

The simple structure of this integral is because the added terms are linear in q and p.

More examples:

$$\frac{1}{i}\partial_{f(t_1)}\frac{1}{i}\partial_{f(t_2)}\left\langle q^{''},t^{''}\middle|q^{'},t^{'}\right\rangle_{f,h}=\int\mathcal{D}p\mathcal{D}q\,q\left(t_1\right)q\left(t_2\right)\exp\left[i\int dt\left(p\dot{q}-H+fq+hp\right)\right]. \tag{4.7.27}$$

$$\frac{1}{i}\partial_{h(t_1)}\left\langle q^{''},t^{''}\middle|q^{'},t^{'}\right\rangle_{f,h}=\int\mathcal{D}p\mathcal{D}q\,p\left(t_1\right)\exp\left[i\int dt\left(p\dot{q}-H+fq+hp\right)\right]. \tag{4.7.28}$$

If we have many such variational derivatives, then we can write the (unnormalized; to normalize divide by $\left\langle q^{''},t^{''}\middle|q^{'},t^{'}\right\rangle_{f,h}$) correlation function in the following way:

$$\left\langle q^{''},t^{''}\middle|T\hat{q}\left(t_1\right)...\hat{p}\left(t_n\right)...\middle|q^{'},t^{'}\right\rangle=\frac{1}{i}\partial_{f(t_1)}...\frac{1}{i}\partial_{h(t_n)}...\left\langle q^{''},t^{''}\middle|q^{'},t^{'}\right\rangle_{f,h}\Big|_{f=g=0}. \tag{4.7.29}$$

Take the limit that $t^{'}\to-\infty$ and $t^{''}\to\infty$ (for most problems we wish to solve, this is not an issue). Sometimes, how we take the limit affects the final answer (which might diverge if we are not careful). So to suppress the infinities, let

$$H\to\left(1-i\epsilon\right)H, \tag{4.7.30}$$

where $\epsilon>0$. Later, we will see the meaning of this operation, which will be something like the following figure:

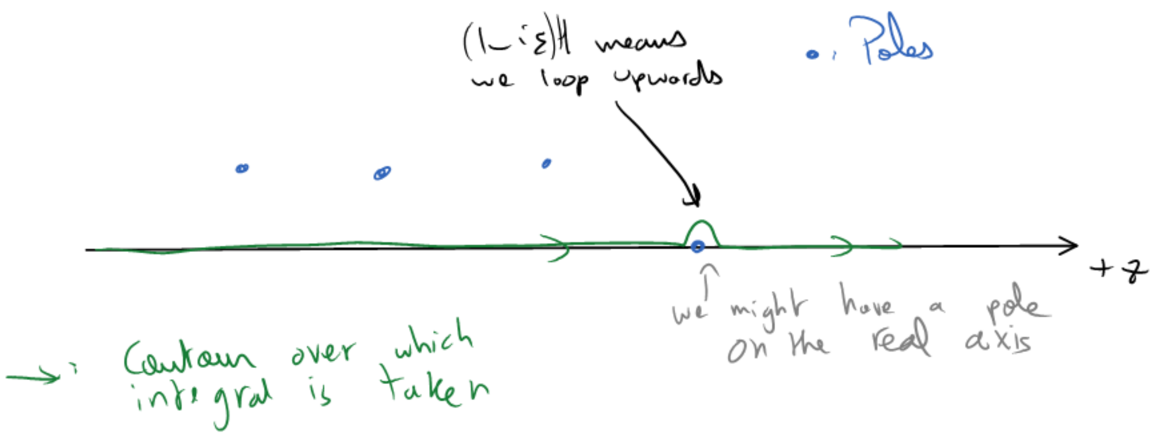


Figure 4.7.3: Integration over poles (note: it is important to loop in the same way; looping one way and then a different way during the same calculation will lead to incorrect results)

4.7.3 Taking the $t\to\infty$ limit

We get

$$\langle 0\,|\,0\rangle_{f,h}=\lim_{\substack{t^{'}\to-\infty\\ t^{''}\to\infty}}\int\mathcal{D}p\mathcal{D}q\left\{dq^{''}dq^{'}\psi_0^*\left(q^{''}\right)\psi_0\left(q^{'}\right)\right\}\exp\left[i\int_{-\infty}^{\infty}dt\left(p\dot{q}-\left(1-i\epsilon\right)H+fq+hp\right)\right]. \tag{4.7.31}$$

When we take the limit that $t\to\infty$, then $\mathcal{D}q$ changes its meaning. Notice that we added $\left\{dq^{''}dq^{'}\psi_0^*\left(q^{''}\right)\psi_0\left(q^{'}\right)\right\}$. When t is finite, we have the boundary conditions that $q\left(t^{'}\right)=q^{'}$ and $q\left(t^{''}\right)=q^{''}$. However, as $t\to\infty$, it does not make a lot of sense to talk about $q\left(\infty\right)=q^{'}$. We would like our integral to be more universal. Hence, we integrate over all initial conditions. Moreover, we weigh each initial condition with the ground state wavefunction ψ_0. In fact, $\langle 0\,|\,0\rangle_{f,h}$ can be written to be

$$\langle 0\,|\,0\rangle_{f,h}=\lim_{\substack{t^{'}\to-\infty\\ t^{''}\to\infty}}\int dq^{''}dq^{'}\psi_0^*\left(q^{''}\right)\left\langle q^{''},t^{''}\middle|q^{'},t^{'}\right\rangle_{f,h}\psi_0\left(q^{'}\right), \tag{4.7.32}$$

as (see equation (4.7.25))

$$\left\langle q^{''},t^{''}\left|q^{'},t^{'}\right.\right\rangle_{f,h}=\int\mathcal{D}p\mathcal{D}q\exp\left[i\int_{t^{'}}^{t^{''}}dt\left(p\dot{q}-H+fq+hp\right)\right]. \tag{4.7.33}$$

Note that $\psi_0^*\left(q^{''}\right)$ is conjugated because we have the $q^{''}$ appearing as a bra.

4.7.3.1 As $t\to\infty$, only the ground sate survives

We will make the connection of $\langle 0\,|\,0\rangle_{f,h}$ with the ground state more explicit. Indeed, (assuming a discrete spectrum of energies)

$$\begin{aligned}\left|q^{'},t^{'}\right\rangle &= e^{iHt^{'}}\left|q^{'}\right\rangle &(4.7.34)\\ &= \sum_{n=0}^{\infty}e^{iHt^{'}}\left|n\right\rangle\left\langle n\left|q^{'}\right.\right\rangle &(4.7.35)\\ &= \sum_{n=0}^{\infty}e^{iE_nt^{'}}\psi_n^*\left(q^{'}\right)\left|n\right\rangle, &(4.7.36)\end{aligned}$$

where $\psi_n(q)=\langle q\,|\,n\rangle$. And the operation $H\to(1-i\epsilon)H$ will pick out the ground state defined to be $|0\rangle$ with energy 0 and with wavefunction $\psi_0(q)=\langle q\,|\,0\rangle$. This is because then E_n has a small imaginary part, which means that $e^{iE_nt^{'}}$ will have a damping term in it: the higher the energy level, the stronger the damping and higher order terms are suppressed. So, it can be shown that[2]

$$\langle 0\,|\,0\rangle_{f,h}=\lim_{t^{'}\to-\infty,t^{''}\to\infty}\int dq^{''}dq^{'}\psi_0^*\left(q^{''}\right)\left\langle q^{''},t^{''}\left|q^{'},t^{'}\right.\right\rangle_{f,h}\psi_0\left(q^{'}\right), \tag{4.7.39}$$

which was what we obtained in equation (4.7.32).

4.7.4 The partition function

We will call $\langle 0\,|\,0\rangle_{f,h}$ the partition function (note that $\langle 0\,|\,0\rangle_{f,h}$ is a universal quantity: it does not depend any specific parameters). We will usually deal with a system of the form

$$H=H_0+H_{int}, \tag{4.7.40}$$

where H_0 is solvable and H_{int} is a perturbation. This is perturbative QFT (and is the only way we can find solutions). Specifically, we will be dealing with- suppressing $i\epsilon$ (for brevity - but ϵ is still there)

$$\begin{aligned}\langle 0\,|\,0\rangle_{f,h} &= \int\mathcal{D}p\mathcal{D}q\exp\left[i\int_{-\infty}^{\infty}dt\left(p\dot{q}-H_0(p,q)-H_{int}(p,q)+fq+hp\right)\right] &(4.7.41)\\ &= \exp\left[-i\int_{-\infty}^{\infty}dtH_{int}\left(\frac{1}{i}\partial_{h(t)},\frac{1}{i}\partial_{f(t)}\right)\right]\times &(4.7.42)\\ &\quad\int\mathcal{D}p\mathcal{D}q\exp\left[i\int_{-\infty}^{\infty}dt\left(\underbrace{p\dot{q}-H_0(p,q)}_{L_0}+fq+hp\right)\right]. &(4.7.43)\end{aligned}$$

Note that $\exp\left[-i\int_{-\infty}^{\infty}dtH_{int}\left(\frac{1}{i}\partial_{h(t)},\frac{1}{i}\partial_{f(t)}\right)\right]$ is like an operator.

Example 5. Let $H_{int}=H_{int}(q)$, $H_0=p^2/2+V(q)$. Then

$$\langle 0\,|\,0\rangle_f=\exp\left[i\int_{-\infty}^{\infty}dtL_{int}\left(\frac{1}{i}\partial_{f(t)}\right)\right]\times\int\mathcal{D}q\exp\left[i\int_{-\infty}^{\infty}dt\left(L_0(\dot{q},q)+fq\right)\right], \tag{4.7.44}$$

where $L_{int}=-H_{int}$.

[2] Using equation (4.7.36) and that $H\to(1-i\epsilon)H$, we have that (note that the f,h does not matter: we only want the energy spectrum to be real)

$$\left\langle q^{''},t^{''}\left|q^{'},t^{'}\right.\right\rangle_{f,h}=\psi_0\left(q^{''}\right)\psi_0^*\left(q^{'}\right)\langle 0\,|\,0\rangle, \tag{4.7.37}$$

because as $t\to\infty$ any small real part in $e^{iE_nt^{'}}$ means that the corresponding term in the sum will be exponentially small:

$$\lim_{t\to\infty}e^{-\alpha_nt}\psi_n^*\left(q^{'}\right)|n\rangle=0 \tag{4.7.38}$$

for any finite positive α_n, entailing that only the ground state survives the limit.

4.7.5 The path integral for the Harmonic Oscillator

Later we will have to use path integrals for calculating concrete things. This section will be practice for such tasks. We have

$$H\left(p,q\right)=\frac{1}{2}p^{2}+\frac{1}{2}\omega^{2}q^{2}. \tag{4.7.45}$$

We have that

$$L\left(q,\dot{q}\right)=\frac{1}{2}\dot{q}^{2}-\frac{1}{2}\omega^{2}q^{2}. \tag{4.7.46}$$

We obtain the following wave equation for the EOM:

$$\left(\partial_{t}^{2}+\omega^{2}\right)q\left(t\right)=0. \tag{4.7.47}$$

Introduce the green's function $G\left(t-t^{'}\right)$:

$$\left(\partial_{t}^{2}+\omega^{2}\right)G\left(t-t^{'}\right)=\delta\left(t-t^{'}\right). \tag{4.7.48}$$

We obtain that

$$G\left(t-t^{'}\right)=\frac{i}{2\omega}\exp\left(-i\omega\left|t-t^{'}\right|\right). \tag{4.7.49}$$

Consider

$$\left\langle 0\,|\,0\right\rangle_{f}=\int\mathcal{D}p\mathcal{D}qe^{iS}, \tag{4.7.50}$$

where

$$S = \int_{-\infty}^{\infty}dt\left[p\dot{q}-\left(1-i\epsilon\right)H+fq\right] \tag{4.7.51}$$

$$(\text{integrating out } p) = \int_{-\infty}^{\infty}dt\left[\frac{1}{2}\dot{q}^{2}\left(1+i\epsilon\right)-\frac{1}{2}\omega^{2}q^{2}\left(1-i\epsilon\right)+fq\right]. \tag{4.7.52}$$

(see 4 for how the integrating out p part is carried out) Next, introduce

$$\tilde{q}\left(E\right) = \int_{-\infty}^{\infty}dte^{iEt}q\left(t\right) \tag{4.7.53}$$

$$q\left(t\right) = \int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iEt}\tilde{q}\left(E\right). \tag{4.7.54}$$

Splitting equation (4.7.52) into symmetric components (example: $fq=\left(fq+qf\right)/2$) and using the Fourier transforms of q

$$\begin{aligned} S &= \int_{-\infty}^{\infty}dt\frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{dE^{'}}{2\pi}\exp\left[-i\left(E+E^{'}\right)t\right]\times \\ &\left[\left(-\left(1+\epsilon\right)EE^{'}-\left(1+i\epsilon\right)\omega^{2}\right)\tilde{q}\left(E\right)\tilde{q}\left(E^{'}\right)+\tilde{f}\left(E\right)\tilde{q}\left(E^{'}\right)+\tilde{f}\left(E^{'}\right)\tilde{q}\left(E\right)\right]\end{aligned}$$

When integrated $dt\exp\left[-i\left(E+E^{'}\right)t\right]$ gives $2\pi\delta\left(E+E^{'}\right)$. Hence,

$$S=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left\{\left[\left(1+i\epsilon\right)E^{2}-\left(1+i\epsilon\right)\omega^{2}\right]\tilde{q}\left(E\right)\tilde{q}\left(-E\right)+\tilde{f}\left(E\right)\tilde{q}\left(-E\right)+\tilde{f}\left(E\right)\tilde{q}\left(-E\right)+\tilde{f}\left(-E\right)\tilde{q}\left(E\right)\right\} \tag{4.7.55}$$

We can write that

$$\left\langle 0\,|\,0\right\rangle_{f}=\int\mathcal{D}\tilde{q}\left(E\right)e^{iS} \tag{4.7.56}$$

because there is a 1-1 correspondence between q and $\tilde{q}$ (??) Let

$$\tilde{x}\left(E\right)\equiv\tilde{q}\left(E\right)+\frac{\tilde{f}\left(E\right)}{E^{2}-\omega^{2}+i\epsilon}. \tag{4.7.57}$$

The second term is like a constant shift to $\tilde{q}$ (as we are only performing the integral on $\tilde{q}$). We can write

$$S=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left[\tilde{x}\left(E\right)\left(E^{2}-\omega^{2}+i\epsilon\right)\tilde{x}\left(-E\right)-\frac{\tilde{f}\left(E\right)\tilde{f}\left(-E\right)}{E^{2}-\omega^{2}+i\epsilon}\right] \tag{4.7.58}$$

(note that ϵ is redefined: for instance $\omega^2\epsilon$ is still very small and can be called ϵ - ??).

Using $\mathcal{D}q = \mathcal{D}x$, we obtain that

$$\langle 0\,|\,0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\,\tilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right]\times \tag{4.7.59}$$

$$\int\mathcal{D}x\exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{x}(E)\left(E^2-\omega^2+i\epsilon\right)\tilde{x}(-E)\right]. \tag{4.7.60}$$

$\int\mathcal{D}x\exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{x}(E)\left(E^2-\omega^2+i\epsilon\right)\tilde{x}(-E)\right]$ is unimportant because it is just a normalization factor: let $\langle 0\,|\,0\rangle_{f=0} = 1$; it does not play a role in calculating correlation functions. So

$$\langle 0\,|\,0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\,\tilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right] \tag{4.7.61}$$

$$= \exp\left[\frac{i}{2}\int_{-\infty}^{\infty}dtdt^{'}\,f(t)\,G\left(t-t^{'}\right)f\left(t^{'}\right)\right], \tag{4.7.62}$$

where

$$G(t) = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{-iEt}}{-E^2+\omega^2-i\epsilon}. \tag{4.7.63}$$

Using correlation functions

We have seen that (equation (4.7.29) with $t^{'}\to\infty$ and $t^{''}\to-\infty$)

$$\langle 0\,|\,T\hat{q}(t_1)\ldots|\,0\rangle = \frac{1}{i}\partial_{f(t_1)}\ldots\langle 0\,|\,0\rangle_f\Big|_{f=0}. \tag{4.7.64}$$

Take a specific example:

$$\langle 0\,|\,T\hat{q}(t_1)\,\hat{q}(t_2)\,|\,0\rangle = \frac{1}{i}\partial_{f(t_1)}\frac{1}{i}\partial_{f(t_2)}\langle 0\,|\,0\rangle_f\Big|_{f=0} \tag{4.7.65}$$

$$= \frac{1}{i}\partial_{f(t_1)}\left[\int_{-\infty}^{\infty}dt^{'}\,G\left(t_2-t^{'}\right)f\left(t^{'}\right)\right]\langle 0\,|\,0\rangle_f\Big|_{f=0} \tag{4.7.66}$$

$$= \left[\frac{1}{i}G(t_2-t_1)+\underbrace{\text{(terms with } f)}_{\text{do not matter as we will set } f=0}\right]\langle 0\,|\,0\rangle_f\Big|_{f=0} \tag{4.7.67}$$

$$= \frac{1}{i}G(t_2-t_1). \tag{4.7.68}$$

So the green function controls the 2 point correlation function for the solvable part of the Hamiltonian, H_0. In fact, green function also controls more general correlation functions. More generally (Wick theorem):

$$\langle 0\,|\,T\hat{q}(t_1)\ldots\hat{q}(t_{2m})\,|\,0\rangle = \frac{1}{i^m}\sum_{\substack{\text{pairings}\\ S_{2m}/\sim}}G\left(t_{i_1}-t_{i_2}\right)\ldots G\left(t_{i_{2m-1}}-t_{i_{2m}}\right), \tag{4.7.69}$$

where $S_{2m}/\sim$ is the set of splittings of $\{1,\ldots,2m\}$ into pairs and there are

$$\frac{(2m)!}{2^m m!} \tag{4.7.70}$$

such splittings - (we chose an even number of $\hat{q}$ because otherwise the correlation function would be equal to 0, by wick's theorem).

Example. The 4 point function is given by

$$\langle 0\,|\,T\hat{q}(t_1)\,\hat{q}(t_2)\,\hat{q}(t_3)\,\hat{q}(t_3)\,|\,0\rangle = \frac{1}{i^2}\left[G(t_1-t_2)\,G(t_3-t_4)+G(t_1-t_3)\,G(t_2-t_4)+G(t_1-t_4)\,G(t_2-t_3)\right]. \tag{4.7.71}$$

4.7.6 Exercises

4.7.6.1 Normalization Factor of $\mathcal{D}q$ after integrating out p for $H = p^2/2m + V(q)$

Exercise 4. Problem 6.1 (a) of [1]. (This will help explain how 4.7.52 was obtained)

We want to write

$$\mathcal{D}q = C\prod_{j=1}^{N} dq_j. \tag{4.7.72}$$

Our starting point is that

$$\left\langle q^{''}, t^{''} \middle| q^{'}, t^{'} \right\rangle = \int \mathcal{D}q\mathcal{D}p \exp\left[i\int_{t'}^{t''} dt\left(p(t)\dot{q}(t) - H(p(t), q(t))\right)\right], \tag{4.7.73}$$

where we assume that $H(p,q)$ is no more than quadratic in the momenta and the term that is quadratic in p is independent of q. Hence, H can be written in the following form:

$$H = a(q) + b(q)p + cp^2. \tag{4.7.74}$$

Thus, we obtain that

$$\left\langle q^{''}, t^{''} \middle| q^{'}, t^{'} \right\rangle = \int \mathcal{D}q\mathcal{D}p \exp\left[-i\int_{t'}^{t''} dt\left\{a(q) + (b(q) - \dot{q})p + cp^2\right\}\right]. \tag{4.7.75}$$

The next step is to complete the squares - using Mathematica*:

$$\left\langle q^{''}, t^{''} \middle| q^{'}, t^{'} \right\rangle = \int \mathcal{D}q\mathcal{D}p \exp\left[-i\int_{t'}^{t''} dt\left\{\left(c - \frac{(b-\dot{q})^2}{4a}\right) + a\left(\frac{b-\dot{q}}{2a} + p\right)^2\right\}\right]. \tag{4.7.76}$$

We now perform the integral by treating $\dot{q}$ and p as separate variables to obtain that

$$\begin{aligned}
\left\langle q^{''}, t^{''} \middle| q^{'}, t^{'} \right\rangle &= \int \mathcal{D}q\mathcal{D}p \exp\left[-i\int_{t'}^{t''} dt\left\{\left(a - \frac{(b-\dot{q})^2}{4c}\right) + c\left(\frac{b-\dot{q}}{2c} + p\right)^2\right\}\right] && (4.7.77)\\
&= \int \mathcal{D}q\mathcal{D}p \exp\left[-i\int_{t'}^{t''} dt\left\{\left(a - \frac{(b-\dot{q})^2}{4c}\right) + cp^2\right\}\right] && (4.7.78)\\
&= \int \mathcal{D}q\left(\int \mathcal{D}p \exp\left[-i\int_{t'}^{t''} dtcp^2\right]\right)\exp\left[-i\int_{t'}^{t''} dt\left(a - \frac{(b-\dot{q})^2}{4c}\right)\right] && (4.7.79)\\
&= \lim_{N\to\infty}\int\prod_{i=1}^{N} dq_i\left(\prod_{i=0}^{N}\int\frac{dp_i}{2\pi}\exp\left[-i\delta tcp_i^2\right]\right)\exp\left[-i\int_{t'}^{t''} dt\left(a - \frac{(b-\dot{q})^2}{4c}\right)\right] && (4.7.80)
\end{aligned}$$

because $\frac{b-\dot{q}}{2c}$ is like a constant functional shift. To evaluate $\int\frac{dp_i}{2\pi}\exp\left[-i\delta tcp_i^2\right]$, we will add a small real part to the integrand to make it converge:

$$\int\frac{dp_i}{2\pi}\exp\left[-i\delta tcp_i^2\right] = \int\frac{dp_i}{2\pi}\exp\left[-i\delta tcp_i^2 - \epsilon p_i^2\right]\Big|_{\epsilon=0}, \tag{4.7.81}$$

where $\epsilon > 0$. Because δt is real, we get that (Mathematica*)

$$\begin{aligned}
\int\frac{dp_i}{2\pi}\exp\left[-i\delta tcp_i^2\right] &= \sqrt{\frac{\pi}{i\delta tc + \epsilon}}\Bigg|_{\epsilon=0} && (4.7.82)\\
&= \sqrt{\frac{\pi}{i\delta tc}}. && (4.7.83)
\end{aligned}$$

We get that

$$\begin{aligned}
\left\langle q^{''}, t^{''} \middle| q^{'}, t^{'} \right\rangle &= \lim_{N\to\infty}\int\prod_{i=1}^{N} dq_i\left(\frac{1}{2\pi}\sqrt{\frac{\pi}{i\delta tc}}\right)^{N+1}\exp\left[-i\int_{t'}^{t''} dt\left(a - \frac{(b-\dot{q})^2}{4c}\right)\right] && (4.7.84)\\
&= \lim_{N\to\infty}\int\prod_{i=1}^{N} dq_i\left(\sqrt{\frac{(N+1)}{4\pi iTc}}\right)^{N+1}\exp\left[-i\int_{t'}^{t''} dt\left(a - \frac{(b-\dot{q})^2}{4c}\right)\right] && (4.7.85)
\end{aligned}$$

where we have used that

$$\int e^{-icp^2}dp = \sqrt{\frac{\pi}{ic}}. \tag{4.7.86}$$

Letting $a = V(q)$, $b = 0$ and $c = 1/2m$. We have that

$$\begin{aligned}\left\langle q^{''}, t^{''}\middle| q^{'}, t^{'}\right\rangle &= \lim_{N\to\infty}\int\prod_{i=0}^{N} dq_i\left(\sqrt{\frac{m(N+1)}{2\pi i T}}\right)^N \exp\left[-i\int_{t'}^{t''} dt\left(V(q) - \frac{m\dot{q}^2}{2}\right)\right] &(4.7.87)\\ &= \int \mathcal{D}q\exp\left[-i\int_{t'}^{t''} dt\left(V(q) - \frac{m\dot{q}^2}{2}\right)\right] = \int\mathcal{D}q\exp\left[i\int_{t'}^{t''} dtL(\dot{q}(t), q(t))\right] &(4.7.88)\end{aligned}$$

Hence, we have that

$$\mathcal{D}q = \prod_{i=1}^{N} dq_i\left(\sqrt{\frac{m(N+1)}{2\pi i T}}\right)^{N+1} = \prod_{i=1}^{N} dq_i\left(\sqrt{\frac{m}{2\pi i\delta t}}\right)^{N+1}. \tag{4.7.89}$$

Continuing to part (b),

Exercise 5. Problem 6.1 (b) of [1]. Evaluate 4.7.88 with $V(q) = 0$.
We obtain that for a Hamiltonian of the $P^2/2m$ that

$$\left\langle q^{''}, t^{''}\middle| q^{'}, t^{'}\right\rangle = \left(\sqrt{\frac{m}{2\pi i\hbar(t''-t')}}\right)\exp\left[\frac{im}{2\hbar}\frac{\left(q^{''}-q^{'}\right)^2}{t''-t'}\right]. \tag{4.7.90}$$

We want to evaluate

$$\begin{aligned}\xi &= \int\mathcal{D}q\exp\left[i\int_{t'}^{t''} dt\frac{m\dot{q}^2}{2}\right] &(4.7.91)\\ &= \lim_{N\to\infty} C\int\prod_{i=1}^{N}\left(dq_i\exp\left[i\delta t\frac{m(q_{i+1}-q_i)^2}{2\delta t^2}\right]\right) &(4.7.92)\\ &= \lim_{N\to\infty} C\int\prod_{i=1}^{N}\left(dq_i\exp\left[i\frac{m(q_{i+1}-q_i)^2}{2\delta t}\right]\right) &(4.7.93)\\ &= \lim_{N\to\infty} C\int\prod_{i=2}^{N}\left(dq_i\exp\left[i\frac{m(q_{i+1}-q_i)^2}{2\delta t}\right]\right)\underbrace{\int dq_1\exp\left[im\frac{(q_2-q_1)^2+\left(q_1-q^{''}\right)^2}{2\delta t}\right]}_{\equiv\alpha}, &(4.7.94)\end{aligned}$$

where $C = \left(\sqrt{\frac{m}{2\pi i\delta t}}\right)^{N+1}$. To perform the integral in the last line, we will add a small regulating parameter:

$$\begin{aligned}\alpha &= \int dq_1\exp\left[im\frac{(q_2-q_1)^2+\left(q_1-q^{''}\right)^2}{2\delta t} - \epsilon q_1^2\right]\Bigg|_{\epsilon=0} &(4.7.95)\\ &= \sqrt{i\frac{\pi\delta t}{m}}\exp\left[\frac{im}{4\delta t}\left(q^{''}-q_2\right)^2\right], &(4.7.96)\end{aligned}$$

where we used Mathematica*. Hence, we obtain that

$$\xi = \lim_{N\to\infty} C\sqrt{i\frac{\pi\delta t}{m}}\int\prod_{i=3}^{N}\left(dq_i \exp\left[i\frac{m\left(q_{i+1}-q_i\right)^2}{2\delta t}\right]\right)\int dq_2 \exp\left[\frac{im}{4\delta t}\left\{\left(q^{''}-q_2\right)^2+2\left(q_3-q_2\right)^2\right\}\right] \quad (4.7.97)$$

$$(\text{Mathematica}) = \lim_{N\to\infty} C\sqrt{i\frac{\pi\delta t}{m}}\sqrt{i\frac{4\pi\delta t}{3m}}\int\prod_{i=3}^{N}\left(dq_i \exp\left[i\frac{m\left(q_{i+1}-q_i\right)^2}{2\delta t}\right]\right)\exp\left[\frac{im}{6\delta t}\left(q^{''}-q_3\right)^2\right] \quad (4.7.98)$$

$$= \lim_{N\to\infty} C\sqrt{i\frac{\delta t\pi}{m}}\sqrt{i\frac{\delta t4\pi}{3m}}\sqrt{i\frac{\delta t3\pi}{2m}}\int\prod_{i=4}^{N}\left(dq_i \exp\left[i\frac{m\left(q_{i+1}-q_i\right)^2}{2\delta t}\right]\right)\exp\left[\frac{im}{8\delta t}\left(q^{''}-q_4\right)^2\right] \quad (4.7.99)$$

$$= \lim_{N\to\infty} C\sqrt{i\frac{\delta t\pi}{m}}\sqrt{i\frac{\delta t4\pi}{3m}}\sqrt{i\frac{\delta t3\pi}{2m}}\int\prod_{i=4}^{N}\left(dq_i \exp\left[i\frac{m\left(q_{i+1}-q_i\right)^2}{2\delta t}\right]\right)\exp\left[\frac{im}{8\delta t}\left(q^{''}-q_4\right)^2\right] \quad (4.7.100)$$

$$= \vdots \quad (4.7.101)$$

$$= \lim_{N\to\infty} C\sqrt{i\frac{\pi\delta t}{m}}\sqrt{i\frac{4\pi\delta t}{3m}}...\sqrt{i\frac{2\left(k-1\right)\delta t\pi}{km}}\int\prod_{i=k}^{N}\left(dq_i \exp\left[i\frac{m\left(q_{i+1}-q_i\right)^2}{2\delta t}\right]\right)\exp\left[\frac{im}{2k\delta t}\left(q^{''}-q_k\right)^2\right] \quad (4.7.102)$$

$$= \vdots \quad (4.7.103)$$

$$= \lim_{N\to\infty} C\prod_{j=1}^{N-1}\sqrt{i\frac{2j\pi\delta t}{\left(j+1\right)m}}\int\left(dq_N \exp\left[i\frac{m\left(q^{'}-q_i\right)^2}{2\delta t}\right]\right)\exp\left[\frac{im}{2N\delta t}\left(q^{''}-q_N\right)^2\right] \quad (4.7.104)$$

$$= \lim_{N\to\infty} C\prod_{j=1}^{N-1}\sqrt{i\frac{2j\pi\delta t}{\left(j+1\right)m}}\int dq_N \exp\left[i\frac{m\left(q^{'}-q_i\right)^2}{2\delta t}\right]\exp\left[\frac{im}{2N\delta t}\left(q^{''}-q_N\right)^2\right] \quad (4.7.105)$$

$$(\text{Mathematica}) = \lim_{N\to\infty} C\prod_{j=1}^{N-1}\sqrt{i\frac{2j\pi\delta t}{\left(j+1\right)m}}\exp\left[\frac{im}{2\left(N+1\right)\delta t}\left(q^{''}-q^{'}\right)^2\right]\sqrt{i\frac{2N}{N+1}\frac{\pi}{m}} \quad (4.7.106)$$

$$= \lim_{N\to\infty} C\prod_{j=1}^{N}\sqrt{i\frac{2j\pi\delta t}{\left(j+1\right)m}}\exp\left[\frac{im}{2\left(N+1\right)\delta t}\left(q^{''}-q^{'}\right)^2\right] \quad (4.7.107)$$

$$= \lim_{N\to\infty} C\sqrt{i\frac{2\pi}{m}}^{N}\left(\prod_{j=1}^{N}\sqrt{\frac{j\delta t}{j+1}}\right)\exp\left[\frac{im}{2\left(N+1\right)\delta t}\left(q^{''}-q^{'}\right)^2\right] \quad (4.7.108)$$

$$= \lim_{N\to\infty} C\sqrt{i\frac{\delta t2\pi}{m}}^{N}\left(\frac{1}{\sqrt{\left(N+1\right)!}}\prod_{j=1}^{N}\sqrt{j}\right)\exp\left[\frac{im}{2\left(N+1\right)\delta t}\left(q^{''}-q^{'}\right)^2\right] \quad (4.7.109)$$

$$= \lim_{N\to\infty} C\sqrt{i\frac{\delta t2\pi}{m}}^{N}\left(\frac{\sqrt{N!}}{\sqrt{\left(N+1\right)!}}\right)\exp\left[\frac{im}{2\left(N+1\right)\delta t}\left(q^{''}-q^{'}\right)^2\right] \quad (4.7.110)$$

$$= \lim_{N\to\infty} C\sqrt{i\frac{\delta t2\pi}{m}}^{N}\left(\frac{1}{\sqrt{N+1}}\right)\exp\left[\frac{im}{2\left(N+1\right)\delta t}\left(q^{''}-q^{'}\right)^2\right] \quad (4.7.111)$$

We had that

$$C = \left(\sqrt{\frac{m}{2\pi i\delta t}}\right)^{N+1} = \left(\sqrt{\frac{m\left(N+1\right)}{2\pi iT}}\right)^{N+1} \quad (4.7.112)$$

So $(T = t^{''} - t^{'})$

$$\xi = \lim_{N\to\infty}\left(\sqrt{\frac{m}{2\pi i\delta t}}\right)^{N+1}\sqrt{i\frac{2\pi\delta t}{m}}^{N}\frac{1}{\sqrt{N+1}}\exp\left[\frac{im}{2(N+1)\delta t}\left(q^{''}-q^{'}\right)^2\right] \quad (4.7.113)$$

$$= \lim_{N\to\infty}\left(\sqrt{\frac{m}{2\pi i\delta t}}\right)\frac{1}{\sqrt{N+1}}\exp\left[\frac{im(N+1)}{2(N+1)T}\left(q^{''}-q^{'}\right)^2\right] \quad (4.7.114)$$

$$= \lim_{N\to\infty}\left(\sqrt{\frac{m}{2\pi i(t''-t')}}\right)\exp\left[\frac{im}{2}\frac{\left(q^{''}-q^{'}\right)^2}{t''-t'}\right] \quad (4.7.115)$$

$$= \left(\sqrt{\frac{m}{2\pi i(t''-t')}}\right)\exp\left[\frac{im}{2}\frac{\left(q^{''}-q^{'}\right)^2}{t''-t'}\right] \quad (4.7.116)$$

How do we restore $\hbar$?

We have that $\hbar$ is in units of $m^2 kg/s$. First, we ask what is dimension of an inner product? We know that we have that

$$\left\langle q^{''}\middle| q^{'}\right\rangle = \delta\left(q^{''}-q^{'}\right), \quad (4.7.117)$$

because we are working in one dimension. Delta functions are in units of length inverse. Hence, ξ must have units of length, and so

$$\sqrt{\frac{m}{(t''-t')}} \quad (4.7.118)$$

has dimensions of length so we must add a term that is proportional $1/(\text{length})\times\sqrt{s/kg} = 1/\sqrt{l^2 kg/s}$ which is just $\sqrt{\hbar^2}$. So

$$\sqrt{\frac{m}{(t''-t')}} \to \sqrt{\frac{m}{\hbar^2(t''-t')}}. \quad (4.7.119)$$

We now turn our attention to what is inside the exponential. It must be dimensionless, now

$$\frac{im}{2}\frac{\left(q^{''}-q^{'}\right)^2}{t''-t'} \quad (4.7.120)$$

has dimensions of $kg \times m^2/s$ which are just the units of $\hbar$. So, we must have that

$$\frac{im}{2}\frac{\left(q^{''}-q^{'}\right)^2}{t''-t'} \to \frac{im}{2\hbar}\frac{\left(q^{''}-q^{'}\right)^2}{t''-t'}. \quad (4.7.121)$$

Hence,

$$\left\langle q^{''},t^{''}\middle| q^{'},t^{'}\right\rangle = \left(\sqrt{\frac{m}{2\pi i\hbar(t''-t')}}\right)\exp\left[\frac{im}{2\hbar}\frac{\left(q^{''}-q^{'}\right)^2}{t''-t'}\right]. \quad (4.7.122)$$

4.7.6.2 The Euclidean Harmonic oscillator: Finding H.O. ground state using Path integrals

Problem 1. (From Gukov's course) After the Wick rotation $t \to -it$ [[$idt_{old} \to -dt_{new}$*, which implies that $t_{old} = it_{new}$]], the Euclidean Lagrangian of a harmonic oscillator is

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2. \quad (4.7.123)$$

and the path integral looks like

$$\left\langle q^{''}\middle| e^{-HT}\middle| q^{'}\right\rangle = \int \mathcal{D}q\exp\left(-\frac{1}{2}\int_0^T dt\left(\dot{q}^2+\omega^2 q^2\right)\right). \quad (4.7.124)$$

(1)Perform the path integral by summing over all "field" configurations $q(t)$ which satisfy the boundary conditions $q(t=0) = q^{'}$ and $q(t=T) = q^{''}$. Specifically, make the time periodic by identifying $t \sim t+T$ and follow the sequence of steps: (1) find the classical solution $q_{class}(t)$ satisfying these boundary conditions; (2) by expanding $q(t)$ around the classical solution $q_{class}(t)$,

$$q(t) = q_{class}(t) + \sum_k q_k \sin\frac{\pi k t}{T} \quad (4.7.125)$$

perform the integral over q_k's; (3) analyze the result in the large T limit and read off the energy E_0 of the ground state and the ground state wave function (up to normalization)

$$\psi_0(q) \sim e^{-\omega q^2/2}. \tag{4.7.126}$$

The Euclidean Lagrangian We originally had that

$$\begin{aligned} L &= \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 && (4.7.127)\\ &= \frac{1}{2}\left(\frac{d}{dt_{old}}q\right)^2 - \frac{1}{2}\omega^2 q^2. && (4.7.128)\end{aligned}$$

This becomes

$$\begin{aligned} L &= \frac{1}{2}\left(\frac{d}{idt_{new}}q\right)^2 - \frac{1}{2}\omega^2 q^2 && (4.7.129)\\ &= -\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2. && (4.7.130)\end{aligned}$$

We can multiply a Lagrangian by -1 because first notice that we the same equations of motion ($\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$) because the minus signs on both sides. More fundamentally, we get the equations of motions by checking when $\delta S = 0$. This procedure does not care whether the function we are solving for gives as a local minimum or maximum (we usually minimize the action but when we multiply by -1 we get the local maximum). The statement of "the principle of minimum action" actually means just extremizing the action.

Part (1) The Euler Lagrange equations tell us that

$$\frac{d}{dt_{new}}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}. \tag{4.7.131}$$

Remark 1. Note that we used t_{new} instead of t_{old} because when we extremize the action, we do so in the new time coordinates: We orginally have that

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t_{old})\, dt_{old}, \tag{4.7.132}$$

which we initially transformed into the t_{new} coordinates, which gave us

$$\delta S = \delta \int_{-t_1}^{-t_2} L(q, \dot{q}, t_{new})\, dt_{new}, \tag{4.7.133}$$

(this introduced a constant proportional to i which doesn't affect the extremization, we also multiplied the Lagrangian with -1, which, as we shown in the previous paragraph, does NOT affect the physics; also note that the boundaries do not change, this is part of the Wick rotation theory) then we did the extremization procedure in the new coordinates to obtain the Euler-Lagrange equations which, thus, will be given in the new coordinates.

We obtain that

$$\begin{aligned} \frac{d}{dt}\dot{q} &= \omega^2 q && (4.7.134)\\ \Longrightarrow \ddot{q} &= \omega^2 q. && (4.7.135)\end{aligned}$$

This can be easily solved with

$$q(t) = C_1 e^{\omega t} + C_2 e^{-\omega t}. \tag{4.7.136}$$

We next impose periodic boundary conditions using Mathematica*:

$$q(t) = \frac{q'' \sinh \omega t - q' \sinh[(t-T)\omega]}{\sinh \omega t}. \tag{4.7.137}$$

Is this condition periodic? We have that

$$q(t+T) = \frac{q'' \sinh \omega(t+T) - q' \sinh[\omega t]}{\sinh \omega(t+T)}. \tag{4.7.138}$$

This is not the same as $q(t)$ so I guess we impose periodicity.

Part (2) Write $q(t)$ as a Fourier series:

$$q(t) = q_{class}(t) + \sum_k q_k \sin\frac{\pi k t}{T} \tag{4.7.139}$$
$$= q_{class}(t) + \delta q(t). \tag{4.7.140}$$

The motivation behind this is that we expect that the system will follow the classical trajectory.

Then, we have that

$$S[q+\delta q] = \int_0^T dt \left\{\left(\dot{q}^2_{class}(t) + \omega^2 q^2_{class}\right) + \left(\dot{\delta q}^2(t) + \omega^2\delta q^2\right) + 2\left(\dot{\delta q}\dot{q}_{class}(t) + \omega^2 \delta q q_{class}\right)\right\} \tag{4.7.141}$$
$$= S[q_{class}] + S[\delta q] + 2\int_0^T dt\left(\dot{\delta q}\dot{q}_{class}(t) + \omega^2\delta q q_{class}\right) \tag{4.7.142}$$
$$= S[q_{class}] + S[\delta q] - 2\int_0^T dt\delta q\ddot{q}_{class}(t) + 2\dot{\delta q}\dot{q}_{class}(t)\Big|_0^T + \int_0^T dt\omega^2\delta q q_{class}. \tag{4.7.143}$$

But the boundaries are fixed: $q(0) = q^{'}$ and $q(T) = q^{''}$ and q_{class} satisfies the boundary conditions (i.e. $q_{class}(0) = q^{'}$ and $q_{class}(T) = q^{''}$). Hence,

$$q(0) = q^{'} = q_{class}(0) + \delta q(0) = q^{'} + \delta q(0) \tag{4.7.144}$$
$$= q^{'}. \tag{4.7.145}$$

Hence, we must have that $\delta q(0) = 0$. By a similar argument we can show that $\delta q(T) = 0$. Hence, $\dot{\delta q}(0) = \dot{\delta q}(T) = 0$. We can now write that

$$S[q+\delta q] = S[q_{class}] + S[\delta q] - 2\int_0^T dt\delta q\ddot{q}_{class}(t) + 2\dot{\delta q}\dot{q}_{class}(t)\Big|_0^T + \int_0^T dt\omega^2\delta q q_{class} \tag{4.7.146}$$
$$= S[q_{class}] + S[\delta q] - 2\int_0^T dt\delta q\ddot{q}_{class}(t) + \int_0^T dt\omega^2\delta q q_{class} \tag{4.7.147}$$
$$= S[q_{class}] + S[\delta q] - 2\delta q\int_0^T dt\left(\ddot{q}_{class}(t) + \omega^2 q_{class}(t)\right) \tag{4.7.148}$$
$$= S[q_{class}] + S[\delta q], \tag{4.7.149}$$

because q_{class} satisfies the classical Equations of motion: $\ddot{q} = \omega^2 q$. We have that

$$S[q_{class}] = \int_0^T dt\left(\dot{q}^2_{class}(t) + \omega^2 q^2_{class}\right) \tag{4.7.150}$$
$$= \omega\frac{\left(q^{'2} + q^{''2}\right)\cosh(\omega T) - 2q^{'}q^{''}}{\sinh(\omega T)}, \tag{4.7.151}$$

where we have used Mathematica*. In addition,

$$\dot{\delta q} = \sum_k q_k\left(\frac{\pi k}{T}\right)\cos\frac{\pi k t}{T} \tag{4.7.152}$$
$$\Longrightarrow \left(\dot{\delta q}\right)^2 = \sum_{kl} q_k q_l\left(\frac{\pi k}{T}\right)\left(\frac{\pi l}{T}\right)\cos\frac{\pi k t}{T}\cos\frac{\pi l t}{T}. \tag{4.7.153}$$

In addition, we have that

$$q^2 = \sum_{kl} q_k q_l \sin\frac{\pi k t}{T}\sin\frac{\pi l t}{T}. \tag{4.7.154}$$

We will also use that

$$\int_0^T \cos\frac{\pi k t}{T}\cos\frac{\pi l t}{T}dt = \frac{T}{\pi}\int_0^\pi \cos kx\cos lx dx \tag{4.7.155}$$
$$= \frac{T}{\pi}\frac{k\cos(l\pi)\sin(k\pi) - l\cos(k\pi)\sin(l\pi)}{k^2 - l^2} \tag{4.7.156}$$
$$= 0 \tag{4.7.157}$$

for $k \neq l$ and where we have used Mathematica in the second line*. When $k = l$, we have that

$$\frac{T}{\pi}\int_0^{\pi}\cos kx\cos lx dx = \frac{T}{\pi}\frac{2k\pi+\sin(2k\pi)}{4k} \quad (4.7.158)$$
$$= \frac{T}{\pi}\frac{2k\pi+\sin(2k\pi)}{4k} \quad (4.7.159)$$
$$= \frac{2kT}{4k} = \frac{T}{2}, \quad (4.7.160)$$

where we have Mathematica in the first line*. Hence, we have that

$$\int_0^T \cos\frac{\pi kt}{T}\cos\frac{\pi lt}{T}dt = \frac{T}{2}\delta_{kl}. \quad (4.7.161)$$

In a similar fashion, we can show that

$$\int_0^T \sin\frac{\pi kt}{T}\sin\frac{\pi lt}{T}dt = \frac{T}{2}\delta_{kl}. \quad (4.7.162)$$

Hence, we have that

$$S[\delta q] = \int_0^T dt\left(\dot{\delta q}^2(t) + \omega^2\delta q^2\right) \quad (4.7.163)$$
$$= \sum_{kl}\int_0^T dt q_k q_l\left(\left(\frac{\pi k}{T}\right)\left(\frac{\pi l}{T}\right)\cos\frac{\pi kt}{T}\cos\frac{\pi lt}{T} + \omega^2\sin\frac{\pi kt}{T}\sin\frac{\pi lt}{T}\right) \quad (4.7.164)$$
$$= \sum_{kl}\frac{Tq_kq_l}{2}\left(\left(\frac{\pi k}{T}\right)\left(\frac{\pi l}{T}\right)\delta_{kl} + \omega^2\delta_{kl}\right) \quad (4.7.165)$$
$$= \sum_k \frac{Tq_k^2}{2}\left(\left(\frac{\pi k}{T}\right)^2 + \omega^2\right). \quad (4.7.166)$$

So

$$\left\langle q^{''}\left|e^{-HT}\right|q^{'}\right\rangle = \int\mathcal{D}q\exp\left(-\frac{1}{2}\int_0^T dt\left(\dot{q}^2+\omega^2q^2\right)\right) \quad (4.7.167)$$
$$= \int\mathcal{D}q\exp\left(-\frac{1}{2}\omega\frac{\left(q^{'2}+q^{''2}\right)\cosh(\omega T) - 2q^{'}q^{''}}{\sinh(\omega T)}\right)\exp\left(-\frac{1}{2}\sum_k\frac{Tq_k^2}{2}\left(\left(\frac{\pi k}{T}\right)^2+\omega^2\right)\right) \quad (4.7.168)$$
$$= \int\mathcal{D}q\exp\left(-\omega\frac{\left(q^{'2}+q^{''2}\right)\cosh(\omega T) - 2q^{'}q^{''}}{2\sinh(\omega T)}\right)\exp\left(-\sum_k\frac{Tq_k^2}{4}\left(\left(\frac{\pi k}{T}\right)^2+\omega^2\right)\right) \quad (4.7.169)$$

What is $\mathcal{D}q$ equal to? We have that

$$\mathcal{D}q(t) = \mathcal{D}\left[q_{class}(t) + \sum_k q_k\sin\frac{\pi kt}{T}\right] \quad (4.7.170)$$
$$= \mathcal{D}\left[\sum_k q_k\sin\frac{\pi kt}{T}\right] \quad (4.7.171)$$

because q_{class} if fixed. Hence,

$$\left\langle q^{''}\left|e^{-HT}\right|q^{'}\right\rangle = \exp\left(-\omega\frac{\left(q^{'2}+q^{''2}\right)\cosh(\omega T) - 2q^{'}q^{''}}{2\sinh(\omega T)}\right)\int\mathcal{D}q\exp\left(-\sum_k\frac{Tq_k^2}{4}\left(\left(\frac{\pi k}{T}\right)^2+\omega^2\right)\right) \quad (4.7.172)$$

To switch to the q_k variables, we need to calculate the Jacobian: we have that $q(t_i) = \sum_k\left[q_k\sin\frac{\pi kt}{T}\right]$. The Jacobian is equal to

$$J = \begin{pmatrix} \partial_{q_1}(q(t_1)) & \partial_{q_2}(q(t_1)) & ... & \partial_{q_N}(q(t_1)) \\ \partial_{q_1}(q(t_2)) & \partial_{q_2}(q(t_2)) & ... & \partial_{q_N}(q(t_2)) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{q_1}(q(t_N)) & \partial_{q_2}(q(t_N)) & ... & \partial_{q_N}(q(t_N)) \end{pmatrix} \tag{4.7.173}$$

$$= \begin{pmatrix} \sin\left(\frac{\pi t_1}{T}\right) & \sin\left(\frac{2\pi t_1}{T}\right) & ... & \sin\left(\frac{N\pi t_1}{T}\right) \\ \sin\left(\frac{\pi t_2}{T}\right) & \sin\left(\frac{2\pi t_2}{T}\right) & ... & \sin\left(\frac{N\pi t_2}{T}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \sin\left(\frac{\pi t_N}{T}\right) & \sin\left(\frac{2\pi t_N}{T}\right) & ... & \sin\left(\frac{N\pi t_N}{T}\right) \end{pmatrix}. \tag{4.7.174}$$

This is very difficult to calculate but we do not have to as the above is T independent (but depends on N). Indeed, we have that $t_i = iT/(N+1)$ as $i = 0, 1, ..., N+1$ with $q(0) = q^{'}$ and $q(T) = q^{''}$. Hence, we have that

$$\mathcal{D}q = f(N)\, C \prod_{i=1}^{N} dq_k, \tag{4.7.175}$$

where $f(N)$ is a normalization factor that only depends on N, and from equation (4.7.89) (we have a Hamiltonian of the form $P^2/2m + V(q)$, so we can use equation (4.7.89))

$$C = \left(\sqrt{\frac{m(N+1)}{2\pi i T_{old}}}\right)^{N+1}. \tag{4.7.176}$$

In our exercise, we have that $m = 1$ and we did the Wick rotation so that $t_{new} = -it_{old}$. Hence, we have that

$$C = \left(\sqrt{\frac{(N+1)}{2\pi T_{new}}}\right)^{N+1} \tag{4.7.177}$$

$$= \left(\frac{N+1}{2\pi T_{new}}\right)^{\frac{N+1}{2}}. \tag{4.7.178}$$

So, we have, by denoting

$$e_{class} = \exp\left(-\omega \frac{\left(q^{'2} + q^{''2}\right)\cosh(\omega T) - 2q^{'}q^{''}}{2\sinh(\omega T)}\right), \tag{4.7.179}$$

that

$$\left\langle q^{''} \middle| e^{-HT} \middle| q^{'} \right\rangle = \exp\left(-\omega \frac{\left(q^{'2} + q^{''2}\right)\cosh(\omega T) - 2q^{'}q^{''}}{2\sinh(\omega T)}\right) \int \mathcal{D}q \exp\left(-\sum_k \frac{Tq_k^2}{4}\left(\left(\frac{\pi k}{T}\right)^2 + \omega^2\right)\right) \tag{4.7.180}$$

$$= e_{class} \times C \times f(N) \int \left(\prod_{i=1}^{N} dq_k\right) \exp\left(-\sum_k \frac{Tq_k^2}{4}\left(\left(\frac{\pi k}{T}\right)^2 + \omega^2\right)\right) \tag{4.7.181}$$

$$= e_{class} \times \left(\frac{N+1}{2\pi T}\right)^{\frac{N+1}{2}} \times f(N) \int \left\{\prod_{i=1}^{N} dq_k \exp\left(-\frac{Tq_k^2}{4}\left(\left(\frac{\pi k}{T}\right)^2 + \omega^2\right)\right)\right\} \tag{4.7.182}$$

$$= e_{class} \times \left(\frac{1}{2\pi T}\right)^{\frac{N+1}{2}} \times f(N) \int \left\{\prod_{i=1}^{N} dq_k \exp\left(-\frac{Tq_k^2}{4}\left(\left(\frac{\pi k}{T}\right)^2 + \omega^2\right)\right)\right\}, \tag{4.7.183}$$

where in the last line, we have absorbed $(N+1)^{(N+1)/2}$ into the definition of $f(N)$. Now, we have that

$$\int dq_k \exp\left(-\frac{Tq_k^2}{4}\left(\left(\frac{\pi k}{T}\right)^2 + \omega^2\right)\right) = \frac{2}{\sqrt{\frac{k^2\pi}{T} + \frac{T\omega^2}{\pi}}} \tag{4.7.184}$$

$$= \frac{2\sqrt{\pi T}}{\sqrt{k^2\pi^2 + T^2\omega^2}}, \tag{4.7.185}$$

where we have used Mathematica in the first line*. Hence,

$$\begin{aligned}
\left\langle q^{''}\left|e^{-HT}\right|q^{'}\right\rangle &= e_{class}\times\left(\frac{1}{2\pi T}\right)^{\frac{N+1}{2}}\times f(N)\left(\prod_{k=1}^{N}\frac{2\sqrt{\pi T}}{\sqrt{k^2\pi^2+T^2\omega^2}}\right) && (4.7.186)\\
&= e_{class}\times\sqrt{\frac{1}{2\pi T}}\times f(N)\left(\prod_{k=1}^{N}\frac{\sqrt{2}}{\sqrt{k^2\pi^2+T^2\omega^2}}\right) && (4.7.187)\\
&= e_{class}\times\sqrt{\frac{1}{2\pi T}}\times f(N)\left(\prod_{k=1}^{N}\sqrt{\frac{2}{k^2\pi^2+T^2\omega^2}}\right) && (4.7.188)\\
&= e_{class}\times\sqrt{\frac{1}{2\pi T}}\times f(N)\left(\prod_{k=1}^{N}\sqrt{\frac{1}{k^2\pi^2+T^2\omega^2}}\right), && (4.7.189)
\end{aligned}$$

where in the last equality, we have absorbed $\sqrt{2}^N$ into $f(N)$. Continuing, we have that

$$\begin{aligned}
\left\langle q^{''}\left|e^{-HT}\right|q^{'}\right\rangle &= e_{class}\times\sqrt{\frac{1}{2\pi T}}\times f(N)\left(\prod_{k=1}^{N}\left(k^2\pi^2+T^2\omega^2\right)\right)^{-1/2} && (4.7.190)\\
&= e_{class}\times\sqrt{\frac{1}{2\pi T}}\times f(N)\left(\prod_{k=1}^{N}k^2\pi^2\left(1+\frac{T^2\omega^2}{k^2\pi^2}\right)\right)^{-1/2} && (4.7.191)\\
&= e_{class}\times\sqrt{\frac{1}{2\pi T}}\times f(N)\sqrt{\omega T}\left(\omega T\prod_{k=1}^{N}k^2\pi^2\left(1+\frac{T^2\omega^2}{k^2\pi^2}\right)\right)^{-1/2} && (4.7.192)\\
&= e_{class}\times\sqrt{\omega}\times f(N)\left(\omega T\prod_{k=1}^{N}\left(1+\frac{T^2\omega^2}{k^2\pi^2}\right)\right)^{-1/2}, && (4.7.193)
\end{aligned}$$

where in the last line we incorporated $\left(\prod_{k=1}^{N}\sqrt{k^2\pi^2}^{-1}\right)/2\pi$ into $f(N)$. We then use equation (5.4.2) to write that

$$\lim_{N\to\infty}\omega T\prod_{k=1}^{N}\left(1+\frac{T^2\omega^2}{k^2\pi^2}\right)=\sinh\omega T. \qquad (4.7.194)$$

Thus, we obtain

$$\left\langle q^{''}\left|e^{-HT}\right|q^{'}\right\rangle = e_{class}\times f(N)\times\sqrt{\frac{\omega}{\sinh\omega T}}. \qquad (4.7.195)$$

Apparently, the normalization can be fixed by imposing that $\left\langle q^{''}\middle|q^{'}\right\rangle=\delta^3\left(q^{''}-q^{'}\right)$.

Part (3) We will pull off a trick that was introduced in section 4.7.3.1:

Denote the eigenstates of H by $|E_n\rangle$, then assuming a discrete spectrum, we have that

$$\begin{aligned}
\left\langle q^{''}\left|e^{-HT}\right|q^{'}\right\rangle &= \sum_{nl}\left\langle q^{''}\middle|n\right\rangle\left\langle l\middle|q^{'}\right\rangle\left\langle E_n\left|e^{-HT}\right|E_l\right\rangle && (4.7.196)\\
&= \sum_{nl}\left\langle q^{''}\middle|n\right\rangle\left\langle l\middle|q^{'}\right\rangle\left\langle E_n\left|e^{-E_lT}\right|E_l\right\rangle. && (4.7.197)
\end{aligned}$$

As $T\to\infty$, only the ground state remains as all other states have higher energies and so are more strongly damped. Hence,

$$\begin{aligned}
\lim_{T\to\infty}\left\langle q^{''}\left|e^{-HT}\right|q^{'}\right\rangle &= \lim_{T\to\infty}\sum_{n}\left\langle q^{''}\middle|n\right\rangle\left\langle 0\middle|q^{'}\right\rangle\left\langle E_n\middle|0\right\rangle e^{-E_0T} && (4.7.198)\\
&= \lim_{T\to\infty}\sum_{n}\left\langle q^{''}\middle|n\right\rangle\left\langle 0\middle|q^{'}\right\rangle\delta_{n0}e^{-E_0T} && (4.7.199)\\
&= \lim_{T\to\infty}\left\langle q^{''}\middle|0\right\rangle\left\langle 0\middle|q^{'}\right\rangle e^{-E_0T} && (4.7.200)\\
&= \lim_{T\to\infty}\psi_0\left(q^{''}\right)\psi_0^*\left(q^{'}\right)e^{-E_0T} && (4.7.201)\\
&\propto \lim_{T\to\infty}e_{class}\times\sqrt{\frac{\omega}{\sinh\omega T}}, && (4.7.202)
\end{aligned}$$

where we used equation (4.7.195) in the last equality, and where (from equation (4.7.179))

$$e_{class} = \exp\left(-\omega\frac{\left(q^{'2}+q^{''2}\right)\cosh(\omega T)-2q^{'}q^{''}}{2\sinh(\omega T)}\right) \tag{4.7.203}$$

$$= \exp\left(-\omega\frac{\left(q^{'2}+q^{''2}\right)\cosh(\omega T)}{2\sinh(\omega T)}+\omega\frac{q^{'}q^{''}}{\sinh(\omega T)}\right), \tag{4.7.204}$$

so

$$\lim_{T\to\infty} e_{class} = \lim_{T\to\infty}\exp\left(-\frac{\omega\left(q^{'2}+q^{''2}\right)}{2}+\omega\frac{q^{'}q^{''}}{\sinh(\omega T)}\right) \tag{4.7.205}$$

and

$$\lim_{T\to\infty}\left\langle q^{''}\middle|e^{-HT}\middle|q^{'}\right\rangle \propto \exp\left(-\frac{\omega\left(q^{'2}+q^{''2}\right)}{2}\right)\lim_{T\to\infty}\sqrt{\omega}\frac{\exp\left(\omega q^{'}q^{''}/\sinh\omega T\right)}{\sqrt{\sinh\omega T}} \tag{4.7.206}$$

$$= \lim_{T\to\infty}\sqrt{\omega}\exp\left(-\frac{\omega\left(q^{'2}+q^{''2}\right)}{2}\right), \tag{4.7.207}$$

as $\sin\omega T\to\infty$ as $T\to\infty$ and so

$$\lim_{T\to\infty}\frac{\exp\left(\omega q^{'}q^{''}/\sinh\omega T\right)}{\sqrt{\sinh\omega T}} = \lim_{T\to\infty}\frac{1}{\sqrt{\sinh\omega T}} \tag{4.7.208}$$

$$\propto \lim_{T\to\infty}\frac{1}{\sqrt{e^{\omega T}}} = \lim_{T\to\infty}e^{-\omega T/2}. \tag{4.7.209}$$

Thus,

$$\lim_{T\to\infty}\left\langle q^{''}\middle|e^{-HT}\middle|q^{'}\right\rangle \propto \lim_{T\to\infty}\exp\left(-\frac{\omega\left(q^{'2}+q^{''2}\right)}{2}\right)e^{-\omega T/2} \tag{4.7.210}$$

$$\propto \lim_{T\to\infty}\psi_0\left(q^{''}\right)\psi_0^*\left(q^{'}\right)e^{-E_0T}. \tag{4.7.211}$$

Thus, we identify the ground state energy to be

$$\boxed{E_0=\frac{\omega}{2}} \tag{4.7.212}$$

and the ground state wavefunction to be real and proportional to

$$\boxed{\psi_0(q)\propto e^{-\omega q^2/2}.} \tag{4.7.213}$$

4.8 The Path Integral for the free field theory

Consider the following Hamiltonian density:

$$\mathcal{H}=\frac{1}{2}\Pi^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m\phi^2, \tag{4.8.1}$$

where$\phi(x)$ is the analogue of $q(t)$ and is the "field". The source terms will now be denoted with $J(x)$:

$$J(x)\leftrightarrow f(t). \tag{4.8.2}$$

To regularize concrete expressions, the equivalent of $\mathcal{H}\to(1-i\epsilon)\mathcal{H}$ will be $m^2\to m^2-i\epsilon$. $m^2\to m^2-i\epsilon$ will be **assumed** in what follows.

We have that

$$\mathcal{L}=-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{1}{2}m^2\phi^2. \tag{4.8.3}$$

Moreover,

$$Z(J) \equiv \langle 0\,|\,0\rangle_J = \int \mathcal{D}\phi \underbrace{\exp\left(i\int d^4x\,[\mathcal{L}+J\phi]\right)}_{e^{iS}}, \tag{4.8.4}$$

where Z is the partition function.

Introduce the Fourier transform of ϕ:

$$\tilde{\phi}(k) = \int d^4x e^{-ikx}\phi(x), \tag{4.8.5}$$

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\tilde{\phi}(k). \tag{4.8.6}$$

And so we can write that (the procedure is nearly identical to what was done in section 4.7.5)

$$S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[-\tilde{\phi}(k)\left(k^2+m^2\right)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)\right] \tag{4.8.7}$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2} - \tilde{\chi}(k)\left(k^2+m^2\right)\tilde{\chi}(-k)\right], \tag{4.8.8}$$

where

$$\tilde{\chi}(k) = \tilde{\phi}(k) - \frac{\tilde{J}(k)}{k^2+m^2}. \tag{4.8.9}$$

Note that $\mathcal{D}\phi = \mathcal{D}\chi$ because χ is ϕ with an added (functional) constant. Normalizing (i.e. $Z(0) = \langle 0\,|\,0\rangle_{J=0} = 1$), we get that

$$Z(J) = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\epsilon}\right] \tag{4.8.10}$$

$$= \exp\left[\frac{i}{2}\int d^4x d^4x' J(x)\Delta\left(x'-x\right)J\left(x'\right)\right], \tag{4.8.11}$$

where

$$\Delta\left(x-x'\right) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{\exp\left[ik\left(x-x'\right)\right]}{k^2+m^2-i\epsilon} \tag{4.8.12}$$

and is called the Feynman propagator. Indeed,

$$\left(-\partial_x^2+m^2\right)\Delta\left(x-x'\right) = \delta^4\left(x-x'\right), \tag{4.8.13}$$

where $\partial_x = \partial_{x^\mu}$ and $\partial_x^2 = \partial_\mu\partial^\mu$.

Calculating correlation functions We have that

$$\langle 0\,|\,T\phi(x_1)\ldots\,|\,0\rangle = \frac{1}{i}\frac{\partial}{\partial J(x_1)}\ldots Z(J)\bigg|_{J=0}. \tag{4.8.14}$$

In particular,

$$\langle 0\,|\,T\phi(x_1)\phi(x_2)\,|\,0\rangle = \frac{1}{i}\frac{\partial}{\partial J(x_1)}\frac{1}{i}\frac{\partial}{\partial J(x_2)}Z(J)\bigg|_{J=0} \tag{4.8.15}$$

$$= \frac{1}{i}\frac{\partial}{\partial J(x_1)}\left[\int d^4x'\Delta(x_2-x_1)J\left(x'\right)\right]Z(J)\bigg|_{J=0} \tag{4.8.16}$$

$$= \left(\frac{1}{i}\Delta(x_2-x_1) + \text{terms with Js}\right)Z(J)\bigg|_{J=0} \tag{4.8.17}$$

$$= \frac{1}{i}\Delta(x_2-x_1). \tag{4.8.18}$$

More generally, using Wick's theorem, we have that

$$\langle 0\,|\,T\phi(x_1)\ldots\phi(x_2)\,|\,0\rangle = \frac{1}{i^m}\sum_{S_{2m}/\sim\text{"}pairing\text{"}}\Delta\left(x_{S(1)}-x_{S(2)}\right)\times\ldots\times\Delta\left(x_{S(2m-1)}-x_{S(2m)}\right). \tag{4.8.19}$$

Example. Consider

$$\langle 0\,|\,T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,|\,0\rangle = \frac{1}{i^2}\left[\Delta(x_1-x_2)\Delta(x_3-x_4)+\Delta(x_1-x_3)\Delta(x_2-x_4)+\Delta(x_1-x_4)\Delta(x_2-x_3)\right]. \tag{4.8.20}$$

4.9 Feynman Diagrams

4.9.1 Normalizations convention

We will be making use of the following normalizations:

1. We have that
$$\langle 0 \,|\, \phi(x) \,|\, 0 \rangle = 0, \tag{4.9.1}$$
(this effectively saying as $\phi(x)$ is a linear combination of creation and annihilation operators with no added constant - e.g. $\phi(x) = \sum a(k) e^{ikx} + h.c.$), where $|0\rangle$ denotes vacuum state.

2. Letting $|k\rangle$ being a one-particle state ($|k\rangle \sim a^{\dagger}(k)|0\rangle$), we have that
$$\langle k \,|\, \phi(x) \,|\, 0 \rangle = e^{-ikx}. \tag{4.9.2}$$
Moreover,
$$\left\langle k' \,\middle|\, k \right\rangle = (2\pi)^3 \, 2k^0 \delta^3 \left(\vec{k} - \vec{k}'\right). \tag{4.9.3}$$

4.9.2 "ϕ^3 theory"

Consider
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \frac{1}{3!} g \phi^3, \tag{4.9.4}$$
where g is called the coupling constant and the ϕ^3 term represents the interaction term in our exactly solvable model (i.e. the ϕ^2 free field model). The Hamiltonian density is given by
$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2} m^2 \phi^2 - \frac{1}{3!} g \phi^3. \tag{4.9.5}$$
Notice that there is an instability because $\mathcal{H} \to -\infty$ as $g\phi^3 \to \infty$. However, this instability will not be visible in the perturbation theory that we will be doing. We will assume that $g \ll 1$ in this perturbative approach.

Write the full path integral
$$\begin{aligned} Z(J) &= \langle 0 \,|\, 0 \rangle_J & (4.9.6)\\ &= \int \mathcal{D}\phi \exp\left[i \int d^4x \left(\mathcal{L}_0 + \mathcal{L}_{int} + J\phi\right)\right], & (4.9.7)\end{aligned}$$
where
$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2, & (4.9.8)\\ \mathcal{L}_{int} &= \frac{1}{3!} g \phi^3. & (4.9.9)\end{aligned}$$
Define W to be the logarithm of the partition function:
$$Z(J) = \exp\left[iW(J)\right]. \tag{4.9.10}$$
We can write
$$\begin{aligned} Z(J) &= \exp\left[i \int d^4x \mathcal{L}_{int}\left(\frac{1}{i}\frac{\partial}{\partial J(x)}\right)\right] \times \int \mathcal{D}\phi e^{i\int d^4x[\mathcal{L}_0 + J\phi]} & (4.9.11)\\ &\sim \exp\left[i \int d^4x \mathcal{L}_{int}\left(\frac{1}{i}\frac{\partial}{\partial J(x)}\right)\right] \times Z_0(J), & (4.9.12)\end{aligned}$$
where (from equation (4.8.11))
$$Z_0(J) = \exp\left[\frac{i}{2} \int d^4x d^4x' \, J(x) \Delta\left(x - x'\right) J\left(x'\right)\right]. \tag{4.9.13}$$
Expanding $\exp\left[i \int d^4x \mathcal{L}_{int}\left(\frac{1}{i}\frac{\partial}{\partial J(x)}\right)\right]$ and Z_0
$$\begin{aligned} Z(J) &= \sum_{v=0}^{\infty} \frac{1}{v!}\left[\frac{ig}{3!}\int d^4x \left(\frac{1}{i}\frac{\partial}{\partial J(x)}\right)^3\right]^v \times & (4.9.14)\\ &\sum_{p=0}^{\infty} \frac{1}{p!}\left[\frac{i}{2}\int d^n y d^n z J(y) \Delta(y - z) J(z)\right]^p. & (4.9.15)\end{aligned}$$

We have $3v$ functional derivatives acting on $2p$ sources. Organize these terms with the use of graphical tools: "Feynman diagrams": see 4.9.1.

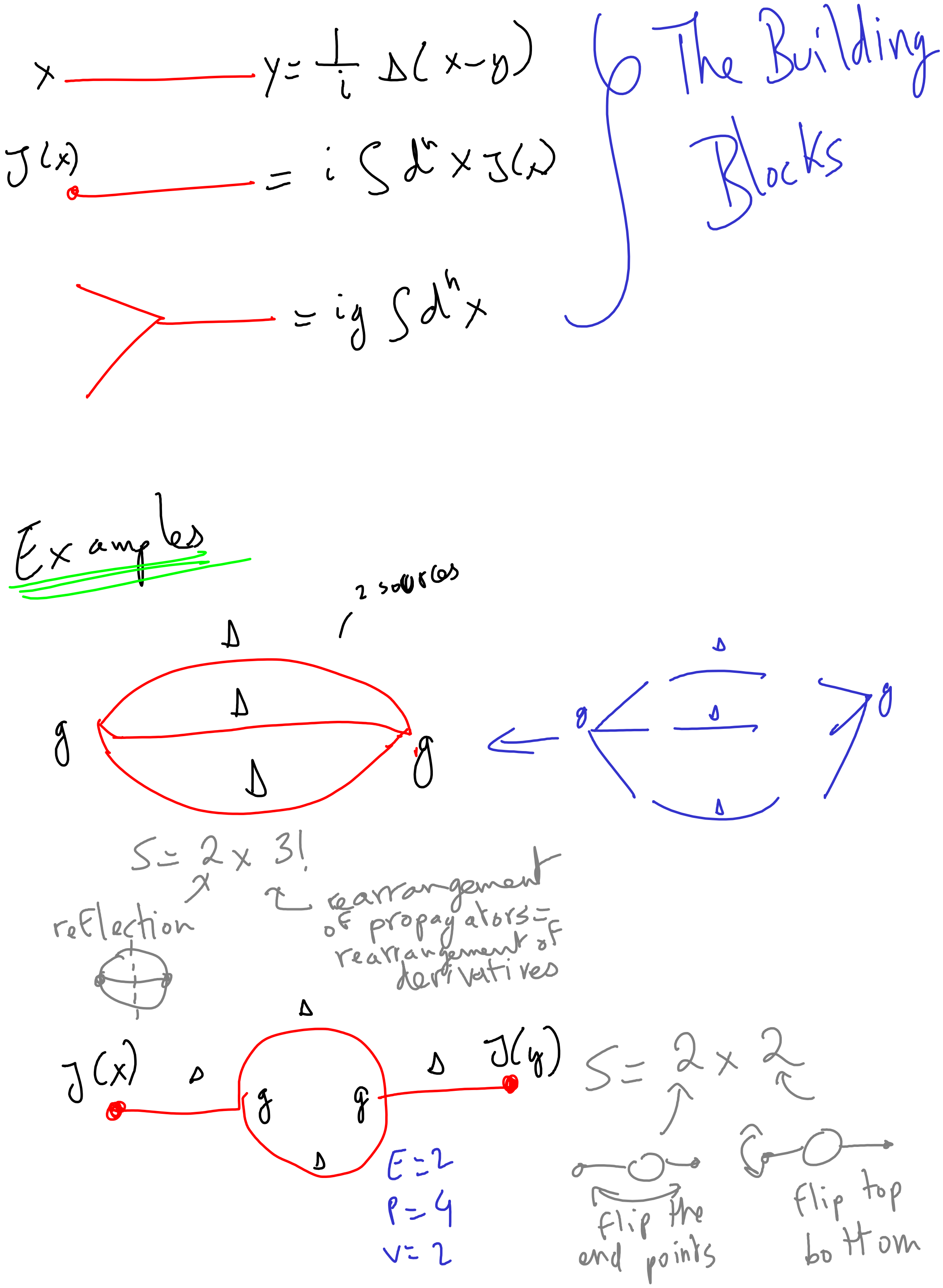


Figure 4.9.1: The building blocks for Feynman diagrams

p is the number of propagators and v is the number of vertices. Not all sources might be matched, let the number of external sources be denoted by

$$E = 2p - 3v \tag{4.9.16}$$

How do we know the numerical factors associated with each diagram? Consider a term with v vertices and p propagators. The overall phase factor is

$$i^v \left(\frac{1}{i}\right)^{3v} i^p = i^{v+E-p}. \tag{4.9.17}$$

We then deal with symmetries (there is more than one way to obtain a certain graph):

1. Each vertex has a factor of 3! (rearrangement of functional derivatives).
2. We have another factor of $v!$ (rearrangement of vertices).
3. To each propagator:
   a) there is a factor of 2! because we can switch the endpoints of a propagator.
   b) Finally, there is a factor of $p!$ for (rearrangement of propagators in the diagram).

All of these numerical factors almost cancel the factors in the expansion of the exponential. There is a slight over-counting (not all diagrams are indepedent) which results in a symmetry factor S (this is one of most confusing and annoying parts of the calculation - proceed carefully). This factor encodes the symmetries of the diagram.

Note that P, V, E do not uniquely determine a Feynman diagram.

Suppose we have a diagram D (which can be disconnected) with many vertices

$$D = \prod \frac{(C_I)^{n_I}}{n_I!}, \tag{4.9.18}$$

where C_I is connected of type I, the superscript n_I means replicating the C_I diagram n_I times. We can conclude that

$$\begin{aligned} Z(J) &\sim \sum_{\{n_I\}} D \sim \sum_{\{n_I\}} \prod_I \frac{(C_I)^{n_I}}{n_I!} && (4.9.19)\\ &= \prod_I \sum_{n_I=0}^{\infty} \frac{(C_I)^{n_I}}{n_I!} && (4.9.20)\\ &\sim \prod_I \exp(C_I) = \exp\left(\sum_I C_I\right) && (4.9.21) \end{aligned}$$

where D now also stands for the contribution of the diagram to the partition function and the $\{n_I\}$ refers to the different possible diagrams. In the last equality, $\sum_I$ means we are summing is over connected diagrams.

The normalization convention is $Z(0) = 1$ which can be reproduced by omitting "vacuum diagrams" (with no source terms). Moreover,

$$Z(J) = \exp(iW(J)). \tag{4.9.22}$$

Hence, we can say that

$$iW(J) = \sum_I C_I, \tag{4.9.23}$$

where the sum is over connected diagrams or we can sum over all connected diagrams which do not include the vacuum diagrams for the above normalization (i.e. $Z(0) = 1$).

The computation of the feynman diagrams will lead to a modification of the Lagrangian:

$$\mathcal{L}_{ct} = -\frac{1}{2}(Z_\phi - 1)(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}(Z_m - 1) m^2 \phi^2 - \frac{g}{3!}(Z_g - 1)\phi^3 + Y\phi, \tag{4.9.24}$$

where ct stands for counter terms.

4.9.3 Divergence Terminology

Example 6. Let us try to calculate the value of the 1-loop diagram ("Loop correction to propagator (Euclidean version)")- the end points are x and y- usually the propagator between x and y is a straight line. See the following figure:

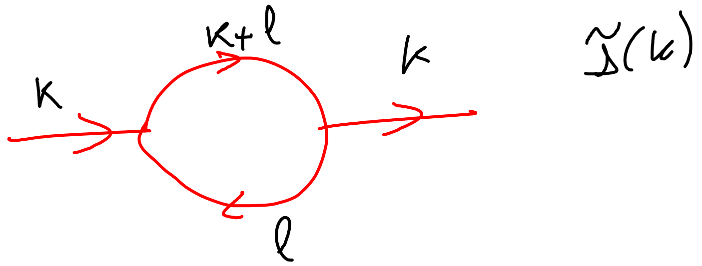


Figure 4.9.2: "Loop correction to propagator (Euclidean version)"

It can be shown that the result is given by

$$g^2 \frac{i}{2} \int \frac{d^n l}{(l^2 - m^2 + i\epsilon)\left((k+l)^2 - m^2 - i\epsilon\right)}, \tag{4.9.25}$$

as $1/\left(l^2-m^2+i\epsilon\right)$ is the form of the propagator ($a \xrightarrow{l} b$). This integral goes like

$$\sim\int\frac{d^nl}{l^4} \tag{4.9.26}$$

which is divergent for $n=4$. It is such divergences that will result in counter terms. How do we handle the divergences?

Definition 1. [*superficial degree of divergence*] The superficial degree of divergence -denoted by

$$div\left(D\right) \tag{4.9.27}$$

- is the difference of the degree of the numerator and the denominator in the integrand corresponding to D.

Example. $\int d^ny/y^4$ has superficial degree of divergence given by $n-4$. Another example: By definition, the degree of l or dl is 1.

Definition 2. A Feynman diagram is called superficially divergent if $div\left(D\right)\geq0$ and superficially convergent if $div\left(D\right)<0$.

Another definition:

Definition 3. A Feynman diagram D is called logarithmically, linearly, quadratically, ... divergent if

$$div\left(D\right)=0,1,2,... \tag{4.9.28}$$

Claim 1. We will later see that

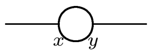
$$div\left(D\right)=nL-2I, \tag{4.9.29}$$

where

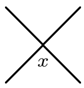
$$\begin{aligned} L &\equiv \text{number of loops}, && (4.9.30)\\ I &\equiv \text{Internal lines in a Feynman diagram}. && (4.9.31)\end{aligned}$$

(as a propagator like $a \xrightarrow{p} b$ will have a $1/p^2$ like dependence, hence the -2 contribution to $div\left(D\right)$)

Note that we will see that the number of external lines (E) help us characterize Feynman and the theory and plays an important role. For instance, for



(4.9.32)

, assume that the circle contains some very complicated internal dynamics, but since there are two external legs, we have $\mathcal{L}_{int}\sim\phi^{2=E}$ (???). Also for example,  would correspond to a ϕ^4 theory.

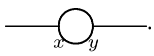
4.9.3.1 Example of ϕ^3 theory

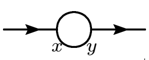
(ϕ^3 theory - see section 4.9.2) We have that

$$\begin{aligned} L &= \frac{V+2-R}{2}, && (4.9.33)\\ I &= \frac{3V-E}{2}. && (4.9.34)\end{aligned}$$

Hence,

$$div\left(D\right)=\left(n-6\right)L+6-2E. \tag{4.9.35}$$

- As $n>6$, the divergence worsens as the number of vertices grows (as then we are likely to have a greater number of loops). (so at very high dimensions, we cannot have interactions of type ϕ^3)
- For $n=6$, all Feynman graphs with $E=2,3$ are equally bad (as we have $div\left(D\right)=6-2E$), while for $E\geq4$, the graphs are superficially convergent. $n=6$ is called the critical dimension as below $n=6$, we would have finitely many superficially divergent graphs.
- For $n=5$, there are only finitely many superficially divergent graphs.
- For $n=4$, the only superficially divergent graph with $E\geq2$ is .
- For $n\leq3$, all graphs with $E\geq2$ are superficially convergent.

Remark 2. A superficially divergent diagram D is **not** necessarily divergent. We will see examples of this. For instance,  in $n=5$ is $\sim l$ (as d^nl/l^4). But if we have parity invariance, then we only expect parity invariant terms like p^2 so we expect that there is a term that cancels the divergent $\sim l$ term.

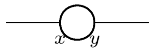
Another remark,

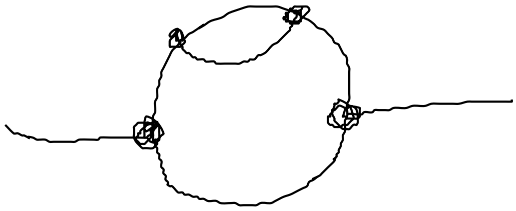
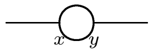
Remark 3. Superficial convergence is **not** sufficient for convergence (for example, we might have a complicated diagram, and we count the number of propagator, etc ... and we calculate $D<0$, but there might be subdiagrams that diverge)

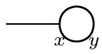
4.9.4 Weinberg Theorem

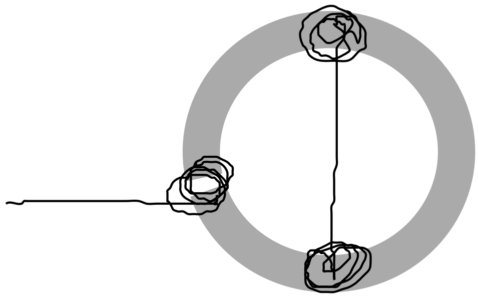
Theorem 1. *Let D be a diagram such that the integral of the corresponding function over any subset of the set of loops of D is superficially convergent. Then the integral corresponding to D is convergent.*

I.e. analyze subdiagrams of D and have them take the $div\,(D)$ test; if the subdiagrams pass the test then we have that D is superficially convergent.

Example 7. In the ϕ^3 theory (section 4.9.2), all integrals are convergent for $n \leq 3$. For $n = 4$, 

is superficially divergent. Another possible graph is  which contains 

so it could be divergent. The former 2 graphs were $E = 2$ examples. For $E = 1$, we can have something like 

which is quadratically divergent, we can also have something like,  which is logarithmically divergent.

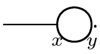
4.9.5 Counter terms

To resolve the divergence, add "counter terms": The coefficients of terms in the Lagrangian can be anything:

$$\mathcal{L}_{ct} = -\frac{1}{2}\left(Z_\phi - 1\right)\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}\left(Z_m - 1\right)m^2\phi^2 + Y\phi, \tag{4.9.36}$$

where the Zs (not to be confused with the partition function) and Ys are called the counter terms, and can be anything. HOWEVER, we are not allowed to add new terms in the Lagrangian like $Z_4\phi^4$.

The idea is to add $-\infty$ from the beginning to cancel the ∞ we get later on. This is renomarlization. We can let the Zs be infinite! This is a feature and not a bug of QFT! It is something deep and measurable. The changed terms have a physical meaning.

What we are doing is like adding Feynman diagrams to cancel diverging Feynman diagrams. For instance, the $Y\phi$ might cancel something like 

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4.10 Dimensional analysis

We should obtain the classical limit once we take $\hbar \to 0$. We set $c = 1$ and $\hbar = 1$. How do we retrieve them? There is a unique way of doing so in a unique way.

Also note that the path integral is dominated by the saddle point approximation (because the path integral is of the form $\int \mathcal{D}\phi \exp\left(iS\left[\phi\right]/\hbar\right)$ and $\hbar$ is small ??).

Setting $c = 1$ means that "time"="length".

Setting $\hbar = 1$ means that "length"=1/"energy".

$E = mc^2$ so when $c = 1$, energy has the units as mass. All of "this allows us to convert a time T to a length L via $T = c^{-1}L$, and a length L to an inverse mass M^{-1} via $L = \hbar c M^{-1}$ [3].Thus any quantity A can be thought of as having units of mass to some power (positive, negative, or zero) that we will call [A]." Hence, in units of mass to some power, we have that

$$[m] = 1 \tag{4.10.1}$$
$$[x^\mu] = -1 \tag{4.10.2}$$
$$[\partial^\mu] = 1 \tag{4.10.3}$$

[3]because $L = Tc$ and indeed T has dimensions of $\hbar M^{-1}/c^2$ and we set $c = 1$.

(as you think of momentum as $\hbar\partial^{\mu}$ and we set $\hbar = 1$). Also,

$$[d^n x] = -n. \tag{4.10.4}$$

The action is given by $S = \int d^n x \mathcal{L}$. Hence,

$$\begin{aligned} [S] &= 0 \quad (4.10.5)\\ [\mathcal{L}] &= 0 \quad (4.10.6) \end{aligned}$$

All of this is important because, given a problem, it will allow us to roughly obtain the answer in dimensions of mass or eV. Numerical factors do not usually accumulate (an example of where they do accumulate is the problem of muon decay).

Example 8. Consider (in n space-time dimensions)

$$\mathcal{L} = -\frac{1}{2}\left(\partial_{\mu}\phi\right)\left(\partial^{\mu}\phi\right) - \frac{1}{2}m^2\phi^2 - \frac{1}{k!}g\phi^k, \tag{4.10.7}$$

so

$$[\phi] = \frac{n-2}{2}. \tag{4.10.8}$$

This can be seen from $m^2\phi^2$ which should have dimension n. We know that m has dimension 2 so ϕ^2 has dimension $n-2$. Next, consider the dimension of g. We have that dimension of $g\phi^k$ is n and the dimension of ϕ^k is $k\left(n-2\right)/2$. Hence,

$$[g] = n - \frac{k\left(n-2\right)}{2}. \tag{4.10.9}$$

So for $k = 3$ (ϕ^3 theory), we have that $[g] = \left(6-n\right)/2$. Hence, $n = 6$ is the critical dimension of the ϕ^3 theory. We determined this critical dimension with the use of Feynman diagrams (see 4.9.3.1). However, our analysis here is much simpler.

In general, $gE^{-[g]}$ is dimensionless. When does $gE^{-[g]} \to 0$? This occurs at high energies (i.e. $E \to \infty$) if $[g] > 0$ and at low energies for $[g] < 0$. In these two regimes, we have non-renormalizability.

Definition 4. A theory is (non-)renormalizable if it needs (in)finitely many counter terms.

Remark 4. We saw that (4.9.29)

$$div\left(Diagram\right) = nL - 2I. \tag{4.10.10}$$

Here we are talking about something different. Imagine we have that $\mathcal{L}_{int} \sim g_E\phi^E$. Now count the units of each diagram, $[\text{diagram}] = [g_E]$, where E is the number of external sources -and- $[\text{diagram}] = nL - 2I + V\left(g\right)$. More generally,

$$[diagram] = nL - 2I + \sum_{k=3}^{\infty} V_k\left[g_k\right], \tag{4.10.11}$$

(each L is an integral over $\int d^n k$) where V_k is the number of $k-valent$ vertices (g_k refers to ϕ^k interaction). We have that

$$div\left(diagram\right) = [g_E] - \sum_{k=3}^{\infty} V_k\left[g_k\right]. \tag{4.10.12}$$

A theory with any $[g_k] < 0$ is already non-renormalizable.

We know that $[g_k] < 0$ if $k > 2n/\left(n-2\right)$. So this is the maximum k-term we can have in the Lagrangian before we run into problems.

Proposition 1. *A theory is non-renormalizable if any coefficient of any term in the Lagrangian has negative mass dimension. (there are some exceptions but they are exotic systems).*

4.11 The LS2 reduction formula

Now focus on Feynman diagrams that converge: Focus on correlation functions

$$\left\langle 0\left|T\phi\left(x_1\right)...\phi\left(x_n\right)\phi\left(x_{1'}\right)...\phi\left(x_{n'}\right)\right|0\right\rangle. \tag{4.11.1}$$

This measures correlations, but in actual experiments what is measured is scattering amplitudes: Consider a collision experiment with a bunch of particles:

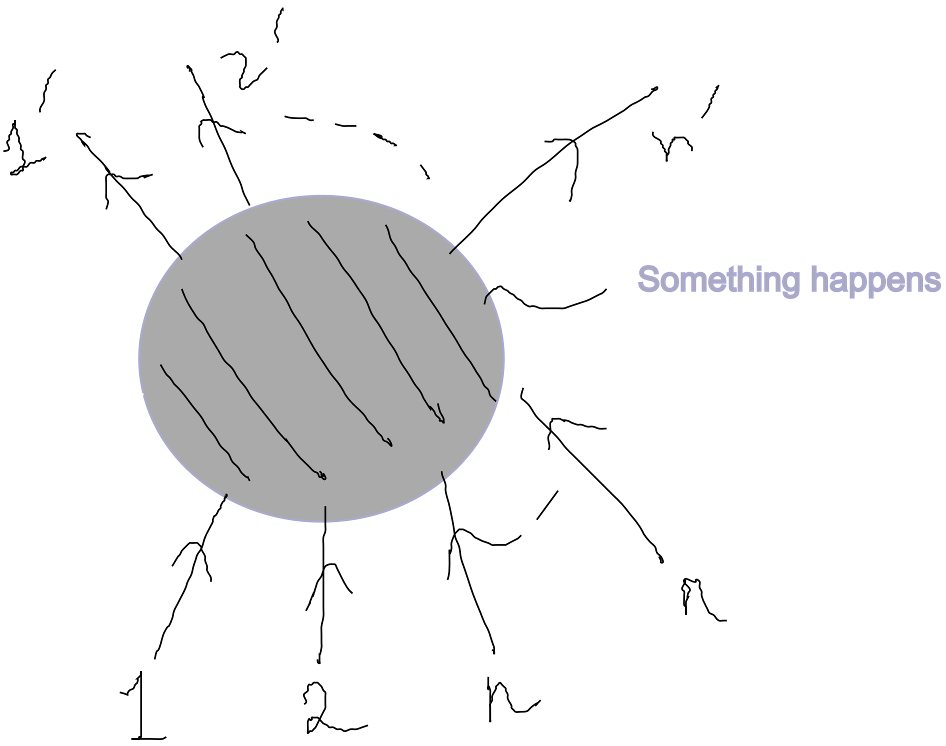
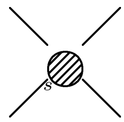


Figure 4.11.1: Model for scattering

. LS2 tells us that there is a link between this scattering model and $\langle 0 \,|\, T\phi(x_1)\ldots\phi(x_n)\,\phi(x_{1'})\ldots\phi(x_{n'})\,|\, 0\rangle$.

Example 9. We will show a connection between the following setup: 

and $\langle 0 \,|\, T\phi(x_1)\,\phi(x_2)\,\phi(x_{1'})\,\phi(x_{2'})\,|\, 0\rangle$

Consider:

$$a\left(\vec{k}\right)|0\rangle = 0, \tag{4.11.2}$$

where $|0\rangle$ is the *vacuum state*. We also assumed that $\langle 0 \,|\, 0\rangle = 1$.

Define the *one-particle state*:

$$|k\rangle = a^{\dagger}\left(\vec{k}\right)|0\rangle\,. \tag{4.11.3}$$

Our choice of normalization will be that

$$\left\langle k \,\middle|\, k^{'}\right\rangle = (2\pi)^3\, 2\omega\delta^{(3)}\left(\vec{k}-\vec{k}^{'}\right). \tag{4.11.4}$$

We will take the same relations (shown above) that we derived for the free field and promote to it to a definition in the case of interacting Hamiltonians! Recall that

$$a^{\dagger}\left(\vec{k}\right) = -i\int d^3x e^{ikx}\overleftrightarrow{\delta_0}\phi(x)\,, \tag{4.11.5}$$

where $\overleftrightarrow{\delta_0}$ is the 2-sided derivative and is defined by

$$f\overleftrightarrow{\delta_\mu}g \equiv f\left(\partial_\mu g\right) - \left(\partial_\mu f\right)g. \tag{4.11.6}$$

Behind the scenes, we have that

$$\text{fields} \quad \leftrightarrow \quad \text{particles} \tag{4.11.7}$$

$$\phi(x) \quad \leftrightarrow \quad \text{particle with } \vec{k} \text{ vector} \tag{4.11.8}$$

(for ϕ with a more complicated structure than a scalar, its interpretation will not be as straightforward).

Consider the following new operator, which is a modification of $a^{\dagger}\left(\vec{k}_1\right)$:

$$\boxed{a_1^{\dagger} \equiv \int d^3k f_1\left(\vec{k}\right)a^{\dagger}\left(\vec{k}\right)}, \tag{4.11.9}$$

where

$$f_1\left(\vec{k}\right) \propto \exp\left[-\frac{\left(\vec{k}-\vec{k}_1\right)^2}{4\sigma^2}\right]. \tag{4.11.10}$$

Choose the normalization such that when $\sigma \to 0$, we have that

$$f_1\left(\vec{k}\right) \to \delta^{(3)}\left(\vec{k}-\vec{k}_1\right). \tag{4.11.11}$$

(when this happens, we have that $a_1^\dagger = a^\dagger\left(\vec{k}_1\right)$). We have that $a_1^\dagger$ creates a particle localized in momentum space near $\vec{k}_1$. (And in position space near the origin).

We have that

- $a^\dagger\left(\vec{k}_1\right)$ and $a_1^\dagger$ are time-independent in free theory.
- $a^\dagger\left(\vec{k}_1\right)$ and $a_1^\dagger$ are time-dependent in interacting theory.

Consider the following *initial state* which consists of many incoming particles:

$$|i\rangle \equiv \lim_{t\to-\infty} a_1^\dagger(t)\, a_2^\dagger(t)\,|0\rangle \tag{4.11.12}$$

Note that we want initial states with definite momentum but we are considering a general state ($a_1^\dagger$ instead of $a^\dagger\left(\vec{k}_1\right)$ because it will make taking the limit $\sigma \to 0$ more easy and explicit). Likewise we can define the *final state*

$$|f\rangle \equiv \lim_{t\to\infty} a_{1'}^\dagger(t)\, a_{2'}^\dagger(t)\,|0\rangle \tag{4.11.13}$$

for $\vec{k}_1 \neq \vec{k}_2$ and $\vec{k}_1' \neq \vec{k}_2'$ that describe two localized widely separated particles in the past/far future.

Remark 5. The interaction happens to be tuned off at time $\pm\infty$ because of taking the limit (if the interaction is localized in one region or dies off with distance, then eventually the particles will no longer interact). We will later revisit this assumption.

Moreover, choose the following normalization

$$\boxed{\langle i\,|\,i\rangle = 1 = \langle f\,|\,f\rangle}. \tag{4.11.14}$$

Finally, scattering amplitudes are calculated via

$$\langle f\,|\,i\rangle = \left\langle 0\middle|\, a_{2'}(\infty)\, a_{1'}(\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)\,\middle|\, 0\right\rangle \tag{4.11.15}$$

(non-trivial formula because we have all of the $a^\dagger$ on one side and the as on the other side; LSZ will be about commuting the as and $a^\dagger$s to calculate the amplitude) We have that

$$\langle f\,|\,i\rangle = \left\langle 0\middle|\, T a_{2'}(\infty)\, a_{1'}(\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)\,\middle|\, 0\right\rangle \tag{4.11.16}$$

(T does not do much because the states were already time-ordered; but it will be needed later on). The key ingredient is the following: Using

$$\boxed{a_1^\dagger(-\infty) = a_1^\dagger(\infty) + i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{ikx}\left(-\partial^2+m^2\right)\phi(x)} \tag{4.11.17}$$

(trying to extract fourier coefficients; slightly more sophisticated form of a Fourier transform; not too surprising as ϕ depends linearly on a). We have that $\left(-\partial^2+m^2\right)\phi(x) = 0$ in free field and we get that a_1 is time-independent. Moreover, its Hermitian conjugate expresses analogous time-dependence:

$$a_1(-\infty) = a_1(\infty) - i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{-ikx}\left(-\partial^2+m^2\right)\phi(x) \tag{4.11.18}$$

and so

$$\boxed{a_1(+\infty) = a_1(-\infty) + i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{-ikx}\left(-\partial^2+m^2\right)\phi(x)} \tag{4.11.19}$$

Proof. (of equation (4.11.17)) We have that

$$\begin{aligned} a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt \partial_0 a_1^\dagger(t) && (4.11.20) \\ &= -i\int d^{3k} f_1\left(\vec{k}\right)\int d^4x \partial_0\left(e^{ikx}\overleftrightarrow{\partial_0}\phi(x)\right), && (4.11.21) \end{aligned}$$

where in the second line we used equation (4.11.5) and equation (4.11.9). Continuing we have that

$$\begin{aligned} a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= -i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{ikx}\left(\partial_0^2+\omega^2\right)\phi(x) && (4.11.22) \\ &= -i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{ikx}\left(\partial_0^2+\underbrace{\vec{k}^2+m^2}_{\omega^2}\right)\phi(x) && (4.11.23) \\ (k^2 \text{ acting on } e^{ikx} \text{ is like taking } \nabla^2) &= -i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{ikx}\left(\partial_0^2-\overleftarrow{\nabla}^2+m^2\right)\phi(x) && (4.11.24) \\ (\text{integration by parts}) &= -i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{ikx}\left(\partial_0^2-\vec{\nabla}^2+m^2\right)\phi(x) && (4.11.25) \\ &= -i\int d^3k f_1\left(\vec{k}\right)\int d^4x e^{ikx}\left(-\partial^2+m^2\right)\phi(x) && (4.11.26) \end{aligned}$$

(f_1 takes care of boundary conditions when we integrate by parts; the $\overleftarrow{\nabla}^2$ means the ∇^2 is acting on e^{ikx}) Now take f_1 to be a delta function. So

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = -i\int d^4x e^{ik_1x}\left(-\partial^2+m^2\right)\phi(x) \qquad (4.11.27)$$

□

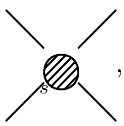
In general, a and $a^\dagger$ have a non-trivial dependence on time and it is very difficult to extract that time-dependence. We then get the LSZ reduction formula by plugging back into $\langle f\,|\,i\rangle$ (and letting the time ordering put $a^\dagger$ next to a): (for general n incoming particles and n' outgoing particles and taking $f \to \delta^3\left(\vec{k}-\vec{k}_1\right)$ - for now we do not want to deal with general wave packets - we are considering particles with definite momentum)

$$\langle f\,|\,i\rangle = i^{n+n'}\int d^n x_1 e^{ik_1x_1}\left(-\partial_1^2+m^2\right)\ldots d^4x_1' e^{-ik_1'x_1'}\left(-\partial_{1'}^2+m^2\right)\ldots\times\left\langle 0\left|T\phi(x_1)\ldots\phi\left(x_1'\right)\ldots\right|0\right\rangle \qquad (4.11.28)$$

(note that all of these arguments are valid provided that $a_1^\dagger(\pm\infty)$, as in free theory, create only single-particle states, i.e.

$$\begin{aligned} \langle 0\,|\,\phi(x)\,|\,0\rangle &= 0 && (4.11.29) \\ \langle k\,|\,\phi(x)\,|\,0\rangle &= e^{-ikx}. && (4.11.30) \end{aligned}$$

We will later come back to these conditions. So to calculate $\langle f\,|\,i\rangle$ from time-ordered correlation functions, one first applies klein-gordon equation on it "amputating the correlation function", then multiplies by complex exponential factor and finally apply the integrals

Example 10. For  , we have that

$$\begin{aligned} \langle f\,|\,i\rangle &= i^4\int d^4x_1 d^4x_2 d^4x_1' d^4x_2' \exp\left(i\left[k_1x_2+k_2x_2-k_1'x_1-k_2'x_2'\right]\right)\times && (4.11.31) \\ &\quad \left(-\partial_{1'}^2+m^2\right)\left(-\partial_{2'}^2+m^2\right)\left\langle 0\left|T\phi(x_1)\phi(x_2)\phi\left(x_1'\right)\phi\left(x_2'\right)\right|0\right\rangle && (4.11.32) \end{aligned}$$

LSZ does two things: it transforms correlation functions (which we calculate with Feynman diagrams) into scattering amplitudes and it does it in Fourier space! This will allow us to interpert Feynman diagrams in terms of scatterings.

5 Misc

5.1 Fudging in QFT

TODO: FILL

5.2 Notation

1. This course uses the Einstein summation convention.
2. ∂_μ is an abbreviation of ∂_{x^μ}.
3. Contractions:

$$A_{\mu\alpha_1\ldots\alpha_n}B^{\mu\beta_1\ldots\beta_n} = A_{\mu\alpha_1\ldots\alpha_n}g^{\mu\nu}B_{\nu\beta_1\ldots\beta_n}, \tag{5.2.1}$$

$$g^{\mu\nu} = \left(g^{-1}\right)_{\mu\nu} \tag{5.2.2}$$

   where $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$ and

4. We also have

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

5. See section §3.1 for more on notations in the context of special relativity.
6. Time ordering of operators: equation (4.7.23)
7. We have that

$$e^{ikx} = e^{i\vec{k}.\vec{x}-i\omega t} \tag{5.2.3}$$

8. $div\left(D\right)$: see 1
9. Divergence terminology: section 4.9.3
10. We have that

$$f\overleftrightarrow{\delta_\mu}g \equiv f\left(\partial_\mu g\right) - \left(\partial_\mu f\right)g. \tag{5.2.4}$$

5.3 Restoring physical constants

1. Restoring factors of $\hbar$: 5

5.4 Random math formulas

1. We have that

$$\int_0^T \cos\frac{\pi kt}{T}\cos\frac{\pi lt}{T}dt = \int_0^T \sin\frac{\pi kt}{T}\sin\frac{\pi lt}{T}dt = \frac{T}{2}\delta_{kl}. \tag{5.4.1}$$

2. We have that (http://dlmf.nist.gov/4.36.E1)

$$\sinh z = z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2\pi^2}\right) \tag{5.4.2}$$

5.5 Books

Bibliography

[1] Mark Srednicki. *Quantum Field Theory*. Cambridge University Press, 1 edition, 2 2007.