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For 22.211

Fundamental Definitions

The purpose of this paper is to define the many fundamental quantities associated with neutron transport, relate them, and comment on them. Most of this is in Duderstadt (p. 105-110), but with some additional comments here. Most importantly, units are given, and several expressions that Duderstadt skims over are covered in more detail.

Most of these definitions and expressions are fundamental to neutron transport, and one should probably memorize them and understand them. Read Duderstadt p. 105-110 and/or see the TAs if you have any questions.

Definitions of Independent Variables

The state of a given neutron, for the purposes of neutron transport analysis, is a function of 7 scalar variables. These variables are: time t at which the state is evaluated, position of the neutron (in geometric space) $\vec{\mathbf{r}}$, neutron's energy E, and neutron's direction of travel $\hat{\mathbf{\Omega}}$. These quantities, and their corresponding differentials, are described below.

Position

Position $\vec{\mathbf{r}}$ of a neutron is a 3-vector, normally written in terms of Cartesian coordinates as follows:

$$\vec{\mathbf{r}} = \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{Cartesian} = \underbrace{\begin{bmatrix} \rho \\ \varphi \\ z \end{bmatrix}}_{Cylindrical} = \underbrace{\begin{bmatrix} r \\ \varphi \\ \theta \end{bmatrix}}_{Spherical} \tag{1}$$

In some cases, such as spherical reactors, it's more convenient to use cylindrical or spherical coordinates to denote the position of a neutron. In this case, $\vec{\mathbf{r}}$ is still a 3-vector, but now it is written differently; see Eq. (1). Notice, that the symbols θ and φ are also used as components of the direction vector $\hat{\mathbf{\Omega}}$, but **those are different, unrelated variables**! Here, when $\vec{\mathbf{r}}$ and $\hat{\mathbf{\Omega}}$ are used in the same expression, and $\vec{\mathbf{r}}$ is in the spherical or cylindrical form, subscript p will be used to refer to position components, and subscript d will be used to refer to direction components. Most texts, including Duderstadt, do not use these subscripts, so one has to look at the context in which a variable is used to know what it refers to. The position of a neutron is in the geometric 3D space, which can also be thought of as "position space."

The conversions between the cylindrical and Cartesian coordinates are given in Eq. (2). Here the function $\operatorname{atan2}(y, x)$ is the 4-quandrant inverse tangent function. The corresponding MATLAB function is `atan2`.

$$\underbrace{\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \rho\cos(\varphi) \\ \rho\sin(\varphi) \\ z \end{bmatrix}}_{Cartesian \leftarrow Cylindrical} \quad \underbrace{\vec{\mathbf{r}} = \begin{bmatrix} \rho \\ \varphi \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \operatorname{atan2}(y, x) \\ z \end{bmatrix}}_{Cylindrical \leftarrow Cartesian} \tag{2}$$

The conversions between the spherical and Cartesian coordinates are given in Eq. (3).

$$\underbrace{\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin(\theta)\cos(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\theta) \end{bmatrix}}_{Cartesian \leftarrow Spherical} \quad \underbrace{\vec{\mathbf{r}} = \begin{bmatrix} r \\ \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \sqrt{x^2+y^2+z^2} \\ \mathrm{atan2}(y,x) \\ \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{bmatrix}}_{Spherical \leftarrow Cartesian} \tag{3}$$

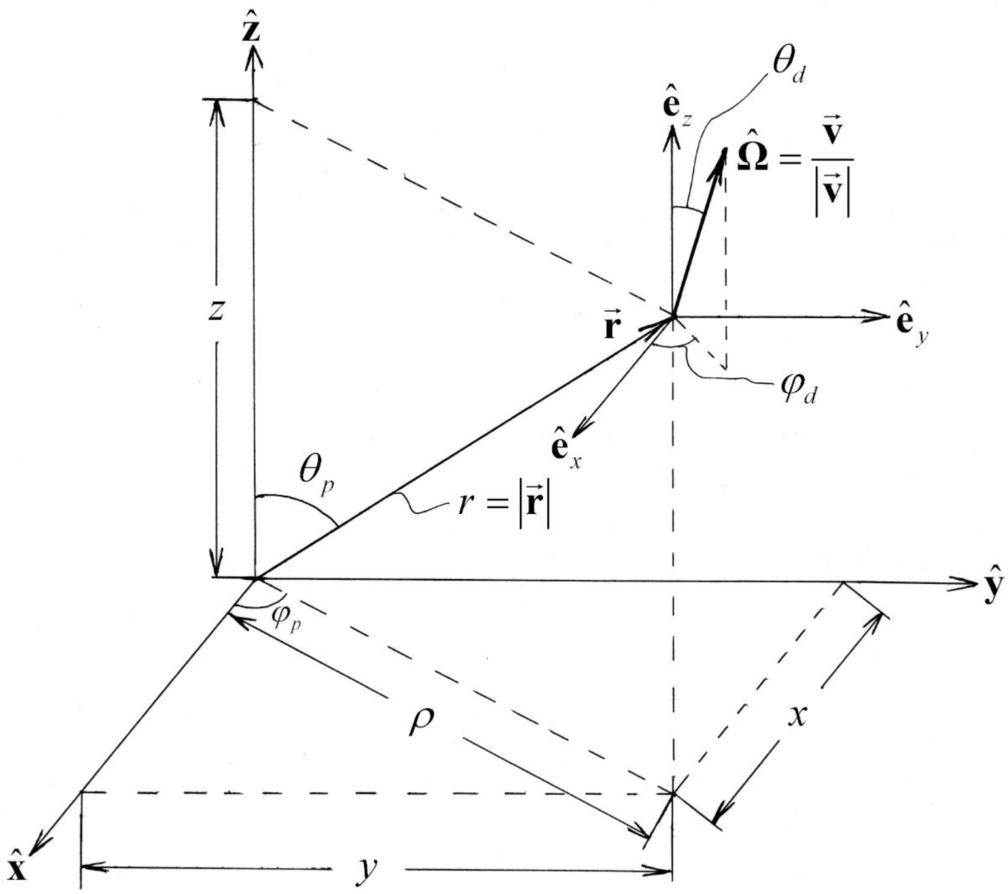

Figure 1. Notation Summary[1]

Figure 1 illustrates the position and direction vectors in a combined plot, and shows the symbols used in these expressions. The neutron velocity vector $\vec{\mathbf{v}}$ (discussed below) points along the direction vector $\hat{\mathbf{\Omega}}$, but unlike $\hat{\mathbf{\Omega}}$, $\vec{\mathbf{v}}$ is not a unit vector. Instead, the magnitude of $\vec{\mathbf{v}}$ is determined by the neutron energy E.

Notice, that with these definitions, the angle θ is the "inclination angle" (also known as the "zenith angle") between the position vector $\vec{\mathbf{r}}$ and the z-axis. Most texts on neutron transport use this convention for both the position and the direction vectors. For the direction vectors, the θ_d is the "inclination angle" between the direction vector $\hat{\mathbf{\Omega}}$ and the z-axis. There exists an alternative convention, in which the angle θ is the "elevation angle" between the position vector $\vec{\mathbf{r}}$ (or the direction vector $\hat{\mathbf{\Omega}}$, depending on which vector is treated), and the xy-plane. In this text, and in

[1] This took way longer to make than I had anticipated.

Duderstadt, only the inclination angle is used, but keep in mind that an alternative definition is possible (often used in antenna theory).

Regardless of the convention used, for the treatment of densities in geometric space (usually called "volume densities"), it is often necessary to define a differential element of the geometric space in which $\vec{\mathbf{r}}$ specifies a unique location. Such an element is usually denoted dV, $d\forall$ or d^3r, sometimes $d^3\vec{\mathbf{r}}$. Duderstadt uses the symbol d^3r. It should be noted that **the differential element of geometric space is not a vector**, despite the occasionally misleading notation. The expressions for the differential element of geometric space in the 3 coordinate systems can be obtained by differentiating Eqs. (2) and (3), and are given by:

$$d^3r = \underbrace{dxdydz}_{Cartesian} = \underbrace{\rho d\rho d\varphi dz}_{Cylindrical} = \underbrace{r^2 \sin(\theta) drd\theta d\varphi}_{Spherical} \tag{4}$$

An integral over geometric space is a triple integral, since it's a 3D space, but is usually written as $\int_V d^3r$. In theory, the geometric 3-space is unbounded, and so the limits for an integral over a section of the position space are set by the dimensions of the volume of interest. The size of the geometric space is measured in $\left[m^3\right]$ or $\left[cm^3\right]$.

Direction and Energy

Direction of travel $\hat{\mathbf{\Omega}}$ of a neutron is a unit vector, pointed in the direction that the neutron is flying in. It is almost always written in terms of spherical coordinates:

$$\hat{\mathbf{\Omega}} = \begin{bmatrix} \varphi \\ \theta \end{bmatrix} \tag{5}$$

As was stated before, the direction of travel of a neutron is a quantity **unrelated** to its position, despite using similar symbols. The position vector $\vec{\mathbf{r}}$ exists in the geometric ("position") 3-space. In the same way, the direction vector $\hat{\mathbf{\Omega}}$ exists in the "direction" 2-space. Together these two spaces form a 5-space.

Also notice that Duderstadt uses the symbol ϕ instead of φ to denote the azimuthal angle.

The magnitude of the velocity vector $\vec{\mathbf{v}}$ is the speed of the neutron $v(E)$, given by:

$$v(E) = \sqrt{\frac{2E}{m}} \tag{6}$$

Here m is the mass of the neutron. v is always positive, since it is a magnitude of a vector.

Typically, the velocity vector $\vec{\mathbf{v}}$ of a neutron is characterized in terms of the direction of travel $\hat{\mathbf{\Omega}}$ and its energy E (or, alternatively, its speed v). The velocity vector itself can be expressed (here, in spherical coordinates) in terms of the direction of travel and the energy E:

$$\vec{\mathbf{v}}\left(E,\hat{\mathbf{\Omega}}\right) = \vec{\mathbf{v}}\left(E,\varphi_d,\theta_d\right) = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \sqrt{\frac{2E}{m}} \begin{bmatrix} \sin(\theta_d)\cos(\varphi_d) \\ \sin(\theta_d)\sin(\varphi_d) \\ \cos(\theta_d) \end{bmatrix} \tag{7}$$

For the treatment of densities in the direction space (usually called "angular densities"), it is often necessary to define a differential element of the direction space in which $\hat{\mathbf{\Omega}}$ specifies a unique position. Such an element is usually denoted $d\hat{\mathbf{\Omega}}$, sometimes $d\Omega$ or $d\hat{\Omega}$. Duderstadt uses the symbol $d\hat{\mathbf{\Omega}}$. It should be noted that **the differential element of the direction space is not a vector**, despite the misleading notation. An integral over a section of the direction space can be thought of as the surface area of the unit sphere that this section spans.

The expression for the differential element $d\hat{\mathbf{\Omega}}$ is given by:

$$d\hat{\mathbf{\Omega}} = \sin(\theta) d\theta d\varphi \tag{8}$$

Unlike the unbounded position space, the direction space is bounded – the entire direction space is the set of all possible flight directions that a neutron can take. An integral over a section of the direction space, or over the entire direction space, is a double integral, since it's a 2-space. The integral over the entire direction space, or the integral "over all angles/directions" is usually written as $\int_{4\pi} d\hat{\mathbf{\Omega}}$, and is given by:

$$\int_{4\pi} d\hat{\mathbf{\Omega}} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin(\theta) = 4\pi \tag{9}$$

This makes sense, since the surface area of a unit sphere is 4π. The size of a section of the direction space is measured in $[sr]$, or "steradians," which can be thought of as being $\left[rad^2\right]$. The size of the entire direction space is $4\pi\, sr$. The density of a quantity in direction space should therefore have units $\left[\frac{n}{sr}\right]$ (in the case of neutrons), but the "per sr" is almost always omitted, since steradians are dimensionless.

The limits given in Eq. (9) span the entire surface of the unit sphere; sometimes, only a part of that surface is of interest – for example, when calculating partial currents. The integral over all "upward" (from $-z$ to $+z$) directions is given by:

$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \sin(\theta) = 2\pi \tag{10}$$

The energy of a neutron E is a measure of the neutron's speed, related by Eq. (6). For the treatment of densities in the energy space ("energy densities," sometimes simply referred to as "densities"), the differential element dE is used. The energy space is semi-bounded, since a neutron's energy is in principle infinite, but cannot be negative. So, the integral over all energies is written as: $\int_0^\infty dE$.

Notice, that a quantity can be position, angle (direction) or energy dependent, without being a density in one or more of those spaces. Together the position, direction and energy form a 6-space, which is usually called "phase space." The neutron densities (or fluxes) commonly analyzed are densities in this phase 6-space.

Definitions of Quantities

In this section all quantities that characterize the neutron population in a system are listed and described. These definitions can be related to each other, which is done in the next section.

All scalar quantities described in this section are densities, which makes them necessarily nonnegative.

Neutron density

Symbol: $N(t,\vec{\mathbf{r}})$, units: $\left[\frac{n}{cm^3}\right]$, usually written as $\left[\frac{1}{cm^3}\right]$.

Neutron density, sometimes called "scalar neutron density," is the density of neutrons in the position space at a given location $\vec{\mathbf{r}}$. It is only a function of position and time. The $[n]$ in the units stands for $[neutrons]$, which is a dimensionless unit, and are typically omitted. It is only a function of position and time.

Scalar neutron flux

Symbol: $\phi(t,\vec{\mathbf{r}})$, units: $\left[\frac{n}{cm^2 s}\right]$, usually written as $\left[\frac{1}{cm^2 s}\right]$.

Scalar neutron flux is the density of neutrons in the position space at a given location $\vec{\mathbf{r}}$, multiplied by their average speed $\bar{v}$. It is only a function of position and time.
Generally in physics, a "flux" is a vector quantity, characterizing the density of flow of some quantity through a given oriented surface at some location and orientation. **This is not the case for neutron flux – they only have the same units** (hence the name)! Neutron flux is a scalar quantity, and is better thought of a derived quantity, characterizing neutron speed and density in volume.

Energy-dependent neutron density

Symbol: $N(t,\vec{\mathbf{r}},E)$, units: $\left[\frac{n}{cm^3 eV}\right]$, usually written as $\left[\frac{1}{cm^3 eV}\right]$.

Energy-dependent neutron density, usually simply called "neutron density," is the density of neutrons in the combined position and energy 4-space, at a given location $\vec{\mathbf{r}}$ and energy E. It is a function of position, energy and time.
Neutron density can be derived from energy-dependent neutron density.

Energy-dependent scalar neutron flux density

Symbol: $\phi(t,\vec{\mathbf{r}},E)$, units: $\left[\frac{n}{cm^2 s\, eV}\right]$, usually written as $\left[\frac{1}{cm^2 s\, eV}\right]$.

Energy-dependent scalar neutron flux density, usually simply called "neutron flux density" or "flux," is the density of neutrons in the combined position and energy 4-space at a given location $\vec{\mathbf{r}}$ and energy E, multiplied by the neutron speed $v(E)$ from Eq. (6). It is a function of position, energy and time.
Scalar flux can be derived from the scalar flux density.

Angular neutron density

Symbol: $n(t,\vec{\mathbf{r}},\hat{\mathbf{\Omega}})$, units: $\left[\frac{n}{cm^3 sr}\right]$, usually written as $\left[\frac{1}{cm^3}\right]$.

Angular neutron density is the density of neutrons in the combined position and direction (angle) 5-space, at a given location $\vec{\mathbf{r}}$ and direction $\hat{\mathbf{\Omega}}$. It is a function of position, direction (angle) and time. This is a scalar quantity, despite being a density in direction.
Neutron density can be derived from the angular neutron density, which can be derived from the energy-dependent angular neutron density.

Angular neutron flux

Symbol: $\varphi(t,\vec{\mathbf{r}},\hat{\mathbf{\Omega}})$, units: $\left[\frac{n}{cm^2 s\, sr}\right]$, usually written as $\left[\frac{1}{cm^2 s}\right]$.

Angular neutron flux is the density of neutrons in the combined position and direction (angle) 5-space at a given location $\vec{\mathbf{r}}$ and direction $\hat{\mathbf{\Omega}}$, multiplied by the average neutron speed $\bar{v}$. It is a function of position, direction (angle) and time. This is a scalar quantity, despite being a density in direction.
Scalar flux can be derived from the angular neutron flux, which can be derived from the energy-dependent angular flux density.

Energy-dependent angular neutron density

Symbol: $n(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}})$, units: $\left[\frac{n}{cm^3\, eV\, sr}\right]$, usually written as $\left[\frac{1}{cm^3\, eV}\right]$.

Energy-dependent angular neutron density, usually simply called "angular neutron density," is the density of neutrons in the combined position, direction (angle) and energy 6-space, at a given

location $\vec{\mathbf{r}}$, direction $\hat{\mathbf{\Omega}}$ and energy E. It is a function of position, direction (angle), energy and time. This is a scalar quantity.
All other neutron densities can be derived from this fundamental neutron density.

Energy-dependent angular neutron flux density

Symbol: $\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$, units: $\left[\frac{n}{cm^2 s\, eV\, sr}\right]$, usually written as $\left[\frac{1}{cm^2 s\, eV}\right]$.

Energy-dependent angular neutron flux density, usually simply called "angular neutron flux density" or "angular flux," is the density of neutrons in the combined position, direction (angle) and energy 6-space, at a given location $\vec{\mathbf{r}}$, direction $\hat{\mathbf{\Omega}}$ and energy E, multiplied by the neutron speed $v\left(E\right)$ from Eq. (6). It is a function of position, energy, direction (angle) and time. This is a scalar quantity.
All other neutron densities can be derived from this fundamental neutron flux. Energy-dependent angular flux can also be thought of as the magnitude of the energy-dependent angular current density vector.

Energy-dependent angular current density

Symbol: $\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$, units: $\left[\frac{n}{cm^2 s\, eV\, sr}\right]$, usually written as $\left[\frac{1}{cm^2 s\, eV}\right]$.

Energy-dependent angular current density, usually simply called "angular current density," is the vector version of the angular flux. However, it is still a density in energy and direction spaces, but now it plays more of a role of a direction-dependent flow density of particles, closer to the traditional definition of flux vector in physics.
Other current densities can be derived from this fundamental current density, which can be derived from the energy dependent angular flux density.

Energy-dependent current density

Symbol: $\vec{\mathbf{J}}\left(t,\vec{\mathbf{r}},E\right)$, units: $\left[\frac{n}{cm^2 s\, eV}\right]$, usually written as $\left[\frac{1}{cm^2 s\, eV}\right]$.

Energy-dependent angular current density is a vector, however, it is not a vector version of the scalar flux. It is still a density in energy, but represents the net flow density of neutrons in the direction the vector points at, thus being identical to the traditional definition of flux vector. Notice, that while angular current density can be related to angular flux, energy-dependent current density cannot be directly related to the scalar flux density.

Current density

Symbol: $\vec{\mathbf{J}}\left(t,\vec{\mathbf{r}}\right)$, units: $\left[\frac{n}{cm^2 s}\right]$, usually written as $\left[\frac{1}{cm^2 s}\right]$.

Neutron current density is a vector, and is identical to the energy-dependent neutron current density, except being summed up over all possible energies. Notice, that while angular current density can be related to angular flux, current density cannot be directly related to the scalar flux.

Fundamental Relations

In this section the fundamental quantities are related to each other via several expressions.
The fundamental energy-dependent angular neutron density can be used to derive all other neutron densities:

$$n\left(t,\vec{\mathbf{r}},\hat{\mathbf{\Omega}}\right)=\int_0^\infty dEn\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$$
$$N\left(t,\vec{\mathbf{r}},E\right)=\int_{4\pi} d\hat{\mathbf{\Omega}}n\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right) \tag{11}$$
$$N\left(t,\vec{\mathbf{r}}\right)=\int_{4\pi} d\hat{\mathbf{\Omega}}\int_0^\infty dEn\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$$

Energy-dependent angular flux can be related to energy-dependent neutron density:

$$\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)=v\left(E\right)n\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right) \tag{12}$$

Fluxes can be related similarly to densities:

$$\varphi\left(t,\vec{\mathbf{r}},\hat{\mathbf{\Omega}}\right)=\int_0^\infty dE\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$$
$$\phi\left(t,\vec{\mathbf{r}},E\right)=\int_{4\pi} d\hat{\mathbf{\Omega}}\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right) \tag{13}$$
$$\phi\left(t,\vec{\mathbf{r}}\right)=\int_{4\pi} d\hat{\mathbf{\Omega}}\int_0^\infty dE\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$$

Notice, that from these expressions, the average neutron speed is really a weighted average, weighted by the density of neutrons of the respective speeds.
If a density or flux is equal in all directions ("isotropic"), it can be given by its scalar analog divided by 4π:

$$n\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)=\frac{1}{4\pi}N\left(t,\vec{\mathbf{r}},E\right)$$
$$\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)=\frac{1}{4\pi}\phi\left(t,\vec{\mathbf{r}},E\right) \tag{14}$$

The energy dependent angular current density can be related to energy-dependent angular flux density:

$$\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)=\hat{\mathbf{\Omega}}\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$$
$$\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)=\left|\vec{\mathbf{j}}\right| \tag{15}$$

Notice, that while $\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$ can be directly related to $\varphi\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$, $\vec{\mathbf{J}}\left(t,\vec{\mathbf{r}},E\right)$ cannot be directly related to $\phi\left(t,\vec{\mathbf{r}},E\right)$.
Other current densities can be expressed in terms of the energy dependent angular current density:

$$\vec{\mathbf{J}}\left(t,\vec{\mathbf{r}},E\right)=\int_{4\pi} d\hat{\mathbf{\Omega}}\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)$$
$$\vec{\mathbf{J}}\left(t,\vec{\mathbf{r}}\right)=\int_0^\infty dE\int_{4\pi} d\hat{\mathbf{\Omega}}\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right) \tag{16}$$

The current densities can be used to express several useful quantities. Consider an oriented surface $\overrightarrow{d\mathbf{A}}=dA\hat{\mathbf{e}}_s$, where $\hat{\mathbf{e}}_s$ is the surface orientation:

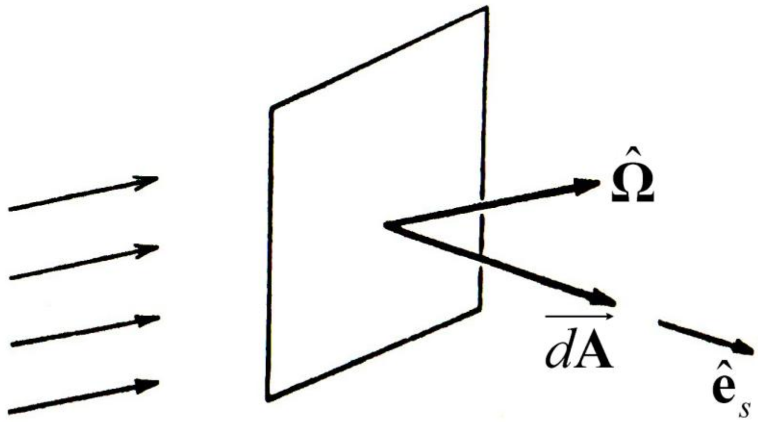

Figure 2. Oriented Surface Illustration[2]

We can now interpret the energy dependent angular current density using this expression:

$$\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)\cdot\overrightarrow{d\mathbf{A}} \tag{17}$$

This quantity is the density in energy and direction of neutrons passing through the oriented surface $\overrightarrow{d\mathbf{A}}$ (which is located at $\vec{\mathbf{r}}$). It is dependent on E and $\hat{\mathbf{\Omega}}$, so to get the net rate of flow through the oriented surface $\overrightarrow{d\mathbf{A}}$ we have to integrate over energy and angle:

$$\int_0^\infty dE\int_{4\pi} d\hat{\mathbf{\Omega}}\,\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)\cdot\overrightarrow{d\mathbf{A}}=\vec{\mathbf{J}}\left(t,\vec{\mathbf{r}}\right)\cdot\overrightarrow{d\mathbf{A}} \tag{18}$$

Partial currents through a given surface are obtained using the expressions (17) and (18). Generally, the sequence is the following:

1. To define the normal vector $\hat{\mathbf{e}}_s$ for the surface.
2. To define the differential element $\overrightarrow{d\mathbf{A}}$ for the surface.
3. To define the appropriate direction limits for the desired partial current.
4. To integrate expression (17) over these limits.

For example, if the normal vector $\hat{\mathbf{e}}_s$ is parallel to the $+z$ direction, then $\hat{\mathbf{e}}_s=\hat{\mathbf{z}}$ and the upward partial current density in energy through the differential surface element $\overrightarrow{d\mathbf{A}}$ will be given by:

$$J_+\left(t,\vec{\mathbf{r}},E\right)=\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta\sin\left(\theta\right)\vec{\mathbf{j}}\left(t,\vec{\mathbf{r}},E,\hat{\mathbf{\Omega}}\right)\cdot dA\hat{\mathbf{z}} \tag{19}$$

It is important to recognize that if, for example, at a given location 10 neutrons are moving in the $+z$ direction, and 10 neutrons are moving in the $-z$ direction, then the net current in either direction is zero. However, the total flux in this case is not zero – since the neutron density at this location is 20 neutrons in some volume. Therefore, while net current density is cancelled out by opposing partial currents, net flux only goes up with the addition of more particles.

The total (not net!) current density through a differential surface element $\overrightarrow{d\mathbf{A}}$ is found by adding up the two partial current densities through that differential element:

$$J_{total}=J_+ + J_- \tag{20}$$

Here J_+ is the partial current density in the direction parallel to $\overrightarrow{d\mathbf{A}}$, and J_- is in the antiparallel direction to $\overrightarrow{d\mathbf{A}}$.

[2] From Duderstadt, Figure 4-3.