Neutron Transport Equation Derivation

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1 Introduction

This paper presents three forms of the neutron transport equation (NTE), from the simplest to the most complicated one:

- NTE with a combined source, which includes neutrons born from fission.
- NTE with a prompt (fission) neutron source.
- NTE with both prompt and delayed neutron sources, and thermal feedback.

Only the transient version of the NTE is discussed in this text. The diffusion equation and the Legendre expansions of the streaming and scattering terms are not discussed in this paper.

The derivations presented here are not mathematically rigorous, and treat differentials as very small numbers/vectors, which is not strictly correct. Nevertheless, this treatment is insightful, and it provides the correct result in this case. Many texts on neutron transport can be consulted for more rigorous derivations.

Several assumptions are made in all derivations presented in this paper. Many of the derivations also have additional assumptions, discussed in their corresponding chapters, but the general ones are:

- Neutrons can be treated as point particles with no wave-like quantum mechanical effects, which is valid on the macroscopic scale considered.
- The neutron density is high enough for the deterministic approach to be valid. This assumption is always valid in nuclear reactors, both at power and at cold shutdown.
- There are no neutron-to-neutron interactions. This is valid because the neutron-to-neutron interaction potential, due to neutrons being neutral particles, is very weak, and atom density is orders of magnitude greater than the neutron density.
- Neutron to nucleus collisions are well-defined 2-body events which occur instantaneously. This is an experimentally validated fact.
- Between collisions, neutrons stream with constant velocity. Valid because neutrons are neutral elementary particles, which only undergo weak nuclear, strong nuclear and gravitational interactions, and therefore are only measurably slowed down through collisions with nuclei.
- The fissionable materials being modeled are treated as stationary; that is, precuror nuclei cannot move. This is generally true for all reactors with solid stationary fuel

(excluding meltdown), but not true for reactors with moving fuel, such as pebble bed and molten salt reactors.

- The time scales are sufficiently short to neglect changes in isotopic composition (other than the delayed neutron precursors) of the materials, and to assume that delayed neutron precursors only appear immediately as daughter nuclei from fissions, and not from decay of other isotopes. For relatively short transients, this is an excellent assumption.
- The (n,xn) (with x indicating 2 or more neutrons) reactions are neglected. In practice, they are not negligible, and are only neglected here for simplicity. They work very similarly to the prompt neutron source terms.

Finally, some significant simplifying assumptions, not made during the derivation in this paper, are presented in section 6.

2 Required Quantities

Many of the quantities used in this paper were first mentioned in the *Fundamental Definitions* document, distributed with this one. The independent variables t, x, y, z, θ_d, φ_d and E are the same in this text as in *Fundamental Definitions*; some of the other notation differs. Table 1 summarizes the notation used in this paper.

All other quantities are defined when first used.

3 NTE with a Combined Source Term

Consider a differential element of the phase 6-space:

$$d\forall dE d\Omega. \tag{1}$$

Let this element be located at location in geometric space $\vec{\mathbf{x}}$, location in direction space (i.e., "at direction") $\hat{\mathbf{\Omega}}$ and at location in energy space E.

The number of neutrons in this differential element is, by definition of $n\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$, given by:

$$n\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right) d\forall dE d\Omega. \tag{2}$$

The rate of change of this number can be expressed as:

$$\frac{\partial}{\partial t} n\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right) d\forall dE d\Omega = \frac{1}{V_n(E)} \varphi\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right) d\forall dE d\Omega, \tag{3}$$

because $V_n(E)$ is not time-dependent.

To construct the NTE, we find expressions for all of the loss and gain mechanisms for neutrons in the above differential element of the phase 6-space. We can then balance them, which yields the time-dependent NTE.

There are three gain mechanisms for neutrons in the differential element of the phase 6-space:

Table 1: Notation Summary

Quantity	**Notation**	**Expression**
Position vector	$\vec{\mathbf{x}}$	$[x, y, z]^{\mathrm{T}}$
Direction unit vector	$\hat{\mathbf{\Omega}}$	$\begin{bmatrix} \sin(\theta_d)\cos(\varphi_d) \\ \sin(\theta_d)\sin(\varphi_d) \\ \cos(\theta_d) \end{bmatrix}$
Volume differential	$d\forall$	$dx\,dy\,dz$
Direction differential	$d\Omega$	$\sin(\theta_d)d\theta_d d\varphi_d$
Surface of phase space differential element	∂S	
Outward normal-oriented differential surface element	$d\vec{\mathbf{S}}$	$\hat{\mathbf{e}}_s dS$
Integral over all directions	$\iint_{4\pi} d\Omega$	$\int_0^{2\pi} d\varphi_d \int_0^{\pi} d\theta_d \sin(\theta_d)$
Integral over surface of phase space differential element	$\oiint_{\partial S} d\vec{\mathbf{S}}$	
Integral over a volume	$\iiint_{\forall} d\forall$	
Neutron mass	m_n	$939.565\,378(21)\,\mathrm{MeV/c^2}$
Neutron speed	$V_n(E)$	$\sqrt{2E/m_n}$
Energy-dependent angular neutron density	$n\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$	
Energy-dependent angular neutron flux density	$\varphi\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$	$V_n(E)\, n\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$
Energy-dependent angular neutron current density	$\vec{\mathbf{j}}\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$	$\hat{\mathbf{\Omega}}\varphi\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$
Energy-dependent scalar neutron flux density	$\phi(t, \vec{\mathbf{x}}, E)$	$\iint_{4\pi} d\Omega\varphi\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$

- Neutrons born through the combined source, which includes prompt and delayed neutrons, as well as neutrons from an external source.
- Neutrons that "stream" (fly) into $d\forall$, the differential volume element, through its differential surface element ∂S.
- Neutrons that "scatter" (change direction and/or energy by hitting a nucleus) into the differential phase 6-space element.

There are two loss mechanisms for neutrons in the differential element of the phase 6-space:

- Neutrons stream (or "leak") out of $d\forall$ through its differential surface element ∂S.

- Neutrons that collide with a nucleus, and either scatter (change direction and/or energy) or get absorbed, therefore leaving the differential phase 6-space element.

Because streaming in and out are physically identical, they are treated together below, along with the rest of the mechanisms. An expression for it is formulated below, along with the rest of the gain and loss mechanisms. Lastly, note, that we are looking for rates of change of the number of neutrons in the differential element of the phase 6-space due to specific gain and loss mechanisms, therefore all their corresponding rates must have the same units as Eq. (3): neutrons per second.

3.1 Gain Through Combined Source

For the purposes of the combined source NTE, the source term is assumed known. We define the energy-dependent angular source density $s\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$, which, because it is a rate density in phase 6-space, can be multiplied by the differential phase 6-space element to yield the combined source term:

$$s\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right) d\forall dE d\Omega. \tag{4}$$

3.2 Gain Through Inscattering

The rate of scattering, like any other interaction of a neutron with a nucleus, is characterized in terms of energy-dependent angular neutron flux density and macroscopic cross-section. The rate of interactions of type R with isotope j in the differential element of phase 6-space is therefore given by:

$$\Sigma_R^j \varphi\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right) d\forall dE d\Omega, \tag{5}$$

in which Σ_R^j is the macroscopic cross-section of reaction type R of isotope j. For scattering events, the interaction type is denoted with a subscript s.

Any time a neutron at position $\vec{\mathbf{x}}$ with energy E' and direction $\hat{\mathbf{\Omega}}'$ undergoes a scattering collision, it changes energy and direction. Because this change is random — there is always a range of new energies and a range of new directions that a neutron can take after a scattering collision — for every initial energy E' and initial (or "source") direction $\hat{\mathbf{\Omega}}'$, there is a probability distribution in final (or "target") energies E_{final} and directions $\hat{\mathbf{\Omega}}_{final}$. Because we are only interested in collisions that end up in the differential phase 6-space element with energy E and direction $\hat{\mathbf{\Omega}}$, we can take:

$$E_{final} = E, \tag{6a}$$

$$\hat{\mathbf{\Omega}}_{final} = \hat{\mathbf{\Omega}}. \tag{6b}$$

Let this scattering probability distribution be denoted $P_s^j\left(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\right)$. In its discrete form it is frequently known as "scattering kernel." Generally, the reachable energies and directions depend on the isotope being collided with, denoted j here: a neutron can only

lose all of its kinetic energy when colliding with a Hydrogen-1 nucleus, otherwise energies below a certain point cannot be reached.

Because $P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big)$ is a probability distribution in E and $\hat{\mathbf{\Omega}}$, it must be normalized:

$$\int_0^\infty dE \iint_{4\pi} d\Omega P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big) = 1, \tag{7}$$

which physically means that any time a neutron scatters, it always has some resulting energy E and direction $\hat{\mathbf{\Omega}}$.

Let $dE'd\Omega'$ denote the differential element of the phase 3-space (energy and direction) from which the neutron with energy E' and direction $\hat{\mathbf{\Omega}}'$ scatters to energy E and direction $\hat{\mathbf{\Omega}}$. The differential element of the source phase 6-space (position, energy and direction) then becomes $d\forall dE'd\Omega'$, since a collision, including a scattering collision, does not change the neutron's position, so the volume differential element is the same as of the target differential element of phase 6-space. From Eq. (5), the rate of scattering in differential element of phase 6-space $d\forall dE'd\Omega'$ with isotope j at position $\vec{\mathbf{x}}$, with energy E' and direction $\hat{\mathbf{\Omega}}'$ then becomes:

$$\Sigma_s^j(\vec{\mathbf{x}}, E')\, \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big)\, d\forall dE'd\Omega'. \tag{8}$$

Multiplying this quantity by the scattering probability distribution yields the rate density of scattering off isotope j in target energies and directions (so density in phase 3-space):

$$\Sigma_s^j(\vec{\mathbf{x}}, E')\, P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big)\, \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big)\, d\forall dE'd\Omega'. \tag{9}$$

The product of $\Sigma_s^j(\vec{\mathbf{x}}, E')$ and $P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big)$ is referred to as double differential scattering cross-section for isotope j:

$$\Sigma_s^j\Big(\vec{\mathbf{x}}, E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big) = \Sigma_s^j(\vec{\mathbf{x}}, E')\, P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big). \tag{10}$$

To evaluate the gain through inscattering into the target differential element of phase 6-space, all possible isotopes j, source energies E' and source directions $\hat{\mathbf{\Omega}}'$ must be considered. To do so, we sum Eq. (9) over all isotopes j and integrate it over all possible source energies and directions:

$$\left\{ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j} \Sigma_s^j(\vec{\mathbf{x}}, E')\, P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big) \right] \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big) \right\} d\forall. \tag{11}$$

Equation (11) yields the inscattering rate density in target energies and directions. Because it is a density in phase 3-space, to get the overall rate of inscattering into the target differential phase 6-space element, we multiply it by $dEd\Omega$:

$$\left\{ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j} \Sigma_s^j(\vec{\mathbf{x}}, E')\, P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big) \right] \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big) \right\} d\forall dEd\Omega. \tag{12}$$

This yields the gain through inscattering term of the NTE. While Eq. (12) is strictly correct, in practice, a combined scattering kernel is usually used instead, which is a sum of isotope-specific scattering kernels:

$$\begin{aligned}\int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s\Big(\vec{\mathbf{x}}, E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big)\, \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big) = \\ = \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j} \Sigma_s^j(\vec{\mathbf{x}}, E')\, P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big)\right] \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big)\,.\end{aligned} \tag{13}$$

The scattering kernel is generally a fairly complicated function, because hundreds of isotopes may be present in the reactor after a period of operation, and each isotope has a different scattering probability function. For this reason, approximations are frequently made to $\Sigma_s\Big(\vec{\mathbf{x}}, E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big)$; the most frequent approximation is the neglect of upscattering; i.e., no neutron may scatter to $E > E'$. This is physically correct except for thermal neutrons, for which upscattering is non-negligible.

Lastly, in almost all scattering situations, except some highly structured materials, the dependence of the scattering probability on $\hat{\mathbf{\Omega}}'$ and $\hat{\mathbf{\Omega}}$ can be expressed as:

$$P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big) = P_s^j\Big(E' \to E, \cos(\theta_s)\Big) = P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \cdot \hat{\mathbf{\Omega}}\Big)\,, \tag{14}$$

in which θ_s is the angle between the initial and final directions (the "scattering angle") in lab coordinate system, equal to the dot product of the two directions. The scattering angle cosine is frequently denoted as μ_s. This representation frequently simplifies the scattering kernel, because it reduces the directional dependence to one independent variable (μ_s) instead of two $\Big(\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}\Big)$.

3.3 Net Loss Through Streaming

Consider a differential surface element dS, which is part of ∂S, the surface of the differential volume element $d\forall$, located at position $\vec{\mathbf{x}}$. Let $\hat{\mathbf{e}}_s$ be the outward-pointing normal unit vector field for all of ∂S. The oriented differential surface element $d\vec{\mathbf{S}}$ then becomes:

$$d\vec{\mathbf{S}} = \hat{\mathbf{e}}_s dS. \tag{15}$$

With these definitions, the net (that is, the difference between outstreaming and instreaming) rate of leakage of neutrons with energy E and direction $\hat{\mathbf{\Omega}}$ from $d\forall$ through the differential surface element dS is given by:

$$d\vec{\mathbf{S}} \cdot \vec{\mathbf{j}}\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big)\, dE d\Omega = d\vec{\mathbf{S}} \cdot \hat{\mathbf{\Omega}} \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big)\, dE d\Omega. \tag{16}$$

To evaluate the overall net rate of leakage of neutrons with energy E and direction $\hat{\mathbf{\Omega}}$ from $d\forall$, we integrate over the surface of $d\forall$, that is, over ∂S:

$$\left\{ \oiint_{\partial S} d\vec{\mathbf{S}} \cdot \hat{\mathbf{\Omega}} \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \right\} dE d\Omega. \tag{17}$$

Next, using the Gauss-Ostrogradsky theorem (often called the divergence theorem), the surface integral in Eq. (17) is transformed into a volume integral:

$$\left\{ \oiint_{\partial S} d\vec{\mathbf{S}} \cdot \hat{\mathbf{\Omega}} \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \right\} dE d\Omega = \left\{ \iiint_{d\forall} d\forall \, \nabla \cdot \hat{\mathbf{\Omega}} \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \right\} dE d\Omega, \tag{18}$$

which, because a volume integral of a quantity over a differential element $d\forall$ is simply the product of that quantity and $d\forall$, simplifies into:

$$\nabla \cdot \hat{\mathbf{\Omega}} \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \, d\forall dE d\Omega. \tag{19}$$

Equation (19) models the net rate of neutron leakage from the differential volume element of interest.

3.4 Loss Through Collisions

Any interaction with a nucleus will move a neutron out of the differential phase 6-space element. As shown in Eq. (5), the total rate of collisions in the differential phase 6-space element is:

$$\Sigma_t(\vec{\mathbf{x}}, E) \, \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \, d\forall dE d\Omega, \tag{20}$$

in which $\Sigma_t(\vec{\mathbf{x}}, E)$ is the total macroscopic cross-section.

Equation (20) models the loss rate through collisions from the differential phase 6-space element. This concludes the modeling of individual gain and loss mechanisms, and the NTE can now be constructed.

3.5 Rate Balance Expression

Balancing the neutron gain and loss rates through Eqs. (3), (4), (12), (19) and (20) yields the NTE with a combined source term:

$$\begin{aligned}
&\frac{\partial}{\partial t} n\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \, d\forall dE d\Omega = s\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \, d\forall dE d\Omega + \\
&+ \left\{ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j} \Sigma_s^j(\vec{\mathbf{x}}, E') \, P_s^j\Big(E' \to E, \hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}\Big) \right] \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big) \right\} d\forall dE d\Omega - \\
&- \nabla \cdot \hat{\mathbf{\Omega}} \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \, d\forall dE d\Omega - \Sigma_t(\vec{\mathbf{x}}, E) \, \varphi\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big) \, d\forall dE d\Omega.
\end{aligned} \tag{21}$$

Lastly, we recognize that the phase 6-space differential multiplies all terms of the NTE,

and so can be dropped. Doing so, and rearranging the equation yields:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}n\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) &= -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) - \Sigma_t(\vec{\mathbf{x}},E)\,\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) + \\ &+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j}\Sigma_s^j(\vec{\mathbf{x}},E')\,P_s^j\Big(E'\to E,\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}}\Big)\right]\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ s\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big).\end{aligned}} \tag{22}$$

This concludes the derivation of the NTE with a combined source term. It is more commonly written with the combined scattering kernel, as follows:

$$\begin{aligned}\frac{\partial}{\partial t}n\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) &= -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) - \Sigma_t(\vec{\mathbf{x}},E)\,\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) + \\ &+ \int_0^\infty dE' \iint_{4\pi} d\Omega'\Sigma_s\Big(\vec{\mathbf{x}},E'\to E,\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}}\Big)\,\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + s\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big)\end{aligned} \tag{23}$$

4 NTE with a Prompt Source Term

When a nucleus fissions, three things happens. First, a significant amount of energy is released, primarily in the form of the kinetic energy of the daughter nuclei. Second, several neutrons are immediately produced, called "prompt neutrons." Third, two or more daughter nuclei are produced. These nuclei are usually unstable, and will tend to decay further, sometimes releasing neutrons in the process. These neutrons, which are not released until the daughter nuclei (called the "delayed neutron precursors") decay, are known as "delayed neutrons."

In this section, we split the combined source term $s\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big)$ into two components: the prompt fission neutrons and the remaining source terms. Consider a fissionable isotope j_f, and a differential phase 6-space element $d\forall dE'd\Omega'$, with position $\vec{\mathbf{x}}$, energy E' and direction $\hat{\mathbf{\Omega}}'$. Again following Eq. (5), the rate of fission reactions occuring in this differential phase 6-space element is given by:

$$\Sigma_f^{j_f}(\vec{\mathbf{x}},E')\,\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big)\,d\forall dE'd\Omega', \tag{24}$$

in which $\Sigma_f^{j_f}(\vec{\mathbf{x}},E)$ is the macroscopic fission cross-section of isotope j_f.

Let $\nu_p^{j_f}(E')$ be the average number of prompt neutrons born when a nucleus of isotope j_f is fissioned by a neutron with energy E'. For most fissile isotopes, this quantity is usually about 2.5 for most energies E'. This quantity is tabulated, like microscopic cross-sections (MT = 456 in ENDF-6 format). The rate of prompt neutron generation in the differential phase 6-space element above is then given by:

$$\nu_p^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E')\,\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big)\,d\forall dE'd\Omega'. \tag{25}$$

Similarly to when a neutron is scattered, when a prompt neutron is generated, it has an associated probability distribution function in energy, called "prompt fission yield spectrum." Like all probability distribution functions, the prompt fission yield spectrum must be normalized to 1. Prompt fission yield spectrum for isotope j_f is denoted $\chi_p^{j_f}(E', E)$, where E' is the energy of the fissioning neutron, and E is the energy of a generated prompt neutron. Note, that the dependence of $\chi_p^{j_f}(E', E)$ on E' is generally very weak, and it is frequently ignored, therefore resulting in $\chi_p^{j_f}(E)$.

Unlike scattered neutrons, whose scattering kernel is highly forward-biased (a neutron is much more likely to change trajectory but keep going forward, than it is to be reflected backward), both prompt and delayed neutrons are born isotropically in directions. This can be viewed as a flat, energy-independent probability distribution function in target direction that normalizes to 1. The total direction space has the size of 4π sr, therefore, the overall probability distribution function in target energies and directions is $\chi_p^{j_f}(E', E)/4\pi$.

The rate density in target energies and directions of prompt neutrons generated from fissions of isotope j_f in the differential phase 6-space element above is now given by:

$$\frac{\chi_p^{j_f}(E', E)}{4\pi} \nu_p^{j_f}(E')\, \Sigma_f^{j_f}(\vec{\mathbf{x}}, E')\, \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big)\, d\forall dE' d\Omega'. \tag{26}$$

Similarly to scattering, to evaluate the rate of prompt neutron generation in the target differential phase 6-space element, all fissionable isotopes j_f, fissioning neutron energies E' and directions $\hat{\mathbf{\Omega}}'$ must be accounted for. To do so, we sum Eq. (26) over all fissionable isotopes and integrate it over all possible source neutron energies and directions:

$$\frac{1}{4\pi}\left\{ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j_f} \chi_p^{j_f}(E', E)\, \nu_p^{j_f}(E')\, \Sigma_f^{j_f}(\vec{\mathbf{x}}, E') \right] \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big) \right\} d\forall. \tag{27}$$

Equation (27) yields the prompt neutron generation rate density in energies and directions. Only the energy-dependent angular flux density is direction-dependent, so we can factor the rest out of the double integral. Because this is a density in phase 3-space, to get the overall rate of prompt neutron generation in the differential phase 6-space element, we multiply it by $dEd\Omega$:

$$\frac{1}{4\pi}\left\{ \int_0^\infty dE' \left[\sum_{\text{all } j_f} \chi_p^{j_f}(E', E)\, \nu_p^{j_f}(E')\, \Sigma_f^{j_f}(\vec{\mathbf{x}}, E') \right] \iint_{4\pi} d\Omega' \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big) \right\} d\forall dE d\Omega. \tag{28}$$

Equation (28) is the strictly correct version of the prompt fission source term. Let the remaining (delayed and external) source term be denoted $s_{d,ex}\Big(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\Big)$. Substituting

these two terms into Eq. (22) yields the NTE with a prompt source term:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}n\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) &= -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) - \Sigma_t(\vec{\mathbf{x}},E)\,\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) + \\ &+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j}\Sigma_s^j(\vec{\mathbf{x}},E')\,P_s^j\Big(E'\to E,\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}}\Big)\right]\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ \frac{1}{4\pi}\int_0^\infty dE'\left[\sum_{\text{all } j_f}\chi_p^{j_f}(E',E)\,\nu_p^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E')\right]\iint_{4\pi} d\Omega'\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ s\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big)\,.\end{aligned}} \tag{29}$$

If we ignore the dependence of $\chi_p^{j_f}(E',E)$ on E', and make an assumption about the shape of $\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big)$, an effective (prompt neutron production rate-weighted) prompt fission yield spectrum $\chi_p(\vec{\mathbf{x}},E)$, along with an effective production cross-section can be computed. The position dependence here is due to the position dependence of $\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big)$. The prompt fission yield spectrum is often further simplified to $\chi_p(E)$. Lastly, the isotope-specific fission production cross-sections (note: $\nu\Sigma_f$ is frequently referred to as a "production cross-section") can also be simplified, by summing them over the fissionable isotopes to get total production cross-sections. With these simplifications, and the combined scattering kernel from Eq. (23), Eq. (29) becomes:

$$\begin{aligned}\frac{\partial}{\partial t}n\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) &= -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) - \Sigma_t(\vec{\mathbf{x}},E)\,\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) + \\ &+ \int_0^\infty dE' \iint_{4\pi} d\Omega'\Sigma_s\Big(\vec{\mathbf{x}},E'\to E,\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}}\Big)\,\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ \frac{\chi_p(E)}{4\pi}\int_0^\infty dE'\nu_p(E')\,\Sigma_f(\vec{\mathbf{x}},E')\iint_{4\pi} d\Omega'\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + s_{d,ex}\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big)\,.\end{aligned} \tag{30}$$

Lastly, it is worth noting that frequently, instead of working with $\nu_p^{j_f}(E')$, we are instead provided with two other quantities: *a)* $\nu^{j_f}(E')$, the total average number of neutrons per fission, including both prompt and delayed, and *b)* $\beta^{j_f}(E')$, the delayed neutron fraction for isotope j_f.

They are related as follows:

$$\nu_p^{j_f}(E') = \big(1-\beta^{j_f}(E')\big)\,\nu^{j_f}(E')\,. \tag{31}$$

$\beta(E')$, the total delayed neutron fraction, is obtained similarly to $\nu_p(E')$ — as a production rate-weighted average of the sum over all isotopes j_f.

This concludes the derivation of the NTE with a combined source term.

5 NTE with Prompt and Delayed Source Terms and Thermal Feedback

In this section, $s_{d,ex}\left(t, \vec{\mathbf{x}}, E, \hat{\mathbf{\Omega}}\right)$ will first be split into delayed and external source terms. Next, thermal dependence of various neutronic parameters will be discussed.

5.1 Delayed Source Term

As stated in section 4, delayed neutrons come from decay of certain daughter nuclei, which appear as products of fission reactions. There are many such nuclei, they have different degrees of stability, and therefore all have different, by many orders of magnitude, decay constants.

It would be almost impossible to treat all such nuclei for an average transient problem, so they are usually grouped into M^{j_f} "delayed neutron precursor families." All precursor nuclei in a family m born from fissions of isotope j_f are assumed to have the same decay constant $\lambda_m^{j_f}$, and the same average number of precursors in that family born per fission at energy E', $\nu_{d,m}^{j_f}(E')$. As with prompt neutrons, these quantities can alternatively be characterized through the family-specific delayed neutron fractions:

$$\nu_{d,m}^{j_f}(E') = \beta_m^{j_f}(E')\,\nu^{j_f}(E')\,, \tag{32}$$

and the total delayed neutron fraction of a specific isotope is composed of the family-specific ones:

$$\beta^{j_f}(E') = \sum_{m=1}^{M^{j_f}} \beta_m^{j_f}(E')\,. \tag{33}$$

Note, that the precursor families (often called "delayed groups") are weakly isotope-specific — $M^{j_f} = 6$ families are almost always used regardless of the isotope, but the decay constants do vary slightly depending on the fissionable isotope j_f. The delayed neutron fractions $\beta^{j_f}(E')$ are still kept as isotope specific though, because they vary strongly between important fissionable isotopes.

Below, both the strict fissionable isotope-dependent and the more common combined treatment are considered.

Let $c_m^{j_f}(t, \vec{\mathbf{x}})$ denote the density in space of delayed neutron precursors of family m born from fissions of isotope j_f. Let $\lambda_m^{j_f}$ be the decay constant for such precursors. The rate density of decay of these precursor nuclei is then given by:

$$\lambda_m^{j_f} c_m^{j_f}(t, \vec{\mathbf{x}})\,, \tag{34}$$

by law of exponential decay.

The delayed neutrons come from the decay of the precursor nuclei. Just like prompt neutrons, they are born isotropically in all directions, and have an associated delayed yield spectrum $\chi_{d,m}^{j_f}(E)$, which is a probability distribution function in energies at which the

delayed neutrons are born from precursor family m of isotope j_f. To evaluate the rate density in target energies and directions of delayed neutrons generated from decay of precursors of family m of isotope j_f in differential volume element $d\forall$, we multiply Eq. (34) by the probability distribution function in energy and direction:

$$\frac{\chi_{d,m}^{j_f}(E)}{4\pi}\lambda_m^{j_f}c_m^{j_f}(t,\vec{\mathbf{x}})\,d\forall. \tag{35}$$

Next, to evaluate the overall rate of delayed neutron generation in the target differential phase 6-space element, all fissionable isotopes j_f and isotope-specific delayed neutron precursor families m must be accounted for. Additionally, we multiply the resulting term by $dEd\Omega$, to get the rate in differential phase 6-space element:

$$\frac{1}{4\pi}\left\{\sum_{\text{all } j_f}\left[\sum_{m=1}^{M^{j_f}}\chi_{d,m}^{j_f}(E)\,\lambda_m^{j_f}c_m^{j_f}(t,\vec{\mathbf{x}})\right]\right\}d\forall\,dE\,d\Omega. \tag{36}$$

Dropping the 6-phase differential yields the total delayed neutron source:

$$\frac{1}{4\pi}\sum_{\text{all } j_f}\left[\sum_{m=1}^{M^{j_f}}\chi_{d,m}^{j_f}(E)\,\lambda_m^{j_f}c_m^{j_f}(t,\vec{\mathbf{x}})\right]. \tag{37}$$

The remaining component of $s_{d,ex}\left(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\right)$, the external source term, will be denoted $s_{ex}\left(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\right)$. Unlike the prompt and delayed source terms, the external source term is independent of the solution, and therefore must be known to solve the NTE. It is frequently negligible, and assumed zero.

Replacing the combined delayed and external source term in Eq. (22) with Eq. (37) and $s_{ex}\left(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\right)$ yields the full NTE with prompt, delayed and external source terms:

$$\begin{aligned}
&\frac{\partial}{\partial t}n\left(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\right) = -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\left(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\right) - \Sigma_t(\vec{\mathbf{x}},E)\,\varphi\left(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\right) + \\
&\quad + \int_0^\infty dE'\iint_{4\pi}d\Omega'\left[\sum_{\text{all } j}\Sigma_s^j(\vec{\mathbf{x}},E')\,P_s^j\left(E'\to E,\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}}\right)\right]\varphi\left(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\right) + \\
&\quad + \frac{1}{4\pi}\int_0^\infty dE'\left[\sum_{\text{all } j_f}\chi_p^{j_f}(E',E)\,\nu_p^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E')\right]\iint_{4\pi}d\Omega'\varphi\left(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\right) + \\
&\quad + \frac{1}{4\pi}\sum_{\text{all } j_f}\left[\sum_{m=1}^{M^{j_f}}\chi_{d,m}^{j_f}(E)\,\lambda_m^{j_f}c_m^{j_f}(t,\vec{\mathbf{x}})\right] + s_{ex}\left(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\right).
\end{aligned} \tag{38}$$

The only remaining complication is the calculation of the delayed neutron precursor densities, which are not explicitly known. The only gain mechanism for a delayed neutron

precursor family m from fissionable isotope j_f is the yield from fissions of this isotope, which can be adapted from Eq. (25). Substituting $\nu_p^{j_f}(E')$ with $\nu_{d,m}^{j_f}(E')$ and integrating over all directions and energies at which fission occurs yields the rate density of delayed neutron precursor family m generation from fissionable isotope j_f:

$$\int_0^\infty dE' \nu_{d,m}^{j_f}(E')\, \Sigma_f^{j_f}(\vec{\mathbf{x}}, E') \iint_{4\pi} d\Omega' \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big)\,. \tag{39}$$

Two potential loss mechanisms exist for the precursor nuclei: absorption of neutrons and decay. Because the precursor families are, by themselves, an approximation, comprised of many isotopes, and they are generally not heavy nuclei, most precursor families have very low absoprtion cross-sections, and therefore the loss through absorption can be safely neglected. The rate density of decay of precursor family m from isotope j_f is given by Eq. (34). Let J_f be the number of fissionable isotopes in the reactor. Balancing the gain and loss mechanisms for all precursor families m of all isotopes j_f yields the delayed neutron precursor equation system:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t} c_m^{j_f}(t, \vec{\mathbf{x}}) = &\int_0^\infty dE' \nu_{d,m}^{j_f}(E')\, \Sigma_f^{j_f}(\vec{\mathbf{x}}, E') \iint_{4\pi} d\Omega' \varphi\Big(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\Big) - \\ &- \lambda_m^{j_f} c_m^{j_f}(t, \vec{\mathbf{x}}) \quad \forall m \in \left[1, \ldots, M^{j_f}\right], j_f \in [1, \ldots, J_f]\,.\end{aligned}} \tag{40}$$

Equation (38) is the most general form of the temperature-independent NTE, with minimum assumptions. It is a linear (but not constant-coefficient), 7-dimensional (phase 6-space and time), partial integro-differential equation. Equations (40) are, on the other hand, an infinite, unless space is discretized, system of ODEs, because no spatial derivative terms are present in it, and the source terms are not dependent on the precursor equations' unknowns. Their system is closed, and, under the assumptions in section 1 and in this subsection, models the neutron population in a nuclear reactor.

Fortunately, many dependencies listed in these equations exist, but are very weak, and therefore can almost always be safely neglected. The most common simplification, frequently present in databases of nuclear properties (but not in ENDF/B-VII), is the assumption that delayed neutron precursor families' decay constants and delayed neutron energy yield spectra (but not their fission yields) are not isotope-dependent. There are almost always $M = 6$ of these combined precursor families. Let their shared decay constants and delayed neutron energy yield spectra be denoted λ_m and $\chi_{d,m}(E)$, respectively.

This greatly simplifies the delayed neutron source in Eq. (38):

$$\begin{aligned}\frac{\partial}{\partial t}n\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) &= -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) - \Sigma_t(\vec{\mathbf{x}},E)\,\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) + \\ &+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j}\Sigma_s^j(\vec{\mathbf{x}},E')\,P_s^j\Big(E'\to E,\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}}\Big)\right]\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ \frac{1}{4\pi}\int_0^\infty dE' \left[\sum_{\text{all } j_f}\chi_p^{j_f}(E',E)\,\nu_p^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E')\right]\iint_{4\pi} d\Omega'\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ \frac{1}{4\pi}\sum_{m=1}^{M=6}\chi_{d,m}(E)\,\lambda_m c_m(t,\vec{\mathbf{x}}) + s_{ex}\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big)\,.\end{aligned} \tag{41}$$

The delayed neutron precursor Eqs. (40), on the other hand, are slightly complicated, although greatly reduced in number. All fissionable isotopes must now be considered as sources:

$$\begin{aligned}\frac{\partial}{\partial t}c_m(t,\vec{\mathbf{x}}) = &\int_0^\infty dE' \left[\sum_{\text{all } j_f}\nu_{d,m}^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E')\right]\iint_{4\pi} d\Omega'\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) - \\ &- \lambda_m c_m(t,\vec{\mathbf{x}}) \quad \forall m\in[1,\ldots,M=6]\,.\end{aligned} \tag{42}$$

These equations are simpler, but are still clearly very complicated. Section 6 lists additional assumptions, and a final, most simplified version of the NTE.

Other than the assumptions listed in section 1, the only other assumption restricting Eq. (38) (and by extension, Eqs. (40)), is the lack of temperature dependence of the cross-sections. In reality, cross-sections depend strongly on temperature, which is discussed in the next subsection.

5.2 Thermal Feedback

The details of the physics of cross-section dependence on temperature are a major topic by themselves, and will not be covered here. For the purposes of the formulation and solution of the NTE, the summary, in terms of the cross-sections encountered in the NTE, is the following:

- The microscopic total cross-section $\sigma_t(\vec{\mathbf{x}},E)$ has the strongest dependence on temperature, particular at higher energies, because the rate of resonant capture greatly increases as the material heats up.
- The microscopic scattering cross-section $\sigma_s(\vec{\mathbf{x}},E)$ has a weaker dependence by itself, but the scattering kernel at low energies changes significantly with higher temperatures because the rate and magnitude of upscattering increases.
- The microscopic fission cross-section also has some dependence on temperature at higher energies, but less than the radiative capture cross-sections.

- Prompt and delayed energy yield spectra, as well as the average numbers of neutrons/precursors generated, are not measurably temperature-dependent.
- Macroscopic cross-sections are dependent not only on the microscopic cross-sections, but also on atom densities. Generally, materials, including nuclear materials, do expand with temperature; coolant boiling is the most significant expansion caused by temperature increase. These dependencies are physically very different from the microscopic cross-sections' temperature dependencies, but also contribute to macroscopic cross-sections' temperature dependencies.

With these dependencies in mind, denoting the material/fluid temperature at location $\vec{\mathbf{x}}$ as T, Eq. (38) becomes:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}n\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) &= -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) - \Sigma_t(\vec{\mathbf{x}},E,T)\,\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) + \\ &+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \left[\sum_{\text{all } j}\Sigma_s^j(\vec{\mathbf{x}},E',T)\,P_s^j\Big(E'\to E,\hat{\mathbf{\Omega}}'\to\hat{\mathbf{\Omega}},T\Big)\right]\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ \frac{1}{4\pi}\int_0^\infty dE' \left[\sum_{\text{all } j_f}\chi_p^{j_f}(E',E)\,\nu_p^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E',T)\right]\iint_{4\pi} d\Omega'\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ \frac{1}{4\pi}\sum_{\text{all } j_f}\left[\sum_{m=1}^{M^{j_f}}\chi_{d,m}^{j_f}(E)\,\lambda_m^{j_f}c_m^{j_f}(t,\vec{\mathbf{x}})\right] + s_{ex}\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big)\,.\end{aligned}} \tag{43}$$

Equation (40) is similarly adjusted:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}c_m^{j_f}(t,\vec{\mathbf{x}}) &= \int_0^\infty dE'\nu_{d,m}^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E',T)\iint_{4\pi}d\Omega'\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) - \\ &\quad - \lambda_m^{j_f}c_m^{j_f}(t,\vec{\mathbf{x}}) \quad \forall m\in\left[1,\ldots,M^{j_f}\right], j_f\in[1,\ldots,J_f]\,.\end{aligned}} \tag{44}$$

Fissions in a nuclear reactor are a very powerful heat source, which must always be present in the thermal energy conservation equation that models the temperature profile of a nuclear reactor. Various forms of this equation will not be discussed here, but the fission heat source term will be discussed.

Let the density of fission-generated thermal power in volume (units of $\mathrm{W\,cm^{-3}}$) be denoted $s_\kappa(t,\vec{\mathbf{x}})$. Let $\kappa^{j_f}(E')$ denote the average amount of thermal energy released per fission of fissionable isotope j_f with a neutron of energy E'; it is usually about 192 MeV. The expression for the fission power density then becomes:

$$\boxed{s_\kappa(t,\vec{\mathbf{x}}) = \int_0^\infty dE'\left[\sum_{\text{all } j_f}\kappa^{j_f}(E')\,\Sigma_f^{j_f}(\vec{\mathbf{x}},E',T)\right]\iint_{4\pi}d\Omega'\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big)\,.} \tag{45}$$

While most of the energy released during a fission is released as thermal energy in the fuel, a small part of fission energy can also be released as γ-radiation, which can then be

absorbed by the reactor materials and fluids, therefore contributing to the thermal energy source. This energy is not shown in Eq. (45), but can be significant enough to modify some transients.

While the total amount of energy released per fission varies weakly (2 to 3 MeV), the share of the thermal energy is generally higher at lower fissioning neutron energies E', while the prompt neutron energy is generally slightly higher with higher E'. These dependencies are almost always neglected, with κ assumed a fixed value (or, sometimes, isotope-dependent), which simplifies Eq. (45):

$$s_\kappa(t, \vec{\mathbf{x}}) = \kappa \int_0^\infty dE' \Sigma_f(\vec{\mathbf{x}}, E', T) \iint_{4\pi} d\Omega' \varphi\left(t, \vec{\mathbf{x}}, E', \hat{\mathbf{\Omega}}'\right), \tag{46}$$

in which $\Sigma_f(\vec{\mathbf{x}}, E', T)$ is the sum of all fisionable isotopes' macroscopic fission cross-sections.

Because the temperature depends on the flux through Eq. (45), and the flux depends on the temperature through Eqs. (43) and (44), the overall system of equations becomes coupled and nonlinear. Together, they are often referred to as the "reactor multiphysics equations." Almost always, when solving either the NTE (with or without the delayed neutron precursor equations) or the thermal hydraulics equations, either the temperature or the thermal power density profiles, respectively, are assumed. In principle, iterating between the two sets of physics while assuming the other physics as known is possible; this approach is known as "operator splitting."

In the last section, further simplifications of the NTE and the delayed neutron precursor equations are discussed.

6 A Note on Further Simplifications

In this section, the temperature-independent forms of the NTE and the delayed neutron precursor equations (DNPEs), are simplified further. Equations (38) and (40) are almost never seen in those forms; multiple additional assumptions are usually made during the derivation. The notation is also sometimes different. These simplifying assumptions and notational modifications are listed below:

- Energy-dependent angular neutron density is rarely seen explicitly in the NTE. Instead, the rate of change of the angular flux is usually present, shown in this paper in Eq. (3). This is a convention, not an assumption; using it does not introduce an inaccuracy.
- As discussed in subsection 3.2, the combined scattering kernel is usually used. With no additional simplifications, this quantity is again, simply a convention, and does not introduce an inaccuracy.
- Additionally, as discussed in subsection 3.2, usually the scattering kernel is made dependent only on the scattering angle cosine $\hat{\mathbf{\Omega}}' \cdot \hat{\mathbf{\Omega}}$, and not on the two directions independently. This *can* introduce an inaccuracy for scattering in a highly structured material.
- As shown in Eq. (30), a simple combined prompt fission yield spectrum is frequently used.

- Because the average number of prompt neutrons born per fission $\nu_p(E')$ is nearly constant, except at very high energies E', at which it jumps up, and most fissions in light water reactors happen at low energies, a one-group approximation ν_p is frequently used. A one-group approximation of any quantity is a weighted average of this quantity. Typically, it is flux-weighted; in this case it's weighted by both the fission cross-section and the flux. This assumption does introduce some inaccuracy, because the dependence of $\nu_p^{j_f}(E')$ on the isotope and fissioning neutron energy is not negligible.
- Additionally, as discussed in section 4, an effective term $\beta(E')$ is usually used together with an isotope-independent total $\nu(E')$, in place of $\nu_p^{j_f}(E')$. For the same reasons as the one-group approximation of ν_p, the one-group approximations of ν and β are also frequently constructed. The one-group approximation of β is normally called β_{eff}, particularly for reactors with more than one significant fissionable isotope. These approximations are only good if an accurate position-dependent energy spectrum (i.e., an accurate weighting energy-dependent scalar flux density) is used, which is usually based on experience with the corresponding reactor type, and is usually sufficient.
- The integral over all directions in the fission source terms in both the NTE and the DNPEs is usually replaced with the scalar flux $\phi(t, \vec{\mathbf{x}}, E)$. This is only a convention, not an approximation.
- As discussed in subsection 5.1, the precursor families' properties (delayed neutron yield energy spectrum and decay constants) are almost always treated as fissionable isotope-independent, although their delayed neutron fractions $\beta_m^{j_f}(E')$ are still usually viewed as isotope-dependent. The notion of 6 precursor families is by itself an approximation, so this additional approximation is almost always quite good.
- Like with the total β, the isotope dependence of $\beta_m^{j_f}(E')$ is usually dropped (through an appropriate weighted average), and the one-group approximations β_m are usually used. The 6 precursor families' delayed neutron fractions still add up to the total β. Dropping the isotope dependence usually is accurate only when there is one significant fissionable isotope, but is less accurate if others are present. The one-group approximation is, again, as good as the quality of the position-dependent weighting spectrum.
- The delayed neutron energy yield spectrum is can be assumed to be family-independent as well, through an appropriate weighted average. It is then denoted $\chi_d(E)$. This is not always a good assumption, but it usually has a limited effect on the reactor transient, because the energy spectrum in a reactor is dominated by the prompt neutrons, because β is usually very small (under 1 %).
- The external source is usually neglected, except in specific (subcritical) reactor types, or at startup. In all other cases, this is a very good assumption, because the rate density of neutron generation by fission for a reactor at power is always far higher than an external, i.e. plutonium-beryllium, source can provide.

With all of the above simplifications and conventions, the NTE with prompt and delayed sources becomes:

$$\boxed{\begin{aligned}\frac{1}{V_n(E)}\frac{\partial}{\partial t}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) &= -\nabla\cdot\hat{\mathbf{\Omega}}\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) - \Sigma_t(\vec{\mathbf{x}},E)\,\varphi\Big(t,\vec{\mathbf{x}},E,\hat{\mathbf{\Omega}}\Big) + \\ &+ \int_0^\infty dE' \iint_{4\pi} d\Omega'\Sigma_s\Big(\vec{\mathbf{x}},E'\to E,\hat{\mathbf{\Omega}}'\cdot\hat{\mathbf{\Omega}}\Big)\,\varphi\Big(t,\vec{\mathbf{x}},E',\hat{\mathbf{\Omega}}'\Big) + \\ &+ \frac{\chi_p(E)}{4\pi}(1-\beta)\,\nu\int_0^\infty dE'\Sigma_f(\vec{\mathbf{x}},E')\,\phi(t,\vec{\mathbf{x}},E') + \frac{\chi_d(E)}{4\pi}\sum_{m=1}^{M=6}\lambda_m c_m(t,\vec{\mathbf{x}})\,.\end{aligned}} \tag{47}$$

The corresponding simplified DNPE is:

$$\boxed{\frac{\partial}{\partial t}c_m(t,\vec{\mathbf{x}}) = \beta_m\nu\int_0^\infty dE'\Sigma_f(\vec{\mathbf{x}},E')\,\phi(t,\vec{\mathbf{x}},E') - \lambda_m c_m(t,\vec{\mathbf{x}}) \quad \forall m\in[1,\ldots,M=6]\,.} \tag{48}$$

Of the approximations made in this section, some are ubiquitous, and others are questionable, and therefore avoided in typical analysis. They are generally not dependent on each other, and so an appropriate combination of approximations can often be made. It is important to realize that normally, a group quantity (such as the one-group approximations of ν, β and others) is only as good as the weighting flux, and an isotope-weighted quantity is as good as the weighting flux and relative importance of corresponding isotopes. Benchmark problems frequently provide these weighted quantities, to ensure consistency between results. In practice, they are often determined by a code prior to a transient analysis.