# MODULI SPACES IN $p$-ADIC NON-ABELIAN HODGE THEORY

BEN HEUER

ABSTRACT. We propose a new moduli-theoretic approach to the $p$-adic Simpson correspondence for any smooth proper rigid space over $\mathbb{C}_p$ with coefficients in any rigid analytic group $G$. For its formulation, we introduce the class of "smoothoid spaces" which are perfectoid families of smooth rigid spaces. We then prove a generalisation of Faltings' local $p$-adic Simpson correspondence to $v$-topological $G$-bundles on smoothoid spaces. We use this to show that there are small moduli $v$-stacks for both sides of the correspondence. Second, we use it to construct an analogue of the Hitchin morphism on either side. This allows us to give a conjectural reformulation of the $p$-adic Simpson correspondence in a more geometric and canonical way: The moduli stack of $v$-topological $G$-bundles is a twist of the moduli stack of $G$-Higgs bundles over the Hitchin base.

## 1. INTRODUCTION

Let $K$ be a complete algebraically closed extension of $\mathbb{Q}_p$. The conjectural $p$-adic Corlette–Simpson correspondence for a connected smooth proper rigid space $X$ over $K$ aims to relate $K$-linear continuous representations of the profinite étale fundamental group $\pi_1^{\text{ét}}(X)$ of $X$ to a full subcategory of the Higgs bundles on $X$. By a reinterpretation of Faltings' pioneering work on generalised representations [Fal05] in terms of Scholze's diamonds [Sch], this should go through a $p$-adic non-abelian Hodge correspondence, namely an equivalence of categories

$$(1) \qquad \{\text{vector bundles on } X_v\} \xrightarrow{\sim} \{\text{Higgs bundles on } X_{\text{ét}}\},$$

via a natural fully faithful embedding of representations of $\pi_1^{\text{ét}}(X)$ into vector bundles on $X_v$.

The aim of this article is to lay the foundations for a new conceptual approach to $p$-adic non-abelian Hodge theory, in particular to (1), which puts moduli spaces at its centre: Indeed, in [Heub], we will construct the equivalence (1) based on ideas of this article. Second, we suggest a new conjectural geometric formulation of the equivalence (1) in terms of a comparison of moduli stacks, which will be proved in [HX24] in the case when $X$ is a curve. Third, this makes it possible to explain the relation to representations of $\pi_1^{\text{ét}}(X)$ in terms of a $p$-adic character variety, which opens up new geometric ways to study the essential image of representations of $\pi_1^{\text{ét}}(X)$ under (1), generalising the approach for line bundles from [Heu24a]. Fourth, we develop these foundations not just for $\mathrm{GL}_n$, but for general rigid groups $G$: This opens up a new line of investigation into generalisations of (1) to $G$-torsors, which we continue in [HWZ].

Besides, the technical foundations of $p$-adic Hodge theory of perfectoid families of rigid spaces that we provide in this article are of broader interest to relative $p$-adic Hodge theory: For example, in [Heu24b], we use them to construct the relative Hodge–Tate spectral sequence.

### 1.1. The sheafified non-abelian Hodge correspondence.
In the spirit of Simpson's non-abelian Hodge theory [Sim92], the starting point of this article is the idea to realise the $p$-adic non-abelian Hodge correspondence (1) by finding a non-abelian generalisation of Scholze's approach to the Hodge–Tate spectral sequence:

To explain this, let us begin by considering the $p$-adic Hodge–Tate short exact sequence

$$(2) \qquad 0 \to H^1_{\text{an}}(X, \mathcal{O}) \to \mathrm{Hom}_{\text{cts}}(\pi_1^{\text{ét}}(X), K) \to H^0(X, \Omega^1_X(-1)) \to 0.$$

The fundamental idea of Scholze's construction of (2) is to realise it as a Leray sequence for the morphism of sites $\nu : X_v \to X_{\text{ét}}$, where $X_v$ is the $v$-site of the diamond associated to $X$. The key result that leads to (2) is then the natural isomorphism [Sch13b, Proposition 3.23]

$$(3) \qquad\qquad \text{HT} : R^1\nu_*\mathcal{O} \xrightarrow{\sim} \Omega_X^1(-1).$$

In this article, we give an analogue of (2) in which the middle term is replaced by continuous representations $\pi_1^{\text{ét}}(X) \to \text{GL}_n(K)$, following Scholze's strategy with $\mathcal{O}$ replaced by $\text{GL}_n(\mathcal{O})$.

In a generalisation that so far has not been studied in the $p$-adic setting, we more generally pass from $\text{GL}_n$ to any rigid analytic group variety $G$ over $K$. Such $G$ are the $p$-adic analogues of complex Lie groups, and we simply refer to them as "rigid groups". For example, $G$ could be the analytification of an algebraic group, or any open subgroup thereof. In order to describe the correct replacement for the right hand side of (3), we need the following:

**Definition 1.1.** A $G$-Higgs bundle on $X$ is a pair $(E, \theta)$ of a $G$-torsor $E$ on $X_{\text{ét}}$ and a section $\theta \in H^0(X, \text{ad}(E) \otimes \Omega_X^1(-1))$ such that $\theta \wedge \theta = 0$, where $\text{ad}(E)$ is the adjoint bundle.

Towards a non-abelian analogue of Scholze's construction, our first main result is now the following "sheafified $p$-adic non-abelian Hodge correspondence" for rigid analytic spaces:

**Theorem 1.2.** *Let $X$ be any smooth rigid space over $K$ and let $\nu : X_v \to X_{\text{ét}}$ be the natural morphism of sites. Let $G$ be a rigid group over $K$, considered as a sheaf on $X_v$. For example, $G$ could be the analytification of any algebraic group. Then there is a canonical isomorphism*

$$\text{HTlog} : R^1\nu_*G \xrightarrow{\sim} \text{Higgs}_G$$

*of sheaves of pointed sets on $X_{\text{ét}}$ that is functorial in $X$, $G$ and $K$. Here*

$$\text{Higgs}_G := (\text{Lie}(G) \otimes_K \Omega_X^1(-1))^{\wedge=0}/G$$

*is the sheaf of isomorphism classes of $G$-Higgs bundles on $X_{\text{ét}}$: Explicitly, $\text{Lie}(G)$ is the Lie algebra of $G$ as a $K$-vector space, $(-)^{\wedge=0}$ denotes the subspace of elements $\theta$ satisfying $\theta \wedge \theta = 0$, and the sheaf quotient is formed with respect to the adjoint action of $G$ on $\text{Lie}(G)$.*

This is a vast generalisation of Scholze's isomorphism (3), which we recover as the special case of $G = \mathbb{G}_a$. In earlier work, we had studied $G = \mathbb{G}_m$, but already for $G = \text{GL}_n$, the theorem is new. In other words, HTlog gives a *canonical* correspondence between isomorphism classes of $v$-topological $G$-bundles on $X$ and $G$-Higgs bundles on $X$ after étale sheafification.

As in Scholze's strategy for $\mathbb{G}_a$, the Leray sequence of $\nu$ now yields a short exact sequence

$$(4) \qquad\qquad 0 \to H^1_{\text{ét}}(X, G) \to H^1_v(X, G) \xrightarrow{\text{HTlog}} \text{Higgs}_G(X)$$

of pointed sets. Of course, for non-abelian $G$, such a sequence gives less structure than a short exact sequence of abelian groups as in (2). The second main idea of this article is therefore to pass from sheaves of isomorphism classes to moduli spaces, and to turn the last morphism in (4) into a morphism of $v$-stacks. The exactness of the Hodge–Tate sequence can then find its generalisation in geometric properties of this morphism, such as being a fibration. As we will illustrate below at the hand of examples, this strategy indeed turns out to be fruitful.

1.2. **Moduli spaces in $p$-adic non-abelian Hodge theory.** In order to define $p$-adic analytic moduli spaces of $v$-topological $G$-torsors and $G$-Higgs bundles on $X$, we need new technical foundations to formulate relative $p$-adic Hodge theory. For this, our first step in this article is to introduce and study a new class of perfectoid families of smooth rigid spaces:

**Definition 1.3.** A **smoothoid space** is an analytic adic space over a perfectoid field $K$ that locally admits a smooth morphism of adic spaces to a perfectoid space over $K$.

An example would be the product $X \times T$ of the smooth rigid space $X$ with a perfectoid space $T$, or any object of its étale site. There is a reasonable notion of differentials on a smoothoid space: For any smoothoid space $Y$ we consider the map $\nu : Y_v \to Y_{\text{ét}}$ and define

$$\widetilde{\Omega}_Y^n := R^n \nu_* \mathcal{O}_Y.$$

We show that this is a vector bundle on $Y$. The cup product induces a natural wedge product on this, yielding a good functorial definition of $G$-Higgs bundles, exactly as in Definition 1.1.

Towards a moduli-theoretic non-abelian Hodge correspondence, we suggest to study the functors fibred in groupoids on the $v$-site $\mathrm{Perf}_K$ of affinoid perfectoid spaces $T$ over $K$

$$\mathscr{B}un_{G,v} : T \mapsto \{G\text{-torsors on } (X \times T)_v\},$$

$$\mathscr{H}iggs_G : T \mapsto \{G\text{-Higgs bundles on } (X \times T)_{\text{ét}}\}.$$

Our next result says that these can reasonably be regarded as geometric objects:

**Theorem 1.4.** $\mathscr{B}un_{G,v}$ and $\mathscr{H}iggs_G$ are small $v$-stacks in the sense of Scholze [Sch, §12].

While for $\mathscr{B}un_{G,v}$ only the smallness requires work, the result for $\mathscr{H}iggs_G$ is much deeper. We deduce it from $v$-descent criteria, namely our main result is that Theorem 1.2 generalises:

**Theorem 1.5.** Let $X$ be a smoothoid space over $K$, then there is a canonical isomorphism

$$\mathrm{HTlog} : R^1 \nu_* G \xrightarrow{\sim} (\mathrm{Lie}(G) \otimes_K \widetilde{\Omega}_X^1)^{\wedge=0}/G$$

of sheaves of pointed sets on $X_{\text{ét}}$, where $\nu : X_v \to X_{\text{ét}}$. It is functorial in $X$, $G$ and $K$. For commutative $G$, there is for any $n \in \mathbb{N}$ a canonical isomorphism $R^n \nu_* G \xrightarrow{\sim} \mathrm{Lie}(G) \otimes_K \widetilde{\Omega}_X^n$.

1.3. **The Hitchin morphism on the Betti side.** The moduli stack $\mathscr{H}iggs_G$ gives rise to a $p$-adic incarnation of the **Hitchin morphism** which plays an important role in complex non-abelian Hodge theory. In the $p$-adic setting, this is a morphism of $v$-stacks

(5) $$\mathcal{H} : \mathscr{H}iggs_G \to \mathcal{A}_G$$

to the Hitchin base $\mathcal{A}_G$ over $X$. Here $\mathcal{A}_G$ is in general a $v$-sheaf, but if $X$ is proper and $G$ is split reductive or commutative, then $\mathcal{A}_G$ is represented by an affine rigid space, for example

$$\mathcal{A}_{\mathrm{GL}_n} = \bigoplus_{k=1}^{n} H^0(X, \mathrm{Sym}^k \widetilde{\Omega}_X) \otimes_K \mathbb{G}_a.$$

The construction of $\mathcal{H}$ in this case is essentially the same as the classical one due to Hitchin [Hit87], given by sending a Higgs bundle $(E, \theta)$ to the characteristic polynomial of $\theta$.

On the "Betti" side, Theorem 1.4 now allows us to construct an analogous morphism:

**Definition 1.6.** For any rigid group $G$, the **Hitchin morphism on the Betti side**

(6) $$\widetilde{\mathcal{H}} : \mathscr{B}un_{G,v} \to \mathcal{A}_G$$

is defined by sending a $v$-topological $G$-torsor on $Y = X \times T$ to the associated class in $R^1 \nu_* G(Y)$ and using HTlog to pass to the Higgs side, where we use the Hitchin map (5).

Assume now that $X$ is a smooth proper rigid space over $K$ and that $G$ is reductive. In this setting, we envision the $p$-adic Simpson correspondence to have a geometric incarnation in terms of a comparison of the Hitchin morphisms (5) and (6). While $\mathscr{B}un_{G,v}$ and $\mathscr{H}iggs_G$ are not in general isomorphic, we conjecture that one is a *twist* of the other in a canonical way, via the Hitchin fibrations. Indeed, building on this article, the following is proved in [HX24]:

**Theorem 1.7** ([HX24]). Let $X$ be a smooth projective curve over $K$ and let $G$ be a reductive group. Then there is a Picard groupoid $\mathscr{P}$ on $\mathcal{A}_{G,v}$ that acts on both $\mathcal{H} : \mathscr{H}iggs_G \to \mathcal{A}_G$ and $\widetilde{\mathcal{H}} : \mathscr{B}un_{G,v} \to \mathcal{A}_G$, and a $\mathscr{P}$-torsor $\mathscr{H}$ for which there is a canonical equivalence of $v$-stacks

$$\mathscr{B}un_{G,v} \xrightarrow{\sim} \mathscr{H} \times^{\mathscr{P}} \mathscr{H}iggs_G.$$

Here $\mathscr{P}$ is the stack of line bundles on the spectral curve. In particular, this shows that for $G = \mathrm{GL}_n$, up to connected components, the morphism $\widetilde{\mathcal{H}} : \mathscr{B}un_{G,v} \to \mathcal{A}_G$ is in this case a fibration in abelian varieties. To illustrate this phenomenon further, let us also mention the easier case of $G = \mathbb{G}_m$, for which [Heua, Theorem 1.3] immediately implies the following:

**Theorem 1.8** ([Heua])**.** *For any smooth proper rigid space $X$ over $K$, the sequence*

$$(7) \qquad 0 \to \mathscr{B}un_{\mathbb{G}_m, \text{ét}} \to \mathscr{B}un_{\mathbb{G}_m, v} \xrightarrow{\widetilde{\mathcal{H}}} H^0(X, \widetilde{\Omega}^1_X) \otimes \mathbb{G}_a \to 0$$

*is short exact. In particular, both $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are torsors under $\mathscr{B}un_{\mathbb{G}_m, \text{ét}}$.*

1.4. **Towards a $p$-adic Corlette–Simpson correspondence for $\pi_1^{\text{ét}}(X)$.** Assume now that $K$ is algebraically closed and that $X$ is a connected smooth proper rigid space over $K$. Fix a base-point $x \in X(K)$ and let $\pi_1^{\text{ét}}(X) := \pi_1^{\text{ét}}(X, x)$ be the profinite étale fundamental group of $X$, i.e. the symmetry group of all pointed connected finite étale covers of $(X, x)$ (rather than the de Jong fundamental group). Then the moduli-theoretic approach also seems fruitful for studying continuous representations of $\pi_1^{\text{ét}}(X)$: We show that the Hitchin morphism on the Betti side $\widetilde{\mathcal{H}}$ induces a geometric generalisation of the map $\mathrm{HT} : \mathrm{Hom}_{\text{cts}}(\pi_1(X), K) \to H^0(X, \widetilde{\Omega})$ in the Hodge–Tate sequence (2), namely a morphism

$$\widetilde{H} : \mathscr{H}om(\pi_1^{\text{ét}}(X), G) \to \mathcal{A}_G,$$

from the representation variety parametrising continuous $G$-representations of $\pi_1^{\text{ét}}(X)$ to the Hitchin base. In contrast to $\widetilde{\mathcal{H}}$, this has the advantage that it is represented by a morphism of rigid spaces. We call this the "**Hitchin–Hodge–Tate morphism**". The name reflects that $\widetilde{H}$ is simultaneously an analogue of the Hitchin morphism and a generalisation of the Hodge–Tate map HT. Indeed, for $G = \mathbb{G}_a$, it is the morphism of rigid spaces associated to HT by tensoring with $\mathbb{G}_a$. As we will explain, the image of a representation $\pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(K)$ under $\widetilde{H}$ is a close analogue of the Hodge–Tate–Sen weights of a local Galois representation.

That $\widetilde{H}$ is rigid analytic is relevant as the exactness of (2) can now find its non-abelian generalisation in geometric properties of $\widetilde{H}$: For $G = \mathbb{G}_m$, [Heu24a, Theorem 4.1] shows that (7) induces on coarse moduli spaces a short exact sequence of rigid groups

$$(8) \qquad 0 \to \mathbf{Pic}^{\text{tt}}_{X, \text{ét}} \to \mathscr{H}om(\pi_1^{\text{ét}}(X), \mathbb{G}_m) \xrightarrow{\widetilde{H}} H^0(X, \widetilde{\Omega}^1_X) \otimes \mathbb{G}_a \to 0,$$

where $\mathbf{Pic}^{\text{tt}}_{X, \text{ét}}$ is the topological torsion Picard functor, so here $\widetilde{H}$ is indeed a fibration. For $G = \mathbb{G}_m$, this was our crucial input to answer in [Heu24a, Theorem 1.1] the question which Higgs bundles correspond to continuous characters of $\pi_1^{\text{ét}}(X) \to \mathbb{G}_m(K)$ under the equivalence (1). We envision that the geometric study of $\widetilde{H}$ leads to an answer of this question for general $G$.

In summary, with the construction of the moduli stacks $\mathscr{B}un_{G,v}$ and $\mathscr{H}iggs_G$ and their Hitchin morphisms $\widetilde{\mathcal{H}}$ and $\mathcal{H}$, we have thus established the technical foundations for our moduli theoretic approach to $p$-adic non-abelian Hodge theory.

1.5. **Application: $G$-torsors on rigid spaces in the $v$-topology.** Theorem 1.2 is of independent interest beyond (1): The sequence (4) describes how far the fully faithful functor

$$\{G\text{-torsors on } X_{\text{ét}}\} \hookrightarrow \{G\text{-torsors on } X_v\}$$

is from being an equivalence, in terms of the explicit set $\mathrm{Higgs}_G(X)$. This is already very interesting for $G = \mathrm{GL}_n$, where it describes the difference between étale and $v$-vector bundles on $X$, which had so far been insufficiently understood: The question when a $v$-vector bundle is already étale appears naturally for example in the context of automorphic sheaves defined via descent from perfectoid Shimura varieties [CHJ17]. Using (4), we can now give satisfactory answers to this question. For example, we deduce the following a priori surprising criterion:

**Corollary 1.9.** *Let $X$ be a smooth rigid space and $V$ a $G$-torsor on $X_v$. Then for any étale map $f : U \to X$ with Zariski-dense image, $V$ is étale-locally trivial if and only if $f^*V$ is.*

This extends the case $G = \mathbb{G}_m$ from [Heu22b] which we had used to simplify the proof that the automorphic sheaves of [CHJ17] are analytic vector bundles. Theorem 1.2 now improves this application: Setting $G := \mathcal{O}^{+,\times}$, we see that the natural integral structure on the automorphic sheaves descend as well, i.e. these are finite locally free $\mathcal{O}^+$-modules.

For applications to $G$-torsors on $X_v$, also the following more categorical result is useful:

**Theorem 1.10.** *Any étale morphism $f : X \to \mathbb{T}^d$ induces an equivalence of categories*

$$\{\text{small } G\text{-torsors on } X_v\} \xrightarrow{\sim} \{\text{small } G\text{-Higgs bundles on } X_{\text{ét}}\}$$

*that is natural in $G$ and $f$, but in general depends on the choice of $f$.*

As we explain in §6.3, this is a generalisation of a rigid analytic version of Faltings' "local $p$-adic Simpson correspondence" [Fal05, Theorem 3]. Other instances of such a correspondence have previously been given by Abbes–Gros and Tsuji [AGT16], Tsuji [Tsu18], Wang [Wan23] and Morrow–Tsuji [MT21], all in the case of $G = \mathrm{GL}_n$. Equipped with our technical preparations on smoothoid spaces, our proof of Theorem 1.10 in the case of $\mathrm{GL}_n$ is similar to the strategy of Faltings. However, the case of general $G$ requires new ideas.

The "smallness" in Theorem 1.10 roughly means that the $G$-torsor has a reduction of structure group to a certain open subgroup of $G$ depending on $X$. As every $G$-torsor on $X_v$ becomes small étale-locally on $X$, Theorem 1.10 always applies locally on $X_{\text{ét}}$. Hence $p$-adic non-abelian Hodge theory is a good framework to study $G$-torsors on $X_v$.

1.6. **Relation to Faltings' global $p$-adic Simpson correspondence.** We now elaborate on the discussion surrounding (1), and explain how our work fits into the historic context: The original goal of $p$-adic non-abelian Hodge theory is to study the $K$-linear representations in $G$ of the étale fundamental group $\pi_1^{\text{ét}}(X)$ of a smooth proper rigid variety $X$ with the methods of $p$-adic Hodge theory. As this has so far almost exclusively been done for $G = \mathrm{GL}_n$, we restrict to this case for the moment. Following Faltings [Fal05], the basic idea is to embed representations into a category of "generalised representations", which can be interpreted as vector bundles on $X_v$ ([Heu22a, §2]). Namely, there is a fully faithful functor

$$\left\{ \begin{array}{c} \text{fin.-dim. continuous } K\text{-linear} \\ \text{representations of } \pi_1^{\text{ét}}(X) \end{array} \right\} \hookrightarrow \left\{ \text{vector bundles on } X_v \right\},$$

defined by regarding a representation as a descent datum for the trivial bundle along the pro-finite-étale universal cover $\widetilde{X} \to X$, which is a $v$-topological $\pi_1^{\text{ét}}(X)$-torsor over $X$.

Based on Faltings' influential work in the case of curves, we have the following:

**Theorem 1.11** ($p$-adic Simpson correspondence)**.** *Assume that $K$ is algebraically closed and $X$ is a smooth proper rigid space over $K$. Then there is an equivalence of categories*

$$\{\text{vector bundles on } X_v\} \xrightarrow{\sim} \{\text{Higgs bundles on } X_{\text{ét}}\}.$$

*It is non-canonical and depends on choices of a $B_{\mathrm{dR}}^+/\xi^2$-lift of $X$ and an exponential on $K$.*

We prove this in [Heub] based on the preparations in this article, a crucial role being played by the Hitchin fibration on the Betti side $\widetilde{\mathcal{H}}$. The conjecture was previously known for curves [Fal05] and line bundles [Heu22b], and there were many partial results e.g. in good reduction settings [Wan23] and arithmetic situations over discretely valued fields [LZ17][MW].

On the other hand, moduli spaces can be used to strengthen this statement: As a first example, by testing on perfectoid spaces associated to profinite sets, we can endow the sets of isomorphism classes $|\mathscr{B}un_{\mathrm{GL}_n,v}(K)|$ and $|\mathscr{H}iggs_{\mathrm{GL}_n}(K)|$ on either side with a natural topology. One can then hope to refine Theorem 1.11 in a topological way that is very close to the complex Corlette–Simpson correspondence:

**Conjecture 1.12.** *The equivalence of Theorem 1.11 induces a homeomorphism*

$$|\mathscr{B}un_{\mathrm{GL}_n,v}(K)| \xrightarrow{\sim} |\mathscr{H}iggs_{\mathrm{GL}_n}(K)|.$$

When $X$ is a curve, this conjecture is proved in [HX24, Theorem 1.1.1]: Indeed, the choices in Theorem 1.11 induce a trivialisation of the torsor $\mathscr{P}$ from Theorem 1.7 on $K$-points. Note that Theorem 1.7 is completely canonical, so this gives a conceptual explanation of the choices in Theorem 1.11 in a geometric fashion. We believe that this perspective should generalise.

1.7. **Relation to other previous works, and outlook.** There has recently been a great deal of activity in $p$-adic non-abelian Hodge theory, and we now sketch how the results of this article relate to some of these recent works, further to the relation to the local $p$-adic Simpson correspondence discussed in detail in Section 6.3.

The first works in the area were due to Deninger–Werner [DW05][DW20], who define a functor from certain Higgs bundles on $X$ with vanishing Higgs field to representations, thus giving a $p$-adic analogue to the complex theory of Narasimhan–Seshadri [NS65]. This was extended by Würthen to the rigid setting [Wü23]. From our perspective, this treats those $v$-vector bundles that are trivial étale-locally, and thus go to 0 under the Hitchin fibration. So this theory plays out in the fibre over 0 of $\widetilde{\mathcal{H}}$, the analogue of the "nilpotent cone".

The same is true for the works of Liu–Zhu [LZ17] and Min–Wang [MW], who study $\mathbb{Q}_p$-local systems, respectively $v$-vector bundles, on a smooth rigid space $X_0$ over a discretely valued field $k$, and relate these to Higgs bundles over the base-change to the completed algebraic closure $K$ of $k$. In both of these cases, the associated Higgs bundles are nilpotent, which from our perspective can be explained as follows: Functoriality in Theorem 1.5 implies that HTlog is Galois-equivariant, and the same is true for the morphism $\widetilde{\mathcal{H}}$. Since the Galois invariants of $K(-1)$ are trivial, it follows that $\widetilde{\mathcal{H}}$ sends the associated $v$-vector bundles to 0.

Regarding $v$-vector bundles in $p$-adic Hodge theory, let us also mention the relation to Sen theory, which is concerned with $v$-vector bundles on rigid spaces over discretely valued fields $L$: The classical theory of Sen [Sen81] essentially describes $v$-vector bundles on $\mathrm{Spa}(L)$. Recently, Sen theory has been further developed for rigid spaces over $L$ by Shimizu [Shi18, §3] based on [LZ17], Min–Wang [MW], Pan [Pan22, §3.2] and Rodríguez Camargo [RC23, §3]. The relation of our work to that of Rodríguez Camargo will be explained in [Heub, §4.2].

The generalisation from vector bundles to $G$-torsors in $p$-adic non-abelian Hodge theory has, to the best of our knowledge, so far only been explored by one previous work: In the case of Higgs bundles on curves with Higgs field $\theta = 0$, Hackstein [Hac08] has generalised the functor of Deninger–Werner to a functor on $G$-bundles for a reductive group $G$, in analogy to the complex theory by Ramanathan [Ram75]. As one considers reductive $G$ in the complex theory, it is arguably a bit surprising that Theorems 1.5 and 1.10 allow any rigid group $G$.

Regarding moduli functors in $p$-adic non-abelian Hodge theory, the only case that has been studied before is the case of $G = \mathbb{G}_m$ mentioned above, which we have studied in [Heua] in terms of the rigid analytic Picard variety, the coarse moduli space of the $v$-stack $\mathscr{B}un_{\mathbb{G}_m,v}$.

As already mentioned, a major open question raised by Faltings is which Higgs bundles correspond to representations of $\pi_1^{\text{ét}}(X)$ under the $p$-adic Simpson correspondence. Cases in which this is known include line bundles [Heu22b] and abeloid varieties [HMW23]. Very recently, Xu [Xu22] has extended the construction of Deninger–Werner in the case of curves, by constructing an equivalence between representations and "potentially Deninger–Werner Higgs bundles" over $\mathbb{C}_p$. Little is known beyond these cases, and it currently seems difficult to even formulate a conjecture in general. We have shown in [Heua] that for $G = \mathbb{G}_m$, the answer is already quite subtle and best described in terms of moduli spaces. For general $G$,

this suggest studying the locus of pro-finite-étale bundles inside $\mathscr{B}un_{G,v}$, which is started in [HX24, §9]. This is another main motivation to study moduli spaces in this context.

Regarding Theorem 1.5, already the case of $G = \mathbb{G}_a$ is new for smoothoid spaces and has interesting applications: As we will explain in detail in [Heu24b], it yields a relative version of the Hodge–Tate spectral sequence for smooth proper morphisms $f : \mathcal{X} \to \mathcal{S}$ of rigid spaces. Related results have been obtained by Abbes–Gros [AG24], Caraiani–Scholze [CS17, Corollary 2.2.4], He [He24, Theorem 12.2], and Gaisin–Koshikawa [GK22, Theorem 1.5].

In a different direction, and inspired by our article as well as [LZ17], Yu has recently shown that for $G = \mathrm{GL}_n$, our sheafified correspondence of Theorem 1.5 can be lifted to the de Rham period sheaf, with Higgs bundles replaced by vector bundles with $t$-connections [Yu24].

1.8. **Overview.** In §2, we introduce smoothoid spaces and their key technical properties. In §3, we present some preparations on rigid groups $G$ and the notion of $G$-Higgs bundles. We start the proof of Theorems 1.2 and 1.5 in §4, by constructing a canonical and functorial map

$$\Psi : \mathrm{Higgs}_G \to R^1\nu_*G.$$

In §5, we show that $\Psi$ is an isomorphism. For this we can work locally. Indeed, in §6 we show that the results of §5 also show the local $p$-adic Simpson correspondence, Theorem 1.10.

In §7, we discuss applications, in particular v-descent criteria, which culminate in the proof of Theorem 1.4. In §8, we combine all of these results to define the two Hitchin fibrations (5) and (6). We also discuss the relation to the continuous character variety.

## Setup, conventions and notation

Let $K$ be a perfectoid field over $\mathbb{Q}_p$. One example that will appear is the field $\mathbb{Q}_p^{\mathrm{cyc}}$ obtained by adjoining to $\mathbb{Q}_p$ all $p$-power roots of unity and completing $p$-adically. We fix a ring of integral elements $K^+ \subseteq K$, e.g. the ring of integers $K^+ = \mathcal{O}_K$. Let $\mathfrak{m} \subseteq K^+$ be the ideal of topologically nilpotent elements. For any $\alpha \in \mathbb{R}_{\geq 0}$, we write $p^\alpha\mathfrak{m}$ for the subset of $x \in K$ with $|x| < |p|^\alpha$.

Throughout we work with analytic adic spaces over $(K, K^+)$ in the sense of Huber [Hub94]. We take it as part of the definition that adic spaces are sheafy. In the very few cases where we consider non-sheafy affinoids, we use the functor of points of Scholze–Weinstein [SW13, Definition 2.1.5], and we call this a pre-adic space following [KL15, Definition 8.2.3].

By a rigid space we mean an adic space locally of topologically finite type over $(K, K^+)$, in the sense of [Hub94, §3]. Since for a given $K$ the ring $K^+$ won't change throughout, we often suppress $K^+$ from notation, and simply speak of rigid and adic spaces over $K$. Associated to any rigid space $X$ over $K$ we have the pro-étale site $X_{\mathrm{proét}}$ in the sense of [Sch13a], which is now sometimes called the "flattened" pro-étale site (although the difference between different versions of the pro-étale site is not crucial for our purposes). For any adic space over

$K$, we denote by $X_{\text{ét}}$ the étale site of Kedlaya–Liu [KL15, Definition 8.2.16/19](cf [Sch12, Definition 7.1]): In particular, this may in general contain non-sheafy pre-adic spaces, but we usually work in the setting of sousperfectoid spaces [HK][SW20, §6.3] where all objects of $X_{\text{ét}}$ are sheafy. We call a morphism $f : X \to Y$ of adic spaces standard-étale if $X$ and $Y$ are affinoid and $f$ is a finite chain of compositions of finite étale morphisms and rational opens. It is immediate from the definition of étale morphisms that standard-étale morphisms form a basis of $X_{\text{ét}}$.

We use perfectoid spaces in the sense of [Sch12] and write $\text{Perf}_K$ for the category of affinoid perfectoid spaces over $K$, or equivalently of perfectoid $(K, K^+)$-algebras. On this we have the étale and $v$-topology in the sense of [Sch, §7], and we denote the corresponding sites by $\text{Perf}_{K,\text{ét}}$ and $\text{Perf}_{K,v}$. To any adic space $X$ over $K$, Scholze associates a diamond $X^\diamond$ over $K$ [Sch, §11,§15]: For our purposes, as we fix the perfectoid base field $K$, we can identify this with a sheaf on $\text{Perf}_{K,v}$. The functor $-^\diamond$ is fully faithful on all subcategories of adic spaces that we consider, and moreover identifies the étale sites [Sch, Lemma 15.6]. We therefore often drop the $-^\diamond$ from notation and switch back and forth freely between $X$ and $X^\diamond$. In particular, we denote by $X_{\text{qproét}}$ and $X_v$ the quasi-pro-étale and the $v$-site of $X^\diamond$ defined in [Sch, §14]. If $X$ is perfectoid, we also use the site $X_{\text{proét}}$ from [Sch, §8].

## 2. Smoothoid spaces

In this section we introduce the class of smoothoid spaces and prove some basic properties.

### 2.1. Definition of smoothoid spaces and toric charts.
We start by fixing the notion of smooth morphisms we work with, following Huber:

**Definition 2.1** ([Hub96, Corollary 1.6.10]). A morphism of adic spaces $X \to S$ is **smooth** if locally on source and target it admits a factorisation $X \xrightarrow{h} S \times_K \mathbb{B}^d \xrightarrow{\pi_1} S$ where $\mathbb{B}^d$ is the unit ball over $K$ of some dimension $d$, and $h$ is an étale morphism.

It is possible to further extend the definition using the pre-adic étale site, but we shall in this section restrict attention to (sheafy) adic spaces. In particular, it is part of our definition that $S \times_K \mathbb{B}^d$ exists as an adic space. This always holds if $S$ is sousperfectoid.

**Definition 2.2.** We say that an adic space over $\text{Spa}(K, K^+)$ is **smoothoid** if it admits an open cover by subspaces $U$ that admit a smooth morphism of adic spaces $U \to Y$ to a perfectoid space $Y$ over $K$. We call such a morphism a **smooth chart**. A morphism of smoothoid spaces is simply a morphism of adic spaces between smoothoid spaces.

**Definition 2.3.** We denote by $\text{Smd}_{K,\text{ét}}$ the category of smoothoid spaces over $K$ endowed with the étale topology. Note that any adic space étale over a smoothoid is again smoothoid.

For any $d \in \mathbb{N}$, let $\mathbb{T}^d = \text{Spa}(K\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle)$ be the $d$-dimensional affinoid torus over $(K, K^+)$. Let

$$(9) \qquad\qquad \mathbb{T}^d_\infty := \text{Spa}(K\langle T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}\rangle) \to \mathbb{T}^d,$$

a pro-étale affinoid perfectoid cover. If $\mathbb{Q}_p^{\text{cyc}} \subseteq K$, this is Galois with group $\Delta := \mathbb{Z}_p^d(1) = \varprojlim_n \mu_{p^n}^d(K)$, i.e. it is a pro-étale torsor under $\Delta$ regarded as a pro-finite adic group.

**Definition 2.4.**    (1) For a smoothoid space $X$, a **toric chart** is a standard-étale map $f : X \to \mathbb{T}^d \times Y$ where $Y$ is an affinoid perfectoid space. We call $X$ **toric** if it admits a toric chart. As standard-étale maps form a basis of the étale site, any smoothoid space can be covered by toric open subspaces.

(2) Let $f' : X' \to \mathbb{T}^{d'} \times Y'$ be a toric chart for a second smoothoid space $X'$. Then by a morphism of toric charts we mean a commutative diagram of morphisms of adic spaces

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathbb{T}^{d'} \times Y' & \overset{\phi}{\longrightarrow} & \mathbb{T}^d \times Y \end{array}$$

where $\phi$ is a product of a morphism $Y' \to Y$ with a homomorphism $\mathbb{T}^{d'} \to \mathbb{T}^d$ of rigid groups. Then $\phi$ lifts canonically to a morphism of perfectoid spaces $\mathbb{T}^{d'}_\infty \times Y' \to \mathbb{T}^d_\infty \times Y$.

**Lemma 2.5.** *Any toric smoothoid space $X$ fits into a Cartesian diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle g} & & \downarrow \\ Z & \overset{f}{\longrightarrow} & S \end{array}$$

*where $f$ is a smooth morphism of smooth rigid spaces and $Y$ is an affinoid perfectoid space.*

*Proof.* Let $h : X \to Y \times \mathbb{T}^d$ be a toric chart where $Y$ is affinoid perfectoid. Let $(Y_i)_{i \in I}$ be the inverse system of all smooth rigid spaces over $(K, K^+)$ with compatible morphisms from $Y$, then by [Heua, Proposition 3.17], we have $Y \approx \varprojlim Y_i$, and thus $Y \times \mathbb{T}^d \approx \varprojlim Y_i \times \mathbb{T}^d$. As $h$ is standard-étale, hence quasi-compact quasi-separated, this implies by [Sch, Proposition 11.23] that there is some $i \in I$ for which $h$ descends to an étale map $Z \to Y_i \times \mathbb{T}^d$. $\square$

We now collect some technical properties saying that smoothoid spaces are well-behaved as adic spaces. These are all immediate applications of known results about adic spaces:

**Lemma 2.6.** *Let $X$ be a smoothoid adic space over $K$.*

 *(1) $X$ is sousperfectoid. If $X$ is affinoid, then the structure sheaf $\mathcal{O}$ is acyclic on $X_{\text{ét}}$.*
 *(2) Diamondification defines an equivalence $X^\diamondsuit_{\text{ét}} = X_{\text{ét}}$ that identifies the structure sheaves, where we see $X^\diamondsuit$ as a v-sheaf on $\text{Perf}_K$ with structure sheaf induced from that of $\text{Perf}_K$.*
 *(3) For the natural map $\nu : X^\diamondsuit_v \to X_{\text{ét}}$, we have $\nu_* \mathcal{O}^+ = \mathcal{O}^+$.*
 *(4) Diamondification on smoothoid adic spaces defines a fully faithful functor*

$$\text{Smd}_K \to \{\text{diamonds over } \text{Spd}(K)\}$$

 *(5) Any finite locally free $\mathcal{O}$-module on $X_{\text{ét}}$ is already locally trivial in the analytic topology.*

We may therefore freely switch between regarding $X$ as a diamond or as an adic space.

*Proof.* Part 1 follows from [SW20, Prop. 6.3], see also [HK, Remark 10.6], or Lemma 2.16 below for an explicit argument. Acyclicity then holds for any analytic sheafy affinoid adic space by [KL15, Theorem 2.4.23].

For 2, the equivalence of étale sites is [Sch, Lemma 15.6]. The identification of structure sheaves boils down to the statement that $\nu_* \mathcal{O} = \mathcal{O}$: This follows from [HK, Theorem 11.18]. Alternatively, we will see an independent proof in Proposition 2.9 below.

Part 3 also follows from [HK, Theorem 11.18]. Alternatively, it follows from $\nu_* \mathcal{O} = \mathcal{O}$ and the fact that any $f \in \mathcal{O}(X)$ lies in $\mathcal{O}^+(X)$ if and only if for any morphism of adic spaces $\text{Spa}(C, C^+) \to X$ valued in an algebraically closed field $C$, we have $x^* f \in C^+$.

Part 4 is immediate from part 2.

Part 5 follows from [KL15, Theorem 8.2.22.(d)]. $\square$

2.2. **Differentials on smoothoid spaces via Hodge–Tate comparison.** Our next goal is to show that there is an intrinsic notion of "global" Kähler differentials $\Omega^1_X$ on a smoothoid adic space $X$: From the perspective of Hodge cohomology, it makes sense to postulate that perfectoid spaces should have no differentials. This suggests that for a smooth morphism $X \to Y$ over a perfectoid space $Y$, the global differentials of $X$ should be given by the sheaf of relative differentials $\Omega^1_{X|Y}$ defined by Huber [Hub96, (1.6.2)].

The issue with this definition is that it is not immediately obvious that it is independent of the smooth chart of $X$, and hence that it glues. It is therefore better to have a more intrinsic definition of $\Omega^1_X$. For this we build on the following result mentioned in the introduction, which is crucial in Scholze's perspective on the Hodge–Tate comparison isomorphism:

**Proposition 2.7** ([Sch13b, Proposition 3.23][Heu22b, Proposition 2.25]). *Let $X$ be a smooth rigid space over $K$ and consider the natural map $\nu : X_{\text{proét}} \to X_{\text{ét}}$. Then there is for any $n \in \mathbb{N}$ a natural isomorphism*
$$R^n\nu_*\mathcal{O} = \Omega^n_X\{-n\}.$$
*Here $\{-n\}$ denotes a Breuil–Kisin–Fargues twist, i.e. a Tate twist by $\mathbb{Z}_p(-n)$ if $\mathbb{Q}_p^{\text{cyc}} \subseteq K$. The isomorphism identifies the cup product on $R\nu_*\mathcal{O}$ with the wedge product on differentials.*

**Remark 2.8.**    (1) One can always make choices to identify $\Omega^n_X\{-n\} \cong \Omega^n_X$. But it is often more natural to keep it, for example to keep track of Galois actions.
   (2) Since $H^i_v(Y,\mathcal{O}) = 0$ for any affinoid perfectoid space $Y$, one can equivalently formulate Proposition 2.7 in terms of any finer site over $X_{\text{proét}}$, in particular for $X_v \to X_{\text{ét}}$. This is the perspective we use in the following as it can be adapted to smoothoid spaces.

The main goal of this subsection is to prove the following generalisation of Proposition 2.7:

**Proposition 2.9.** *Let $X$ be a smoothoid space. Let $\nu : X_v \to X_{\text{ét}}$ be the natural map. Then*
   *(1) $R^n\nu_*\mathcal{O}$ is a vector bundle on $X_{\text{ét}}$.*
   *(2) Suppose that there is a smooth morphism $f : Z \to S$ of rigid spaces (we do not require $Z$ and $S$ to be smooth over $K$) such that $X$ fits into a Cartesian diagram*

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle g} & & \downarrow \\
Z & \xrightarrow{\ f\ } & S
\end{array}
$$

   *where $Y$ is perfectoid. Then there is a canonical and functorial isomorphism*
$$R^n\nu_*\mathcal{O} = g^*\Omega^n_{Z|S}\{-n\}.$$

Before we give a proof, we discuss some consequences. Firstly, we can use this to define:

**Definition 2.10.** Let $X$ be a smoothoid space. For any $n \in \mathbb{N}$, we set $\widetilde{\Omega}^n_X := R^n\nu_*\mathcal{O}$.

We also set $\widetilde{\Omega}_X := \widetilde{\Omega}^1_X$. It follows from the local description in Proposition 2.9 that $\widetilde{\Omega}^n_X$ enjoys all usual compatibilities of the Kähler differentials. For example, the cup product induces a natural isomorphism $\wedge^n\widetilde{\Omega}_X \xrightarrow{\sim} \widetilde{\Omega}^n_X$ that we can use to define a wedge product

$$(10) \qquad\qquad\qquad\qquad \wedge : \widetilde{\Omega}^i_X \otimes \widetilde{\Omega}^j_X \to \widetilde{\Omega}^{i+j}_X.$$

**Definition 2.11.** Let $X$ be a smoothoid adic space. Then the number $d := \text{rk}_{\mathcal{O}_X}\widetilde{\Omega}_X$ is locally constant on $X$, and we refer to it as the **smooth dimension** $\dim X$ of $X$.

A second advantage of the definition via $R^1\nu_*\mathcal{O}$ is that by the usual cohomological comparison between the big and small étale sites, we immediately obtain:

**Lemma 2.12.** *The functor $\widetilde{\Omega}^n : X \mapsto H^0(X, \widetilde{\Omega}^n_X)$ is a sheaf on the big étale site $\text{Smd}_{K,\text{ét}}$ of smoothoid spaces over $K$. Explicitly, it is given by $R^n\mu_*\mathcal{O}$ for $\mu : \text{Smd}_{K,v} \to \text{Smd}_{K,\text{ét}}$.*

**Lemma 2.13** (cotangent sequence)**.** *Let $f : X \to Y$ be a morphism of smoothoid spaces that is of topologically finite presentation. Then there is a natural right-exact sequence of sheaves on $X_{\text{ét}}$*

$$f^* \widetilde{\Omega}_Y \to \widetilde{\Omega}_X \to \Omega_{X|Y}\{-1\} \to 0$$

*where $\Omega_{X|Y}$ is Huber's sheaf of relative Kähler differentials [Hub96, (1.6.2)]. If $f$ is smooth, this is an exact sequence of vector bundles. If $f$ is étale, the first map is an isomorphism.*

*Proof.* The first map exists by adjunction via Lemma 2.12. To prove the lemma, we can work locally and may thus assume by rigid approximation that $f$ is the base-change of a morphism $f_0 : X_0 \to Y_0$ of smooth rigid spaces along a morphism $Y \to Y_0$. For $f_0$, the exact sequence translates by Proposition 2.7 to the $\{-1\}$-twist of the usual cotangent sequence, so the Lemma holds by [Hub96, Proposition (1.6.3), Corollary (1.6.10)]. By Proposition 2.9 we get the desired sequence via pullback. If $f$ is smooth we can as in the proof of Lemma 2.5 arrange for $f_0$ to be smooth. $\qquad\square$

2.3. **Computations with toric charts.** For the proof of Proposition 2.9, we can work locally and assume that we have a toric chart $f : X \to \mathbb{T}^d \times Y$ for some affinoid perfectoid space $Y$. We fix $f$ for the rest of this subsection.

We start by adapting some technical results from [Sch13a, §4-5] to smoothoid spaces: The chart $f$ induces a perfectoid cover $X_\infty \to X$ by pullback of $X$ along the toric cover (9). Throughout this subsection, we use the following notation for the associated Huber pairs:

$$
\begin{array}{ccccc}
X_\infty \longrightarrow \mathbb{T}^d_\infty \times Y \longrightarrow \mathbb{T}^d_\infty & & (R_\infty, R_\infty^+) \leftarrow (B_\infty, B_\infty^+) \leftarrow (A_\infty, A_\infty^+) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow & \overset{\mathcal{O}(-)}{\rightsquigarrow} & \uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \\
X \longrightarrow \mathbb{T}^d \times Y \longrightarrow \mathbb{T}^d & & (R, R^+) \longleftarrow (B, B^+) \longleftarrow (A, A^+).
\end{array}
$$

If $\mathbb{Q}_p^{\text{cyc}} \subseteq K$, then the vertical maps are $\Delta$-torsors. For $n \in \mathbb{N}$, let $\mathbb{T}^d_n$ be the torus in the variables $T_1^{1/p^n}, \dots, T_d^{1/p^n}$, then $\mathbb{T}^d_\infty \sim \varprojlim_{n \in \mathbb{N}} \mathbb{T}^d_n$. Let $X_n = \mathrm{Spa}(R_n, R_n^+)$ be the pullback of $X \to \mathbb{T}^d$ to $\mathbb{T}^d_n$, then $X_\infty \sim \varprojlim X_n$. As $X_\infty$ is standard-étale over $\mathbb{T}^d_\infty \times Y$, it is perfectoid.

Finally, we fix any real number $1 \geq \alpha \geq \frac{1}{p-1}$. This will be required to kill some torsion. The following is the analogue of [Sch13a, Lemmas 4.5 and 5.5] in our setting.

**Lemma 2.14.** *(1) There is $\beta \in \mathbb{R}_{>0}$ such that the map $R^+ \hat{\otimes}_{A^+} A_\infty^+ \to R_\infty^+$ is injective with $p^\beta$-torsion cokernel.*

*(2) Let $\epsilon > 0$, then after replacing $X$ by the cover $X_n \to X$ for $n \gg 0$, we can take $\beta < \epsilon$.*

*(3) Assume that $\mathbb{Q}_p^{\text{cyc}} \subseteq K$. Then for any $s \in \mathbb{N}$ and $i \geq 0$, the kernel and cokernel of*

$$H^i_{\text{cts}}(\Delta, R^+/p^s) \to H^i_{\text{cts}}(\Delta, R_\infty^+/p^s)$$

*are killed by $p^\gamma$ where $\gamma := 2\beta + \alpha$. The same is true for $H^i_{\text{cts}}(\Delta, R^+) \to H^i_{\text{cts}}(\Delta, R_\infty^+)$.*

*Proof.* We first treat the case $X = \mathbb{T}^d \times Y$. Write $S^+ = \mathcal{O}^+(Y)$ and consider the map $g : S^+ \hat{\otimes}_{K^+} A^+ \to B^+$. Its domain can be described as $S^+\langle T_1^\pm, \dots, T_d^\pm \rangle$ which is a ring of definition of $B = S\langle T_1^\pm, \dots, T_d^\pm \rangle$, so $g$ is an isomorphism after inverting $p$. It thus has bounded $p$-torsion cokernel as $B$ is uniform. On the other hand, after applying $\hat{\otimes}_{A^+} A_\infty^+$ we have maps

$$S^+ \hat{\otimes}_{K^+} A^+ \hat{\otimes}_{A^+} A_\infty^+ \to B^+ \hat{\otimes}_{A^+} A_\infty^+ \to B_\infty^+ \overset{a}{=} S^+ \hat{\otimes}_{K^+} A_\infty^+$$

where the last almost isomorphism comes from [Sch12, Proposition 6.18]. The composition is clearly an isomorphism. Since the map $A^+/p^n \to A_\infty^+/p^n$ is flat for all $n$, the first map is still injective with bounded $p$-torsion cokernel. Again by flatness, the middle term is $p$-torsionfree. It follows that all of the above maps are almost isomorphisms, thus so is $g$.

We now add a standard-étale map $X \to \mathbb{T}^d \times Y$. For this, the argument from [Sch13a, Lemma 4.5] goes through: Let $Z$ be a smoothoid for which we already know the statement, e.g. $\mathbb{T}^d \times Y$, and consider a standard-étale map $h : X \to Z$. Write $Z = \mathrm{Spa}(B, B^+)$ and $X = \mathrm{Spa}(R, R^+)$ with toric covers $X_\infty = \mathrm{Spa}(R_\infty, R_\infty^+)$ and $Z_\infty = \mathrm{Spa}(B_\infty, B_\infty^+)$.

**Claim 2.15.** *The map $R^+ \hat{\otimes}_{B^+} B_\infty^+ \to R_\infty^+$ has bounded $p$-torsion cokernel, and is an isomorphism after inverting $p$.*

*Proof.* If $h$ is a rational localisation defined by some $f_1, \dots, f_r, g \in B$, then a ring of definition $R_0 := B^+\langle \frac{f_1}{g}, \dots, \frac{f_r}{g} \rangle$ of $(R, R^+)$ is given by the $p$-adic completion of the sub-algebra of $B^+[\frac{1}{g}]$ generated by $\frac{f_1}{g}, \dots, \frac{f_r}{g}$. Similarly, $B_\infty^+\langle \frac{f_1}{g}, \dots, \frac{f_r}{g} \rangle$ is a ring of definition of $(R_\infty, R_\infty^+)$. Consider now $T_0 := R_0 \hat{\otimes}_{B^+} B_\infty^+$ and let $T^+$ be the integral closure of the image of $T_0$ in $T := T_0[\frac{1}{p}]$, then by construction $\mathrm{Spa}(T, T^+) = X \times_Z Z_\infty = \mathrm{Spa}(R_\infty, R_\infty^+)$. It follows that the natural map $T_0 \to R_\infty^+$ is an isomorphism after inverting $p$, and that both its image and $R_\infty^+$ are rings of definition of $R_\infty$, hence $T_0 \to R_\infty^+$ has bounded $p$-torsion cokernel. The claim follows by considering the composition $T_0 \to R^+ \hat{\otimes}_{B^+} B_\infty^+ \to R_\infty^+$ since $R_0 \subseteq R^+$ has bounded $p$-torsion cokernel, both being rings of definition of $R$.

If instead $h$ is finite étale, then $B_\infty \to R_\infty = R \otimes_B B_\infty$ is finite étale, and the image of $R^+ \hat{\otimes}_{B^+} B_\infty^+ \to R_\infty$ is a ring of definition of $R_\infty$, so the statement follows by uniformity. $\square$

Combining the first part of the proof and the claim, we see that in the composition

$$R^+ \hat{\otimes}_{A^+} A_\infty^+ \to R^+ \hat{\otimes}_{B^+} B_\infty^+ \to R_\infty^+,$$

both maps have bounded $p$-torsion cokernel and become an isomorphism after inverting $p$. Since the first term is $p$-torsionfree by the same flatness argument as before, it follows that the composition is injective already before inverting $p$. This proves part (1).

Part (2) follows exactly as in [Sch13a, Lemma 4.5]: By the explicit description of the integral subrings of rational localisations [Sch12, Lemma 6.4], respectively finite étale extensions of a perfectoid algebra, there is for any $\epsilon > 0$ a subalgebra $R_{\infty,\epsilon}^+ \subseteq R_\infty^+$ that is topologically finitely generated over $B_\infty^+$ and such that the cokernel $R_\infty^+ / R_{\infty,\epsilon}^+$ is $p^\epsilon$-torsion.

To deduce part (3) from (1), consider the diagram

$$
\begin{array}{ccc}
H^i_{\mathrm{cts}}(\Delta, R^+/p^s) & \longrightarrow & H^i_{\mathrm{cts}}(\Delta, R_\infty^+/p^s) \\
\uparrow & & \uparrow \\
H^i_{\mathrm{cts}}(\Delta, A^+/p^s) \otimes_{A^+} R^+ & \longrightarrow & H^i_{\mathrm{cts}}(\Delta, A_\infty^+ \otimes_{A^+} R^+/p^s).
\end{array}
$$

By part (1), the right vertical map has $p^{2\beta}$-torsion kernel and cokernel. The left vertical map is an isomorphism: $\Delta$ acts trivially on $R^+$ and $A^+$, so $H^i_{\mathrm{cts}}(\Delta, -) = \mathrm{Hom}_{\mathrm{cts}}(\wedge^i_{\mathbb{Z}_p} \Delta, -)$.

The bottom map has $p^\alpha$-torsion kernel and cokernel by [Sch13a, Lemma 5.5], which says that

$$H^i_{\mathrm{cts}}(\Delta, A_\infty^+ \otimes_{A^+} R^+/p^s) = H^i_{\mathrm{cts}}(\Delta, A_\infty^+) \otimes_{A^+} R^+/p^s$$

and that $H^i_{\mathrm{cts}}(\Delta, A^+) \to H^i_{\mathrm{cts}}(\Delta, A_\infty^+)$ is injective with $p^\alpha$-torsion cokernel.

It follows that the top map has $p^{2\beta+\alpha}$-torsion kernel and cokernel. The same works for $H^i_{\mathrm{cts}}(\Delta, R_\infty^+)$ when we instead use the last part of [Sch13a, Lemma 5.5]. $\square$

As an immediate application, this shows more directly that $X$ is sousperfectoid:

**Lemma 2.16.** *The normalised traces $A_\infty^+ \to A_n^+$ induce canonical $R_n$-linear and continuous splittings $\mathrm{tr}_n : R_\infty \to R_n$ such that for any $x \in R_\infty$, we have $\mathrm{tr}_n(x) \to x$ for $n \to \infty$.*

*Proof.* By Lemma 2.14.1, we get such splittings by applying $- \hat{\otimes}_{A^+} R^+$ and inverting $p$. $\square$

*Proof of Proposition 2.9.* For part 1, we may work locally and assume that $X$ is toric. Let us first assume $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$. Then the Cartan–Leray sequence [Heu22b, Proposition 2.18] for the $\Delta$-torsor $X_\infty \to X$ induces isomorphisms $H_v^n(X, \mathcal{O}) = H_{\mathrm{cts}}^n(\Delta, \mathcal{O}(X_\infty))$. By Lemma 2.14.3 and Proposition 2.7,

$$H_{\mathrm{cts}}^n(\Delta, \mathcal{O}(X_\infty)) = H_{\mathrm{cts}}^n(\Delta, \mathcal{O}(X)) = H_{\mathrm{cts}}^n(\Delta, \mathcal{O}(\mathbb{T}^d)) \hat{\otimes}_{\mathcal{O}(\mathbb{T}^d)} \mathcal{O}(X) = H^0(\mathbb{T}^d, \widetilde{\Omega}^n) \hat{\otimes} \mathcal{O}(X).$$

This in particular shows that the cup product induces an isomorphism

$$\bigwedge^n R^1 \nu_* \mathcal{O} \xrightarrow{\sim} R^n \nu_* \mathcal{O}.$$

Next, we prove part 2. For this we use the following Lemma:

**Lemma 2.17** ([Bha17, §3.4]). *Let $X$ be any stably uniform rigid space over $K$. Then there is a canonical and functorial morphism of sheaves on $X_{\mathrm{ét}}$*

$$\mathcal{H}^0(\widehat{L_{\mathcal{O}_{X_{\mathrm{ét}}}^+ | \mathbb{Z}_p}})[\tfrac{1}{p}] \to R^1 \nu_* \mathcal{O}\{1\}.$$

*The first term is $= \Omega_{X|K}$. Under this identification, when $X$ is moreover smooth, then this morphism agrees with the isomorphism from Proposition 2.7.*

*Proof.* When $K$ is algebraically closed, this is proved in [Bha17, §3.4]: The reference assumes that $X$ is a smooth rigid space, but the construction of the morphism goes through mutatis mutandis when we replace the formal model $\mathfrak{X}_{aff}$ by the site $(X_{\mathrm{an}}, \mathcal{O}^+)$.

The case of general $K$ can be seen in the same way: We only need to see that

$$\widehat{L_{K^+ | \mathbb{Z}_p}}[\tfrac{1}{p}] = K\{1\}[1].$$

Let $C$ be the completion of an algebraic closure of $K$, then the transitivity triangle for $\mathbb{Z}_p \to K^+ \to C^+$ yields a canonical isomorphism, compatible with Galois actions

$$\widehat{L_{K^+ | \mathbb{Z}_p}}[\tfrac{1}{p}] \otimes_K C = \widehat{L_{C^+ | \mathbb{Z}_p}}[\tfrac{1}{p}].$$

The statement follows by taking Galois invariants, like in [Heu22b, Proposition 2.25]. $\qquad\square$

Applying Lemma 2.17 to $Z$, we obtain a natural morphism

$$g^* \Omega_{Z|K}^1 \to g^* R^1 \nu_* \mathcal{O}_Z \to R^1 \nu_* \mathcal{O}_X,$$

and similarly for $S$. By functoriality, the following diagram commutes:

$$\begin{array}{ccc} g^* \Omega_{Z|K}^1 & \longrightarrow & R^1 \nu_* \mathcal{O}_X \\ \uparrow & & \uparrow \\ g^* f^* \Omega_{S|K}^1 & \longrightarrow & g^* R^1 \nu_* \mathcal{O}_Y \end{array}$$

But $Y$ is perfectoid, so the bottom right term vanishes. By the cotangent sequence, we thus obtain a natural morphism

$$g^* \Omega_{Z|S}^1 \to R^1 \nu_* \mathcal{O}_X.$$

To see that this is an isomorphism, we can work locally on $Z$ and thus assume that $f$ factors into an étale morphism $Z \to \mathbb{T}^d \times S$ and the projection $\mathbb{T}^d \times S \to S$. Then $X$ is étale over $\mathbb{T}^d \times Y$. The statement is then immediate from comparing Cartan–Leray sequences for the toric towers $\mathbb{T}_\infty^d \times S \to \mathbb{T}^d \times S$ and $\mathbb{T}_\infty^d \times Y \to \mathbb{T}^d \times Y$.

For general $K$, we deduce the results via Galois descent, exactly as in [Heu22b, Proposition 2.25]: Let $(C, C^+)$ be the completion of an algebraic closure of $(K, K^+)$, let $Q = \mathrm{Gal}(C|K)$ be the Galois group and $X_C$ the base-change. Then for any $i, j \geq 0$ we have

$$H_{\mathrm{cts}}^i(Q, H_v^j(X_C, \mathcal{O})) = \begin{cases} H^0(X, \widetilde{\Omega}_X^j) & i = 0 \\ 0 & i > 0, \end{cases}$$

since $H_v^j(X_C, \mathcal{O}) = \widetilde{\Omega}_{X_C}^j(X_C) \cong \mathcal{O}(X_C)^k$ as $Q$-modules for $k = \binom{d}{j}$ by part 1, and the map $H^i_{\mathrm{cts}}(Q, \mathcal{O}(X_C)) \hookrightarrow H^i_{\mathrm{cts}}(Q, \mathcal{O}(X_{C,\infty})) = H_v^i(X_\infty, \mathcal{O}) = 0$ for $i > 0$ is injective due to Lemma 2.16. The Cartan–Leray sequence for $X_C \to X$ now implies $H_v^n(X, \mathcal{O}) = H^0(X, \widetilde{\Omega}_X^n)$ for any $n \in \mathbb{N}$. Part 2 follows as the map $h$ is an isomorphism after base-change to $C$. $\qquad \square$

For a toric smoothoid space $X$, this gives an explicit description of $\widetilde{\Omega}_X$ depending on $f$:

**Lemma 2.18.** *Assume $K$ contains $\mathbb{Q}_p^{\mathrm{cyc}}$. Then the chart $f$ induces an isomorphism*

$$\mathrm{HT}_f : \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}(X)) \to H^1_{\mathrm{cts}}(\Delta, \mathcal{O}(X_\infty)) \to H_v^1(X, \mathcal{O}) \to H^0(X, \widetilde{\Omega}_X)$$

*where the second map is from the Cartan–Leray sequence (Proposition A.4) for $X_\infty \to X$ and the last is from the Leray sequence for the morphism $\nu : X_v \to X_{\mathrm{ét}}$, using Proposition 2.9.*

*Proof.* The first map is an isomorphism by Lemma 2.14.3. The second is an isomorphism as $H_v^1(X_\infty, \mathcal{O}) = 0$. The third is an isomorphism as $H^i_{\mathrm{an}}(X, \mathcal{O}) = 0$ for $i \geq 1$ by Lemma 2.6. $\qquad \square$

**Definition 2.19.** Let $\widetilde{\Omega}_X^+ \subseteq \widetilde{\Omega}_X$ be the finite free $\mathcal{O}^+$-submodule (depending on $f$) given by the image of $\mathrm{HT}_f : \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}^+(X)) \otimes_{\mathcal{O}^+(X)} \mathcal{O}^+ \to \widetilde{\Omega}_X$.

2.4. **Non-cyclotomic base fields.** Rather than deducing the case of general $K$ from that where $K$ contains $\mathbb{Q}_p^{\mathrm{cyc}}$ by Galois descent, one can also give a more direct argument. We record it since we need it later:

Assume that $K$ does not contain all $p$-power roots of unity and consider the cyclotomic extension $K^{\mathrm{cyc}}|K$ obtained by adjoining them. For simplicity, let us assume that $K$ contains at least $\zeta_p$. Then $\mathrm{Spa}(K^{\mathrm{cyc}}) \to \mathrm{Spa}(K)$ is a torsor under the profinite group $Q := \mathrm{Gal}(K^{\mathrm{cyc}}|K) \subset \mathbb{Z}_p^\times$ such that $Q = 1 + p^m\mathbb{Z}_p \cong \mathbb{Z}_p$ for some $m \in \mathbb{N}$.

Consider the base-change $X^{\mathrm{cyc}} \to \mathrm{Spa}(K^{\mathrm{cyc}})$ of $X \to \mathrm{Spa}(K)$ along this extension. Denote by $(R^{\mathrm{cyc}}, R^{\mathrm{cyc}+})$ its global sections. Similarly, let $(R_\infty^{\mathrm{cyc}}, R_\infty^{\mathrm{cyc}+})$ be the global sections of the base-change $X_\infty^{\mathrm{cyc}} \to \mathrm{Spa}(K^{\mathrm{cyc}})$ of $X_\infty \to \mathrm{Spa}(K)$, as in the last section. Then

$$X_\infty^{\mathrm{cyc}} \to \mathrm{Spa}(K^{\mathrm{cyc}}) \to \mathrm{Spa}(K)$$

is again a pro-finite-étale Galois torsor for a group $\Lambda$ which is canonically a split extension

$$0 \to \Delta \to \Lambda \to Q \to 0,$$

where $\Delta$ is the Galois group of $X_\infty^{\mathrm{cyc}} \to \mathrm{Spa}(K^{\mathrm{cyc}})$ as in the last section. Namely, $\Lambda = \Delta \rtimes Q$ is the semi-direct product given by the action of $Q \subseteq \mathbb{Z}_p^\times$ by multiplication on $\Delta$. Writing elements of $\Lambda$ as tuples $(\gamma, \sigma)$, these act on $K^{\mathrm{cyc}}\langle T^{\pm 1/p^\infty}\rangle$ via $(\gamma, \sigma) \cdot T^m = \sigma(a_m)\zeta^{m\gamma} T^m$.

We can now describe $\mathrm{R\Gamma}_{\mathrm{cts}}(\Lambda, R^{\mathrm{cyc}+})$, as in Lemma 2.14. While calculations like these are standard in $p$-adic Hodge theory, there is a major difference to the classical setting as in [Tat67, §3]: For $0 \neq i \in \mathbb{Z}$ we have $\mathbb{Q}_p^{\mathrm{cyc}}(i)^{\mathbb{Z}_p^\times} = 0$, but for perfectoid $K$ we instead have:

**Lemma 2.20.** *For $i \in \mathbb{Z}$, we have $K^{+\mathrm{cyc}}(i)^Q \stackrel{a}{=} K^+$ and $H^j_{\mathrm{cts}}(Q, K^{+\mathrm{cyc}}(i)) \stackrel{a}{=} 0$ for $j > 0$.*

*Proof.* $K^{+\mathrm{cyc}}(i)$ with its $Q$-action defines an invertible $\mathcal{O}^+$-module $L_i$ on $X := \mathrm{Spa}(K, K^+)$ by descent from $\mathrm{Spa}(K^{\mathrm{cyc}})$, and $H^j_{\mathrm{cts}}(Q, K^{+\mathrm{cyc}}(i))$ computes $H_v^j(X, L_i)$ by the Cartan–Leray sequence. But any such $L_i$ is trivial by [Heu22a, Corollary 2.27], so $H_v^j(X, L_i) \stackrel{a}{=} 0$. $\qquad \square$

**Lemma 2.21.** (1) *There is $\gamma > 0$ such that for any $s, i \geq 0$, the kernel and cokernel of*

$$H^i_{\mathrm{cts}}(\Lambda, R^{\mathrm{cyc}+}/p^s) \to H^i_{\mathrm{cts}}(\Lambda, R_\infty^{\mathrm{cyc}+}/p^s)$$

*are annihilated by $p^\gamma$. The same is true for the map $H^i_{\mathrm{cts}}(\Lambda, R^{+\mathrm{cyc}}) \to H^i_{\mathrm{cts}}(\Lambda, R_\infty^{+\mathrm{cyc}})$.*
(2) *The restriction $H^1_{\mathrm{cts}}(\Lambda, R^{\mathrm{cyc}+}/p^s) \to H^1_{\mathrm{cts}}(\Delta, R^{\mathrm{cyc}+}/p^s)^Q$ is split surjective: a splitting $r$ is given by sending $\rho : \Delta \to R^{\mathrm{cyc}+}/p^s$ to the 1-cocycle $r(\rho) : (\gamma, \sigma) \mapsto \rho(\gamma)$. In the limit over $s$, this also defines a splitting of $H^1_{\mathrm{cts}}(\Lambda, R^{+\mathrm{cyc}}) \to H^1_{\mathrm{cts}}(\Delta, R^{+\mathrm{cyc}})^Q$.*

(3) As an $R^+$-module, $H^1_{\mathrm{cts}}(\Delta, R^{+\mathrm{cyc}})^Q$ is almost finite free of rank $d$.

*Proof.* Let $A^{\mathrm{cyc}} = A \hat{\otimes}_K K^{\mathrm{cyc}}$. The main calculation is that for any $i, j \geq 0$ we have

$$H^j_{\mathrm{cts}}(Q, H^i_{\mathrm{cts}}(\Delta, A^{\mathrm{cyc}+}/p^s)) \overset{a}{\cong} \begin{cases} A^+/p^s & j = 0, \\ 0 & j > 0. \end{cases}$$

Indeed, writing $A^{\mathrm{cyc}+}/p^s = \oplus_{i \in \mathbb{Z}} K^{\mathrm{cyc}+}/p^s \cdot T^i$, this follows directly from Lemma 2.20. Let now $c > 0$ be such that kernel and cokernel of the map $A^{\mathrm{cyc}+} \otimes_{A^+} R^+/p^s \to R^{+\mathrm{cyc}}/p^s$ are killed by $p^c$ for all $s$. Then $H^j_{\mathrm{cts}}(Q, H^i_{\mathrm{cts}}(\Delta, R^{\mathrm{cyc}+}/p^s))$ is $p^{2c}$-torsion for $j = 1$ and any $i$, and vanishes for $j > 1$ since $Q \cong \mathbb{Z}_p$. By inflation-restriction, it follows that for any $n \geq 1$, the map

$$H^n_{\mathrm{cts}}(\Lambda, R^{\mathrm{cyc}+}/p^s) \to H^n(\Delta, R^{\mathrm{cyc}+}/p^s)^Q$$

is surjective with $p^{2c}$-torsion kernel. The exact same argument also works for $R^{\mathrm{cyc}+}_\infty/p^s$, so we can assume that there is $c > 0$ independent of $s$ such that in the commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{cts}}(\Lambda, R^{+\mathrm{cyc}}/p^s) & \longrightarrow & H^i_{\mathrm{cts}}(\Delta, R^{+\mathrm{cyc}}/p^s)^Q \\ \downarrow & & \downarrow \\ H^i_{\mathrm{cts}}(\Lambda, R^{+\mathrm{cyc}}_\infty/p^s) & \longrightarrow & H^i_{\mathrm{cts}}(\Delta, R^{+\mathrm{cyc}}_\infty/p^s)^Q, \end{array}$$

the top, bottom and right map have $p^c$-torsion kernel and cokernel, thus also the left map.

To see part (3), we use that $\mathrm{Hom}_{\mathrm{cts}}(\Delta, R^{\mathrm{cyc}+}) \cong R^{\mathrm{cyc}+}(-1)^d$ as $R^+$-modules with $Q$-action. We can regard $R^{\mathrm{cyc}+}(-1)$ with its $Q$-action as a descent datum for an invertible $\mathcal{O}^+$-module $L$ along the cover $X^{\mathrm{cyc}} \to X$ which has Galois group $Q$. In fact, via the natural map $R^+ \hat{\otimes}_{K^+} K^{\mathrm{cyc}+}(-1) \to R^{\mathrm{cyc}+}(-1)$, this comes via pullback from the invertible $\mathcal{O}^+$-module $L_0$ on $\mathrm{Spa}(K)_v$ associated to the descent datum $K^{\mathrm{cyc}+}(-1)$ for $\mathrm{Spa}(K^{\mathrm{cyc}}) \to \mathrm{Spa}(K)$. But $L_0$ is free by [Heu22a, Corollary 2.27], hence the global sections $R^{\mathrm{cyc}+}(-1)^Q$ of $L$ are a free $R^+$-module.

It remains to see part (2): As the elements of $H^1_{\mathrm{cts}}(\Delta, R^{+\mathrm{cyc}}/p^s)^Q = \mathrm{Hom}_{\mathrm{cts}}(\Delta, R^{+\mathrm{cyc}}/p^s)^Q$ are precisely the $Q$-equivariant homomorphisms, this is the elementary observation: $\square$

**Lemma 2.22.** *If $G$ is a topological group with continuous $\Lambda$-action such that $\Delta$ acts trivially, then* res: $H^1_{\mathrm{cts}}(\Lambda, G) \to H^1_{\mathrm{cts}}(\Delta, G)^Q$ *has a natural section sending $\rho$ to $\sec(\rho)\colon (\gamma, \sigma) \mapsto \rho(\gamma)$.*

### 2.5. Higgs bundles on smoothoid spaces.
Having introduced differentials on smoothoid spaces, we obtain a notion of Higgs bundles:

**Definition 2.23.** Let $X$ be a smoothoid space over $K$. A **Higgs bundle** on $X$ is a pair $(E, \theta)$ consisting of a vector bundle $E$ on $X_{\text{ét}}$ and $\theta \in H^0(X, \mathrm{End}(E) \otimes \widetilde{\Omega}^1_X)$ such that $\theta \wedge \theta = 0$ in $\mathrm{End}(E) \otimes \widetilde{\Omega}^2_X$, where $\wedge$ was defined in (10). Explicitly, this means that for any local basis of $\widetilde{\Omega}^1_X$, the coefficients of $\theta$ commute. Any such $\theta$ is called a **Higgs field** and can be written as a map $\theta : E \to E \otimes \widetilde{\Omega}^1$. A morphism of Higgs bundles is a morphism of the underlying bundles that commutes with the Higgs fields written in this form.

**Definition 2.24.** Let $X$ be a smoothoid space and let $(E, \theta)$ be a Higgs bundle on $X$. Again using the wedge product from (10), we can define the associated **Higgs complex**

$$\mathcal{C}^*_{\mathrm{Higgs}}(E, \theta) := \left[ E \xrightarrow{\theta} E \otimes \widetilde{\Omega}^1 \xrightarrow{\theta_1} E \otimes \widetilde{\Omega}^2 \xrightarrow{\theta_2} \ldots \xrightarrow{\theta_{n-1}} E \otimes \widetilde{\Omega}^n \right]$$

where the transition maps are defined as $\theta_k : E \otimes \widetilde{\Omega}^k \xrightarrow{\theta \otimes \mathrm{id}} E \otimes \widetilde{\Omega}^1 \otimes \widetilde{\Omega}^k \xrightarrow{\mathrm{id} \otimes \wedge} E \otimes \widetilde{\Omega}^{k+1}$. Passing to the image of $\mathcal{C}^*_{\mathrm{Higgs}}(E, \theta)$ in the derived category $\mathcal{D}^b(X_{\text{ét}})$, we then set

$$R\Gamma_{\mathrm{Higgs}}(X, (E, \theta)) := R\Gamma_{\text{ét}}(X, \mathcal{C}^*_{\mathrm{Higgs}}(E, \theta)).$$

We call $H^k(X, (E, \theta)) := H^k(R\Gamma_{\mathrm{Higgs}}(X, (E, \theta)))$ the **Dolbeault cohomology** of $(E, \theta)$.

## 3. Rigid groups as $v$-sheaves

In this short section, we briefly collect some background on rigid analytic group varieties:

**Definition 3.1.** A **rigid group** is a group object in the category of rigid spaces over $K$.

Since $K$ has characteristic 0, a rigid group is automatically smooth [Far19, Proposition 1].

**Example 3.2.**   (1) Any algebraic group over $K$ defines a rigid group via analytification.
   (2) If $\mathcal{G}$ is a formal group scheme of topologically finite type over $K^+$, its adic generic fibre is a rigid group. A rigid group over $K$ is said to have **good reduction** if it is isomorphic to the adic generic fibre of a smooth formal group scheme $\mathcal{G}$ over $K^+$.

As always, we identify $G$ with the associated diamond, an abelian $v$-sheaf on $\mathrm{Perf}_K$.

3.1. **The Lie algebra and its exponential.** We now recall some basic constructions on rigid groups, and refer to [Heu22a, §3] for more details.

Rigid groups are the non-archimedean analogues of complex Lie groups. One way in which this analogy manifests itself is by the $p$-adic Lie algebra Lie group correspondence:

Let $G$ be any rigid group, then its tangent space at the identity inherits the structure of a Lie algebra $\mathfrak{g}$ over $K$ of dimension $\dim_K \mathfrak{g} = \dim G$. We consider this as a $v$-sheaf via the associated vector group over $K$, i.e. we set $\mathfrak{g}(Y) = \mathfrak{g} \otimes_K \mathcal{O}(Y)$ on $\mathrm{Perf}_{K,v}$. Then there is a natural adjoint action $\mathrm{ad} : G \to \mathrm{End}(\mathfrak{g})$. Sending a rigid group to its associated Lie algebra defines an equivalence of categories after localising at the class of open subgroups. Moreover, as for complex Lie groups, there is an exponential relating $G$ and $\mathfrak{g}$:

**Proposition 3.3** ([Heu22a, Proposition 3.5]). *There is an open subgroup $\mathfrak{g}^\circ \subseteq \mathfrak{g}$ whose underlying rigid space is isomorphic to $\mathbb{B}^d$, and a rigid open subgroup $G^\circ \subseteq G$ with an isomorphism of rigid spaces*

$$\exp : \mathfrak{g}^\circ \xrightarrow{\sim} G^\circ$$

*that sends open subgroups onto open subgroups, and is functorial in $G$. We denote its inverse by $\log$.*

**Corollary 3.4.** *Any rigid group $G$ has a neighbourhood basis $(U_k)_{k\in\mathbb{N}}$ of $1$ of open subgroups $U_k \subseteq G$ of good reduction whose underlying rigid space is isomorphic to $\mathbb{B}^d$.*

**Lemma 3.5** ([Heu22a, Lemma 3.10]). *Let $Y$ be any adic space over $K$.*
   *(1) If $A, B \in \mathfrak{g}^\circ(Y)$ satisfy $[A, B] = 0$, then $\exp(A)$ and $\exp(B)$ commute and*

$$\exp(A + B) = \exp(A)\exp(B).$$

   *(2) If $g, h \in G^\circ(Y)$ commute, then $[\log(g), \log(h)] = 0$ and $\log(gh) = \log(g) + \log(h)$.*
   *(3) If $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \mathfrak{g}^\circ$ with images $G_1 \subseteq G_2 \subseteq G$ under $\exp$ and $g \in G_1(Y)$ are such that $\mathrm{ad}(g)(\mathfrak{g}_1) \subseteq \mathfrak{g}_2$, then $g^{-1}G_1 g \subseteq G_2$. For $A \in \mathfrak{g}_1(Y)$, we then have*

$$\exp(\mathrm{ad}(g)(A)) = g^{-1}\exp(A)g.$$

**Proposition 3.6** ([Heu22a, §4.2]). *If $G$ has good reduction, then the Lie algebra of the formal model induces a finite free $\mathcal{O}^+$-submodule $\mathfrak{g}^+ \subseteq \mathfrak{g}$. Let $\mathfrak{g}_k^+ := p^k \cdot \mathfrak{m} \cdot \mathfrak{g}^+$, then there is $\alpha > 0$ such that $\exp$ is defined on $\mathfrak{g}_k^+$ for any $k > \alpha$, and $G_k := \exp(\mathfrak{g}_k^+)$ is an open subgroup of $G$ such that for any $\alpha < r < s \in \mathbb{Q}$ with $s \leq 2r - \alpha_0 \in \mathbb{Q}$, the exponential induces isomorphisms*

$$\exp : \mathfrak{g}_r^+/\mathfrak{g}_s^+ \xrightarrow{\sim} G_r/G_s, \quad \exp : \mathfrak{g}_r^+(X)/\mathfrak{g}_s^+(X) \xrightarrow{\sim} G_r(X)/G_s(X)$$

*of abelian sheaves on $\mathrm{Smd}_{K,\text{ét}}$, respectively of abelian groups for any $X \in \mathrm{Smd}_K$.*

The $G_k$ can be described as the kernel of the reduction mod $p^k\mathfrak{m}$. In particular:

**Lemma 3.7** ([Heu22a, Lemma 4.18]). *If $G$ has good reduction, then we have $G = \varprojlim_{k\in\mathbb{N}} G/G_k$.*

3.2. **Torsors under rigid groups.** We now recall the definition of $G$-torsors on diamonds, which we studied in [Heu22a, §3].

**Definition 3.8.** Let $X$ be any diamond and let $\tau \in \{\text{ét}, v\}$. Then a $G$**-torsor** on $X_\tau$ is a sheaf $F$ on $X_\tau$ with a left action $m : G \times F \to F$ by $G$ such that there is a $\tau$-cover of $X' \to X$ where there is a $G$-equivariant isomorphism $G \times_X X' \xrightarrow{\sim} F \times_X X'$. A morphism of $G$-bundles on $X_\tau$ is a $G$-equivariant morphism of sheaves on $X_\tau$.

It is clear that the set of isomorphism classes of $G$-torsors on $X_\tau$ is given by $H^1_\tau(X, G)$.

**Remark 3.9.** Any morphism of $G$-torsors is an isomorphism. So the category of $\mathrm{GL}_n$-torsors has the same objects as the category of vector bundles of rank $n$, but fewer morphisms.

**Lemma 3.10** ([Heu22a, Proposition 3.16]). *Let $X$ be a smoothoid adic space over $K$. Then there is a natural fully faithful functor*

$$\{G\text{-torsors on } X_{\text{ét}}\} \hookrightarrow \{G\text{-torsors on } X_v\}$$

*which on isomorphism classes induces the natural map $H^1_{\text{ét}}(X, G) \to H^1_v(X, G)$.*

Finally, we recall the main technical results from [Heu22a] about $G$-torsors on adic spaces:

**Lemma 3.11** ([Heu22a, Lemma 4.27]). *If $X$ is an affinoid perfectoid space and $G$ is a rigid group of good reduction, then for any $k > \alpha$, we have $H^1_v(X, G_k) = 1$.*

**Proposition 3.12** ([Heu22a, Proposition 4.8]). *Let $G$ be a rigid group and let $U \subseteq G$ be a rigid open subgroup. Let $X$ be any sousperfectoid space and let $\nu : X_v \to X_{\text{ét}}$ be the natural map. Then the morphism*

$$R^1\nu_* U \to R^1\nu_* G$$

*is surjective. If $G$ is commutative, then $R^k\nu_* U = R^k\nu_* G$ for all $k \geq 1$.*

3.3. $G$**-Higgs bundles.** Like in complex geometry, one can generalise the notion of Higgs bundles on smoothoid spaces from vector bundles to $G$-bundles for any rigid analytic group $G$ over $K$.

**Definition 3.13.** Let $X$ be a smoothoid space. For any $G$-bundle $E$ on $X_{\text{ét}}$, one defines the **adjoint bundle** of $E$ to be $\mathrm{ad}(E) := \mathfrak{g} \times^G E$, the associated bundle with respect to the adjoint action $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$. This has the natural structure of a vector bundle on $X_{\text{ét}}$.

**Example 3.14.** If $G = \mathrm{GL}_n$, then $\mathrm{ad}(E) = \mathrm{End}(E)$ is the endomorphism bundle.
If $E = G$ is the trivial bundle, then $\mathrm{ad}(E) = \mathfrak{g}$. This holds for any $E$ if $G$ is commutative.

Since the adjoint action of $G$ commutes with the Lie bracket on $\mathfrak{g}$, we obtain by functoriality a Lie bracket on $\mathrm{ad}(E)$. We can use this to define a natural map

$$\begin{aligned} \wedge : \mathrm{ad}(E) \otimes \widetilde{\Omega}^1_X &\to \mathrm{ad}(E) \otimes \widetilde{\Omega}^2_X, \\ \theta = \sum_{i=1}^n A_i \otimes \delta_i &\mapsto \theta \wedge \theta := \sum_{i<j} [A_i, A_j] \otimes \delta_i \wedge \delta_j. \end{aligned}$$

**Definition 3.15.** Let $X$ be a smoothoid space over $K$.
  (1) A $G$**-Higgs bundle** on $X$ is a pair $(E, \theta)$ of a $G$-bundle $E$ on $X_{\text{ét}}$ and an element $\theta \in H^0(X, \mathrm{ad}(E) \otimes \widetilde{\Omega}_X)$ such that $\theta \wedge \theta = 0$. Such $\theta$ are called **Higgs fields**.
  (2) We denote by $\mathrm{Higgs}_G$ the sheafification of the presheaf of pointed sets on $X_{\text{ét}}$ of isomorphism classes of $G$-Higgs bundles. Using Lemma 2.12, there is for any morphism of smoothoids $f' : X' \to X$ a natural pullback map $\mathrm{Higgs}_G(X) \to \mathrm{Higgs}_G(X')$. We can therefore also regard $\mathrm{Higgs}_G$ as a sheaf on the big étale site $\mathrm{Smd}_{K,\text{ét}}$.

**Remark 3.16.** If $G$ is commutative, then there is no interrelation between $E$ and $\theta$, and the Higgs field condition is vacuous. Therefore $\theta$ is in this case simply any section of $\mathfrak{g} \otimes \widetilde{\Omega}$.

**Lemma 3.17.** *There is a natural isomorphisms of sheaves of pointed sets*

$$\mathrm{Higgs}_G = (\mathfrak{g} \otimes \widetilde{\Omega}_X)^{\wedge=0}/G$$

*given by interpreting the right hand side as a Higgs field on the trivial bundle. Here on the right we form the sheaf quotient by the adjoint action of $G$.*

*Proof.* The map from right to left is injective for presheaves, thus also after the sheafification. It is surjective since for any $G$-Higgs bundle $(E, \theta)$, the $G$-bundle $E$ is trivial étale-locally.   $\square$

**Example 3.18.** Explicitly, for $G = \mathrm{GL}_n$, we have $\mathrm{Higgs}_G = (M_n(K) \otimes_K \widetilde{\Omega}_X)^{\wedge=0}/\mathrm{GL}_n$.

The notion of Higgs bundles is functorial in $G$, namely for any homomorphism $\varphi : G \to G'$ of rigid groups there is a functor from $G$-Higgs bundles to $G'$-Higgs bundles defined by sending $(E, \theta)$ to $(G' \times^G E, \mathfrak{g}' \times^G \theta)$. This defines a morphism of sheaves $\mathrm{Higgs}_G \to \mathrm{Higgs}_{G'}$.

**Example 3.19.** If $G \subseteq G'$ is an open subgroup, then this morphism is clearly surjective. But it might not be injective: For $X = \mathrm{Spa}(K\langle T\rangle)$, the Higgs fields $A_1 dT$ and $A_2 dT$ on $E = \mathcal{O}^2$ for $A_1 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_2 := \left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right)$ are conjugated over $G' = \mathrm{GL}_2(\mathcal{O})$ via $\left(\begin{smallmatrix} p & p \\ 0 & 1 \end{smallmatrix}\right)$, but not over $G = \mathrm{GL}_2(\mathcal{O}^+)$. Hence they are different elements in the same fibre of $\mathrm{Higgs}_G \to \mathrm{Higgs}_{G'}$.

Nevertheless, if we just consider the kernel, i.e. the fibre over 0, we do have the following:

**Lemma 3.20.**     (1) If $f : X' \to X$ is a morphism of smoothoid spaces such that the map $f^* : \widetilde{\Omega}_X \to f_*\widetilde{\Omega}_{X'}$ on $X_{\text{ét}}$ is injective, then $\mathrm{Higgs}_G(X) \to \mathrm{Higgs}_G(X')$ has trivial kernel.
(2) If $\varphi : G \to G'$ is a homomorphism of rigid groups over $K$ such that $\mathfrak{g} \to \mathfrak{g}'$ is injective (e.g. if $\varphi$ is injective), then $\mathrm{Higgs}_G(X) \to \mathrm{Higgs}_{G'}(X)$ has trivial kernel.

**Example 3.21.**     (1) If $f : X' \to X$ is an étale morphism with Zariski-dense image, then $\widetilde{\Omega}_{X'} = f^*\widetilde{\Omega}_X$ and $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective, hence the conditions of Lemma 3.20.(1) hold.
(2) Let $X$ be a smooth rigid space and let $g : Y' \to Y$ be a $v$-cover of perfectoid spaces. Set $f = (\mathrm{id}, g) : X \times Y' \to X \times Y$. Then again $\widetilde{\Omega}_{X'} = f^*\widetilde{\Omega}_X$, and $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective by the $v$-sheaf property. Thus the condition of Lemma 3.20.(1) hold.

*Proof.* We first observe that the surjective morphism of sheaves $(\mathfrak{g} \otimes \widetilde{\Omega}_X)^{\wedge=0} \to \mathrm{Higgs}_G$ has trivial kernel since $x \in \mathfrak{g}$ is conjugated to 0 via the adjoint action if and only if $x = 0$.

Let now $x \in \mathrm{Higgs}_G(X)$ be in the kernel. After passing to an étale cover $\widetilde{X} \to X$ with pullback $\widetilde{X}' \to X'$, we can assume that $x$ lifts to $\widetilde{x} \in (\mathfrak{g} \otimes \widetilde{\Omega}_X)(\widetilde{X})$. Chasing the diagram

$$
\begin{array}{ccc}
(\mathfrak{g} \otimes \widetilde{\Omega}_X)^{\wedge=0}(\widetilde{X}) & \longrightarrow & \mathrm{Higgs}_G(\widetilde{X}) \\
\downarrow & & \downarrow \\
(\mathfrak{g} \otimes \widetilde{\Omega}_{X'})^{\wedge=0}(\widetilde{X}') & \longrightarrow & \mathrm{Higgs}_G(\widetilde{X}')
\end{array}
$$

in which the map on the left is injective by assumption, we see that $\widetilde{x} = 0$, whence $x = 0$.

Part 2 can be seen similarly, using that $(\mathfrak{g} \otimes \widetilde{\Omega}_X)^{\wedge=0} \to (\mathfrak{g}' \otimes \widetilde{\Omega}_X)^{\wedge=0}$ is injective.   $\square$

## 4. From $G$-Higgs bundles to $v$-topological $G$-bundles

With the preparations of the last section, we can now state our main result, the sheafified correspondence between $v$-$G$-bundles and $G$-Higgs bundles:

**Theorem 4.1.** *Let $K$ be a perfectoid field over $\mathbb{Q}_p$. Let $X$ be a smoothoid space over $K$ (for example a smooth rigid space) and let $\nu : X_v \to X_{\text{ét}}$ be the natural morphism of sites. Let $G$*

*be a rigid group over $K$, regarded as a sheaf of groups $G = G(\mathcal{O})$ on $X_v$. Let $\mathfrak{g}$ be the Lie algebra. Then there is a canonical isomorphism of sheaves of pointed sets on $X_{\text{ét}}$*

$$\text{HTlog} : R^1\nu_*G \xrightarrow{\sim} \text{Higgs}_G$$

*which is functorial in $G$, $X$ and $K$. Here $\text{Higgs}_G = (\mathfrak{g} \otimes \widetilde{\Omega}_X)^{\wedge=0}/G$ (see Definition 3.15).*
  *If $G$ is commutative, we more generally have for any $n \geq 1$ an isomorphism*

$$\text{HTlog} : R^n\nu_*G \xrightarrow{\sim} \mathfrak{g} \otimes \widetilde{\Omega}_X^n.$$

**Remark 4.2.**   (1) The notions of $G$-torsors agrees on $X_v$ and $X_{\text{qproét}}$, as well as on $X_{\text{proét}}$ if $X$ is smooth or perfectoid, by [Heu22a, Corollary 4.29]. It follows that in the hierarchy

$$X_v \to X_{\text{qproét}} \to X_{\text{proét}} \to X_{\text{ét}},$$

replacing $X_v$ with $X_{\text{qproét}}$ (or $X_{\text{proét}}$) gives an equivalent formulation of Theorem 4.1.
(2) If $G = \text{GL}_n$, then by Lemma 2.6.5 also the $G$-torsors on $X_{\text{ét}}$ and $X_{\text{an}}$ are equivalent, so we could replace $\nu$ by the projection to $X_{\text{an}}$. But this is not true for general $G$.
(3) As mentioned in the introduction, the case of $G = \mathbb{G}_a$ and smooth rigid $X$ recovers Scholze's result that $R^n\nu_*\mathcal{O} = \widetilde{\Omega}_X^n$ for any $n \geq 1$. In [Heu21], we have studied the case of $G = \mathbb{G}_m$. For all other $G$, the result is new already for smooth rigid $X$.
(4) If $X$ is perfectoid, then $\widetilde{\Omega}_X^n = 0$ for all $n \geq 1$, hence $\text{Higgs}_G = 0$. So this case recovers [Heu22a, Theorem 1.1], which said that $R^1\nu_*G = 0$ in this case. That said, we note that the cited result is used in the proof of Theorem 4.1.
(5) At least for commutative $G$, we can informally remember Theorem 4.1 as saying that one can compute $R^1\nu_*G(\mathcal{O})$ by a "chain rule" applied to the "composition" $G(\mathcal{O})$, with $\mathfrak{g}$ interpreted as the "derivative" of $G$, and $\widetilde{\Omega}_X^1$ as the "derivative" of $\mathcal{O}$.
(6) The functoriality in $X$ and $K$ means in other words that the morphisms HTlog for varying $X$ can be assembled to an isomorphism of sheaves on the big site $\text{Smd}_{K,\text{ét}}$.
(7) That $K$ is perfectoid is necessary: For example, if $K$ is instead discretely valued, then already $\text{Spa}(K)$ has many non-trivial $v$-vector bundles, as described by Sen theory.
(8) Already for algebraic $G$ like $G = \text{GL}_n$ the proof goes by considering analytic open subgroups of $G$, so the perspective of rigid groups is natural already in this case.
(9) Rigid groups are the $p$-adic analogues of complex Lie groups. However, in the context of the complex Simpson correspondence (see Section 8.8), one usually only works with reductive Lie groups, and we are not aware of any version without this assumption.

The proof of Theorem 4.1 will take us two sections. Let us begin with an overview of the strategy: We first treat the case of commutative $G$ in §4.1, which is done by defining an exponential map $\text{Higgs}_G \to R^1\nu_*G$. The main reason why this case is easier is that for commutative $G$, the exponential is a homomorphism, hence it induces a map on cohomology.

The basic idea for generalising this to the non-commutative case is to approximate both $H^0(X, \widetilde{\Omega})$ and $H^1_v(X, G)$ by continuous cocycles associated to perfectoid Galois covers, and to define a map

$$\Psi_G : \text{Higgs}_G \to R^1\nu_*G$$

by exponentiating these cocycles. Carrying this out is the subject of §4. To this end, in §4.2, we need to make certain choices which essentially amount to choices of representative cocycles for $H^0(X, \widetilde{\Omega})$. These allow us to first define a local and integral version $\Psi^+_{\text{grp}}$ of $\Psi$ in §4.3: This is a map of sets defined on a certain integral submodule of $H^0(X, \widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0}$ where we have convergence of exponentials. In §4.4, we explain how to extend this map to a map on the full module

$$\Psi_{X,G} : H^0(X, \widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0} \to R^1\nu_*G(X)$$

by a Galois descent argument. Finally, in §4.5, we explain that this map is independent of all choices, functorial in $X$ and $G$-equivariant. All in all, this allows us to assemble the $\Psi_{X,G}$ for varying $X$ into the desired morphism $\Psi_G$, which is canonical and functorial in $G$.

The goal of §5 is to show that $\Psi_G$ is an isomorphism. For this we can work locally, and as we will explain, one can therefore restrict attention to certain "abelian cocycles" coming from toric charts. In §5.3 we will use this to prove that $\Psi_G$ is surjective, by generalising a lifting argument of Faltings from the local $p$-adic Simpson correspondence [Fal05, §2] from the case of $\mathrm{GL}_n$ to general rigid $G$. When $G$ has good reduction, a similar argument also proves that $\Psi_G$ is injective. The subject of §5.5 is to treat injectivity of $\Psi_G$ for general rigid $G$. For this we need to give a different argument than in [Fal05]: Our approach is to reduce to the case of good reduction by a method which ultimately relies again on a decompletion argument.

Having explained the overall strategy, we now prepare the construction. Let $\mathfrak{g}^\circ \subseteq \mathfrak{g}$ be an open subgroup of the Lie algebra isomorphic as a rigid group to $\mathcal{O}^{+d}$ that admits an exponential map of rigid spaces $\exp : \mathfrak{g}^\circ \to G$. It will in the following be irrelevant how large this subgroup $\mathfrak{g}^\circ$ is, as long as it is open and thus satisfies $\mathfrak{g} = \cup_{k\in\mathbb{N}} p^{-k}\mathfrak{g}^\circ$. It will be convenient to take $\mathfrak{g}^\circ$ small enough so that $\exp$ still converges on $p^{-1}\mathfrak{g}^\circ$. By Proposition 3.3, the image of $\mathfrak{g}^\circ$ under $\exp$ defines a rigid open subgroup

$$G^\circ \subseteq G.$$

For any affinoid adic space $X$ over $K$, the set $G^\circ(X)$ inherits the structure of a topological group in a canonical way, with underlying topological space homeomorphic to $\mathcal{O}^{+d}(X)$ via $\exp$.

4.1. **The morphism $\Psi$ in the commutative case.** By way of motivation, we first consider the much simpler case that $G$ is a commutative rigid group: Apart from $\mathbb{G}_a$ and $\mathbb{G}_m$ and their open subgroups, examples for such $G$ include abelian/abeloid varieties and analytic $p$-divisible groups in the sense of Fargues [Far19].

If $G$ is commutative, then $\exp$ is a homomorphism and restricts to an isomorphism of rigid groups $\exp : \mathfrak{g}^\circ \to G^\circ$. For any $m \geq 0$ we derive from this an isomorphism

$$\exp : R^m\nu_*\mathfrak{g}^\circ \xrightarrow{\sim} R^m\nu_*G^\circ.$$

*Proof of Theorem 4.1 for commutative $G$.* We begin by observing that by Proposition 3.12 applied to $\mathcal{O}^+ \subseteq \mathcal{O}$, we have $R^m\nu_*\mathcal{O}^+ = R^m\nu_*\mathcal{O} = \widetilde{\Omega}^m$. The projection formula then shows

$$R^m\nu_*\mathfrak{g}^\circ = \mathfrak{g}^\circ \otimes R^m\nu_*\mathcal{O}^+ = \mathfrak{g} \otimes \widetilde{\Omega}^m.$$

Composing with $\exp$, we obtain from this a canonical and functorial morphism

$$(11) \qquad \Psi_G : \mathfrak{g} \otimes \widetilde{\Omega}^m = R^m\nu_*\mathfrak{g}^\circ \xrightarrow{\exp} R^m\nu_*G^\circ \to R^m\nu_*G.$$

By Proposition 3.12 applied to $G^\circ \subseteq G$, also the last morphism is an isomorphism. $\qquad\square$

Theorem 4.1 for commutative $G$ is already interesting for smooth rigid $X$, where as a consequence, we get a generalisation of the Hodge–Tate spectral sequence, with $G$-coefficients:

**Corollary 4.3.** *The Leray sequence for $X_v \to X_{\text{ét}}$ induces a first quadrant spectral sequence*

$$E_2^{ij} := \begin{cases} H_{\text{ét}}^i(X, G) & \text{if } j = 0 \\ H_{\text{ét}}^i(X, \widetilde{\Omega}_X^j) \otimes_K \mathfrak{g} & \text{if } j > 0 \end{cases} \Rightarrow H_v^{i+j}(X, G).$$

Of course this simple construction of $\Psi_G$ cannot work for general $G$ since $\exp$ is then not a homomorphism, and cannot be derived. The basic idea, going back to Faltings [Fal05], is to represent both sides as group cohomology and apply $\exp$ to carefully chosen cocycles.

4.2. **Preparations and choices.** As long as we later prove that our construction is canonical and functorial in $X$, it suffices to construct $\Psi_G$ locally. We may therefore assume that $X = \mathrm{Spa}(R, R^+)$ is a toric affinoid smoothoid space in the sense of Definition 2.4. Note that we do not fix a toric chart.

We begin by making some auxiliary choices, we later prove that the construction is independent of these. We first choose any affinoid perfectoid pro-finite-étale Galois cover

$$\widetilde{X} = \mathrm{Spa}(\widetilde{R}, \widetilde{R}^+) \to X = \mathrm{Spa}(R, R^+).$$

**Remark 4.4.** In practice, there are various concrete ways to choose this:

(1) If $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$, we can choose a toric chart and consider the toric tower as in Section 2.3.
(2) If $\mathrm{Spec}(\mathcal{O}(X))$ is connected, then any choice of base point $x \in X(\overline{K})$ induces an affinoid perfectoid universal pro-finite-étale cover $\widetilde{X} \to X$, defined as the inverse limit over Zariski-connected finite étale covers of $X$ together with a fixed lift $\tilde{x}$ of $x$.

However, the greater generality is useful to show that the construction is canonical and functorial. It also allows for a uniform treatment independent of whether $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$ or not.

Let us denote the Galois group of $\widetilde{X} \to X$ by $\pi$, then $\widetilde{X} \to X$ is a pro-finite-étale $\pi$-torsor. As $H^1_v(\widetilde{X}, \mathcal{O}) = 0$, the Cartan–Leray sequence Proposition A.4 induces an isomorphism

$$H^1_{\mathrm{cts}}(\pi, \widetilde{R}) = H^1_v(X, \mathcal{O}) = H^0(X, \widetilde{\Omega}_X).$$

Since $X$ is toric, $H^0(X, \widetilde{\Omega}_X)$ is finite free over $R$. Moreover, by Lemma 2.14 we can up to bounded torsion identify $H^1_v(X, \mathcal{O}^+)$ with an $R^+$-sublattice of rank $d$ of $H^0(X, \widetilde{\Omega})$. Thus

$$H^0(X, \widetilde{\Omega})^\circ := \mathrm{im}\left(H^1_{\mathrm{cts}}(\pi, \widetilde{R}^+) \overset{a}{=} H^1_v(X, \mathcal{O}^+) \to H^0(X, \widetilde{\Omega})\right)$$

is an open $R^+$-submodule of $H^0(X, \widetilde{\Omega})$ that generates the whole module upon inverting $p$. We now choose a basis $\delta = (\delta_1, \ldots, \delta_d)$ of $H^0(X, \widetilde{\Omega})$ as an $R$-module that already lies in $\mathfrak{m} \cdot H^0(X, \widetilde{\Omega})^\circ$. Let

$$H^0(X, \widetilde{\Omega})^+ \subseteq H^0(X, \widetilde{\Omega})^\circ$$

be the finite free $R^+$-sublattice spanned by $\delta$. We also write this as $H^0(X, \widetilde{\Omega})^{+,\delta}$ to indicate the dependence on $\delta$. Third, we now choose for each $\delta_i$ a representative $\rho_i : \pi \to \widetilde{R}^+$ in the set of continuous 1-cocycles $\mathcal{Z}^1_{\mathrm{cts}}(\pi, \widetilde{R}^+)$ that maps to $\delta_i$ under the map

$$\mathrm{HT} \circ [-] : \mathcal{Z}^1_{\mathrm{cts}}(\pi, \widetilde{R}^+) \overset{[-]}{\longrightarrow} H^1_{\mathrm{cts}}(\pi, \widetilde{R}^+) \to H^1_v(X, \mathcal{O}^+) \overset{\mathrm{HT}}{\longrightarrow} H^0(X, \widetilde{\Omega})^\circ.$$

**Choices 4.5.** *In summary, we have made the following choices:*

*(1) an affinoid perfectoid pro-finite-étale cover $\widetilde{X} \to X$ that is Galois with group $\pi$,*
*(2) a basis $\delta = (\delta_1, \ldots, \delta_d)$ of $H^0(X, \widetilde{\Omega})$,*
*(3) a set of representative 1-cocycles $\rho_i \in \mathcal{Z}^1_{\mathrm{cts}}(\pi, \mathcal{O}^+(\widetilde{X}))$ such that $\mathrm{HT}([\rho_i]) = \delta_i$.*

Of course the $\rho_i$ determine the $\delta_i$, but it is later helpful to see this choice as two steps.

**Remark 4.6.** In the special case that $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$ and $\widetilde{X} \to X$ is a toric cover, there are canonical choices for (2) and (3), so no additional choices are necessary: Indeed, by Lemma 2.14 we then have $H^0(X, \widetilde{\Omega}) = \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}(X))$. One can then choose the $\delta_i$ to be the pullbacks of the translation invariant differentials $\frac{dT_1}{T_1}, \ldots, \frac{dT_d}{T_d}$ on $\mathbb{T}^d$ along the chart $X \to \mathbb{T}^d$, and let $\rho_i$ be the associated maps $\Delta = \pi \to R$, which can be shown to always factor through $R^+$.

In fact, one can in this special case construct $\Psi$ without using such bases, cf Theorem 6.5.

That being said, the above generality is useful even in this case, since the proof of independence of the toric chart would lead back to working with more general $\rho_i$. Furthermore, the given generality allows us to treat more general perfectoid $K$ over $\mathbb{Q}_p$.

### 4.3. The integral morphism $\Phi^+$: exponentiating cocycles.
Having made Choices 4.5, we now construct as the first step a map

$$\Phi_{\mathrm{grp}}^+ : (H^0(X,\widetilde{\Omega})^+ \otimes_{R^+} \mathfrak{g}^\circ(R))^{\wedge=0} \to H^1_{\mathrm{cts}}(\pi, G^\circ(\widetilde{R})).$$

Let $\theta$ be an element on the left. Since $\mathfrak{g}^\circ$ is a finite free $\mathcal{O}^+$-module, $H^0(X,\widetilde{\Omega})^+ \otimes_{R^+} \mathfrak{g}^\circ(R)$ is a finite free $R^+$-module. Therefore $\theta$ has a unique expansion $\theta = \sum_{i=1}^d \delta_i \otimes A_i$ in terms of the basis $\delta$ of the free $R^+$-module $H^0(X,\widetilde{\Omega})^+$ for some $A_i \in \mathfrak{g}^\circ(R)$. The condition $\theta \wedge \theta = 0$ means precisely that the $A_i$ commute with each other. We now first define a map

$$\widetilde{\Phi}_{\mathrm{grp}}^+ : H^0(X,\widetilde{\Omega})^+ \otimes_{R^+} \mathfrak{g}^\circ(R) \;\;\to\;\; \mathrm{Map}_{\mathrm{cts}}(\pi, G^\circ(\widetilde{R}))$$
$$\sum_{i=1}^d \delta_i \otimes A_i \;\;\mapsto\;\; \Big(\gamma \mapsto \prod_{i=1}^d \exp(\rho_i(\gamma) \cdot A_i)\Big).$$

This is well-defined as $\rho_i$ has image in $\widetilde{R}^+$ and $A_i \in \mathfrak{g}^\circ(R)$, so their product lies in $\mathfrak{g}^\circ(\widetilde{R})$.

The commutativity condition in Lemma 3.5 is precisely the reason why the definition is only really sensible when we restrict to Higgs fields, where the matrices $A_i$ commute:

**Lemma 4.7.** *If $\theta \wedge \theta = 0$, then $\widetilde{\Phi}_{\mathrm{grp}}^+(\theta)$ is a 1-cocycle. Hence $\widetilde{\Phi}_{\mathrm{grp}}^+$ induces a map*

$$\Phi_{\mathrm{grp}}^+ : (H^0(X,\widetilde{\Omega})^+ \otimes_{R^+} \mathfrak{g}^\circ(R))^{\wedge=0} \to H^1_{\mathrm{cts}}(\pi, G^\circ(\widetilde{R})).$$

*Proof.* Using that $\rho_i \in \mathcal{Z}^1_{\mathrm{cts}}(\pi, \widetilde{R}^+)$, we have by Definition A.1 for any $\gamma_1, \gamma_2 \in \pi$

$$\widetilde{\Phi}_{\mathrm{grp}}^+(\theta)(\gamma_1 \cdot \gamma_2) = \prod_i \exp(\rho_i(\gamma_1)A_i + \gamma_1^* \rho_i(\gamma_2)A_i).$$

Since $\rho_i(\gamma_1)$ and $\gamma_1^*\rho_i(\gamma_2)$ are scalars in $\widetilde{R}^+$, any two elements of the subset of $\mathfrak{g}^\circ(\widetilde{R})$ given by $\rho_i(\gamma_1)A_i$ and $\gamma_1^*\rho_i(\gamma_2)A_i$ for $i = 1,\dots,d$ still commute. We may thus write this as

$$= \prod_i \exp(\rho_i(\gamma_1)A_i) \cdot \gamma_1^* \prod_i \exp(\rho_i(\gamma_2)A_i) = \widetilde{\Phi}_{\mathrm{grp}}^+(\theta)(\gamma_1) \cdot \gamma_1^* \widetilde{\Phi}_{\mathrm{grp}}^+(\theta)(\gamma_2). \qquad \square$$

Via the map $H^1_{\mathrm{cts}}(\pi, G^\circ(\widetilde{R})) \to H^1_v(X, G^\circ)$ of the Cartan–Leray sequence, we get a map

$$\Phi^+ : (H^0(X,\widetilde{\Omega})^+ \otimes_{R^+} \mathfrak{g}^\circ(R))^{\wedge=0} \xrightarrow{\Phi_{\mathrm{grp}}^+} H^1_{\mathrm{cts}}(\pi, G^\circ(\widetilde{R})) \to H^1_v(X, G^\circ) \to H^1_v(X, G).$$

If we want to indicate $X$, $G$ or the choice of $\delta$ and $\rho$, we add this as a subscript, e.g. $\Phi_{X,\delta}^+$.

At this point, we have associated to any "small" $G$-Higgs field on the trivial bundle a "small" $v$-$G$-bundle. One can show that this construction is in fact functorial in $X$ in the sense that one could define a notion of *compatible* choices as in Choices 4.5 with respect to a morphism $X' \to X$ of smoothoid adic spaces, for which $\Phi_X^+$ and $\Phi_{X'}^+$ commute with the pullback maps. But the construction of $\Phi^+$ is not independent of our choices. However, we will later prove that the construction becomes independent of choices as well as functorial after sheafifying. As an intermediate step, we already need the following weaker statement:

**Lemma 4.8.** *For each $i = 1,\dots,d$, let $\rho_i' \in \mathcal{Z}^1_{\mathrm{cts}}(\pi, \widetilde{R}^+)$ be such that $[\rho_i'] = [\rho_i]$ in $H^1_{\mathrm{cts}}(\pi, \widetilde{R}^+) \stackrel{a}{=} H^1_v(X, \mathcal{O}^+)$. Then $\Phi^+$ is the same whether it is computed using $\rho_i$ or $\rho_i'$.*

This is weaker than independence of the choice of $\rho_i$, as $\rho_i$ is by definition also a preimage of $\delta_i$ under the map $\mathrm{HT} : H^1_v(X, \mathcal{O}^+) \to H^0(X,\widetilde{\Omega})$ which is in general not injective.

*Proof.* That $[\rho_i] = [\rho_i']$ means that there is $x_i \in \widetilde{R}^+$ such that $\rho_i'(\gamma) = \gamma^* x_i + \rho_i(\gamma) - x_i$ for all $\gamma \in \pi$. Since $\gamma^* x$ and $x$ are scalars in $\widetilde{R}^+$, all the $\gamma^* x A_i$, $x A_i$ and $\rho_i'(\gamma)A_i$ in $\mathfrak{g}^\circ(\widetilde{R})$ still commute with each other for all $i$. Consequently, for any $\gamma \in \pi$ we have

$$\prod_i \exp(\rho_i'(\gamma)A_i) = \gamma^*\big(\prod_i \exp(x_i A_i)\big) \cdot \prod_i \exp(\rho_i(\gamma)A_i) \cdot \big(\prod_i \exp(x_i A_i)\big)^{-1}.$$

Setting $y := \prod_i \exp(x_i A_i) \in G^\circ(\widetilde{R})$, we see that $\prod_i \exp(\rho_i(\gamma)A_i)$ and $\prod_i \exp(\rho_i'(\gamma)A_i)$ agree up to the $\gamma$-conjugation by $y$ and thus have the same image in $H^1_{\mathrm{cts}}(\pi, G^\circ(\widetilde{R}))$. $\qquad\square$

We will frequently use the following simple observation to make cocycles small:

**Lemma 4.9.** *Let $0 \neq a \in K^+$. Suppose that each $\rho_i$ has image in $a\widetilde{R}^+$. Then $a^{-1}\delta$ is a basis of $H^0(X, \widetilde{\Omega})$ contained in the image of $H^1_v(X, \mathcal{O}^+)$, and $a^{-1}\rho$ is a choice of integral cocycle representatives for $a^{-1}\delta$. Then on $(H^0(X, \widetilde{\Omega})^{+,\delta} \otimes_{R^+} \mathfrak{g}^\circ(R))^{\wedge = 0}$ we have $\Phi^+_{a^{-1}\rho} = \Phi^+_\rho$, but $\Phi^+_{a^{-1}\rho}$ is now defined on the larger space $(H^0(X, \widetilde{\Omega})^{+,\delta} \otimes_{R^+} a^{-1}\mathfrak{g}^\circ(R))^{\wedge = 0}$.*

*Proof.* Clear from the definition by writing $A_i \cdot \rho_i = aA_i \cdot a^{-1}\rho_i$. $\qquad\square$

4.4. **Extension to all Higgs fields.** Next, we extend $\Phi^+$ to a morphism on all of $(H^0(X, \widetilde{\Omega}) \otimes \mathfrak{g})^{\wedge = 0}$. For this, we first note that already the case of $G = \mathbb{G}_m$ and $X = \mathbb{B}^2$ shows that HTlog$\colon H^1_v(X, \mathbb{G}_m) \to H^0(X, \widetilde{\Omega})$ is in general not surjective, see [Heu22b, §6]. We can therefore only expect $\Phi^+$ to extend to all Higgs fields after étale sheafification on $X$, i.e. after passing from $H^1_v(X, G)$ to $R^1\nu_* G$.

We therefore now compose $\Phi^+$ with the map from the Leray sequence of $\nu\colon X_v \to X_{\mathrm{\acute{e}t}}$

$$\Psi^+ : (H^0(X, \widetilde{\Omega})^+ \otimes \mathfrak{g}^\circ(K))^{\wedge = 0} \xrightarrow{\Phi^+} H^1_v(X, G) \to R^1\nu_* G(X).$$

As before, we denote this by $\Psi^+_X$ or $\Psi^+_{X,\delta}$ etc. if we want to indicate $X$, $G$, or our choices. This is the map that we now extend to all Higgs fields: The basic idea for doing so is that every Higgs fields becomes small on a finite Galois cover, by the following two lemmas.

**Lemma 4.10.** *If $\widetilde{X} \to X' \to X$ is a sub-cover such that $f : X' \to X$ is finite étale, then $f^*\widetilde{\Omega}_X = \widetilde{\Omega}_{X'}$ by Lemma 2.13. Therefore Choices 4.5 induce natural choices for $X'$:*

*(1) For the Galois cover we use $\widetilde{X} \to X'$, whose Galois group is an open subgroup $\pi' \subseteq \pi$.*
*(2) For the basis of $H^0(X', \widetilde{\Omega})$ we use the image of $\delta$ under $H^0(X, \widetilde{\Omega}) \to H^0(X', \widetilde{\Omega})$.*
*(3) For the representatives we use the restriction of the $\rho_i$ to $\pi'$.*

**Lemma 4.11.** *Let $k \in \mathbb{N}$. For any small enough normal open subgroup $\pi' \subseteq \pi$ corresponding to a finite étale Galois cover $f : X' \to X$, the induced cocycles $\rho_i$ from Lemma 4.10 have image in $p^k\mathcal{O}^+(\widetilde{R})$. In particular, for any $\theta \in (H^0(X, \widetilde{\Omega}) \otimes_K \mathfrak{g}(K))^{\wedge = 0}$, we can arrange that*

$$f^*\theta \in H^0(X', \widetilde{\Omega})^{+,\delta'} \otimes_{K^+} \mathfrak{g}^\circ(K) \quad \text{where } \delta' := p^{-k}\delta.$$

*Proof.* Since $\rho_i \colon \pi \to \mathcal{O}^+(\widetilde{X})$ is continuous, $\rho_i^{-1}(p^k\mathcal{O}^+(\widetilde{X})) \subseteq \pi$ is an open neighbourhood of the identity. It therefore contains an open normal subgroup $\pi' \subseteq \pi$. This corresponds to a finite étale cover $X' \to X$ on which the restriction of $\rho_i$ to $\pi'$ maps into $p^k\mathcal{O}^+(\widetilde{X})$. $\qquad\square$

The idea is now to pass to such a cover $X' \to X$ and apply $\Psi^+_{X'}$ there. In order to be able to use this to extend $\Psi^+_X$ to $(H^0(X, \widetilde{\Omega}) \otimes_K \mathfrak{g}(K))^{\wedge = 0}$, we need to see that we can descend the image of $\Psi^+_{X'}$ back to $X$. For this we use a first instance of functoriality of $\Psi$ in $X$:

**Lemma 4.12.** *Let $f : \widetilde{X} \to X' \to X$ be a subcover such that $X' \to X$ is finite étale and Galois with group $Q$. Let $\delta'$ and $\rho'$ be the choices induced by Lemma 4.10. Then the map $\Phi^+_{X',\delta',\rho'} : (H^0(X', \widetilde{\Omega})^+ \otimes_{K^+} \mathfrak{g}^\circ(K))^{\wedge = 0} \to H^1_v(X', G)$ is equivariant for the natural $Q$-actions on either side. In particular, so is $\Psi^+_{X'}$.*

*Proof.* Write $\pi'$ for the Galois group of $\widetilde{X} \to X'$, then $Q = \pi/\pi'$. Consider the restriction

$$\mathrm{res} : H^1_{\mathrm{cts}}(\pi, \widetilde{R}^+) \to H^1_{\mathrm{cts}}(\pi', \widetilde{R}^+)^Q,$$

where we recall that the natural action of $Q$ on classes in $H^1_{\mathrm{cts}}(\pi', \widetilde{R}^+)$ is given for $g \in Q$ with any lift $\tilde{g}$ to $\pi$ by $[\rho] \mapsto g[\rho] := [\tilde{g}\rho(\tilde{g}^{-1} - \tilde{g})]$. Using that $[\rho_i] = g[\rho_i]$ in $H^1_{\mathrm{cts}}(\pi', \mathcal{O}^+(\widetilde{X}))^Q$, we may by Lemma 4.8 use either of $\rho$ or $\tilde{g}\rho(\tilde{g}^{-1} - \tilde{g})$ to compute $\Phi^+_{X'}$.

Let $\theta = \sum \delta_i \otimes A_i$ be in the domain of $\Phi^+_{X', \delta', \rho'}$, then for any $\tilde{g} \in \pi$ with image $g$ in $Q$,

$$\Phi^+_{X'}(g\theta) = [\gamma \mapsto \prod_i \exp(\rho_i(\gamma) \cdot gA_i)]$$
$$= [\gamma \mapsto \prod_i \exp(\tilde{g}\rho_i(\tilde{g}^{-1}\gamma\tilde{g}) \cdot \tilde{g}A_i)] = g\Phi^+_{X'}(\theta)$$

because $A_i \in \mathfrak{g}(X') \subseteq \mathfrak{g}(\widetilde{X})$, so we can write $gA_i = \tilde{g}A_i$ inside $\mathfrak{g}(\widetilde{X})$.                    $\square$

We are now prepared to define $\Psi$: For any $\theta = \sum \delta_i \otimes A_i \in (H^0(X, \widetilde{\Omega}) \otimes_K \mathfrak{g}(K))^{\wedge=0}$, let $k \in \mathbb{N}$ be such that $p^k A_i \in \mathfrak{g}^\circ(X)$ for all $i$. Using Lemma 4.11 we find a Galois subcover $\widetilde{X} \to X' \to X$ with Galois group $Q$ where all $p^{-k}\rho_i$ become integral with respect to $p^{-k}\delta$, so that $\theta$ is in the domain of $\Psi^+_{X', p^{-k}\delta}$. Then since $\theta \in (H^0(X', \widetilde{\Omega}) \otimes_K \mathfrak{g}(K))^Q$, Lemma 4.12 implies that

$$\Psi^+_{X', p^{-k}\delta}(\theta) \in (R^1\nu_* G(X'))^Q = R^1\nu_* G(X)$$

as $R^1\nu_* G$ is a sheaf on $X_{\text{ét}}$. Using Lemma 4.9, we see that $\Psi^+_{X', p^{-k}\delta}(\theta) = \Psi^+_{X, \delta}(\theta)$ for any $\theta \in (H^0(X, \widetilde{\Omega})^{+,\delta} \otimes_{K^+} \mathfrak{g}^\circ(K))^{\wedge=0}$, so this construction is compatible with the integral one. Since any two finite étale sub-covers of $\widetilde{X} \to X$ have a common finite étale cover under $\widetilde{X}$, this also shows that for any two covers $X' \to X$ and $X'' \to X$ on which $\theta$ becomes small enough, we have $\Psi^+_{X'}(\theta) = \Psi^+_{X''}(\theta)$ inside of $R^1\nu_* G(X)$. Thus the following is well-defined:

**Definition 4.13.** In the colimit over finite étale sub-covers $\widetilde{X} \to X' \to X$, we obtain a map

$$\Psi_{X,G} : H^0(X, \widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0} \to R^1\nu_* G(X), \quad \theta \mapsto \varinjlim_{X' \to X} \Psi^+_{X'}(\theta).$$

In order to simplify notation, or to emphasize a particular variable, we also simply denote this by $\Psi_X$ or $\Psi_G$ or even just $\Psi$ if $X$ and $G$ are clear from the context.

The following lemma summarises the above discussion of compatibility for varying $X'$:

**Lemma 4.14.** *Let $\widetilde{X} \to X' \to X$ be a finite étale sub-cover. Then using the induced choices of Lemma 4.10 for $X'$ to compute $\Psi_{X'}$, the following diagram commutes:*

$$\begin{array}{ccc}
H^0(X', \widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0} & \xrightarrow{\Psi_{X'}} & R^1\nu_* G(X') \\
\cup| & & \cup| \\
H^0(X, \widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0} & \xrightarrow{\Psi_X} & R^1\nu_* G(X).
\end{array}$$

4.5. **Independence of choice, and functoriality.** While our definition of $\Psi_X$ a priori depends on Choices 4.5, we now check that the result is canonical. We begin by showing that $\Psi_X$ is independent of Choices 4.5.3, strengthening Lemma 4.8:

**Lemma 4.15.** $\Psi_X$ *does not depend on the choice of integral 1-cocycles $\rho_i$ such that $[\rho_i] = \delta_i$.*

*Proof.* Let $\rho'_i \in \mathcal{Z}^1_{\mathrm{cts}}(\pi, \widetilde{R}^+)$ be such that $[\rho_i] = [\rho'_i]$ in $H^1_{\mathrm{cts}}(\pi, \widetilde{R})$. Then there is $x \in \widetilde{R}$ with

$$(12) \qquad\qquad \rho'_i(\gamma) = \gamma^* x + \rho_i(\gamma) - x \quad \text{ for all } \gamma \in \pi.$$

**Claim 4.16.** *For any $\epsilon > 0$, there is a finite étale subcover $\widetilde{X} \to X' \to X$ with Galois group $\pi' \subseteq \pi$ on which we can find $x \in \widetilde{R}$ such that (12) holds for $\gamma \in \pi'$ and such that $p^\epsilon x \in \widetilde{R}^+$.*

*Proof.* Since (12) implies $(\gamma - 1)x \in \widetilde{R}^+$ for all $\gamma \in \pi$, there is $k \in \mathbb{N}$ such that $y := p^k x \in \widetilde{R}^+$ satisfies $(\gamma - 1)y \in p^k \widetilde{R}^+$ for all $\gamma$, so the image $\bar{y}$ of $y$ in $H^0(\widetilde{X}, \mathcal{O}^+/p^k)$ is $\pi$-invariant.

By [Heu22a, Corollary 2.15], $H^0(\widetilde{X}, \mathcal{O}^+/p^k)^\pi = H^0(X, \mathcal{O}^+/p^k)$. The short exact sequences

$$0 \to \mathcal{O}^+(X')/p^k \to \mathcal{O}^+/p^k(X') \to H^1(X', \mathcal{O}^+)[p^k] \to 0$$

in the colimit over all $\widetilde{X} \to X' \to X$ show that $\varinjlim H^1(X', \mathcal{O}^+)[p^k] = H^1(\widetilde{X}, \mathcal{O}^+)[p^k] \overset{a}{=} 0$. Hence there is $X'$ such that $p^\epsilon \overline{y}$ lifts to $\mathcal{O}^+(X')$. Then $p^\epsilon y \in \mathcal{O}^+(X') + p^k \mathcal{O}^+(\widetilde{X})$ which implies $p^\epsilon x \in \mathcal{O}(X') + \mathcal{O}^+(\widetilde{X})$. As $x$ is only determined by (12) up to the difference of an element in $\mathcal{O}(X')$, we can thus change $x$ to arrange for $p^\epsilon x$ to be in $\widetilde{R}^+ = \mathcal{O}^+(\widetilde{X})$. $\qquad\square$

By Lemma 4.14, we may check the independence on $X' \to X$. Let now $\theta = \sum \delta_i \otimes A_i \in (H^0(X, \widetilde{\Omega})^+ \otimes_{R^+} \mathfrak{g}^\circ(R))^{\wedge=0}$, then by the claim, we can assume that $x \cdot A_i, \gamma^* x \cdot A_i \in p^{-\epsilon} \mathfrak{g}^\circ(\widetilde{X})$ where exp still converges. Now the statement follows exactly as in Lemma 4.8. $\qquad\square$

**Lemma 4.17.** $\Psi_X$ *does not depend on the choice of Galois cover* $\widetilde{X} \to X$.

*Proof.* We first note that if $\widetilde{X}' \to \widetilde{X} \to X$ is any dominating pro-finite-étale Galois cover, then it is clear from functoriality of the Cartan–Leray sequence that the construction for $\widetilde{X}$ is compatible with that of $\widetilde{X}'$. Therefore, if $\widetilde{X}_1 \to X$ and $\widetilde{X}_2 \to X$ are any two pro-finite-étale Galois cover such that there exists a pro-finite-étale Galois cover $\widetilde{X}_3 \to X$ that dominates both $\widetilde{X}_1$ and $\widetilde{X}_2$, then the construction for either agrees with that for $\widetilde{X}_3$. It now suffices to observe that such a simultaneous dominating cover always exists in the case that $\widetilde{X}_2 \to X$ is the universal cover from Remark 4.4.2: But here any finite étale cover of $\widetilde{X}_2$ is split, hence again Galois over $X$, from which it is clear that we can find such a cover $\widetilde{X}_3$. $\qquad\square$

**Remark 4.18.** Given that the proof is easy, and that $\widetilde{X} \to X$ could simply be induced by a choice of base-point, part 1 may look like the most innocuous part of Choices 4.5. However, it is this choice that keeps us from upgrading the construction to a *functor* from all Higgs bundles to $v$-bundles: The crucial point is that there is no *canonical* choice of $\widetilde{X}_3$ in the above proof: Suppose we are given two base points $x_1$ and $x_2$ of $X$. Let $\widetilde{X}_1 \to X$ and $\widetilde{X}_2 \to X$ be the associated pro-finite-étale universal covers from Remark 4.4.2. Any choice of lift $\tilde{x}_2$ of $x_1$ to $\widetilde{X}_2$ induces an isomorphism $\phi : \widetilde{X}_1 \xrightarrow{\sim} \widetilde{X}_2$ over $X$ sending $\tilde{x}_1$ to $\tilde{x}_2$, and an isomorphism of Galois groups $\pi_1(X, x_1) \xrightarrow{\sim} \pi_1(X, x_2)$ for which $\phi$ is equivariant. Let $\rho_i : \pi_1(X, x_2) \to \mathcal{O}^+(\widetilde{X}_2)$ be any representative cocycles with respect to $x_2$ for the given basis $\delta_i$. Then $\phi^* \rho_i := [\gamma \mapsto \phi \circ \rho_i(\phi \circ \gamma \circ \phi^{-1})]$ is a cocycle in $\mathcal{Z}^1_{\text{cts}}(\pi_1(X, x_1), \mathcal{O}^+(\widetilde{X}_1))$ that represent the $\delta_i$ with respect to $x_1$. For this choice, we have $\Phi^+_{X, \phi^*\rho, x_1} = \phi^* \circ \Phi^+_{X, \rho, x_2}$ by the Lemma. However, the explicit cocycle $\phi^* \rho_i$ depends on the choice of $\phi$, and thus on $\tilde{x}_2$. Only the associated element in group cohomology is independent of this choice.

**Lemma 4.19.** $\Psi_X$ *is independent of the choice of basis* $\delta$ *of* $H^0(X, \widetilde{\Omega})$, *Choices 4.5.2.*

*Proof.* Let $\delta' = (\delta'_1, \ldots, \delta'_d)$ be any other basis lying in $H^0(X, \widetilde{\Omega})^\circ$. Choose $\rho'_j \in \mathcal{Z}^1_{\text{cts}}(\pi, \widetilde{R}^+)$ representing the $\delta'_j$. Then $\delta_i = \sum_j b_{ij} \delta'_j$ for some $b_{ij} \in \mathcal{O}(X)$, and $\sum_i \delta_i \otimes A_i = \sum_j \delta'_j \otimes A'_j$ for $A'_j := \sum_i b_{ij} A_i$. To see that $\Psi_{X,\delta} = \Psi_{X,\delta'}$, we may by Lemma 4.14 pass to a finite étale cover $X' \to X$, so by Lemma 4.11 we may assume that $b_{ij} \rho'_j \in \mathcal{Z}^1_{\text{cts}}(\pi', \widetilde{R}^+)$ for all $i, j$. Then $\rho_i := \sum_j b_{ij} \rho'_j \in \mathcal{Z}^1_{\text{cts}}(\pi', \widetilde{R}^+)$ represents $\delta_i$, so by Lemma 4.15 we may use it to compute $\Psi_{X',\delta}$. Now since $b_{ij} \rho'_j(\gamma) A_i \in \mathfrak{g}^\circ(\widetilde{R})$ for any $\gamma$ in the Galois group of $\widetilde{X} \to X'$, we have

$$\Phi^+_{\rho,\delta}(\theta)(\gamma) = \prod_i \exp(\rho_i(\gamma) A_j) = \prod_i \exp(\sum_j b_{ij} \rho'_j(\gamma) A_i) = \prod_j \exp(\rho'_j(\gamma) A'_i) = \Phi^+_{\rho',\delta'}(\theta)(\gamma).$$

It follows that the same is true for $\Psi^+_{X'}$ and thus for $\Psi_X$. $\qquad\square$

In summary, Lemmas 4.15, 4.17 and 4.19 show that $\Psi_X$ is independent of Choices 4.5, hence canonical. Next, we show functoriality in $X$:

**Lemma 4.20.** *For any morphism* $f \colon X_2 \to X_1$ *of toric smoothoid spaces, the following diagram commutes:*

$$
\begin{array}{ccc}
H^0(X_1, \widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0} & \xrightarrow{\Psi_{X_1}} & R^1\nu_* G(X_1) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
H^0(X_2, \widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0} & \xrightarrow{\Psi_{X_2}} & R^1\nu_* G(X_2)
\end{array}
$$

*Proof.* Choose a sequential pro-finite-étale Galois cover $\widetilde{X}_1 = \varprojlim_{n \in \mathbb{N}} X_{1,n} \to X_1$ with group $\pi_1$ as in Choices 4.5. The pullback $\widetilde{X}_1 \times_{X_1} X_2 \to X_2$ is a pro-finite-étale $\pi$-torsor over $X_2$, but not necessarily perfectoid. But we can find a sequential affinoid perfectoid Galois cover $\widetilde{X}_2 \to X_2$ with group $\pi_2$ that dominates $\widetilde{X}_1 \times_{X_1} X_2$. We thus get a morphism $\widetilde{f} : \widetilde{X}_2 \to \widetilde{X}_1$ over $f : X_2 \to X_1$, and a quotient map $\phi : \pi_2 \to \pi_1$ with respect to which $\widetilde{f}$ is equivariant. By functoriality of the Cartan–Leray sequence, this induces a commutative diagram

(13)
$$
\begin{array}{ccccc}
H^1_{\mathrm{cts}}(\pi_1, \mathcal{O}(\widetilde{X}_1)) & \longrightarrow & H^1_v(X_1, \mathcal{O}) & \longrightarrow & H^0(X_1, \widetilde{\Omega}) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
H^1_{\mathrm{cts}}(\pi_2, \mathcal{O}(\widetilde{X}_2)) & \longrightarrow & H^1_v(X_2, \mathcal{O}) & \longrightarrow & H^0(X_2, \widetilde{\Omega})
\end{array}
$$

where for any 1-cocycle $\rho$ in the top left, we define $f^*\rho : \pi_2 \xrightarrow{\phi} \pi_1 \xrightarrow{\rho} \mathcal{O}(\widetilde{X}_1) \xrightarrow{\widetilde{f}^*} \mathcal{O}(\widetilde{X}_2)$.

Let $\delta_1$ be any basis in $H^0(X_1, \widetilde{\Omega})^\circ$ and $\delta_2$ any basis in $H^0(X_2, \widetilde{\Omega})^\circ$, and choose representative integral cocycles $\rho_1$ and $\rho_2$. Then we can find $b_{ij} \in \mathcal{O}(X_2)$ such that $f^*\delta_{1,i} = \sum_j b_{ij}\delta_{2,j}$. For any $\theta = \sum_i \delta_{1,i} \otimes A_i$ with $A_i \in \mathfrak{g}(X_1)$ set $A'_j := \sum_i b_{ij} f^* A_i$, then $f^*\theta = \sum_j \delta_{2,j} \otimes A'_j$.

Choose now a finite étale sub-cover $X'_1 \to X_1$ such that we can take $A_i \in \mathfrak{g}^\circ(X'_1)$. Let $X''_2 \to X_2$ be the pullback. Then by Lemma 4.11, we can find a finite étale sub-cover $\widetilde{X}_2 \to X'_2 \to X''_2$ of $X_2$ with Galois group $\pi'_2$ such that $b_{ij}\rho_{2,j} \in \mathcal{Z}^1_{\mathrm{cts}}(\pi'_2, \mathcal{O}^+(\widetilde{X}_2))$ for all $i, j$. The composition $f' : X'_2 \to X''_2 \to X'_1$ is an intermediate morphism over $f$ and under $\widetilde{f}$.

By definition, we can now use $\Phi^+_{X'_1, \rho_1}$ to compute $\Psi_{X_1}(\theta)$: This yields

$$
f'^* \Phi^+_{X'_1, \rho_1}(\theta) = [\gamma \mapsto \textstyle\prod_i \exp(f'^*\rho_{1,i}(\gamma) f^* A_i)].
$$

By the commutative diagram (13) (applied to $f' : X'_2 \to X'_1$), the 1-cocycle $f'^*\rho_{1,i}$, defined as the restriction of $f^*\rho_{1,i}$ to $\pi'_2 \subseteq \pi_2$, represents $f'^*\delta_{1,i} \in H^0(X'_2, \widetilde{\Omega})$. The same is true for $\rho'_{2,i} := \sum_j b_{ij}\rho_{2,j} \in \mathcal{Z}^1_{\mathrm{cts}}(\pi'_2, \mathcal{O}(\widetilde{X}_2))$. It follows that there is $x_i \in \mathcal{O}(\widetilde{X}_2)$ such that

$$
(f'^*\rho_{1,i})(\gamma) = \gamma^* x_i + \rho'_{2,i}(\gamma) - x_i \quad \text{for all } \gamma \in \pi'_2.
$$

Let $k \in \mathbb{N}$ be such that $p^k x_i \in \mathcal{O}^+(\widetilde{X}_2)$ and thus $p^k \gamma^* x_i \in \mathcal{O}^+(\widetilde{X}_2)$ for all $\gamma$. After possibly increasing $X'_1 \to X_1$ and $X'_2 \to X_2$, which by Lemmas 4.11 and 4.19 allows us to replace $\delta_1$ by $p^{-k}\delta_1$, and thus $A_i$ by $p^k A_i$, we can then assume that $\gamma^* x_i \cdot f^* A_i \in \mathfrak{g}^\circ(\widetilde{X}_2)$ for all $\gamma$. Similarly, we can assume that the $b_{ij} f^* A_i$ are in $\mathfrak{g}^\circ(\widetilde{X}_2)$, and then so are the $A'_j$.

The lemma now follows from the same calculation as in the previous lemmas: By the usual commutativity argument we see that inside $H^1_{\mathrm{cts}}(\pi'_2, G^\circ(\widetilde{X}_2))$, we have for any $\gamma \in \pi'_2$:

$$
f'^* \Phi^+_{X'_1, \rho_1}(\theta) = \big[\gamma \mapsto \textstyle\prod_i \exp(\rho'_{2,i}(\gamma) \cdot f^* A_i)\big] = \big[\gamma \mapsto \textstyle\prod_j \exp(\rho_{2,j}(\gamma) \cdot A'_j)\big] = \Phi^+_{X'_2, \rho_2}(f^*\theta)
$$

The functoriality of $\Psi$ follows in the colimit over sub-covers of $\widetilde{X}_1 \to X_1$ and $\widetilde{X}_2 \to X_2$. $\quad\square$

As a consequence of Lemma 4.20, we see that for any smoothoid $X$, the maps $\Psi_{X'}$ for the basis of toric $X' \to X$ in $X_{\text{ét}}$ glue to a morphism of sheaves of pointed sets on $X_{\text{ét}}$

$$
\Psi_X : (\widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0} \to R^1\nu_* G.
$$

**Lemma 4.21.** $\Psi_X$ *factors through the quotient by the adjoint action of $G$ on $(\widetilde{\Omega} \otimes \mathfrak{g})^{\wedge=0}$.*

*Proof.* It suffices to prove this locally. Let thus $X$ be any toric smoothoid space, let $B \in G(X)$, and let $\theta = \sum_{i=1}^{d} \delta_i \otimes A_i$. Let $k \in \mathbb{N}$ be large enough such that $p^k \mathrm{ad}(B)(A_i) \in \mathfrak{g}^\circ(X)$ for all $i$. After choosing $X' \to X$ large enough, we may by Lemma 4.19 replace $\delta$ by $p^{-k}\delta$ and thus $A_i$ by $p^k A_i$, so that $\mathrm{ad}(B)(A_i) \in \mathfrak{g}^\circ(X)$. Then by Lemma 3.5.3,

$$\Phi^+_{X',\rho}(\mathrm{ad}(B)(\theta))(\gamma) = \prod_i \exp(\rho_i(\gamma) \cdot \mathrm{ad}(B)(A_i)) = B\Phi^+_{X',\rho}(\theta)(\gamma)B^{-1}$$

for all $\gamma \in \pi'$, where $\pi'$ is the Galois group of $\widetilde{X} \to X'$. Since $B \in G(X)$, we have $\gamma^* B = B$ for all $\gamma \in \pi'$. If $B \in G^\circ(R)$, this shows that $\Phi^+_{X',\rho}(\mathrm{ad}(B)(\theta))$ is equal to $\Phi^+_{X',\rho}(\theta)$ in $H^1_{\mathrm{cts}}(\pi', G^\circ(\widetilde{X}))$. In general, one easily verifies directly that the images in $H^1_v(X, G)$ agree: Namely, $B$ defines an isomorphism between the associated $v$-$G$-torsors on $X$. $\qquad\square$

Using Lemma 3.17, this finally induces the desired morphism of sheaves on $X_{\text{ét}}$

$$(14) \qquad\qquad \Psi_{X,G} : \mathrm{Higgs}_G \to R^1\nu_* G.$$

Once again we simply denote this by $\Psi_X$ or $\Psi_G$ or $\Psi$ when $X$ and/or $G$ are clear from context.

The functoriality of $\Psi_{X,G}$ in $G$ is clear from functoriality of the exponential. In summary:

**Proposition 4.22.** *The morphism $\Psi_{X,G}$ is canonical and functorial in $G$ and $X$.*

In particular, we can assemble the $\Psi_{X,G}$ to a canonical morphism on $\mathrm{Smd}_{K,\text{ét}}$. At this point, we have constructed a candidate for the isomorphism in Theorem 4.1.

**Remark 4.23.** It would be nice to find a "coordinate-free" definition of $\Psi_{X,G}$ that does not a priori depend on local choices such as Choices 4.5, more along the lines of Lemma 2.17 (which is the case of $G = \mathbb{G}_a$). We do not know such a formulation in general. But at least if $X$ is a rigid space that admits a smooth formal model $\mathfrak{X}$, a potential strategy is to consider $G^+$-torsors on the Hodge–Tate stack of $\mathfrak{X}$ introduced by Bhatt–Lurie [BL22], along the lines of the small $p$-adic Simpson correspondence of [AHL24]. If the geometry of the Hodge–Tate stack could moreover be understood for semi-stable formal schemes over $\mathcal{O}_K$, this might lead to a coordinate-free definition of $\Psi_{X,G}$ in general, because any smooth rigid space étale-locally admits a semi-stable formal model by a result of Temkin [Tem17].

## 5. FROM $v$-TOPOLOGICAL $G$-BUNDLES TO $G$-HIGGS BUNDLES

Let $X$ be any smoothoid space over $K$. The goal of this section is to prove that the morphism $\Psi := \Psi_{X,G}$ from (14) is an isomorphism. For this, we may again localise and assume that $X$ is toric.

Following Faltings in the case of $\mathrm{GL}_n$, the basic idea is as follows: Assume that $X$ admits a pro-finite-étale Galois cover $X_\infty \to X$ with some *abelian* Galois group $\Delta$. Let now $\mathfrak{u} \subseteq \mathfrak{g}^\circ$ be any open subgroup and let $U \subseteq G^\circ$ be its image under exp. Then $U(X) \subseteq G^\circ(X)$ inherits from $\mathfrak{g}^\circ(X)$ the structure of a topological group. We consider the continuous homomorphisms $\rho : \Delta \to U(X)$. The inclusion $U(X) \subseteq U(\widetilde{X})$ and the Cartan–Leray map for $X_\infty \to X$ define natural maps

$$\mathrm{Hom}_{\mathrm{cts}}(\Delta, U(X)) \to H^1_{\mathrm{cts}}(\Delta, U(X_\infty)) \to H^1_v(X, U) \to H^1_v(X, G)$$

that associate a $v$-$G$-bundle $V$ on $X$ to $\rho$. On the other hand, the fact that $\Delta$ is abelian allows us to associate a Higgs bundle to $\rho$ by composing with the logarithm map

$$\rho \mapsto [\log \circ \rho : \Delta \xrightarrow{\rho} U(X) \xrightarrow{\log} \mathfrak{g}(X)] \in H^1_{\mathrm{cts}}(\Delta, \mathfrak{g}(X)) = H^0(X, \widetilde{\Omega} \otimes \mathfrak{g}).$$

By Lemma 3.5, as $\Delta$ is abelian, this satisfies the Higgs field condition, thus defines a Higgs field on the trivial $G$-bundle. This is the Higgs bundle that we would like to associate to $V$.

However, it is not clear whether any $v$-$G$-bundle arises from such an "abelian" cocycle $\rho$ (surjectivity of $\Psi$), and whether this construction is independent of the choice of $\rho$ (injectivity of $\Psi$). The goal of this section is to show that both hold after sheafification.

5.1. **Abelian $G$-bundles.** In order to carry out this strategy, let us first assume that $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$. We relax this condition in the next subsection. We fix a toric chart, by Definition 2.4 this is a standard-étale map

$$f : X \to \mathbb{T}^d \times Y$$

where $Y$ is an affinoid perfectoid space and $\mathbb{T}^d$ is some rigid torus. Using notation as in Section 2.3, $f$ induces a perfectoid cover $X_\infty \to X$ which is an affinoid perfectoid $\Delta := \mathbb{Z}_p^d(1)$-torsor due to the assumption on $K$. We write $X = \mathrm{Spa}(R, R^+)$ and $X_\infty = \mathrm{Spa}(R_\infty, R_\infty^+)$.

**Definition 5.1.** Let us write $\mathcal{Z}^{\mathrm{ab}}(U)$ for the sheaf on $X_{\mathrm{ét}}$ that sends $Y \in X_{\mathrm{ét}}$ to the set $\mathrm{Hom}_{\mathrm{cts}}(\Delta, U(\mathcal{O}(Y)))$. We call this the "sheaf of abelian cocycles on $X_{\mathrm{ét}}$". Consider the map

$$W_{f,U} : \mathcal{Z}^{\mathrm{ab}}(U)(X) \to H^1_{\mathrm{cts}}(\Delta, U(R)) \to H^1_{\mathrm{cts}}(\Delta, U(R_\infty)) \to H^1_v(X, G).$$

Since the action of $\Delta$ on $U(R)$ is trivial, the first map can be described as the quotient with respect to the conjugation action by $U(R)$. The third map is the Cartan–Leray map.

Composing $W_f$ with the sheafification on $X$, we obtain a morphism of sheaves on $X_{\mathrm{ét}}$

$$\mathcal{W}_{f,U} : \mathcal{Z}^{\mathrm{ab}}(U) \to R^1 \nu_* G.$$

It is clear that $W_{f,U}$ is functorial in $f$ and $U$ (for morphisms as in Definition 2.4.2), thus so is $\mathcal{W}_{f,U}$. We will often drop the subscript $U$ from notation when it is clear from context.

We now explain how $\mathcal{W}_f$ allows us to define partial inverses of $\Psi$ on the image of $\mathcal{W}_f$: Recall from Lemma 2.18 that $f$ induces an isomorphism $\mathrm{HT}_f : \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}(X)) \to H^0(X, \widetilde{\Omega})$. By tensoring with the Lie algebra, this induces an isomorphism of sheaves on $X_{\mathrm{ét}}$

$$\mathrm{HT}_f : \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathfrak{g}) \xrightarrow{\sim} \mathfrak{g} \otimes \widetilde{\Omega}.$$

Under $\mathrm{HT}_f$, the subsheaf of Higgs fields $(\mathfrak{g} \otimes \widetilde{\Omega})^{\wedge = 0}$ gets identified with the subsheaf

$$\mathcal{Z}^{\mathrm{ab}}(\mathfrak{g}) \subseteq \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathfrak{g})$$

consisting of those $\varphi : \Delta \to \mathfrak{g}(Y)$ for $Y \in X_{\mathrm{ét}}$ with $[\varphi(x), \varphi(y)] = 0$ for all $x, y \in \Delta$. We thus have an isomorphism

$$\mathrm{HT}_f : \mathcal{Z}^{\mathrm{ab}}(\mathfrak{g}) \xrightarrow{\sim} (\mathfrak{g} \otimes \widetilde{\Omega})^{\wedge = 0}.$$

Consider now the subsheaf $\mathcal{Z}^{\mathrm{ab}}(\mathfrak{u}) \subseteq \mathcal{Z}^{\mathrm{ab}}(\mathfrak{g})$ of homomorphisms $\varphi$ with image in $\mathfrak{u}$. On any such homomorphism $\varphi$, the exponential $\exp \circ \varphi$ converges, and the commutativity condition is by Lemma 3.5 equivalent to $\exp \circ \varphi$ being a homomorphism. We thus have a map

$$\exp : \mathcal{Z}^{\mathrm{ab}}(\mathfrak{u}) \xrightarrow{\sim} \mathcal{Z}^{\mathrm{ab}}(U)$$

which has an inverse defined by log. We will use this to compare $\Psi_G$ to the map $\mathcal{W}_f$.

5.2. **A non-cyclotomic variant.** In order to allow for more general $K$, we also need a non-cyclotomic variant: Assume that $\mathbb{Q}_p^{\mathrm{cyc}} \not\subseteq K$, but $\zeta_p \in K$. With notation as in Section 2.4, there is then a $\Lambda$-torsor

$$X_\infty^{\mathrm{cyc}} = \mathrm{Spa}(R_\infty^{\mathrm{cyc}}, R_\infty^{\mathrm{cyc}+}) \to X^{\mathrm{cyc}} = \mathrm{Spa}(R^{\mathrm{cyc}}, R^{\mathrm{cyc}+}) \to X = \mathrm{Spa}(R, R^+)$$

given by base-change to the cyclotomic extension $K^{\mathrm{cyc}}|K$. Lemma 2.22 gives a natural section

$$\mathrm{sec} : \mathrm{Hom}_{\mathrm{cts}}(\Delta, U(R^{\mathrm{cyc}}))^Q \to H^1_{\mathrm{cts}}(\Lambda, U(R_\infty^{\mathrm{cyc}}))$$

**Definition 5.2.** Write $\mathcal{Z}^{\mathrm{ab}}(U)(X)$ for the set $\mathrm{Hom}_{\mathrm{cts}}(\Delta, U(R^{\mathrm{cyc}}))^Q$. Composing sec with the Cartan–Leray map for the cover $X_\infty^{\mathrm{cyc}} \to X$, we obtain a map $W_{f,U}$ defined as

$$W_{f,U} : \mathcal{Z}^{\mathrm{ab}}(U)(X) \xrightarrow{\mathrm{sec}} H^1_{\mathrm{cts}}(\Lambda, U(R_\infty^{\mathrm{cyc}})) \to H^1_v(X, U) \to H^1_v(X, G).$$

Replacing $X$ by objects in $X_{\mathrm{\acute{e}t}}$, the $\mathcal{Z}^{\mathrm{ab}}(U)$ define a sheaf of pointed sets $\mathcal{Z}^{\mathrm{ab}}(U)$ on $X_{\mathrm{\acute{e}t}}$, and as before, $W_{f,U}$ defines upon sheafification on $X_{\mathrm{\acute{e}t}}$ a morphism $\mathcal{W}_{f,U} : \mathcal{Z}^{\mathrm{ab}}(U) \to R^1\nu_* G$.

Second, for the Lie algebra, we observe that we have an isomorphism

$$\mathrm{HT}_f : \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathfrak{g}(R^{\mathrm{cyc}}))^Q \xrightarrow{\sim} H^1_{\mathrm{cts}}(\Lambda, \mathfrak{g}(R^{\mathrm{cyc}})) \xrightarrow{\sim} \mathfrak{g} \otimes \widetilde{\Omega},$$

this is clear for $\mathfrak{g} = \mathcal{O}$ and in general follows by tensoring with $\mathfrak{g}$. As before, we then write $\mathcal{Z}^{\mathrm{ab}}(\mathfrak{g})(X) \subseteq \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathfrak{g}(R^{\mathrm{cyc}}))^Q$ for those $\varphi : \Delta \to \mathfrak{g}(R^{\mathrm{cyc}})$ such that $[\varphi(x), \varphi(y)] = 0$ for all $x, y \in \Delta$. This defines a sheaf $\mathcal{Z}^{\mathrm{ab}}(\mathfrak{g})$ that $\mathrm{HT}_f$ identifies with $(\mathfrak{g} \otimes \widetilde{\Omega})^{\wedge=0}$. Let $\mathcal{Z}^{\mathrm{ab}}(\mathfrak{u}) \subseteq \mathcal{Z}^{\mathrm{ab}}(\mathfrak{g})$ be the sheaf of those homomorphisms with image in $\mathfrak{u}(R^{\mathrm{cyc}})$. With these definitions, composing with the exponential again defines an isomorphism

$$\exp : \mathcal{Z}^{\mathrm{ab}}(\mathfrak{u}) \xrightarrow{\sim} \mathcal{Z}^{\mathrm{ab}}(U).$$

As the cocycles in $\mathcal{Z}^1_{\mathrm{cts}}(\Lambda, \mathfrak{u}(R^{\mathrm{cyc}}))$ and $\mathcal{Z}^1_{\mathrm{cts}}(\Lambda, U(R^{\mathrm{cyc}}))$ associated via sec still satisfy the commutativity condition, exp and log commute with passing from $\Delta$-cocycles to $\Lambda$-cocycles.

5.3. **Computing $\Psi_G$ with abelian cocycles.** Assume from now on that we are in either of the two setups of the last two subsections. We can now use abelian cocycles depending on the choice of toric chart to compute $\Psi_G$:

**Proposition 5.3.** *The following diagram commutes:*

(15)
$$\begin{array}{ccc}
\mathcal{Z}^{\mathrm{ab}}(\mathfrak{u}) & \xrightarrow[\sim]{\exp} & \mathcal{Z}^{\mathrm{ab}}(U) \\
{\scriptstyle \mathrm{HT}_f}\downarrow & & \downarrow{\scriptstyle \mathcal{W}_{f,U}} \\
(\mathfrak{g} \otimes \widetilde{\Omega})^{\wedge=0}/G & \xrightarrow{\Psi_G} & R^1\nu_* G
\end{array}$$

*The morphism $\mathrm{HT}_f$ on the left has trivial fibre over $0$.*

**Remark 5.4.** Diagram (15) is the reason for the name HTlog for the inverse of $\Psi_G$.

The role of the morphism $\mathcal{W}_{f,U}$ is therefore that it allows us to define a partial inverse of $\Psi$ on the image of $\mathcal{W}_{f,U}$ by applying log to abelian cocycles $\rho \in \mathcal{Z}^{\mathrm{ab}}(U)$.

*Proof.* We first assume that $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$: We may then compute $\Psi$ using the following choices: For the Galois cover, we take the toric cover $\widetilde{X} := X_\infty \to X$. Fix an isomorphism $\Delta \cong \mathbb{Z}_p^d$, then $\mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}^+(X)) = \mathcal{O}^+(X)^d$. This induces an integral basis $\delta$ of $H^0(X, \widetilde{\Omega})$ via the map $\mathrm{HT}_f$ from Lemma 2.18. Moreover, the left hand side defines canonical representatives $\rho_i \in \mathcal{Z}^1_{\mathrm{cts}}(\Delta, \mathcal{O}^+(X_\infty))$ of the $\delta_i$. By Lemma 4.15, we can now use $\Phi^+_{X,\rho}$ to compute $\Psi^+_X$.

With these choices, any $\theta = \sum \delta_i \otimes A_i \in (\mathfrak{u} \otimes \widetilde{\Omega})^{\wedge=0}$ is sent by $\Phi^+_{X,\rho}$ to the cocycle

$$\Delta \to U(X), \quad \gamma \mapsto \prod_i \exp(\rho_i(\gamma)A_i) = \exp(\sum_i \rho_i(\gamma)A_i)$$

The associated $G$-torsor is exactly $\mathcal{W}_{f,U}(\exp(\theta))$ by definition, so the square (15) commutes.

If $\mathbb{Q}_p^{\mathrm{cyc}} \nsubseteq K$, we use the cover $X_\infty^{\mathrm{cyc}} \to X$ from Section 5.2. By Lemma 2.21.3, we have $\mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}^+(R^{\mathrm{cyc}}))^Q \overset{a}{\cong} \mathcal{O}^+(R)^d$ as $\mathcal{O}^+(R)$-modules. Tensor with $\mathfrak{m}$ and let $\rho'_1, \ldots, \rho'_d$ be the images of the standard basis of $p\mathcal{O}^+(R)^d$ on the left hand side, then via the map

$$\mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}^+(R^{\mathrm{cyc}}))^Q \xrightarrow{\mathrm{sec}} H^1_{\mathrm{cts}}(\Lambda, \mathcal{O}^+(R_\infty^{\mathrm{cyc}})) \to H^1_v(X, \mathcal{O}^+) \xrightarrow{\mathrm{HT}} H^0(X, \widetilde{\Omega})^\circ,$$

the images of the $\rho'_i$ define an integral basis. Let $\rho_1, \ldots, \rho_d$ be the images of the $\rho'_i$ in $H^1_{\mathrm{cts}}(\Lambda, \mathcal{O}^+(R_\infty^{\mathrm{cyc}}))$. Computing $\Psi_G$ for these choices proves the statement, as before. $\square$

5.4. **Surjectivity of $\Psi$.** The map $W_{f,U}$ cannot in general be surjective as the map $H^1_v(X, U) \to H^1_v(X, G)$ is not in general surjective. However, it turns out that for $U$ small enough, this is the only obstruction, namely we now prove that any $v$-$G$-bundle on $X$ becomes abelian on an étale cover. For this we adapt Faltings' method from [Fal05, §2], using the technical means prepared in §2.

**Proposition 5.5.** *Assume that $\zeta_p \in K$. Let $G$ be a rigid group of good reduction over $K$. Then there is $c \geq 0$ such that with $U := G_c$ as defined in Proposition 3.6, the following hold for the map $W_f := W_{f,U}$ from Definition 5.1 (if $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$) or Definition 5.2 (if $\mathbb{Q}_p^{\mathrm{cyc}} \not\subseteq K$):*

  *(1) $W_f$ has the same image as the natural map $H^1_v(X, G_c) \to H^1_v(X, G)$.*
  *(2) Assume that $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$. Then for any two homomorphisms $\varphi_1, \varphi_2 : \Delta \to G_c(X)$ with $W_f(\varphi_1) = W_f(\varphi_2)$, there is $g \in G(X)$ such that $\varphi_1(\gamma) = g\varphi_2(\gamma)g^{-1}$ for all $\gamma \in \Delta$.*

*The constant $c$ only depends on $X$, not on $G$, and we can choose $c$ uniformly among all finite étale subcovers of $X_\infty \to X$.*

We will see at the end of Section 6.1 that part 2 also holds for general $K$.

*Proof.* Let us first explain the argument in the simpler case that $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$: Let $\alpha, \beta$ be as in Lemma 2.14 and let $\gamma = 2\beta + \alpha$. We claim that any $c \geq 5\gamma$ does the job. To see this, we can argue essentially as in [Fal05, Lemma 1.(i)], except that further care needs to be taken in our much more general setting, as the "isomorphism up to torsion"-results are weaker.

It is clear from the construction that $W_f$ factors through the image of $H^1_v(X, G_c)$. Conversely, let $x \in H^1_v(X, G_c)$, then by Lemma 3.11, this $x$ becomes trivial in $H^1_v(X_\infty, G_c)$ and thus comes from an element in $H^1_{\mathrm{cts}}(\Delta, G_c(R_\infty))$ via the Cartan–Leray sequence. Let

$$\rho : \Delta \to G_c(R_\infty) \subseteq G(R_\infty)$$

be any 1-cocycle representing $x$. In the following, for any $s \geq 0$ let us write $\rho_s$ for the image of $\rho$ in $H^1_{\mathrm{cts}}(\Delta, G/G_s(R_\infty))$. Then $\rho_c = 1$. We consider for varying $s \geq c$ the natural map

$$h_s : \mathrm{Hom}_{\mathrm{cts}}(\Delta, G(R)/G_s(R)) \to H^1_{\mathrm{cts}}(\Delta, G/G_s(R_\infty))$$

as well as for any $c \leq k < s$ the reduction map

$$r_k : \mathrm{Hom}_{\mathrm{cts}}(\Delta, G(R)/G_s(R)) \to \mathrm{Hom}_{\mathrm{cts}}(\Delta, G(R)/G_k(R)).$$

Set $t := 3\gamma$. We will inductively construct a compatible system of continuous homomorphisms $\varphi_s : \Delta \to G(R)/G_s(R)$ for any $s \geq c$ such that $h_s(\varphi_s) = \rho_s$ and $r_t(\varphi_s) = 1$.

We start with $s = c$, for which we can take $\varphi_s := 1$ to be the trivial representation.

For the induction, we claim that we can find a lift of the reduction $r_{s-2\gamma}(\varphi_s)$ to a homomorphism $\varphi'_{s+\gamma} : \Delta \to G(R)/G_{s+\gamma}(R)$ such that $h_{s+\gamma}(\varphi'_{s+\gamma}) = \rho_{s+\gamma}$. Note that since $s \geq c \geq 5\gamma$, this still satisfies $r_t(\varphi'_{s+\gamma}) = r_t(\varphi_s) = 1$, so that we can then lift $\varphi_s$ inductively.

For the claim, the conditions imply $s + t < 2s - \alpha_0$, so that by Proposition 3.6 we have a short exact sequence

$$0 \to \overline{\mathfrak{g}}_t^+ \to G/G_{s+t} \to G/G_s \to 1,$$

where $\overline{\mathfrak{g}}_t^+ := \mathfrak{g}_0^+/\mathfrak{g}_t^+$ is isomorphic to $(\mathfrak{m}\mathcal{O}^+/p^t\mathfrak{m}\mathcal{O}^+)^{\dim G}$. It therefore stays exact upon evaluation at the affinoid perfectoid $X_\infty$. For the same reason, $\overline{\mathfrak{g}}_t^+(R_\infty) = \mathfrak{g}_0^+(R_\infty)/p^t$.

On the other hand, by the second part of Proposition 3.6, we also have an exact sequence

(16)        $$0 \to \mathfrak{g}_0^+(R)/p^t \to G(R)/G_{s+t}(R) \to G(R)/G_s(R) \to 1$$

Taking continuous $\Delta$-cohomology of both sequences, we thus obtain a commutative diagram

$$\begin{array}{ccccc}
H^1_{\mathrm{cts}}(\Delta, \mathfrak{g}_0^+(R_\infty)/p^t) & \longrightarrow & H^1_{\mathrm{cts}}(\Delta, G/G_{s+t}(R_\infty)) & \longrightarrow & H^1_{\mathrm{cts}}(\Delta, G/G_s(R_\infty)) \\
\uparrow & & \uparrow{\scriptstyle h_{s+t}} & & \uparrow{\scriptstyle h_s} \\
\mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathfrak{g}_0^+(R)/p^t) & \to & \mathrm{Hom}_{\mathrm{cts}}(\Delta, G(R)/G_{s+t}(R)) & \to & \mathrm{Hom}_{\mathrm{cts}}(\Delta, G(R)/G_s(R)).
\end{array}$$

By induction hypothesis, the image of $\rho_{s+t}$ in the top right has a preimage $\varphi_s$ under $h_s$. Hence there is $b = b_s \in G/G_s(R_\infty)$ such that $b^{-1} \cdot \rho_s(\gamma) \cdot \gamma^* b = \varphi_s(\gamma)$ for all $\gamma \in \Delta$.

We now repeatedly use that by Lemma 2.14, the map

$$(17) \qquad H_{\mathrm{cts}}^i(\Delta, \mathfrak{g}_0^+(R)/p^t) \to H_{\mathrm{cts}}^i(\Delta, \mathfrak{g}_0^+(R_\infty)/p^t)$$

has $p^\gamma$-torsion kernel and cokernel for any $i \geq 1$.

The bottom row of the above diagram is still left-exact, but not necessarily right-exact (it is not clear that one can lift the images to $G(R)/G_{s+t}(R)$ in such a way that they commute). However, as the term on the left in (16) is an abelian group, the obstruction to lifting $\varphi_s$ to the middle of the bottom row is a class in $H_{\mathrm{cts}}^2(\Delta, \mathfrak{g}_0^+(R)/p^t)$ by Proposition A.3.3. This class is mapped to 0 under the map (17) for $i = 2$ since $\rho_s$ does lift and the obstruction class is functorial. As the kernel of (17) is killed by $p^\gamma$, it follows that after reducing the whole diagram mod $p^\gamma$, we can find a lift $\widetilde{\varphi}$ of $\varphi_s$ to the middle of the bottom row of the diagram

$$
\begin{array}{ccccc}
H_{\mathrm{cts}}^1(\Delta, \mathfrak{g}_0^+(R_\infty)/p^t) & \longrightarrow & H_{\mathrm{cts}}^1(\Delta, G/G_{s+t-\gamma}(R_\infty)) & \longrightarrow & H_{\mathrm{cts}}^1(\Delta, G/G_{s-\gamma}(R_\infty)) \\
\uparrow & & \uparrow{\scriptstyle h_{s+t-\gamma}} & & \uparrow{\scriptstyle h_{s-\gamma}} \\
\mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathfrak{g}_0^+(R)/p^t) & \to & \mathrm{Hom}_{\mathrm{cts}}(\Delta, G(R)/G_{s+t-\gamma}(R)) & \to & \mathrm{Hom}_{\mathrm{cts}}(\Delta, G(R)/G_{s-\gamma}(R)).
\end{array}
$$

Let us denote by $\phi$ the image of $\widetilde{\varphi}$ under $h_{s+t-\gamma}$. Then by commutativity of the diagram, $\phi$ and $\rho_{s+t-\gamma}$ both define a lift of $\rho_{s-\gamma}$ (i.e. the image of $\rho$ in the top right) along the top right map. By Proposition A.3.2, these differ by a class $\delta$ in $H_{\mathrm{cts}}^1(\Delta, {}_\phi(\mathfrak{g}_0^+(R_\infty)/p^t))$, where ${}_\phi(\dots)$ denotes the module with the $\Delta$-action twisted by $\phi$: Explicitly, let $\widetilde{b}$ be any lift of $b$ to $G/G_{s+t-\gamma}(R_\infty)$. Then there is a unique cocycle $\delta : \Delta \to {}_\phi(\mathfrak{g}_0^+(R_\infty)/p^t)$ such that

$$(18) \qquad \widetilde{b}^{-1} \cdot \rho(\gamma) \cdot \gamma^* \widetilde{b} = \delta(\gamma) \cdot \phi(\gamma) \text{ in } G/G_{s+t-\gamma}(R_\infty)$$

under the identification $\mathfrak{g}_0^+(R_\infty)/p^t = G_{s-\gamma}(R_\infty)/G_{s+t-\gamma}(R_\infty)$. Since $\rho \equiv 1 \mod p^t$, we have $\phi \equiv 1 \mod p^t$. This shows that on the level of cohomology sets, we have

$$H_{\mathrm{cts}}^1(\Delta, {}_\phi(\mathfrak{g}_0^+(R_\infty)/p^t)) = H_{\mathrm{cts}}^1(\Delta, \mathfrak{g}_0^+(R_\infty)/p^t).$$

Therefore, we can again use (17), this time for $i = 1$: The image of $\delta$ in the cokernel of this map is annihilated by $p^\gamma$. We deduce that after reducing the whole diagram mod $p^\gamma$ once again, the cocycle $\delta$ in the top left can be lifted to a class $\delta'$ in the bottom left. Explicitly, this means that there is $a \in \mathfrak{g}_0^+(R_\infty)/p^t = G_{s-2\gamma}(R_\infty)/G_{s+t-2\gamma}(R_\infty)$ such that

$$\delta'(\gamma) := a^{-1} \cdot \delta(\gamma) \cdot \gamma *_\phi a = a^{-1} \cdot \delta(\gamma) \cdot \phi(\gamma) \cdot \gamma^* a \cdot \phi(\gamma)^{-1}$$

defines a homomorphism $\delta' : \Delta \to G_{s-2\gamma}(R)/G_{s+t-2\gamma}(R)$. Then

$$\widetilde{\varphi}' := \delta' \cdot \widetilde{\varphi} : \Delta \to G(R)/G_{s+t-2\gamma}(R)$$

is a homomorphism which also lifts $\varphi_{s-2\gamma}$ due to exactness of the bottom row, and whose image $\phi' := h_{s+t-2\gamma}(\widetilde{\varphi}')$ is equivalent to $\rho_{s+t-2\gamma}$ in $H_{\mathrm{cts}}^1(\Delta, G/G_{s+t-2\gamma}(R_\infty))$: Namely, we can again be more explicit in terms of cocycles. By definition, we have for any $\gamma \in \Delta$

$$\phi'(\gamma) = \delta'(\gamma) \cdot \phi(\gamma) = a^{-1} \cdot \delta(\gamma) \cdot \phi(\gamma) \cdot \gamma^* a \cdot \phi(\gamma)^{-1} \phi(\gamma) \overset{(18)}{=} (\widetilde{b}a)^{-1} \cdot \rho(\gamma) \cdot \gamma^* (\widetilde{b}a).$$

Note that $b_{s+t-2\gamma} := \widetilde{b}a$ is still a lift of the image $b_{s-2\gamma}$ of $b$ in $G/G_{s+t-2\gamma}(R_\infty)$ as $a$ is trivial mod $G_{s-2\gamma}(R_\infty)$. Using $s + t - 2\gamma = s + \gamma$, this shows that $\varphi_{s+\gamma} := \widetilde{\varphi}'$ defines a preimage of $\rho_{s+\gamma}$ under $h_{s+\gamma}$ with the desired properties, completing the induction step.

In the limit $s \to \infty$, the $\varphi_s : \Delta \to G(R)/G_s(R)$ define by Lemma 3.7 a homomorphism $\varphi : \Delta \to G(R)$, and the $b_s$ define an element $b \in G(R_\infty)$ such that

$$\varphi(\gamma) = b^{-1}\rho(\gamma)\gamma^* b.$$

This shows that $\rho$ and $\varphi$ define the same class in $H^1_{\text{cts}}(\Delta, G(R_\infty))$. Thus $W_f(\varphi)$ agrees with the image of $x$ in $H^1_v(X, G)$. This proves part 1 in the case that $\mathbb{Q}_p^{\text{cyc}} \subseteq K$.

To deduce part 2, we run the above argument with $\rho := \varphi_1$ and $\widetilde{\varphi} := \varphi_2$. Inductively, we then see that we may choose $b_s \in G(R)/G_s(R)$: For the induction start $c = s$ this is clear. In the induction step, we may then find $a \in \mathfrak{g}_0^+(R)/p^t$, and thus $\widetilde{b} \in G(R)/G_s(R)$ such that $\varphi_{1,s} = \widetilde{b}^{-1}\varphi_{2,s}\widetilde{b}$. Again using Lemma 3.7, this defines in the limit $s \to \infty$ the desired element $b \in G(R)$. This finishes the proof in the case that $K$ contains $\mathbb{Q}_p^{\text{cyc}}$.

In the general case, we can argue in exactly the same way, but instead use the diagram

$$
\begin{array}{ccccc}
H^1_{\text{cts}}(\Lambda, \mathfrak{g}_0^+(R_\infty^{\text{cyc}})/p^t) & \longrightarrow & H^1_{\text{cts}}(\Lambda, G/G_{s+t}(R_\infty^{\text{cyc}})) & \longrightarrow & H^1_{\text{cts}}(\Lambda, G/G_s(R_\infty^{\text{cyc}})) \\
\uparrow & & \uparrow{\scriptstyle h_{s+t}} & & \uparrow{\scriptstyle h_s} \\
H^1_{\text{cts}}(\Lambda, \mathfrak{g}_0^+(R^{\text{cyc}})/p^t) & \to & H^1_{\text{cts}}(\Lambda, G(R^{\text{cyc}})/G_{s+t}(R^{\text{cyc}})) & \to & H^1_{\text{cts}}(\Lambda, G(R^{\text{cyc}})/G_s(R^{\text{cyc}})).
\end{array}
$$

The map on the the left, and the map on obstruction classes in $H^2$, have $p^\gamma$-torsion cokernel by Lemma 2.21. The same inductive lifting procedure as in the cyclotomic case then shows that we can lift $\rho$ to an element $\varphi \in H^1_{\text{cts}}(\Lambda, G(R^{\text{cyc}}))$. In fact, we can arrange for $\varphi$ to be in the image of the map sec of Lemma 2.22: As in the first part of the proof, we find a lift $\widetilde{\varphi}$ of $\varphi_s$ to the middle of the bottom row. In an additional step, we now apply sec$\circ$res to this: Since in the induction start $\varphi_s$ is trivial, and sec and res are functorial, this shows inductively that sec$\circ$res$(\widetilde{\varphi})$ is also a lift of $\varphi_s$. Second, on $H^1_{\text{cts}}(\Lambda, \mathfrak{g}_0^+(R_\infty^{\text{cyc}})/p^t)$, the map res is an isomorphism up to the $p^\gamma$-torsion kernel, which is a direct factor. We can therefore reduce modulo $p^\gamma$ once more to arrange that the difference $\delta'$ from the last proof is already in the image of sec. This way we ensure inductively that $\varphi_{s+\gamma}$ is in the image of sec.     $\square$

We can now prove the next instance of Theorem 4.1, the case that $G$ has good reduction:

**Proposition 5.6.** *Let $K$ be a perfectoid field over $\mathbb{Q}_p$. Let $G$ be a rigid group over $K$ of good reduction. Then for any smoothoid space $X$ over $K$, the map $\Psi_G$ is surjective. If moreover $\mathbb{Q}_p^{\text{cyc}} \subseteq K$, then $\Psi_G$ is an isomorphism.*

*Proof.* We may work locally in $X_{\text{ét}}$ and assume that there is a toric chart $f : X \to \mathbb{T}^d \times Y$, as well as $\zeta_p \in K$. We then have the commutative diagram Proposition 5.3.

To see that $\Psi_G$ is surjective, let $x \in H^1_v(X, G)$. Let $k \in \mathbb{N}$ and consider the subgroup $G_k \subseteq G$ from Proposition 3.6. By Proposition 3.12, every v-$G$-bundle on $X$ admits a reduction of structure group to $G_k$ on some étale cover $X' \to X$. Going up the toric tower of $X'$, we can for $k \gg 0$ arrange by Lemma 2.14.2 that $k > c$ where $c$ is the constant of Proposition 5.5 for $X'$. Now $x$ is in the image of $H^1_v(X', G_c) \to H^1_v(X', G)$. By Proposition 5.5.1, this implies that $x \in \text{im}(W_f)$. It follows from Proposition 5.3 that $x \in \text{im}(\Psi_G)$.

To see that $\Psi_G$ is injective, let $\theta_1, \theta_2 \in (\mathfrak{g} \otimes \widetilde{\Omega})^{\wedge=0}(X)$ be such that $\Psi_G(\theta_1) = \Psi_G(\theta_2)$. By Proposition 5.3, we can pass to an étale cover $X' \to X$ where we can lift these to homomorphisms $\widetilde{\theta}_1, \widetilde{\theta}_2 \in \mathcal{Z}^{\text{ab}}(\mathfrak{u})$. Then by commutativity, we have $W_f(\exp(\widetilde{\theta}_1)) = W_f(\exp(\widetilde{\theta}_2))$, which by Proposition 5.5.2 implies that $\exp(\widetilde{\theta}_1)$ and $\exp(\widetilde{\theta}_2)$ are conjugated via $G(X')$. By Lemma 3.5.3, it follows that already $\widetilde{\theta}_1, \widetilde{\theta}_2$ are conjugated via $G(X')$, thus so are $\theta_1, \theta_2$.     $\square$

**Corollary 5.7.** *Let $K$ be any perfectoid field over $\mathbb{Q}_p$. Let $G$ be any rigid group over $K$. Then $\Psi_G$ is surjective on any smoothoid space over $K$.*

*Proof.* By Corollary 3.4, there is an open subgroup $G^\circ \subseteq G$ of good reduction. The diagram

$$
\begin{array}{ccc}
\mathrm{Higgs}_{G^\circ} & \xrightarrow{\ \Psi_{G^\circ}\ } & R^1\nu_* G^\circ \\
\downarrow & & \downarrow \\
\mathrm{Higgs}_G & \xrightarrow{\ \Psi_G\ } & R^1\nu_* G
\end{array}
$$

commutes by functoriality of $\Psi$. By Proposition 3.12, the right vertical map is surjective. The top morphism is surjective by Proposition 5.6. Thus the bottom map is surjective.  $\square$

5.5. **Injectivity of $\Psi$.** Throughout this subsection, we assume that $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$ and retain the notation of Section 5.1, i.e. $X = \mathrm{Spa}(R, R^+)$ is a toric smoothoid space with $\Delta$-torsor $X_\infty = \mathrm{Spa}(R_\infty, R_\infty^+) \to X$. In order to prove that $\Psi_G$ is injective, we use the map $W_f \colon \mathrm{Hom}(\Delta, U(R)) \to H^1_{\mathrm{cts}}(X, G(R_\infty))$ from Section 5.1. Unravelling Definition A.1, we see that we have the following:

**Lemma 5.8.** *For any two $\rho_1, \rho_2 \colon \Delta \to U(R)$ in $\mathcal{Z}^{\mathrm{ab}}(G)(X)$, the following are equivalent:*
  *(1) $W_f(\rho_1) = W_f(\rho_2)$*
  *(2) There is $A \in G(R_\infty)$ such that*

$$
(19) \qquad\qquad \rho_1(\gamma) = A^{-1} \cdot \rho_2(\gamma) \cdot \gamma^* A \quad \text{for all } \gamma \in \Delta.
$$

Equivalently, (19) expresses that there is an isomorphism between the $G$-torsors on $X_v$ associated to $\rho_1$ and $\rho_2$. As we will explain in detail in the next section, the following Proposition says that this can be translated into an isomorphism of $G$-Higgs bundles:

**Proposition 5.9.** *Assume that there is $A \in G(R_\infty)$ such that (19) holds. Then $A \in G(R)$.*

For $G = \mathrm{GL}_n$, this is the analogue of [Fal05, Lemma 1.(ii)] in our setting. However, our argument is different to Faltings', which uses the embedding of $\mathrm{GL}_n$ into the ambient ring $M_n$, for which there is no analogue for general rigid groups $G$.

*Proof.* The proof consists of two steps: A "decompletion" showing that $A \in G(X_n)$ for some subcover $X_\infty \to X_n \to X$, and then the descent further to $G(X)$. We begin with the latter:

**Claim 5.10.** *Assume that (19) holds with $A \in G(R_n)$ for some $n \in \mathbb{N}$. Then $A \in G(R)$.*

*Proof.* $A$ is fixed by $\Delta_n := p^n\Delta$, hence $\rho_1(\gamma^{p^n}) = A^{-1} \cdot \rho_2(\gamma^{p^n}) \cdot A$ for all $\gamma \in \Delta$. Let $x := \rho_1(\gamma)$ and $y := \rho_2(\gamma)$. Consider the subgroups $V := U(R) \cap AU(R)A^{-1} \subseteq U(R)$ and $\mathfrak{v} := \mathfrak{u}(R) \cap \mathrm{ad}(A^{-1})(\mathfrak{u}(R)) \subseteq \mathfrak{u}(R)$, then the above equation shows that $y^{p^n} \in V$. Let moreover $V' := U(R) \cap A^{-1}U(R)A \subseteq U(R)$ and $\mathfrak{v}' := \mathfrak{u}(R) \cap \mathrm{ad}(A)(\mathfrak{u}(R)) \subseteq \mathfrak{u}(R)$, then by functoriality of the logarithm, we have a commutative diagram of bijections

$$
\begin{array}{ccc}
V & \xrightarrow{\ g \mapsto A^{-1}gA\ } & V' \\
{\scriptstyle \log}\downarrow & & \downarrow{\scriptstyle \log} \\
\mathfrak{v} & \xrightarrow{\ \mathrm{ad}(A)\ } & \mathfrak{v}'.
\end{array}
$$

It follows that $p^n \log(x) = \log(x^{p^n}) = \log(A^{-1}y^{p^n}A) = \mathrm{ad}(A)(\log(y^{p^n})) = p^n \mathrm{ad}(A)\log(y)$ in $\mathfrak{v}' \subseteq \mathfrak{g}(R)$, which implies $\log(x) = \mathrm{ad}(A)(\log(y))$ in the $K$-vector space $\mathfrak{g}(R)$. It follows from this that we still have $\log x \in \mathfrak{v}'$. We can therefore reverse the calculation using $\exp$,

$$
x = \exp(\mathrm{ad}(A)(\log(y))) = A^{-1} \cdot y \cdot A.
$$

Since $x = A^{-1}y\gamma^*A$ by assumption, this shows that $\gamma^*A = A$. It follows that $A \in G(R)$.  $\square$

**Claim 5.11.** *If $G$ has good reduction, there is $c \geq 0$ depending on $X$ such that for any $\rho_1, \rho_2 : \Delta \to G_c(R)$ in $\mathcal{Z}^{\mathrm{ab}}(G_c)(X)$ and $A \in G(R_\infty)$ such that (19) holds, we have $A \in G(R)$.*

*Proof.* Let $c$ be as in Proposition 5.5, then we already know that there is $B \in G(R)$ such that $\rho_1 = B^{-1}\rho_2 B$. Replacing $A$ by $B^{-1}A$, we may therefore assume that $\rho_1 = \rho_2 =: \rho$.

Consider $G(R_\infty)$ endowed with the action of $\Delta$ twisted by $\rho$ in the sense of Definition A.2, i.e. $\gamma *_\rho x = \rho(\gamma)\gamma^* x \rho(\gamma)^{-1}$, we write this $\Delta$-module as $_\rho G(R_\infty)$. Then (19) can be expressed as $A \in H^0(\Delta, {}_\rho G(R_\infty))$, and we wish to see that this is already in the image of

$$H^0(\Delta, {}_\rho G(R)) \to H^0(\Delta, {}_\rho G(R_\infty)).$$

This can be seen by the same argument as in Proposition 5.5 but in cohomological degree 0:

Let $\gamma$ be as in Lemma 2.14, then $c \geq t := 3\gamma$. We first observe that in the v-sheaf $\overline{G}_c := G/G_c$, where $\rho_1$ becomes trivial, the image $\overline{A}_c$ of $A$ satisfies $\gamma^*\overline{A}_c = \overline{A}_c$ and is thus contained in $\overline{G}_c(R_\infty)^\Delta = \overline{G}_c(R)$. By [Heu22a, Lemma 4.15, Proposition 4.17], we may find an étale cover $X' \to X$ on which $\overline{A}_c$ lifts to $G(X')/G_c(X')$. Moving up the toric tower of $X'$, we may by Lemma 2.14.2 assume after replacing $\Delta$ by an open subgroup $\Delta_n$ that $c$ also works for $X'$. Replacing $X$ by $X'$, this first shows that $A \in G(X'_n)$, and then by Claim 5.10 that $A \in G(X')$ is already $\Delta$-invariant, thus $A \in G(X)$. We may therefore assume that $\overline{A}_c$ lifts to $G(X)/G_c(X)$.

We now show inductively that for any $s \geq c$, the image $\overline{A}_s$ of $A$ in $\overline{G}_s$ is $\Delta$-invariant and lifts to $G(R)/G_s(R)$: If we know this for $s$, then by Proposition 3.6 we have exact sequences

$$0 \to ({}_\rho\mathfrak{g}_0^+(R_\infty)/p^t)^\Delta \to ({}_\rho G/G_{s+t}(R_\infty))^\Delta \longrightarrow ({}_\rho G/G_s(R_\infty))^\Delta \to H^1_{\text{cts}}(\Delta, {}_\rho\mathfrak{g}_0^+(R_\infty)/p^t)$$
$$\uparrow h_t \qquad\qquad \uparrow h_{s+t} \qquad\qquad \uparrow h_s \qquad\qquad \uparrow h_t$$
$$0 \to ({}_\rho\mathfrak{g}_0^+(R)/p^t))^\Delta \to ({}_\rho G(R)/G_{s+t}(R))^\Delta \to ({}_\rho G(R)/G_s(R))^\Delta \to H^1_{\text{cts}}(\Delta, {}_\rho\mathfrak{g}_0^+(R)/p^t).$$

As the first and last vertical maps are isomorphisms up to $p^\gamma$-torsion by Lemma 2.14, the same chase as in Proposition 5.5 shows that there is a preimage of $\overline{A}_{s+t-2\gamma}$ under $h_{s+t-2\gamma}$ that lifts to $G(R)$. This shows that $A \in \varprojlim_{s \in \mathbb{N}} G(X)/G_s(X) = G(X)$. $\square$

We now return to the case of general $G$. To prove Proposition 5.9, it suffices by Claim 5.10 to prove that $A$ is fixed by an open subgroup of $\Delta$. For the proof, we are allowed to make the following two different kinds of localisation steps:

(1) First, we may of course replace $\Delta$ by an open subgroup $\Delta_n$ and $X$ by the corresponding intermediate étale cover $X_\infty \to X_n \to X$.
(2) Second, since the property of an element of $G(R_\infty)$ of being fixed by $\Delta$ can be checked étale-locally on $X$, we may replace $X$ by an étale cover $X' \to X$ and $X_\infty \to X$ by the pullback $X'_\infty := X_\infty \times_X X' \to X'_\infty$. This does not change $\Delta$.

Keeping this in mind, consider the image of $A$ in $G/U(R_\infty)$. By [Heu22a, Proposition 4.1], there is $n \in \mathbb{N}$ such that this comes from $G/U(R_n)$. After replacing $X$ by an étale cover of type (2) above, we can assume that this lifts to an element $A_n \in G(R_n)$. Then $A^\circ := A \cdot A_n^{-1} \in U(R_\infty)$. Consider $\rho_3(\gamma) := A_n \rho_2(\gamma) A_n^{-1}$ defined on the open subgroup $\Delta_n \subseteq \Delta$ that fixes $R_n \subseteq R_\infty$. On the level of Lie algebras we see that we can find a small rigid open subgroup $H \subseteq U$ such that $A_n H(R_\infty) A_n^{-1} \subseteq U(R_\infty)$. After a localisation of type (1) above, we may replace $\Delta$ (and hence $\Delta_n$) by an open subgroup such that $\rho_2$ has image in $H$. Then $\rho_3$ is also a continuous homomorphism of the form $\rho_3 : \Delta \to U(R_n)$. Then for any $\gamma \in \Delta_n$,

$$\rho_1(\gamma) = A^{-1} \cdot \rho_2(\gamma) \cdot \gamma^* A = A^{-1} \cdot A_n^{-1}\rho_3(\gamma)A_n \cdot \gamma^* A = A^{\circ -1}\rho_3(\gamma)\gamma^* A^\circ.$$

We now apply Claim 5.11 to $\rho_1$ and $\rho_3$ to deduce that $A^\circ \in U(R_n)$, hence $A \in G(R_n)$. $\square$

**Proposition 5.12.** *Let $K|\mathbb{Q}_p^{\text{cyc}}$ be a perfectoid field. Let $X$ be any smoothoid over $K$ and let $G$ be any rigid group over $K$. Then $\Psi_G$ is injective.*

*Proof.* We may assume that $X$ is toric and choose a toric chart. We consider diagram (15):

Let $\theta_1, \theta_2 \in (\mathfrak{g} \otimes \widetilde{\Omega})^{\wedge=0}(X)$ be such that $\Psi_G(\theta_1) = \Psi_G(\theta_2)$. As in the last part of the proof of Proposition 5.6, we can pass to an étale cover $X' \to X$ where we can lift these to homomorphisms $\widetilde{\theta}_1, \widetilde{\theta}_2$ in the top left, so that $W_f(\exp(\widetilde{\theta}_1)) = W_f(\exp(\widetilde{\theta}_2))$ inside $H^1_v(X', G)$. By Lemma 5.8, there is $A \in G(R_\infty)$ such that $\exp(\widetilde{\theta}_1(\gamma)) = A^{-1}\exp(\widetilde{\theta}_2(\gamma))\gamma^* A$. By Proposition 5.9, we have $A \in G(X')$, so $\exp(\widetilde{\theta}_1)$ and $\exp(\widetilde{\theta}_2)$ are conjugated via $A$. By Lemma 3.5.3 it follows that already $\widetilde{\theta}_1, \widetilde{\theta}_2$ are conjugated via $G(X')$, thus so are $\theta_1, \theta_2$. $\qquad\square$

This finishes the proof the $\Psi_G$ is an isomorphism if $K$ contains $\mathbb{Q}_p^{\mathrm{cyc}}$. One could prove the case of non-cyclotomic perfectoid $K$ directly with some more work, but what we have shown so far is already enough to deduce this case by descent in the next section.

## 6. THE LOCAL $p$-ADIC SIMPSON CORRESPONDENCE FOR $G$

The technical work of the previous section also proves a new instance of Faltings' "local $p$-adic Simpson correspondence": Via the identification of "generalised representations" with $v$-vector bundles ([Heu22a, Proposition 2.3]), we can now generalise this from $\mathrm{GL}_n$ to rigid groups $G$ and from rigid spaces to perfectoid families of such. For the formulation, we need a notion of "smallness" on either side of the correspondence. In the setting of rigid groups, this is not always intrinsic to $G$. Rather, it depends on an integral structure on $G$: The choice of a rigid open subgroup $G^+ \subseteq G$ of good reduction, which always exists by Corollary 3.4.

**Example 6.1.** For $G = \mathrm{GL}_n$, the canonical choice $G^+ = \mathrm{GL}_n(\mathcal{O}^+)$ recovers the classical setting of the local Simpson correspondence. More generally, there is a canonical choice for $G^+$ when $G$ is the analytification of an algebraic group over $K$ that extends to a group scheme $\mathcal{G}$ over $\mathcal{O}_K$: Namely, we can then take $G^+ := \mathcal{G}(\mathcal{O}^+)$, which is represented by the generic fibre of the $p$-adic completion of $\mathcal{G}$. This works for any split reductive groups.

Recall from Proposition 3.6 that given $G^+$, we get a canonical system of open subgroups $G_k^+$ for $k \geq 0$, as well as the integral subgroups $\mathfrak{g}_k^+ \subseteq \mathfrak{g}$ of the Lie algebra. By Proposition 3.3, there is $\alpha > 0$ such that for any $k > \alpha$, we have an exponential map $\exp : \mathfrak{g}_k^+ \xrightarrow{\sim} G_k^+$.

**Definition 6.2.** Let $X$ be a toric smoothoid space and fix a toric chart $f : X \to \mathbb{T}^d \times Y$ (Definition 2.4). Let $G$ be a rigid group and fix an open subgroup $G^+ \subseteq G$ of good reduction.
  (1) A $v$-$G$-bundle $V$ on $X_v$ is called **small** (with respect to $G^+$ and $f$) if it admits a reduction of structure group to $G_c^+$ for some $c$ such that Proposition 5.5 holds.
  (2) A $G$-Higgs bundle $(E, \theta)$ on $X$ is called **small** (with respect to $G^+$ and $f$) if for some $c > \alpha$ such that Proposition 5.5 holds, there is a $G_c^+$-torsor $E^+$ with an isomorphism $E = G \times^{G_c^+} E^+$ with respect to which $\theta$ is a section of the integral $\mathcal{O}^+$-submodule

$$\mathrm{ad}(E)^+ \otimes_{\mathcal{O}^+} \widetilde{\Omega}^+ \subseteq \mathrm{ad}(E) \otimes_{\mathcal{O}} \widetilde{\Omega},$$

  where $\mathrm{ad}(E)^+ := \mathfrak{g}_c^+ \times^{G_c^+} E^+ \subseteq \mathrm{ad}(E)$ is the $\mathcal{O}^+$-sublattice induced by $E$, and $\widetilde{\Omega}^+ \subseteq \widetilde{\Omega}$ is the locally free integral submodule induced by the chart $f$, see Definition 2.19.

For $G = \mathrm{GL}_n$ and $G^+ := \mathrm{GL}_n(\mathcal{O}^+)$, we also define a $v$-vector bundle $V$ to be small if it is the change of fibre of a $v$-topological $1 + p^c \mathfrak{m} M_n(\mathcal{O}^+)$-bundle $V_c$ to $\mathbb{G}_a^d$, i.e. $V \cong \mathbb{G}_a^d \times^{G_c} V_c$. By a small Higgs bundle, we then mean the change of fibre of a small Higgs bundle to $\mathbb{G}_a^d$.

**Lemma 6.3.** *Any small $v$-$G$-bundle on $X_v$ that is étale-locally trivial is already globally trivial on $X$. In particular, the $G$-bundle underlying any small Higgs bundle is trivial.*

*Proof.* By Proposition 5.5, any element $x$ in the image of $H^1_v(X, G_c^+) \to H^1_v(X, G)$ is in the image of $W_f$. On the other hand, if $x$ comes from $H^1_{\mathrm{ét}}(X, G)$, it is sent to 0 in $R^1\nu_* G(X)$, thus the image under $\mathcal{W}_f : \mathcal{Z}^{\mathrm{ab}}(G_c)(X) \to R^1\nu_* G(X)$ is trivial. But the fibre over 0 of this map is trivial by Proposition 5.3 and Corollary 6.12. Hence $x = 0$. $\qquad\square$

An important fact is that all $v$-$G$-bundles and $G$-Higgs bundles are étale-locally small:

**Lemma 6.4.** *Fix any open subgroup $G^+ \subseteq G$ of good reduction. Then for any $v$-$G$-bundle $V$ on $X_v$, there is an étale cover $(h_i : X_i \to X)_{i \in I}$ by toric smoothoids with toric charts $f_i$ for $X_i$ such that $h_i^* V$ is small with respect to $G^+$ and $f_i$.*

*The analogous statement for Higgs bundles also holds.*

*Proof.* By Proposition 3.12, we can after replacing $X$ by an étale cover find a reduction of structure group of $V$ to $G_k^+$ for some $k > \alpha$. This does not prove the lemma yet since passage to an étale cover might change the required bound $c$, which depends on $X$. However, we now can argue exactly as in the proof of [Heu22a, Lemma 2.27]: By Lemma 3.11, the $v$-$G$-bundle $V$ becomes trivial on $X_\infty \to X$. It therefore corresponds to a class $[\rho]$ in $H^1_{\mathrm{cts}}(\Delta, G_k^+(X_\infty))$. Going up the toric tower, which does not change $c$ by Proposition 5.5, we can replace $\Delta$ by an open subgroup that is sent into $G_c^+(X_\infty)$ by $\rho$.

For $G$-Higgs bundles, it remains to prove that we can make $\theta$ small by going up the toric tower. This follows from Lemma 4.11: More explicitly, in Definition 2.19, replacing $\Delta$ by the subgroup $p^n\Delta$ has the effect of replacing $\widetilde{\Omega}^+$ by $p^{-n}\widetilde{\Omega}^+$.                                    $\square$

**Theorem 6.5** (Local $p$-adic Simpson correspondence for $G$-bundles in perfectoid families)**.** *Let $X$ be a toric smoothoid space over $K$ and fix a toric chart $f : X \to \mathbb{T}^d \times Y$ (see Definition 2.4). Let $G$ be a rigid group over $K$. Fix an open subgroup $G^+ \subseteq G$ of good reduction. This induces a notion of smallness of $G$-bundles (see Definition 6.2).*

(1) *There is an equivalence of categories*

$$LS_f : \{\textit{small } G\textit{-bundles on } X_v\} \xrightarrow{\sim} \{\textit{small } G\textit{-Higgs bundles on } X_{\text{ét}}\}$$

*that is natural in $G$, but not in general independent of the chart $f$.*

(2) *In the case of $G = \mathrm{GL}_n$, this extends to an exact equivalence of categories*

$$\{\textit{small vector bundles on } X_v\} \xrightarrow{\sim} \{\textit{small Higgs bundles on } X_{\text{ét}}\}.$$

(3) *Let $g : X' \to X$ be any morphism of smoothoids with toric charts $f$ of $X$ and $f'$ of $X'$. Then any morphism of toric charts $\widetilde{g} : f' \to f$ over $g$ induces a natural equivalence*

$$
\begin{array}{ccc}
\{\textit{small } G\textit{-bundles on } X'_v\} & \xrightarrow{LS_{f'}} & \{\textit{small } G\textit{-Higgs bundles on } X'_{\text{ét}}\} \\
g^* \big\uparrow & \overset{\widetilde{g}}{\Longleftarrow} & g^* \big\uparrow \\
\{\textit{small } G\textit{-bundles on } X_v\} & \xrightarrow{LS_f} & \{\textit{small } G\textit{-Higgs bundles on } X_{\text{ét}}\}
\end{array}
$$

*where the vertical functors are defined only on the respective subcategories of small objects with respect to $X$ for which the pullback along $g$ is small with respect to $X'$.*

(4) *Let $V$ be a small $v$-vector bundle on $X_v$ and let $LS_f(V) = (E, \theta)$. Then there is a natural isomorphism $R\Gamma_v(X, V) = R\Gamma_{\text{Higgs}}(X, (E, \theta))$ in $\mathcal{D}(\mathcal{O}(X))$. In particular, for $\nu : X_v \to X_{\text{ét}}$ we get a natural isomorphism $R\nu_* V = R\Gamma_{\text{Higgs}}(E, \theta)$ in $\mathcal{D}(X_{\text{ét}})$.*

**Remark 6.6.**    (1) For functoriality in $f$ in part 3, we use the notion of morphisms of toric charts from Definition 2.4.2. We caution that for a given morphism $X' \to X$, it is not always possible to find charts $f$ and $f'$ that admit a morphism of charts between them.

(2) Our notion of "smallness" is more restrictive than that in other setups like [Fal05] [AGT16] [Tsu18] [Wan23]. This seems necessary to treat general rigid groups $G$. We discuss the precise comparison to these other works in Section 6.3.

That being said, our notion of "smallness" does not aim to be optimal. As we will see in §7, what is usually relevant for applications of the local correspondence is that there is *some* explicit notion of smallness for which both Lemma 6.4 and Theorem 6.5.(1) hold. See also Remark 6.14 for some comments on other notions of smallness.

(3) The last sentence in (3) is due to the fact that in general, $g^*$ may not preserve smallness, as the implicit constant $c$ in Proposition 5.5 depends on $X$. However, in many concrete situations of interest, $g^*$ does preserve smallness, for example when $X = Y$ and $g$ describes some Galois action.

(4) Naturality in $G$ in part 1 means the following: Let $\varphi : G \to H$ be a homomorphism of rigid groups and choose subgroups of good reduction $G^+ \subseteq G$ and $H^+ \subseteq H$ such that $\varphi(G^+) \subseteq H^+$. Then for any $v$-$G$-bundle that is small with respect to $G^+$ and $f$, the pushout along $\varphi$ is a small $v$-$H$-bundle with respect to $H^+$ and $f$: This follows from the fact that $c$ in Proposition 5.5 does not depend on $G$. The analogous statement holds for Higgs bundles. The statement is then that $LS_f$ commutes with pushout along $\varphi$.

(5) For $G = \mathrm{GL}_n$, apart from exactness, the difference between Theorem 6.5.(1) and (2) is that in (2), we allow morphisms that are not necessarily isomorphisms, see Remark 3.9.

(6) Even if $G$ is reductive and $X$ is a rigid space, we do not see how Theorem 6.5.(1) could simply be deduced from $G = \mathrm{GL}_n$ by the Tannakian formalism, due to the smallness conditions.

(7) The cohomological comparison in Theorem 6.5.(4) is the generalisation to smoothoids of results that are known for smooth rigid spaces at least in the arithmetic setup: For Faltings local correspondence, this is due to Faltings [Fal05, p852], see also [AGT16, IV]. It is also closely related to [LZ17, Theorem 2.1.(v)] and [MW, Theorem 3.13] which give analogous comparison results in the analytic setting over discretely valued fields. Our proof of (4) follows a similar strategy. Apart from the different base field, our result can be interpreted as giving a relative version for families of rigid spaces.

## 6.1. **Proof of the local correspondence for general $G$.**

*Proof of Theorem 6.5.* We first assume that $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$, then the toric cover $X_\infty \to X$ associated to $f$ in Section 2.3 is an affinoid perfectoid $\Delta$-torsor. In this setting, our technical preparations so far allow us to essentially follow Faltings' construction [Fal05, Theorem 3]:

We will define a functor $LS_f^{-1}$ from right to left, and show that this is fully faithful and essentially surjective. Let $(E, \theta)$ be a small $G$-Higgs bundle and let $E^+ \subseteq E$ be a reduction of structure group to $G_c^+$ with respect to which $\theta$ has coefficients in $\mathrm{ad}(E)^+$. As explained in the beginning of Section 5, $\theta$ can be written via Lemma 2.18 as a continuous homomorphism $\rho : \Delta \to \mathrm{ad}(E)^+(X)$ with commutative image. By Lemma 6.3 and Lemma 3.5, we can thus apply exp to turn this into a continuous 1-cocycle $\exp(\rho) : \Delta \to \underline{\mathrm{Aut}}_{G_c^+}(E^+)(X)$. This defines via Cartan–Leray a small $v$-$G$-bundle $V_\rho$ on $X_v$ that sends $W \in X_v$ to

$$V_\rho(W) := \{s \in E(X_\infty \times_X W) | \gamma \cdot s = \exp(-\rho(\gamma))s \text{ for all } \gamma \in \Delta\}.$$

(The sign in front of $\rho(\gamma)$ is usually required due to our conventions on cocycles Definition A.1, to make the cocycle condition translate into $\gamma_1\gamma_2 \cdot x = \gamma_1^* \exp(-\rho(\gamma_2)) \exp(-\rho(\gamma_1))x$. Here it is of minor importance as $\rho$ has commutative image). We now set $LS^{-1}(E, \theta) := V_\rho$.

We note that $E$ is a trivial $G$-bundle on $X$ by Lemma 6.3, and $V_\rho$ is given by regarding $\exp(\rho)$ as a descent datum for $E$ along $q : X_\infty \to X$. In particular, $V_\rho$ becomes trivial on $X_\infty$. We also note that this immediately gives a natural isomorphism of $G$-bundles on $X_\infty$

$$(20) \qquad q^*E \xrightarrow{\sim} q^*V_\rho.$$

We claim that the construction $(E, \theta) \mapsto V_\rho$ is functorial: A morphism of $G$-Higgs bundles $(E_1, \theta_1) \to (E_2, \theta_2)$ is the same as a morphism of $G$-bundles $\varphi : E_1 \to E_2$ such that for the homomorphisms $\rho_1 : \Delta \to \mathrm{ad}(E_1)^+(X)$ and $\rho_2 : \Delta \to \mathrm{ad}(E_2)^+(X)$ associated to $\theta_1, \theta_2$ we have

$$(21) \qquad \varphi^{-1}\rho_2(\gamma)\varphi = \rho_1(\gamma) \quad \forall \gamma \in \Delta.$$

By Lemma 3.5.3, it follows that inside $E_1(X)$, we have

$$(22) \qquad \varphi^{-1} \exp(-\rho_2(\gamma))\varphi = \exp(-\rho_1(\gamma)) \quad \forall \gamma \in \Delta,$$

which implies that for any $s \in E(X_\infty \times_X W)$, we have $\gamma \cdot \varphi(s) = \exp(-\rho_2(\gamma))\varphi(s)$ for all $\gamma \in \Delta$. This shows that the natural homomorphism $\varphi : E_1(X_\infty \times_X W) \to E_2(X_\infty \times_X W)$ restricts to the desired map $V_{\rho_1}(W) \to V_{\rho_2}(W)$.

$LS^{-1}$ **is essentially surjective:** Let $V$ be a small $v$-$G$-bundle on $X$. Then by Proposition 5.5.1, we can find $\rho : \Delta \to G_c^+(X)$ such that the class of $V$ in $H_v^1(X, G)$ is equal to $W_f(\rho)$. Then the $G$-Higgs bundle $(G, \theta)$ with $\theta := \log(\rho)$ is such that $LS^{-1}(G, \theta) \cong V$.

$LS^{-1}$ **is fully faithful:** Let $(E_1, \theta_1)$ and $(E_2, \theta_2)$ be two small $G$-Higgs bundles on $X$. By Lemma 6.3, we can find isomorphisms $E_1 = G$ and $E_2 = G$. Let $\phi : V_{\rho_1} \to V_{\rho_2}$ be any morphism between the associated $v$-topological $G$-torsors. Since $V_{\rho_1}$ and $V_{\rho_2}$ become trivial on $X_\infty$ by construction, this is by descent the same as a $\Delta$-invariant $G$-linear homomorphism $\phi : V_{\rho_1}(X_\infty) \to V_{\rho_2}(X_\infty)$. Choosing generators $x$ and $y$ on either side, this is a $\Delta$-equivariant homomorphism $\phi : G(X_\infty)x \to G(X_\infty)y$. Let $A \cdot y$ be the image of $1 \cdot x \in G(X_\infty)x$, then the $\Delta$-equivariance means that for all $\gamma \in \Delta$, we have:

$$\begin{aligned} \gamma \cdot \phi(x) = \phi(\gamma \cdot x) &\Rightarrow \gamma^* A \exp(-\rho_2(\gamma)) \cdot y = \exp(-\rho_1(\gamma))A \cdot y \\ &\Rightarrow A \exp(\rho_2(\gamma))\gamma^* A^{-1} = \exp(\rho_1(\gamma))A \end{aligned}$$

By Proposition 5.9, this implies $A \in G(X)$. We thus have (22) with $\varphi = A$, which again by Lemma 3.5.3 is equivalent to (21), expressing that $A$ defines a uniquely determined morphism of Higgs bundles $A : (E_1, \theta_1) \to (E_2, \theta_2)$. Hence $LS^{-1}$ is an equivalence.

**Naturality in $f$ and $G$:** Naturality in $G$ is clear. Functoriality in $f$ can be seen exactly as in Lemma 4.20: Any morphism $\widetilde{g}$ of toric charts between $f'$ and $f$ as in Definition 2.4 induces a morphism $X'_\infty \to X_\infty$ which is equivariant with respect to the induced map $\Delta' \to \Delta$ between the Galois groups. The pullback of $G$-Higgs bundles is then given by sending $\rho$ to $\rho' : \Delta' \to \Delta \xrightarrow{\rho} \mathrm{ad}(E^+)(X) \to \mathrm{ad}(E^+)(X')$ and exp of this agrees with the cocycle

$$\exp(\rho') : \Delta' \to \Delta \xrightarrow{\exp(\rho)} \underline{\mathrm{Aut}}_{G_c^+}(E^+)(X) \to \underline{\mathrm{Aut}}_{G_c^+}(E^+)(X').$$

Thus the pullback of $V_\rho$ along $X' \to X$ agrees with $V_{\rho'}$ defined with respect to $f'$.

This finishes the proof of Theorem 6.5.(1). For part 2, we additionally need to treat general morphisms $\varphi : E_1 \to E_2$ which are not necessarily isomorphisms. Following the above proof of functoriality, we then instead of (21) arrive at the equation

$$(23) \qquad \rho_2(\gamma)\varphi = \varphi\rho_1(\gamma) \quad \forall \gamma \in \Delta.$$

By [Heu22a, Lemma 3.11], this is again equivalent to

$$(24) \qquad \exp(-\rho_2(\gamma))\varphi = \varphi \exp(-\rho_1(\gamma)) \quad \forall \gamma \in \Delta,$$

replacing (22). From here the proof of functoriality goes through verbatim, as does the proof of essential surjectivity.

For part 2, it remains to prove exactness. For this we use that a complex $C^\bullet$ of vector bundles on $X_{\text{ét}}$ is exact if and only if the associated complex $\mu^* C^\bullet$ on $X_{\text{proét}}$ is exact, where $\mu : X_{\text{proét}} \to X_{\text{ét}}$ denotes the natural morphism. Indeed, this follows from the fact that $\mu^*$ is exact on short exact sequences of coherent $\mathcal{O}_X$-modules and satisfies $\mu_* \mu^* = \mathrm{id}$ on coherent $\mathcal{O}_X$-modules (for example by [Heub, Proposition 2.9]).

Assume now we are given a complex of Higgs bundles, then exactness just means that the underlying complex of vector bundles $E^\bullet$ is exact. Since $q : X_\infty \to X$ is a pro-étale cover, it follows that $E^\bullet$ is exact if and only if $q^* E^\bullet$ is exact. The analogous statement also holds for the associated complex of $v$-vector bundles. Hence the statement now follows from (20).

We end this subsection with two remarks on globalisation:

**Remark 6.7.** One key difference between Theorem 4.1 and Theorem 6.5 is that the former works for any smooth rigid space $X$, while the latter depends on a toric chart $f : X \to \mathbb{T}^d$. For general smooth $X$, one might try to globalise the construction by choosing a cover by affinoids that admit a toric chart. However, Theorem 6.5.3 says that it is only possible to compare the equivalences on overlaps in a canonical way after the choice of a morphism of toric charts. It is usually not possible to find these in a way that preserves gluing data.

This should not be regarded as a flaw of the local correspondence, but rather as a meaningful conceptual barrier: In fact, the global correspondences of Faltings [Fal05], Abbes-Gros and Tsuji [AGT16], Wang [Wan23] and others (see Section 6.3) all rely on an additional choice, namely the choice of an $A_{\text{inf}}/\xi^2$-lift of a semi-stable model of $X$. Second, they require a stronger smallness condition than the one that works for the local correspondence.

**Remark 6.8.** In this light, it is interesting to ask to what extent it is possible to generalise Theorem 6.5 to a more general class of perfectoid Galois covers, thus allowing better gluing properties in some special cases. We will therefore now sketch how one can use the more general preparations from Section 4 to generalise the local correspondence by axiomatising the class of Galois covers for which the proof still goes through. We will apply this in [HWZ] to prove a small $p$-adic Simpson correspondence for $G$-torsors on abeloid varieties.

As in §4.2, let $X$ be an affinoid smoothoid adic space over a perfectoid field $K$ containing $\mathbb{Q}_p^{\text{cyc}}$ and let

$$f : \widetilde{X} = \operatorname{Spa}(R_\infty, R_\infty^+) \to X = \operatorname{Spa}(R, R^+)$$

be a pro-étale affinoid perfectoid cover. We suppose that $f$ satisfies the following axioms:

(1) $f$ is a pro-finite-étale Galois cover, and the Galois group $\Delta$ is a finite free $\mathbb{Z}_p$-module.
(2) There is $\gamma > 0$ such that for any $s \in \mathbb{N}$ and $i \in \{0, 1, 2\}$, the kernel and cokernel of

$$H_{\text{cts}}^i(\Delta, R^+/p^s) \to H_{\text{cts}}^i(\Delta, R_\infty^+/p^s)$$

are killed by $p^\gamma$. Moreover, the same is true for $H_{\text{cts}}^i(\Delta, R^+) \to H_{\text{cts}}^i(\Delta, R_\infty^+)$.

Note that (1) and (2) imply that $\Delta \cong \mathbb{Z}_p^d$ where $d$ is the smooth dimension of Definition 2.11.

The role of (2) is to replace Lemma 2.14.3. In practice, (2) can often be verified using certain traces that induce a $\Delta$-equivariant $R$-module splitting of $R \to R_\infty$: Indeed, conditions (1) and (2) are closely related to the "Tate–Sen axioms" of Berger–Colmez [BC08, Definition 3.1.3], and (2) is essentially our version of their axiom "TS3". Let us also mention in this context the closely related "Geometric Sen theory" of Pan and Rodríguez Camargo, see [RC23, §2.2].

The case of $i = 1$ implies that the Cartan–Leray map of $f$

$$\operatorname{Hom}_{\text{cts}}(\Delta, \mathcal{O}(R)) \to H_{\text{cts}}^1(\Delta, \mathcal{O}(R_\infty)) \xrightarrow{\sim} H_v^1(X, \mathcal{O}) \xrightarrow{\text{HT}} H^0(X, \widetilde{\Omega}^1)$$

is an isomorphism: Indeed, the second map is an isomorphism because $H_v^1(\widetilde{X}, \mathcal{O}) = 0$ due to the assumption that $\widetilde{X}$ is affinoid perfectoid, and the map HT is an isomorphism because $X$ is affinoid. We can thus deduce from (2) the analogue of Lemma 2.18 in this setting.

Then the analogue of Proposition 5.5 holds for $f$, namely the exact same proof goes through with $c := 5\gamma$ by replacing Lemma 2.14 with axiom (2). We can use this to define a notion of smallness as in Definition 6.2. However, we will have to slightly modify $c$ further:

We also require an analogue of Proposition 5.9 for $f$, but for this we need to be slightly more careful. The proof of this goes through verbatim except for the step in Claim 5.11 where Lemma 2.14.2 is invoked, for which we do not have a direct analogue for $f$. However, this step of the proof can still be generalised with some more work, by replacing the étale localisation argument with a lifting argument at the expense of increasing $c$:

In the notation of the proof, we need to show that $\overline{A}_c \in G/G_c(X)$ lifts to $G(X)$. The obstruction to lifting defines a class in $H_v^1(X, G_c)$. We claim that this class vanishes if we

replace $U = G_c$ in Claim 5.11 by $U := G_{2c}$. Indeed, if $\rho_1, \rho_2$ factor through $G_{2c}(R)$, then we already have $\overline{A}_{2c} \in G/G_{2c}(X)$ and the lifting obstruction thus lies in the image of a map

$$h : H^1_v(X, G_{2c}) \to H^1_v(X, G_c).$$

It thus suffices to see that $h = 1$. But this follows from Lemma 6.3: Unravelling the definitions, we see that for the group $G' := G_c$, a $v$-$G'$-bundle on $X$ is small if it admits a reduction of structure group to $G'_c = G_{2c}$. Hence $h = 1$, so $\overline{A}_c$ lifts uniquely to $G(X)/G_c(X)$ as desired. From here the proof of Proposition 5.9 goes through without further changes.

Adapting the notion of smallness accordingly, so that a $v$-$G$-bundle is small if it admits a reduction of structure group to $G_{2c}$, it follows that also Lemma 6.3 holds in this setting.

With these results at hand, the construction of the functor

$$LS_f : \{\text{small } G\text{-bundles on } X_v\} \to \{\text{small } G\text{-Higgs bundles on } X_{\text{ét}}\}$$

now goes through verbatim: Indeed, the first part of the proof of Theorem 6.5 work in the same way using the analogue of Lemma 6.3. The proof of essential surjectivity works verbatim using the analogue of Proposition 5.5, and the proof of fully faithfulness works verbatim using the analogue of Proposition 5.9.

All in all, this shows that we get a more general version of part 1, 2, 3 of Theorem 6.5 for not necessarily toric covers $\widetilde{X} \to X$ satisfying axioms (1) and (2), but at the expense of a more restrictive smallness assumption (defined in terms of $2c$ instead of $c$, where $c = 5\gamma$). In some special cases where one has global perfectoid covers, this allows for better globalisation than one has for toric charts. That being said, the cohomological correspondence, Theorem 6.5.4, will require more input specific to toric charts. This is our next goal.

6.2. **Cohomology of $v$-vector bundles.** Continuing the proof of Theorem 6.5, we now move on to part 4. For this the main computation is a more general version of part of Lemma 2.14 (which was in turn based on [Sch13a, Lemma 5.5]), describing the $v$-cohomology of small $v$-vector bundles, Lemma 2.14 being the case of the trivial bundle $\mathcal{O}$. As before, we write $X = \mathrm{Spa}(R, R^+)$ and $X_\infty = \mathrm{Spa}(R_\infty, R_\infty^+)$.

**Lemma 6.9.** *Let $\rho : \Delta \to \mathrm{GL}_n(R^+)$ be a continuous representation such that $\rho \equiv 1 \bmod p^\alpha$. Let either $M := R^{+n}$ or $M := R^{+n}/p^k$ for some $k \in \mathbb{N}$, endowed with $\Delta$-action via $\rho$. Endow $M \otimes_{R^+} R_\infty^+$ with the diagonal $\Delta$-action and consider for any $m \geq 0$ the natural map*

$$H^m_{\mathrm{cts}}(\Delta, M) \to H^m_{\mathrm{cts}}(\Delta, M \otimes_{R^+} R_\infty^+).$$

*Then we can find $c > 0$ independent of $k$ such that the kernel and cokernel of this map are annihilated by $p^c$. For $m = 0$ and $M = R^{+n}$, the map is an isomorphism.*

**Remark 6.10.** For $m = 0$, this give an alternative proof of Proposition 5.9 for $G = \mathrm{GL}_n$, by setting $M = \mathrm{End}(\rho_1, \rho_2)$. This is essentially Faltings' original proof in [Fal05, Lemma 1].

*Proof.* We use Lemma 2.14 and its notation: As $R^+ \hat{\otimes}_{A^+} A_\infty^+ \to R_\infty^+$ is injective with $p^\beta$-torsion cokernel, it suffices to see the statement with $R^+ \to R_\infty^+$ replaced by $A^+ \to A_\infty^+$. For $m = 0$ we then get upon inverting $p$ an isomorphism $M^\Delta \otimes_{R^+} R = (M \otimes_{R^+} R_\infty)^\Delta$, and using that $R^+ = R_\infty^+ \cap R$, we see that it is already an isomorphism before inverting $p$.

By a limit argument, it thus suffices to prove the statement for $M/p^k$ for any $k \in \mathbb{N}$. Here we can argue as in [Fal02, p205-206]: Fix a compatible system of $p$-power roots of unity $\zeta$. Write $\gamma_1, \ldots, \gamma_d$ for the induced generators of $\Delta = \mathbb{Z}_p^d$. We then have a decomposition

$$M \otimes_{A^+} A_\infty^+/p^k = \bigoplus_{i=(i_1,\ldots,i_d)\in[0,1)\cap\mathbb{Z}[\frac{1}{p}]} (M/p^k) \cdot T_1^{i_1} \cdots T_d^{i_d}$$

as a $\Delta$-module, where $(M/p^k) \cdot T_1^{i_1} \cdots T_d^{i_d}$ is the $R^+$-module $M/p^k$ with the action of $\rho$ twisted by the character $\gamma_j \mapsto \zeta^{i_j}$. It thus suffices to prove that already $H^m_{\mathrm{cts}}(\Delta, (M/p^k) \cdot T_1^{i_1} \cdots T_d^{i_d})$

is $p^c$-torsion unless $i = 0$. Via the inflation-restriction sequence and Lemma A.5, we can reduce to $\mathbb{Z}_p$-cohomology. We are thus left to describe $R\Gamma_{\mathrm{cts}}(\mathbb{Z}_p, M/p^k)$ with $1 \in \mathbb{Z}_p$ acting as multiplication by $\zeta^i \rho(\gamma_j)$ on $M/p^k$. By Lemma A.5, this is computed by the complex

$$0 \to M/p^k \xrightarrow{\zeta^i \rho(\gamma_j) - 1} M/p^k \to 0.$$

By assumption we have $\rho(\gamma_j) = 1 + p^\alpha A$ for some $A \in M_n(R^+)$, so $(\zeta^i \rho(\gamma_j) - 1)/(\zeta^i - 1)$ is a unit for $i \neq 0$. Hence the above map has the same kernel and cokernel as $\zeta^i - 1$. $\square$

**Lemma 6.11.** *Let $\rho : \Delta \to \mathrm{GL}_n(R^+)$ be a continuous representation such that $\rho \equiv 1 \bmod p^\alpha$. Let $M := R^n$ endowed with $\Delta$-action via $\rho$. Then for $E := M \otimes \mathcal{O}$ and $\theta := \log(\rho)$ interpreted as a section of $M_n \otimes \widetilde{\Omega}$ as in Section 6.1, there is a natural isomorphism*

$$R\Gamma_{\mathrm{cts}}(\Delta, M) \xrightarrow{\sim} R\Gamma_{\mathrm{Higgs}}(X, (E, \theta)).$$

*Proof.* Let $\gamma_1, \ldots, \gamma_d \in \Delta$ be topological generators and let $A_1, \ldots, A_d$ be the images of $\gamma_1, \ldots, \gamma_d$ under $\theta$. Via $\mathrm{Hom}(\Delta, \mathcal{O}(X)) = \widetilde{\Omega}(X)$, the dual basis of $\gamma$ induces a basis $\delta_1, \ldots, \delta_d$ of $\widetilde{\Omega}(X)$ and thus an isomorphism $\widetilde{\Omega}(X) = R^d$. The Higgs complex now evaluates on $X$ to

$$\mathcal{C}^*_{\mathrm{Higgs}} = \left[ M \to M \otimes_R R^d \to M \otimes_R \wedge^2 R^d \to \cdots \to M \otimes_R \wedge^d R^d \right]$$

where the $k$-th transition map sends $m \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_j A_j(m) \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_j$

Comparing the complex $\mathcal{C}^*_{\mathrm{Higgs}}$ to $\mathcal{C}^*_{\mathrm{grp}}$ from Lemma A.5, we see that both have the same terms, but different transition maps. We claim that there is an $R$-linear isomorphism $u : \mathcal{C}^*_{\mathrm{grp}} \to \mathcal{C}^*_{\mathrm{Higgs}}$. For this we first observe that inside $\mathrm{End}_R(M)$, when we identify $\gamma_i$ with its image under $\rho$, then $A_i/(\gamma_i - 1) = u_i := 1 + (\gamma_i - 1) \sum_{w=2}^\infty \frac{(-1)^{w+1}}{w} (\gamma_i - 1)^{w-2} \in \mathrm{End}_R(M)$ is invertible as $\gamma_i - 1$ is topologically nilpotent in $\mathrm{End}_R(M)$. Writing each term in $\mathcal{C}^*_{\mathrm{grp}}$ as $M \otimes_R \wedge^k R^d = \bigoplus_{i_1 < \cdots < i_k} M \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}$, we now define $u$ on each direct summand as

$$u : M \cdot e_{i_1} \wedge \cdots \wedge e_{i_k} \to M \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad m \mapsto u_{i_1} \cdots u_{i_k} m.$$

This gives an isomorphism of complexes as the $\gamma_i$ and $A_i$ all commute with each other. $\square$

In summary, we obtain for the $v$-vector bundle $V$ natural isomorphisms in $\mathcal{D}(R)$

$$R\Gamma_v(X, V) = R\Gamma_{\mathrm{cts}}(\Delta, V(X_\infty)) = R\Gamma_{\mathrm{cts}}(\Delta, M) = R\Gamma_{\mathrm{Higgs}}(X, (E, \theta))$$

where the first morphism comes from the Cartan–Leray sequence [Heu22b, Proposition 2.8].

This finishes the proof of Theorem 6.5 in the case that $K$ contains $\mathbb{Q}_p^{\mathrm{cyc}}$.

It remains to deduce the general case by Galois descent. For this we use the following:

**Corollary 6.12.** *For any perfectoid field $K$, any smoothoid $X$ over $K$ and any rigid group $G$ over $K$, the morphism $\Psi_G$ has trivial fibre over the trivial element.*

*Proof.* Let $X_{K'} \to X$ be the base-change to a completed algebraic closure of $K$. The diagram

$$\begin{array}{ccc} \mathrm{Higgs}_G(X_{K'}) & \xrightarrow{\Psi_G} & R^1\nu_* G(X_{K'}) \\ \uparrow & & \uparrow \\ \mathrm{Higgs}_G(X) & \xrightarrow{\Psi_G} & R^1\nu_* G(X) \end{array}$$

commutes by Proposition 4.22. The top morphism is injective by Proposition 5.12. The left map has trivial kernel by Lemma 3.20. Hence the bottom map has trivial kernel. $\square$

**Corollary 6.13.** *Let $X$ be a smoothoid space over $K$. Let $K'|K$ be any extension of perfectoid fields. Then we have $v$-descent of étale $G$-torsors along the base-change map $X_{K'} \to X$.*

*Proof.* Any descent datum for an étale $G$-torsor along the morphism $X_{K'} \to X$ defines a $v$-$G$-bundle $V$ on $X$ that becomes trivial on $X_{K'}$. This is étale if its class in $R^1\nu_* G(X)$ vanishes. This follows by chasing the above diagram using that $\Psi_G$ is surjective by Proposition 5.6.    $\square$

Let $C$ be the completion of the cyclotomic extension of $K$ obtained by adjoining all $p$-power roots. By almost purity, this is again a perfectoid field. Let $G = \mathrm{Gal}(C|K)$ be the Galois group. Let $f_C : X_C \to \mathbb{T}_C^d \times Y$ be the base-change of $f$ from $K$ to $C$.

Let now $V$ be a $G$-torsor on $X_v$. Let $V_C$ be the pullback to $X_C$ and let $(E, \theta)$ be the Higgs bundle associated to $V_C$ via the local correspondence $LS$ with respect to $f_C$. Then by functoriality in the toric chart, the $v$-descent datum for $V_C$ along $X_C \to X$ induces a descent datum on $(E, \theta)$ for $X_C \to X$. By Corollary 6.13, this is effective and thus defines a Higgs bundle on $X$. The functoriality of this construction and all desired compatibilities now follow from those for the correspondence on $X_C$ using that the functor from étale $G$-torsors to $v$-$G$-torsors is fully faithful by Lemma 3.10, and thus morphisms can be defined $v$-locally.

Similarly, it suffices to check that the morphism between cohomologies in part 3 is an isomorphism after an extension of base-field. This completes the proof of Theorem 6.5 in general.    $\square$

The construction of Theorem 6.5 is clearly compatible with that of Theorem 4.1, in the sense that after passing to isomorphism classes and sheafifying on $X_{\text{ét}}$, the functor $LS$ computes the map $\Psi$. This completes the last missing piece of the proof of Theorem 4.1: The above descent shows that $\Psi$ is an isomorphism over general perfectoid base fields.    $\square$

6.3. **Relation to Faltings' local $p$-adic Simpson correspondence.** We now elaborate on the comparison of our correspondence in the case of smooth rigid spaces and $G = \mathrm{GL}_n$ to Faltings' local correspondence as studied by Abbes–Gros–Tsuji.

The idea to build a global $p$-adic Simpson correspondence out of local correspondences between small objects goes back to [Fal05], and instances of such a correspondence for $G = \mathrm{GL}_n$ are now known in good generality: For example, it is known for a certain class of log schemes defined over discretely valued $k$ due to Faltings [Fal05], Abbes–Gros–Tsuji [AGT16] and completed by Tsuji [Tsu18], as well as in cases of good reduction, due to Wang [Wan23]. Let us also mention in this context the "q-deformed" version due to Morrow–Tsuji [MT21]. We refer to the appendix of [AHL24] for a general overview over different variants of local and global $p$-adic Simpson correspondences.

The first difference of these to our version is the technical setup in which the correspondence is formulated: The approach of Faltings/Abbes–Gros/Tsuji is rooted in the setting of Faltings' $p$-adic Hodge theory, and deals with certain log schemes with toroidal singularities, defined over discretely valued base fields $k$. On these one defines the category of generalised representations, and the local correspondence then essentially happens after the base change to the completed algebraic closure $\overline{K}$ of $k$. Let us refer to this as the "arithmetic" setup.

An alternative technical foundation (of course inspired by Faltings') is that of Scholze's $p$-adic Hodge theory [Sch13a], which instead works with smooth rigid spaces $X$ over algebraically closed $K$. For the Hodge–Tate comparison (on which the $p$-adic Simpson correspondence is based), this is possible without assuming that $X$ admits a model over a discretely valued field. One then uses locally perfectoid constructions, like the pro-étale site, or the category of diamonds, which allow one to reinterpret generalised representations as locally free sheaves on $X$. Let us call this the "geometric" setup.

In many situations, it is possible to pass from the "arithmetic setup" in the above sense to the "geometric setup" by analytifying and passing from $k$ to $K$. This is not to say that one setup is more general than the other, for example Faltings' setting allows for more flexibility in the regularity conditions imposed on $X$. Moreover, there are of course many variations in between, for example many authors use Scholze's technical language but work with varieties

that allow models over discretely valued fields, for various good reasons: For example, one can then expect a "de Rham" side to the non-abelian picture, as in [LZ17].

The second difference is that we use a different notion of smallness, related to the difference between [AGT16, Definition 13.1 vs 13.2]: In terms of generalised representations, the definition used by Faltings/Abbes–Gros/Tsuji is that the underlying module is projective and *generated* by elements on which the action becomes trivial $\mathrm{mod}\,p^\alpha$, whereas the definition that we use here is that the module is already finite free with trivial action $\mathrm{mod}\,p^\alpha$.

The proof of the local Simpson correspondence for $G = \mathrm{GL}_n$ for Faltings' more general notion of smallness is quite subtle, as explored in detail by Abbes–Gros–Tsuji (see in particular [AGT16, II.14], [Tsu18, Remark 2.3]). The major difficulty is that when using their notion of smallness, the construction requires an additional technical result, which in the language of Abbes–Gros is that "small representations are Dolbeault". This was the missing piece provided by [Tsu18]. We choose to sidestep these problems by way of our more restrictive notion of smallness, which also seems more natural in our general setting of $G$-torsors.

If we ignore all of these technical differences, the main novelty in Theorem 6.5 is of course the generalisation from $\mathrm{GL}_n$ to rigid groups $G$ and from smooth rigid spaces $X$ to the relative setting of perfectoid families thereof. A minor additional point is the generalisation from algebraically closed $K$ to any perfectoid base field over $\mathbb{Q}_p$.

**Remark 6.14.** In the work of Faltings, the local $p$-adic Simpson correspondence is the first of three variants of a $p$-adic non-abelian Hodge correspondence: The second is the "global $p$-adic Simpson correspondence of small objects". Instead of a toric chart, it relies on the datum of a lift of $X$ to $\mathbb{B}_{\mathrm{dR}}^+/\xi^2$, and it uses a different notion of smallness that can be formulated in terms of the Hitchin base (cf §8). The third is the "proper $p$-adic Simpson correspondence" which describes all v-vector bundles in the case of smooth projective varieties. Both of the latter two rely on the local correspondence. We refer to [AHL24, §A] for a systematic overview comparing these different types of correspondences, as well as pointers to the literature.

In light of Theorem 6.5.(1), it is natural to ask if there are also analogues of the "global" and "proper" correspondence for general rigid groups $G$ (we refer [HWZ] for some first results in this direction). In particular, it would be interesting to use a different notion of smallness for the global correspondence, which only relies on the image of the Hitchin morphism of §8.

6.4. **Relation between sheafified and local correspondence.** One could in principle deduce a weaker version of Theorem 4.1 from Theorem 6.5 by passing to isomorphism classes, using that $G$-bundles and $G$-Higgs bundles are étale-locally small. But it is important for our purposes that Theorem 4.1 is stronger than this version, as it is "more canonical and more functorial in $X$" in the following ways:

**Dependence on the chart:** As mentioned in the introduction, in contrast to the complex case, the $p$-adic Simpson correspondence is in general non-canonical and will depend on certain choices. The situation for the local $p$-adic Simpson correspondence is slightly better, but still quite subtle: It still depends on the choice of chart up to non-canonical isomorphism (see also [Tsu18, Remark 13.3]). This means that it is in general not clear how to globalize from the local case, since one has no canonical glueing data on overlaps. Other versions of the local correspondence have similar restrictions. Indeed, there are conceptual reasons why one cannot expect a completely canonical correspondence in general: If this existed, then by functoriality one would be able to glue the local correspondences to get a global correspondence without any smallness assumptions, which is known not to exist.

In contrast, the advantage of the "sheafified correspondence" of Theorem 4.1 is that it is completely canonical and functorial with respect to localisation. In particular, the local correspondence, Theorem 4.1 holds for any smooth rigid space. This functoriality is crucial for our purposes in this article, and especially for constructing the Hitchin morphism $\widetilde{\mathcal{H}}$.

**Functoriality:** Given a morphism of smoothoids $h : X_1 \to X_2$, it is not always possible to find compatible toric charts, and hence to compare the local correspondences, even étale locally. The functoriality property of Theorem 4.1 is stronger and includes such morphisms.

**Smallness:** A third way in which Theorem 4.1 is more canonical than Theorem 6.5 is that for general rigid groups $G$, there is no canonical notion of "smallness" of $G$-torsors.

## 7. Applications to $v$-vector bundles on rigid spaces

We now give several immediate applications of Theorem 4.1 and Theorem 6.5. Let $G$ be any rigid group. Our first application is that the explicit description of $R^1\nu_*G$ lets us deduce $v$-descent criteria for $G$-torsors. By this we mean methods to answer the following question:

**Question 7.1.** Let $f : X' \to X$ be a $v$-cover of smoothoid (e.g. smooth rigid or perfectoid) spaces. What are conditions on $f$ that ensure descent of étale $G$-torsors along $f$?

As it is tautological that we have descent of $v$-$G$-torsors along $f$, it is equivalent to ask how we can tell whether $v$-$G$-torsors on $X$ are already étale-locally trivial.

Examples of situations where this question arises include moduli stacks of étale vector bundles, but also the study of automorphic sheaves like bundles of overconvergent $p$-adic modular forms defined via descent from perfectoid Shimura varieties of infinite level.

7.1. **Criteria for $v$-vector bundles to be étale-locally trivial.** Since Question 7.1 is étale-local, we can use either of Theorem 4.1 and Theorem 6.5 to study it. We shall mainly use the former as it is slightly more convenient in this setting, e.g. it does not require additional choices. We begin with the following immediate consequence:

**Corollary 7.2.** *For any smoothoid space $X$ and a rigid group $G$ over $K$, the Leray sequence for $\nu : X_v \to X_{\text{ét}}$ induces an exact sequences of pointed sets, functorial in $X$ and $G$,*

$$1 \to H^1_{\text{ét}}(X, G) \to H^1_v(X, G) \to \mathrm{Higgs}_G(X).$$

We deduce the following criterion for $v$-descent, a generalisation of [Heu22b, Corollary 1.5]:

**Corollary 7.3.** *Let $V$ be a $G$-torsor on $X_v$. Let $U \to X$ be any étale map with Zariski-dense image. Then $V$ is étale, i.e. comes from a $G$-torsor on $X_{\text{ét}}$, if and only if $V|_U$ is.*

In particular, if $X$ is connected, this means that to prove that $V$ is étale on $X$, it suffices to prove this on *any* non-empty open subspace $U \subseteq X$. This is a priori quite surprising.

*Proof.* By functoriality in Theorem 4.1, the restriction to $U$ fits into a commutative diagram

$$
\begin{array}{ccccccc}
1 & \longrightarrow & H^1_{\text{ét}}(X, G) & \longrightarrow & H^1_v(X, G) & \longrightarrow & \mathrm{Higgs}_G(X) \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^1_{\text{ét}}(U, G) & \longrightarrow & H^1_v(U, G) & \longrightarrow & \mathrm{Higgs}_G(U).
\end{array}
$$

Thus it suffices to see that the restriction map on the right has trivial kernel. This is precisely the statement of Lemma 3.20 applied in the case of Example 3.21.1. □

We also get the following $v$-descent criterion, giving a satisfactory answer to Question 7.1:

**Corollary 7.4.** *Let $f : X' \to X$ be a $v$-cover of affinoid smoothoid spaces. Then descent of étale $G$-torsors along $f$ is effective if and only if $f^* : H^0(X, \widetilde{\Omega}_X) \to H^0(X', \widetilde{\Omega}_{X'})$ is injective.*

*Proof.* Any descent datum for an étale $G$-torsor $E$ along $f$ defines a $v$-topological $G$-torsor on $X$ whose pullback to $X'$ is $E$. The result thus follows from a similar diagram as in the proof of Corollary 7.3, again using Lemma 3.20: Here by Proposition 2.9, the sheaf $\widetilde{\Omega}_X$ is a vector bundle, so $\widetilde{\Omega}_X \to f_*\widetilde{\Omega}_{X'}$ is injective if and only if it is injective on global sections. □

This generalises Corollary 6.13, where $f$ is a base-change $X_{K'} \to X_K$. Another case is:

**Corollary 7.5.** *Let $X$ be a smooth rigid space, let $Y$ be a perfectoid space, and let $Z \to X \times Y$ be étale. Then a $G$-torsor on $Z_v$ is étale if it is after pullback to a perfectoid $v$-cover $Y' \to Y$.*

Applying the same argument as in Corollary 7.3 to the case of Example 3.21.2 shows:

**Corollary 7.6.** *Let $G \hookrightarrow G'$ be an injective homomorphism of rigid groups. Let $V$ be a $G$-torsor on $X_v$. Then $V$ is étale-locally trivial if and only if the $G'$-torsor $V \times^G G'$ is.*

A special case of the Corollary is the map $\mathrm{GL}_n(\mathcal{O}^+) \hookrightarrow \mathrm{GL}_n(\mathcal{O})$, for which this shows:

**Corollary 7.7.** *Let $E^+$ be a finite $v$-locally free $\mathcal{O}^+$-module on $Z_v$ such that $E := E^+[\frac{1}{p}]$ is étale-locally free. Then $E^+$ is étale-locally free.*

However, Example 3.19 shows that the map $R^1\nu_* G \to R^1\nu_* G'$ is not in general injective.

**Remark 7.8.** Another application of Corollary 7.6 shows that for a linear algebraic group $G$ with embedding $G \hookrightarrow \mathrm{GL}_n$, a $G$-torsor $V$ on $X_v$ is étale-locally trivial if and only if the associated $v$-vector bundle $\mathrm{GL}_n \times^G V$ is. The more general question whether two $G$-torsors $V_1$, $V_2$ on $X_v$ are étale-locally isomorphic if the associated $v$-vector bundles are isomorphic for every representation of $G$ is more subtle, and Theorem 4.1 translates this into a question about simultaneous conjugation that is related to Steinberg's conjugacy conjecture [Ste78].

We finish with two results about $v$-vector bundles, generalising [Heu22b, Corollary 3.6]:

**Corollary 7.9.** *Let $V_1 \hookrightarrow V_2$ be an injective morphism of $v$-vector bundles on a smoothoid space $Z$. If $V_2$ is étale-locally free, then so is $V_1$.*

*Proof.* We can work locally and assume that $V_1$, $V_2$ are small. Then Theorem 6.5 translates this into an inclusion of Higgs bundles $(E_1, \theta_1) \subseteq (E_2, \theta_2)$, and $\theta_1$ vanishes if $\theta_2$ does. $\square$

Second, as a consequence of Theorem 6.5.(4), we see on the Higgs side:

**Corollary 7.10.** *Let $X$ be a smoothoid space over $K$ and let $V$ be a $v$-vector bundle on $X$. Let $\nu : X_v \to X_{\text{ét}}$ be the natural morphism of sites and let $n \in \mathbb{N}$.*
  *(1) The $\mathcal{O}_X$-module $R^n\nu_* V$ vanishes for $n > \dim X$.*
  *(2) If $X$ is a smooth rigid space, then the $\mathcal{O}_X$-module $R^n\nu_* V$ on $X_{\text{ét}}$ is coherent.*

*Proof.* Both statements are local on $X_{\text{ét}}$. By Lemma 6.4, we can therefore assume that $X$ has a toric chart $f$ and $V$ is small with respect to $f$. Then the statement follows from Theorem 6.5.(4). $\square$

This reproves a result of Kedlaya–Liu [KL16, Theorem 8.6.2.(a)] for pseudo-coherent modules in the special case of $v$-vector bundles. For paracompact rigid $X$, it follows from Grothendieck vanishing [dJvdP96, Corollary 2.5.10] that $H_v^i(X, V) = 0$ for $i > 2 \dim X$. A related result has been obtained in a similar way by Min–Wang [MW, Corollary 1.10] for proper $X$ over a discretely valued field $K_0$. We note that the Corollary also illustrates the failure of $R\nu_*$ to be conservative: If the associated Higgs complex is exact, then $R\nu_* V = 0$.

### 7.2. $v$-stacks of étale $G$-torsors.
Let $f : X \to \mathrm{Spa}(K)$ be a smooth rigid space over $K$. As an application of Corollary 7.5, we can now prove that the prestack of étale vector bundles on $X$ relatively over the $v$-site over $K$ is a stack. This is one of our main motivations for introducing smoothoid spaces.

We shall freely use the language of $v$-stacks from [Sch, §9]. We now recall their definition, adapted to our setting of perfectoid spaces over a fixed perfectoid field $K$. In other words, we will always consider $v$-stacks with a fixed structure map to $\mathrm{Spd}(K)$:

**Definition 7.11.**   (1) A prestack over $K$ is a contravariant functor $F : \mathrm{Perf}_K \to \mathrm{Groupoids}$.

(2) A prestack $F$ is called a $v$-**stack** if it satisfies $v$-descent, i.e. if for each $v$-cover $Y' \to Y$ with projections $\pi_1, \pi_2 : Y' \times_Y Y' \to Y'$, the following functor is an equivalence:

$$F(Y) \to \{(s, \alpha) | s \in F(Y'), \alpha : \pi_1^* s \xrightarrow{\sim} \pi_2^* s \text{ such that cocycle condition holds}\}$$

(3) A $v$-stack $F$ is called **small** if there is a surjection $h : S \to F$ from a perfectoid space for which $R = S \times_F S$ is a small $v$-sheaf, i.e. there is a surjection $S' \to R$ of $v$-sheaves from a perfectoid space $S'$. It is not difficult to see that one gets an equivalent definition if $S$ and $S'$ are just assumed to be diamonds.

**Definition 7.12.** Let $G$ be any rigid group over $K$, for example $G = \mathrm{GL}_n$.

(1) We denote by $\mathscr{B}un_{G,v}$ the prestack on $\mathrm{Perf}_K$ that sends a perfectoid space $Y$ to the groupoid of $v$-$G$-bundles on $(X \times Y)_v$, where as usual the fibre product is over $\mathrm{Spa}(K)$.
(2) We denote by $\mathscr{B}un_{G,\text{ét}}$ the prestack on $\mathrm{Perf}_K$ that sends a perfectoid space $Y$ to the groupoid of étale $G$-bundles on $(X \times Y)_{\text{ét}}$.
(3) We denote by $\mathscr{H}iggs_G$ the prestack on $\mathrm{Perf}_K$ that sends a perfectoid space $Y$ to the groupoid of $G$-Higgs bundles on $(X \times Y)_{\text{ét}}$.

We now use the descent criteria from the last section to prove our next main result:

**Theorem 7.13.** *All of the prestacks $\mathscr{B}un_{G,v}$, $\mathscr{B}un_{G,\text{ét}}$ and $\mathscr{H}iggs_G$ are small $v$-stacks.*

*Proof.* Let $g : Y' \to Y$ be a $v$-cover in $\mathrm{Perf}_K$, then $X \times Y' \to X \times Y$ is also a $v$-cover. We clearly have $v$-descent for $v$-topological $G$-bundles, so $\mathscr{B}un_{G,v}$ is a $v$-stack.

For $\mathscr{B}un_{G,\text{ét}}$, we need to prove that we have descent of étale $G$-bundles along the map $f : X \times Y' \to X \times Y$. By Corollary 7.4 to Theorem 4.1, it suffices to see that the map $\widetilde{\Omega}(X \times Y) \to \widetilde{\Omega}(X \times Y')$ is injective. This follows from Proposition 2.9 which shows $\widetilde{\Omega}_{X \times Y} = \widetilde{\Omega}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, and the fact that $g_* \mathcal{O}_Y \to \mathcal{O}_{Y'}$ is injective as $g$ is a $v$-cover.

The case of $\mathscr{H}iggs_G$ now follows easily: Given a descent datum on a $G$-Higgs bundle $(E', \theta')$, we first descend the étale bundle $E'$ to a $G$-bundle $E$. Then $\mathcal{F} = \mathrm{ad}(E) \otimes \widetilde{\Omega}_X$, being an étale vector bundle by Proposition 2.9, satisfies $\nu_* \nu^* \mathcal{F} = \mathcal{F}$. In other words, $\mathcal{F}$ already satisfies the sheaf property for the $v$-topology on $\mathrm{Perf}_K$, so $\theta$ can be defined $v$-locally. By the same argument, the vanishing of the section $\theta \wedge \theta \in \mathrm{ad}(E) \otimes \widetilde{\Omega}^2$ can be checked $v$-locally.

That the $v$-stack $\mathscr{B}un_{G,\text{ét}}$ is small can be seen similarly as in [FS, Proposition III.1.3]: As explained there, it suffices to prove that for any affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $K$, we have

$$\mathscr{B}un_{G,\text{ét}}(S) = \varinjlim_{i \in I} \mathscr{B}un_{G,\text{ét}}(S_i)$$

where $S_i = \mathrm{Spa}(R_i, R_i^+)$ and $(R_i)_{i \in I}$ ranges through the topologically countably generated perfectoid subalgebras of $R$. To see that this holds, let $E \in \mathscr{B}un_{G,\text{ét}}(S)$ be any étale $G$-torsor, and let $U \to Z := X \times S$ be a trivialising standard-étale cover so that $E$ corresponds to a class in the kernel of $G(U \times_Z U) \to G(U \times_Z U \times_Z U)$. Note that $U$ descends to a standard-étale cover $U_i \to X \times S_i$ for any $i \gg 0$. Then $\mathcal{O}(U \times_Z U) = \varinjlim \mathcal{O}(U_i \times_{X \times S_i} U_i)$ since any function of $U \times_Z U$ can be presented using countably many functions of $R$ which are thus already contained in some $R_i$. After refining $U$, we may assume that $U$ is a finite disjoint union of affinoid opens that factor through an affinoid open $\mathrm{Spa}(A, A^+) \subseteq G$ of topologically finite presentation, so that $E$ is already in the image of

$$G(U_i \times_{X \times S_i} U_i) \to G(U \times_Z U)$$

As $\varinjlim G(U_i \times_{X \times S_i} U_i \times_{X \times S_i} U_i) \to G(U \times_Z U \times_Z U)$ is injective, it follows that $E$ descends to a $G$-torsor $E_i$ on $X \times S_i$ for $i \gg 0$, as desired.

The same approximation arguments show that $\mathrm{ad}(E) \otimes \widetilde{\Omega}(X \times S) = \varinjlim \mathrm{ad}(E) \otimes \widetilde{\Omega}(X \times S_i)$, so that any Higgs field $\theta$ on $E$ already descends to a Higgs field on $E_i$ for $i \gg 0$. Thus

$$\mathscr{H}iggs_{G,\text{ét}}(S) = \varinjlim_{i \in I} \mathscr{H}iggs_{G,\text{ét}}(S_i),$$

which shows that $\mathscr{H}iggs_G$ is small. We deduce the case of $\mathscr{B}un_{G,v}$ from this using Theorem 6.5: Let $V$ be a $v$-$G$-bundle on $X \times S$. By Lemma 6.4, this becomes small after passing to an étale cover of $X$ by toric affinoid subspaces $U \to X$. By functoriality of the local correspondence in the toric chart, we deduce that the restriction of $V$ to $U \times S$ descends to some $U \times S_i$. Since morphisms of Higgs bundles also descend from $U \times S$ to some $U \times S_i$, we see that we can descend the gluing data, and deduce that $V$ descends to some $X \times S_i$. $\qquad\square$

While we believe that the $v$-stack $\mathscr{B}un_{G,\text{ét}}$ is of independent interest, its role in the following is that we use it to understand $\mathscr{H}iggs_G$. For example, for commutative $G$ we have:

**Lemma 7.14.** *Let $G$ be a commutative rigid group. Let $X$ be a proper smooth rigid space. Then we have a canonical isomorphism of $v$-stacks on $\mathrm{Perf}_K$*

$$\mathscr{H}iggs_G = \mathscr{B}un_{G,\text{ét}} \times H^0(X, \mathfrak{g} \otimes \widetilde{\Omega}_X) \otimes_K \mathbb{G}_a,$$

*where the second factor on the right hand side is a rigid vector group. In particular, all of $\mathscr{B}un_{G,\text{ét}}$, $\mathscr{B}un_{G,v}$ and $\mathscr{H}iggs_G$ then have the structure of a group $v$-stack.*

*Proof.* In the proper case, we have $H^0(X \times Y, \widetilde{\Omega}) = H^0(X, \widetilde{\Omega}) \otimes \mathcal{O}(Y)$ by [Heua, Proposition 2.6] (or Lemma 8.4 below). The description therefore follows from Remark 3.16. The group structure on $\mathscr{B}un_{G,\text{ét}}$ and $\mathscr{B}un_{G,v}$ comes from the tensor product on the category of $G$-bundles given by sending $G$-torsors $E_1, E_2$ to the pushout of $E_1 \times E_2$ along the multiplication $G \times G \to G$. $\qquad\square$

A similar description is still possible if we drop the assumption that $X$ is proper. In this case, the second factor might no longer be a rigid group, but it is still a diamond:

**Lemma 7.15.** *Let $f : X \to \mathrm{Spa}(K)$ be a smooth rigid space. Let $E$ be an analytic vector bundle on $X$. Then the $v$-sheaf $f_*E$ on $\mathrm{Perf}_K$ defined by $Y \mapsto E(X \times Y)$ is a diamond.*

*Proof.* Let us write $\mathcal{F}_{X,E} := f_*E$. Given any cover $X = \cup U_i$, the map $\mathcal{F}_{X,E} \hookrightarrow \prod \mathcal{F}_{U_i,E}$ is injective, where $\mathcal{F}_{U_i,E}$ is the restriction to $U_i$. By [Sch, Proposition 11.10], we can thus work locally on $X$ and assume that $E$ is free, reducing to $E = \mathcal{O}$. By a result of Achinger, we can moreover assume that $X$ admits a finite étale map $X \to \mathbb{D}$ to some rigid polydisc [Ach17, Proposition 6.6.1][Zav21, Corollary D.5]. In this case, $\mathcal{O}(X \times Y) = \mathcal{O}(\mathbb{D} \times Y) \otimes_{\mathcal{O}(\mathbb{D})} \mathcal{O}(X)$, so we are reduced to the case that $X = \mathrm{Spa}(K\langle T_1, \ldots, T_n\rangle)$. In this case, the result follows because $\mathcal{F}_{\mathbb{D},\mathcal{O}} \subseteq \prod_{\mathbb{N}} \mathbb{A}^n$, and the latter is a diamond by [Sch, Lemma 11.22]. $\qquad\square$

## 8. The Hitchin morphism for $v$-vector bundles

We now give the second main application of the sheafified correspondence Theorem 4.1: The construction of the Hitchin morphism on the Betti side. To motivate the construction, we reiterate that our goal in this series is to investigate to what extent non-abelian $p$-adic Hodge theory can be understood in terms of a comparison of the moduli $v$-stacks introduced in the last section. As in the complex theory, it is clear from the case of $G = \mathbb{G}_m$ which we explain in detail in §8.5 that we cannot expect $\mathscr{H}iggs_G$ and $\mathscr{B}un_{G,v}$ to be isomorphic as $v$-stacks. Instead, as mentioned in Theorem 1.7 in the introduction, what we will show in [HX24] is that for $G = \mathrm{GL}_n$ and $X$ a smooth proper curve, the two $v$-stacks are twists of each other via two natural morphisms to the same rigid analytic base: the Hitchin base.

The goal of this section is to introduce these morphisms and prove some first properties. We also explain why we believe that the Hitchin morphism on the Betti side is interesting for studying representations of the étale fundamental group.

As before, let $K$ be any perfectoid extension of $\mathbb{Q}_p$. Throughout this section, let $X$ be a smooth rigid space over $K$. We discuss the the Hitchin morphisms on the Higgs side.

8.1. **The Hitchin morphism for Higgs bundles in the case of $G = \mathrm{GL}_n$.** We describe the Hitchin morphism for $G = \mathrm{GL}_n$ because this is the main case studied in the $p$-adic Simpson correspondence so far, and the description simplifies significantly in this case. To simplify notation, let us in this subsection write

$$\mathscr{B}un_{n,v} := \mathscr{B}un_{\mathrm{GL}_n,v}, \quad \mathscr{H}iggs_n := \mathscr{H}iggs_{\mathrm{GL}_n}.$$

In algebraic geometry, the Hitchin morphism for a smooth proper variety $Y$ is a morphism from the stack of Higgs bundles of $Y$ to the Hitchin base, which is a certain affine space depending on $Y$ [Sim94, cf §6]. The construction is straightforward to adapt to the $p$-adic analytic setting, as we now discuss, roughly following [Sim94, p.20].

Let $Z$ be any smoothoid space and let $(E, \theta)$ be a Higgs bundle of rank $n$ on $Z_{\text{ét}}$. We regard $\theta$ as a section of $\mathrm{End}(E) \otimes \widetilde{\Omega}$. On any $U \in Z_{\text{ét}}$ where $E$ becomes trivial, choose an isomorphism $\psi : E|_U \cong \mathcal{O}_U^n$, then $\theta$ defines a homogeneous element of degree 1 in $M_n(H^0(U, \mathrm{Sym}\,\widetilde{\Omega}))$ and we can consider its characteristic polynomial in $H^0(U, \mathrm{Sym}\,\widetilde{\Omega}[T])$. As this is independent of the choice of $\psi$, this glues to a polynomial

$$\chi_{E,\theta} = T^n + a_1 T^{n-1} + \cdots + a_n \in H^0(Z, \mathrm{Sym}\,\widetilde{\Omega}[T])$$

defined over all of $Z$, where $a_k \in H^0(Z, \mathrm{Sym}^k\,\widetilde{\Omega})$ for $k = 1, \ldots, n$.

**Definition 8.1.** Since $\chi_{E,\theta}$ only depends on the isomorphism class of $(E, \theta)$, this defines a natural map from the set of isomorphism classes of Higgs bundles on $Z_{\text{ét}}$ of rank $n$

$$H : \{\text{Higgs bundles on } Z_{\text{ét}} \text{ of rank } n\}/\sim \to \bigoplus_{k=1}^n H^0(Z, \mathrm{Sym}^k\,\widetilde{\Omega}), \quad (E, \theta) \mapsto \chi_{E,\theta}.$$

As $H$ is functorial in $Z$ by construction, it induces a morphism of sheaves on $Z_{\text{ét}}$

$$(25) \qquad\qquad\qquad h_Z : \mathrm{Higgs}_n \to \bigoplus_{k=1}^n \mathrm{Sym}^k\,\widetilde{\Omega}.$$

**Remark 8.2.** If $\dim Z = 1$, then $\mathrm{Sym}^i\,\widetilde{\Omega} = \widetilde{\Omega}^{\otimes i}$, and we get a simpler description of $\chi_{E,\theta}$. For this we consider for each $k = 1, \ldots, n$ the $k$-th wedge product of the Higgs field

$$\wedge^k \theta : \wedge^k E \to \wedge^k(E \otimes \widetilde{\Omega}) = \wedge^k E \otimes \widetilde{\Omega}^{\otimes k},$$

then $a_k := \mathrm{tr}(\wedge^k \theta)$ is the $k$-th coefficient of $\chi_{E,\theta}$. This matches Hitchin's definition [Hit87].

We now return to the smooth rigid space $X$, for which we can assemble the various maps $h_Z$ on $Z = X \times Y$ for $Y \in \mathrm{Perf}_K$ to a Hitchin morphism in terms of moduli stacks, using:

**Definition 8.3.** The **Hitchin base** of $X$ is the $v$-sheaf $\mathcal{A}_n = \mathcal{A}_{X,n}$ on $\mathrm{Perf}_K$ defined by

$$\mathcal{A}_n : Y \mapsto \bigoplus_{k=1}^n H^0(X \times Y, \mathrm{Sym}^k\,\widetilde{\Omega}_{X \times Y}).$$

In general, $\mathcal{A}_n$ is a diamond by Lemma 7.15, but if $X$ is proper, then by Proposition 2.9 and Lemma 8.4 below, $\mathcal{A}_n$ is represented by the rigid space $\bigoplus_{k=1}^n H^0(X, \mathrm{Sym}^k\,\widetilde{\Omega}) \otimes_K \mathbb{G}_a$.

**Lemma 8.4.** *Let $X$ be a smooth proper rigid space and let $Y$ be a perfectoid space. Let $\pi : X \times Y \to X$ be the projection. Then for any vector bundle $E$ on $X_{\text{ét}}$ and $i \geq 0$, we have*

$$H^i_{\text{ét}}(X \times Y, \pi^* E) = H^i_{\text{ét}}(X, E) \otimes_K \mathcal{O}(Y).$$

*Proof.* This follows from a smoothoid version of "cohomology and base change", see [Heu24b, Corollary 3.12], or by a direct computation in Čech-cohomology as in [Heua, Proposition 4.2]. $\square$

**Definition 8.5.** Using the small $v$-stack of Higgs bundles $\mathscr{H}iggs_n$ from Theorem 7.13, we define a morphism of $v$-stacks over $K$, the **Hitchin morphism** for $\mathrm{GL}_n$

(26) $$\mathcal{H} : \mathscr{H}iggs_n \to \mathcal{A}_n$$

as follows: We compose the map $\mathscr{H}iggs_n(X) \to \mathrm{Higgs}_n(X \times Y)$, given by passing to isomorphism classes and sheafifying on $(X \times Y)_{\text{ét}}$, with the map $h_Z$ from (25).

**Remark 8.6.** We think of elements $(a_1, \ldots, a_n) \in \mathcal{A}_n$ of the Hitchin base as monic polynomials $f(T) = T^n + a_1 T^{n-1} + \cdots + a_n$. For example, this defines a natural map of $v$-sheaves

$$\cdot : \mathcal{A}_n \times \mathcal{A}_m \to \mathcal{A}_{n+m}, \quad f(T), g(T) \mapsto (f \cdot g)(T).$$

The image of a direct sum of Higgs bundles $E_1 \oplus E_2$ under $\mathcal{H}$ is then $\mathcal{H}(E_1) \cdot \mathcal{H}(E_2)$.

8.2. **The Hitchin morphism for general $G$.** We now describe the construction of the Hitchin morphism for general rigid groups $G$. Again, this is a fairly straightforward adaptation to $v$-stacks of the algebraic construction, for which we roughly follow [Ngô06]:

Since we will need to form a GIT quotient by $G$, we need to start the discussion in the world of schemes, and only later pass to $v$-sheaves. Let $(R, R^+)$ be any Huber pair over $(K, K^+)$ and let $V$ be a finite free $R$-module of rank $d$. We begin with some recollections on affine vector groups: Write $V \otimes \mathbb{A}_R^1$ for the associated affine space over $\mathrm{Spec}(R)$, where $\mathbb{A}_R^1$ is the affine line over $\mathrm{Spec}(R)$ considered as a scheme: Explicitly, we have $V \otimes \mathbb{A}_R^1 = \mathrm{Spec}(R[V])$ for $R[V] := \mathrm{Sym}_R V^\vee$, where $V^\vee$ is the $R$-linear dual of $V$. Then the functor of points of $V \otimes \mathbb{A}_R^1$ sends any $R$-algebra $S$ to $V \otimes_R S$. The $R$-module structure on $V$ can then be encoded into a natural $\mathbb{G}_m$-action on $V \otimes \mathbb{A}_R^1$, with $S^\times$ acting on $V \otimes_R S$ via the scalar action on $S$. Explicitly, this is defined by the map $R[V] \to R[V] \otimes_R R[T^{\pm 1}]$ sending any $w \in V^\vee$ to $w \otimes T$.

We now first apply this construction to $R = K$ and the $R$-module $\mathfrak{g}(K)$. Let us denote the resulting affine vector group simply by $\mathfrak{g} \otimes \mathbb{A}_K^1$. Recall that $G$ acts on $\mathfrak{g}$ via the adjoint action $\mathrm{ad} : G \to \mathrm{End}(\mathfrak{g})$. This induces an action of the abstract group $G(K)$ on the scheme $\mathfrak{g} \otimes \mathbb{A}_K^1$, and hence on $\mathcal{O}(\mathfrak{g} \otimes \mathbb{A}_K^1) = K[\mathfrak{g}(K)]$. Let us simply denote by $K[\mathfrak{g}]^G \subseteq K[\mathfrak{g}(K)]$ the $K$-subalgebra of $G(K)$-invariants of this action. We can then form "the quotient by $G$", namely the quotient by $G(K)$ in the sense of geometric invariant theory:

(27) $$\mathfrak{g} \otimes \mathbb{A}_K^1 \to \mathfrak{g} \otimes \mathbb{A}_K^1 /\!/ G := \mathrm{Spec}(K[\mathfrak{g}]^G).$$

Since the $G$-action on $\mathfrak{g}$ commutes with the $\mathbb{G}_m$-action (as $G$ acts via linear maps), the quotient inherits a $\mathbb{G}_m$-action and the quotient map is $\mathbb{G}_m$-equivariant. Also, the quotient is obviously $G$-equivariant when the target is endowed with the trivial $G$-action.

Returning to our general $K$-algebra $R$ and finite $R$-module $V$, we note that any element $x \in \mathfrak{g}(R) \otimes_R V$ defines by duality a morphism of $R$-modules $\mathfrak{g}(R)^\vee \to V$, hence a morphism of $R$-algebras $R[\mathfrak{g}(R)] \to R[V^\vee]$, hence a morphism $V^\vee \otimes \mathbb{A}_R^1 \to \mathfrak{g} \otimes \mathbb{A}_R^1$ of affine vector groups over $R$. Composed with the base-change of (27) to $R$, we obtain a $\mathbb{G}_m$-equivariant morphism

$$V^\vee \otimes \mathbb{A}_R^1 \to \mathrm{Spec}(K[\mathfrak{g}]^G \otimes_K R)$$

of affine schemes over $R$, associated to $x$ in a natural way.

**Definition 8.7.** Let $A_{V,G,R}$ be the $v$-sheaf over $\mathrm{Spa}(R, R^+)$ sending a perfectoid algebra $(S, S^+)$ to the set of $\mathbb{G}_m$-equivariant morphisms of $R$-schemes $V^\vee \otimes \mathbb{A}_S^1 \to \mathrm{Spec}(K[\mathfrak{g}]^G \otimes_K R)$. Using that $\mathcal{O}$ on $\mathrm{Perf}_K$ is a $v$-sheaf, one easily verifies that this is a $v$-sheaf.

Passing from schemes to $v$-sheaves, and regarding $\mathfrak{g}$ as an $\mathcal{O}$-module on $\mathrm{Perf}_K$, the above construction now defines a natural morphism of $v$-sheaves on $\mathrm{Spa}(R, R^+)$

$$\mathfrak{g} \otimes_R V \to A_{V,G,R}$$

that is functorial in $R$ and equivariant for the natural $G \times \mathrm{GL}_V$-action where $G$ acts via the adjoint action on $\mathfrak{g}$ on the left and trivially on the right, and $\mathrm{GL}_V$ acts on $V$ on both sides.

Let now $Z$ be any smoothoid space over $K$. We apply this construction to $R = \mathcal{O}(U)$ for any toric open subspace $U \to Z$ in $Z_{\text{ét}}$ and $V := H^0(U, \widetilde{\Omega}_Z)$. We thus obtain natural maps of $v$-sheaves on $U$

$$\mathfrak{g} \otimes_R H^0(U, \widetilde{\Omega}_U) \to A_{V,G,\mathcal{O}(U)}$$

that are still $G \times \mathrm{GL}_V$-equivariant. For varying $U$, the $A_{V,G,\mathcal{O}(U)}$ clearly glue to a sheaf $A_G$ over $Z_{\text{ét}}$. Due to the $G$-invariance, the above maps then glue to a natural morphism of sheaves on $Z_{\text{ét}}$

$$(28) \qquad\qquad\qquad h_Z : \mathrm{Higgs}_G \to A_G.$$

Let now $X$ be a smooth rigid space and consider $Z = X \times Y$ for test objects $Y \in \mathrm{Perf}_K$.

**Definition 8.8.** The $v$-sheaf $\mathcal{A}_G$ (or $\mathcal{A}_{G,X}$) on $\mathrm{Perf}_K$ defined by $\mathcal{A}_G(Y) := A_G(X \times Y)$ is the **Hitchin base for $G$ and $X$**. It is clear from the construction that the formation of $\mathcal{A}_{G,X}$ is functorial in both $G$ and $X$. We then easily verify that $h_{X \times Y}$ is functorial in $Y$.

In general, the $v$-sheaf $\mathcal{A}_G$ is a diamond by Lemma 7.15. However, we have the following:

**Lemma 8.9.** *Assume that $X$ is proper, and that $G$ is a rigid group such that the Lie algebra $\mathfrak{g}$ satisfies $K[\mathfrak{g}]^G = K[u_1, \ldots, u_n]$ for some homogeneous generators $u_i$ of degree $e_i$, for example by Chevalley's Restriction Theorem we can take split reductive $G$, but we can also take commutative $G$. Then $\mathcal{A}_G$ is represented by an affine rigid space over $K$, namely*

$$\mathcal{A}_G \cong \bigoplus_{k=1}^{n} H^0(X, \mathrm{Sym}^{e_k} \widetilde{\Omega}_X) \otimes \mathbb{G}_a.$$

*Proof.* The condition on $G$ ensures that for any toric affinoid $U \subseteq X$ and $R = H^0(U \times Y, \widetilde{\Omega})$, we have $A_{V,G,R} \cong \prod_{k=1}^{n} (\mathrm{Sym}_R^{e_k} V) \otimes \mathbb{A}^1$. Gluing for a cover of $X$ by such $U$, we see that $A_G = \prod_{k=1}^{n} H^0(X \times Y, \mathrm{Sym}^{e_k} \widetilde{\Omega}_{X \times Y}) \otimes \mathbb{A}^1$. We now use that $X$ is proper and therefore $H^0(X, \mathrm{Sym}^{e_k} \widetilde{\Omega}_X)$ is a finite dimensional $K$-vector space. It follows from Lemma 8.4 that

$$H^0(X \times Y, \mathrm{Sym}^{e_k} \widetilde{\Omega}_{X \times Y}) = H^0(X, \mathrm{Sym}^{e_k} \widetilde{\Omega}_X) \otimes \mathcal{O}(Y),$$

and the right hand side is indeed represented by the rigid space $H^0(X, \mathrm{Sym}^{e_k} \widetilde{\Omega}_X) \otimes \mathbb{G}_a$.   $\square$

**Definition 8.10.** Composing the morphism of $v$-sheaves $h_{X \times Y}$ from (28) with the sheafification $\mathscr{H}iggs_G(Y) \to \mathrm{Higgs}_G(X \times Y)$ for varying $Y \in \mathrm{Perf}_K$, we get the **Hitchin morphism**

$$(29) \qquad\qquad\qquad \mathcal{H} : \mathscr{H}iggs_G \to \mathcal{A}_G,$$

a morphism of small $v$-stacks on $\mathrm{Perf}_K$ that is functorial in $G$ and $X$.

For $G = \mathrm{GL}_n$, this recovers the earlier description since the coefficients of the characteristic polynomial generate $k[\mathfrak{g}]^G$. Indeed, these are mapped to the elementary symmetric polynomials under the restriction map $k[\mathfrak{g}]^G \to k[\mathfrak{t}]^W$ in Chevalley's Restriction Theorem.

**Example 8.11.** If $G$ is commutative, then we have $K[\mathfrak{g}]^G = K[\mathfrak{g}]$. If $X$ is proper, we then have $\mathcal{A}_G = \mathfrak{g} \otimes \widetilde{\Omega}_X$. In any case, $\mathcal{H}$ is for such $G$ simply the projection to the second factor $\mathscr{H}iggs_G \xrightarrow{\sim} \mathscr{B}un_{G,\text{ét}} \times \mathcal{A}_G \to \mathcal{A}_G$ where the first map is from Lemma 7.14.

8.3. **The Hitchin morphism on the Betti side.** On the other side of the $p$-adic Simpson correspondence, we now construct the promised Hitchin morphism for the stack of $v$-topological $G$-bundles for any rigid group $G$ over $K$.

Let $Z$ be any smoothoid space. Combining the isomorphism HTlog from Theorem 4.1 with the Hitchin map (25) or (28), we obtain a morphism of sheaves on $Z_{\text{ét}}$

$$R^1 \nu_* G \xrightarrow{\mathrm{HTlog}} \mathrm{Higgs}_G \xrightarrow{h_Z} \mathcal{A}_G$$

Let $X$ be any smooth rigid space over $K$. Using crucially that HTlog is functorial in $Z$, we see that these morphisms for $Z = X \times Y$ glue for varying $Y$ to a morphism of $v$-stacks:

**Definition 8.12.** The **Hitchin morphism on the Betti side** is the morphism of $v$-stacks

$$\widetilde{\mathcal{H}} : \mathscr{B}un_{G,v} \to \mathcal{A}_G$$

defined on $Y \in \mathrm{Perf}_K$ as the composition

$$\mathscr{B}un_{G,v}(Y) \to H^1_v(X \times Y, G) \to R^1\nu_* G(X \times Y) \xrightarrow{\mathrm{HTlog}} \mathrm{Higgs}_G(X \times Y) \xrightarrow{h_{X \times Y}} \mathcal{A}_G(Y)$$

where the first map is the passage from groupoids to sets of isomorphism classes, where $\nu : (X \times Y)_v \to (X \times Y)_{\mathrm{\acute{e}t}}$ is the natural map, where HTlog is the isomorphism from Theorem 4.1, and $h_{X \times Y}$ is the sheafified Hitchin morphism on the Higgs side (25) or (28).

With this definition, the basic idea for our moduli theoretic approach to $p$-adic non-abelian Hodge theory is to compare $\mathscr{B}un_{G,v}$ and $\mathscr{H}iggs_G$ geometrically via the morphisms

(30)
$$
\begin{array}{c}
\mathscr{B}un_{G,v} \\
\qquad\qquad \searrow^{\widetilde{\mathcal{H}}} \\
\qquad\qquad\qquad \mathcal{A}_G \\
\qquad\qquad \nearrow_{\mathcal{H}} \\
\mathscr{H}iggs_G
\end{array}
$$

By Lemma 8.9, this is particularly interesting if $X$ is proper and $G$ is split reductive or commutative, as the Hitchin base $\mathcal{A}_G$ is then represented by an affine rigid space.

**Proposition 8.13.** *The morphism $\widetilde{\mathcal{H}} : \mathscr{B}un_{G,v} \to \mathcal{A}_G$ has the following properties:*

*(1) $\widetilde{\mathcal{H}}$ is functorial in $X \to \mathrm{Spa}(K)$. In particular, if $X$ has a model over a subfield $K_0 \subseteq K$, then $\widetilde{\mathcal{H}}$ is equivariant with respect to the $\mathrm{Aut}(K|K_0)$-actions on both sides.*

*(2) $\widetilde{\mathcal{H}}$ is functorial in $G$. In particular, if $0 \to V_1 \to V \to V_2 \to 0$ is a short exact sequence of $v$-vector bundles, then in the notation of Remark 8.6 we have $\widetilde{\mathcal{H}}(V) = \widetilde{\mathcal{H}}(V_1) \cdot \widetilde{\mathcal{H}}(V_2)$.*

*(3) If $G$ is commutative, then $\widetilde{\mathcal{H}}$ is a homomorphism of group stacks.*

*Proof.* The functoriality follows from the analogous properties for Higgs bundles and from functoriality of HTlog. To see part 2, we apply HTlog for parabolic subgroups $G \subseteq \mathrm{GL}_n$ and use functoriality in $G$. On the Higgs side, the statement then follows from Remark 8.6.

If $G$ is commutative, then HTlog is a homomorphism by Theorem 4.1, and $\mathcal{H}$ is a homomorphism of group stacks by Example 8.11. $\qquad\qquad\square$

The possibility of constructing a Hitchin morphism for $v$-$G$-bundles is a new idea already for the case $G = \mathrm{GL}_n$ of $v$-vector bundles, which was suggested to us by Scholze in reaction to an earlier version of Theorem 4.1. There are however two abelian special cases that can be described explicitly using known results, as we now explain in detail. From now on, we assume that $K$ is algebraically closed and that $X$ is a smooth proper rigid space over $K$.

8.4. **The Hitchin morphism on the Betti side for $G = \mathbb{G}_a$.** We first treat $G = \mathbb{G}_a$: In this case, Lemma 8.9 shows that $\mathcal{A}_{\mathbb{G}_a} = H^0(X, \widetilde{\Omega}) \otimes \mathbb{G}_a$ and by Example 8.11, the map $\mathcal{H} : \mathscr{H}iggs_{\mathbb{G}_a} \to \mathcal{A}_{\mathbb{G}_a}$ is simply the projection

$$\mathcal{H} : \mathscr{B}un_{\mathbb{G}_a, \mathrm{\acute{e}t}} \times H^0(X, \widetilde{\Omega}) \otimes \mathbb{G}_a \to H^0(X, \widetilde{\Omega}) \otimes \mathbb{G}_a.$$

Here $\mathscr{B}un_{\mathbb{G}_a, \mathrm{\acute{e}t}}$ parametrises étale $\mathbb{G}_a$-torsors on $X \times Y$. By Lemma 8.4 for $i = 1$, it can therefore be identified with the $v$-sheaf represented by the rigid vector group $H^1_{\mathrm{\acute{e}t}}(X, \mathcal{O}) \otimes \mathbb{G}_a$.

On the other hand, the Hitchin morphism on the Betti side is of the form

$$\widetilde{\mathcal{H}} : \mathscr{B}un_{\mathbb{G}_a, v} = H^1_v(X, \mathcal{O}) \otimes \mathbb{G}_a \to H^0(X, \widetilde{\Omega}) \otimes \mathbb{G}_a$$

and we easily see by comparing the construction of $\widetilde{\mathcal{H}}$ with Scholze's construction of the Hodge–Tate sequence [Sch13b, §3] that this map is the morphism of rigid varieties associated to the Hodge–Tate map HT from (2). This makes precise the idea that $\widetilde{\mathcal{H}}$ generalises HT.

We deduce from these explicit descriptions that (30) takes the following form:

**Proposition 8.14.** *Any splitting of* (2) *induces an isomorphism of rigid spaces (and thus of v-stacks)* $\mathscr{B}un_{\mathbb{G}_a,v} \xrightarrow{\sim} \mathscr{H}iggs_{\mathbb{G}_a}$ *that commutes with the Hitchin fibrations.*

In summary, the Hitchin morphism for $G = \mathbb{G}_a$ encodes the Hodge–Tate sequence. This is the precise way in which in this moduli-theoretic setting, non-abelian Hodge theory is about generalising a $p$-adic Hodge theoretic result from $\mathbb{G}_a$ to more general rigid groups $G$.

8.5. **The Hitchin morphism on the Betti side for** $G = \mathbb{G}_m$**.** We now turn to $G = \mathbb{G}_m$ to explain the precise relation to [Heua]. In order to get a description in terms of rigid spaces like for $\mathbb{G}_a$, we pass to coarse moduli spaces: Let $\mathbf{Bun}_{n,v}$ be the "coarse moduli sheaf" of $\mathscr{B}un_{n,v} = \mathscr{B}un_{\mathrm{GL}_n,v}$, by which we mean the functor obtained by passing from groupoids to sets of isomorphism classes and $v$-sheafifying on $\mathrm{Perf}_K$. We analogously define the coarse moduli $v$-sheaf $\mathbf{Higgs}_n$ of isomorphism classes of Higgs bundles of rank $n$ on $\mathrm{Perf}_K$. For $n = 1$, we thus obtain on the one hand the $v$-Picard functor

$$\mathbf{Bun}_{1,v} = \mathbf{Pic}_{X,v}$$

introduced in [Heua, §2], and on the other hand the coarse moduli space of Higgs line bundles

$$\mathbf{Higgs}_1 = \mathbf{Pic}_{X,\text{ét}} \times H^0(X, \widetilde{\Omega}) \otimes \mathbb{G}_a.$$

Both are representable by rigid groups over $K$ if the usual rigid analytic Picard functor is representable, by [Heua, Theorem 1.1]. For example, for algebraic $X$, $\mathbf{Pic}_{X,\text{ét}}$ is the diamond-dification of the algebraic Picard functor. There is then a natural short exact sequence

$$0 \to \mathbf{Pic}_{X,\text{ét}} \to \mathbf{Pic}_{X,v} \to H^0(X, \widetilde{\Omega}) \otimes \mathbb{G}_a \to 0$$

of commutative rigid groups. Crucially, this sequence is not split outside of trivial cases. By inspecting the proof in [Heua, §2], we see that in the language of this article, the last map is precisely the Hitchin morphism $\widetilde{\mathcal{H}}$ for $n = 1$, up to composition with $\mathscr{B}un_{1,v} \to \mathbf{Pic}_{X,v}$.

The upshot of this discussion is that in the case of $G = \mathbb{G}_m$, the two Hitchin morphisms over $\mathcal{A}_1 = H^0(X, \widetilde{\Omega}_X) \otimes \mathbb{G}_a$ are both torsors under $\mathbf{Pic}_{X,\text{ét}}$, but one is split while the other is usually not. In particular, in contrast to the case of $G = \mathbb{G}_a$, this shows that in general $\mathbf{Bun}_{n,v} \not\cong \mathbf{Higgs}_n$. Returning to moduli $v$-stacks, this is what we believe might generalise: It seems plausible to us that for any reductive group $G$, the morphism $\widetilde{\mathcal{H}}$ exhibits $\mathscr{B}un_{G,v}$ as a twist of $\mathscr{H}iggs_G$ over $\mathcal{A}_G$. To recover the $p$-adic Simpson correspondence from this geometric statement, one would have to see that the usual choices induce a trivialisation of this twist on $K$-points. This explains the motivation behind Conjecture 1.12.

8.6. **The rigid analytic representation variety.** Our final goal in this article is to obtain from the morphism of $v$-stacks $\widetilde{\mathcal{H}}$ a morphism of rigid analytic spaces by passing from $\mathscr{B}un_{G,v}$ to the rigid analytic representation variety.

**Definition 8.15.** Let $\Gamma$ be a profinite group. Let $G$ be a rigid group over $K$. We denote by

$$\mathscr{H}om(\Gamma, G)$$

the inner Hom sheaf on $\mathrm{Perf}_{K,v}$ where $\Gamma$ is considered as a profinite $v$-sheaf and $G$ as a diamond. This sends a perfectoid space $Y$ to the $Y$-linear homomorphisms $\Gamma \times Y \to G$. Let us say that a homomorphism $\Gamma \to G(Y)$ is continuous if it comes from a morphism of adic spaces of this form. To justify this name, we observe that if $G \subseteq \mathrm{GL}_n$ is a linear algebraic group, then $\mathscr{H}om(\Gamma, G)$ is the sheaf on $\mathrm{Perf}_K$ sending a perfectoid space $Y$ to $\mathrm{Hom}_{\mathrm{cts}}(\Gamma, G(Y))$, where $G(Y)$ is endowed with the subspace topology of $\mathrm{GL}_n(Y) \subseteq M_n(\mathcal{O}(Y))$.

**Proposition 8.16.** *Let* $\Gamma$ *be any profinite group. Let* $G$ *be a rigid group over* $K$*. Then*

(1) *The* $v$-sheaf $\mathscr{H}om(\Gamma, G)$ *is a diamond.*
(2) *If* $\Gamma$ *is topologically finitely generated, then* $\mathscr{H}om(\Gamma, G)$ *is represented by a semi-normal rigid space over* $K$*. We call this the* ***continuous representation variety*** *of* $\Gamma$*.*

*Proof.* Let $\gamma := (\gamma_i)_{i \in I}$ be a set of algebraic generators of $\Gamma$, not necessarily finite. Then we have an injection $\mathrm{ev}(\gamma) : \mathscr{H}om(\Gamma, G) \hookrightarrow \prod_I G$ of $v$-sheaves, where $\mathrm{ev}(\gamma)$ denotes the evaluation map $\rho \mapsto (\rho(\gamma_i))_{i \in I}$. By [Sch, Lemma 11.22], the right hand side is a diamond, thus so is the left hand side by [Sch, Proposition 11.10]. This proves part 1.

For the proof of part 2, let $\Gamma_0$ be a dense subgroup of $\Gamma$ with generators $g = (g_1, \ldots, g_r)$. Step 1: Reduction to the case that $\Gamma = \widehat{\Gamma}_0$ is the profinite completion of $\Gamma_0$. By the Theorem of Nikolov–Segal [NS07], any finite index subgroup of $\Gamma$ is open, so profinite completion induces a surjective, open, continuous homomorphism $\phi : \widehat{\Gamma}_0 \twoheadrightarrow \Gamma$. We deduce that for any set of generators $a = (a_i)_{i \in I}$ of $\ker \phi$, a continuous map from $\Gamma$ is the same as a continuous map from $\widehat{\Gamma}_0$ that vanishes on the $a_i$. We thus have a left-exact sequence

$$0 \to \mathscr{H}om(\Gamma, G) \to \mathscr{H}om(\widehat{\Gamma}_0, G) \xrightarrow{\mathrm{ev}(a)} G^I.$$

If the Proposition holds for $\widehat{\Gamma}_0$, this shows that $\mathscr{H}om(\Gamma, G)$ is the fibre of a morphism from a rigid space to a product of adic spaces, hence is itself represented by a rigid space.
Step 2: Deformations of the trivial representation. By Corollary 3.4 and Proposition 3.6, there is an open subgroup $G^+ \subseteq G$ of good reduction and a basis of neighbourhoods given by open subgroup $(G_n)_{n \in \mathbb{N}}$ of $G^+$ such that $G_n/G_{n+1} = \overline{\mathfrak{g}}_t^+ := \mathfrak{g}_0^+/\mathfrak{g}_t^+$ for some $t > 0$.

**Lemma 8.17.** *Let $(R, R^+)$ be any perfectoid $K$-algebra. Then any group homomorphism $\rho : \Gamma_0 \to G_0(R)$ extends uniquely to a continuous morphism $\widehat{\rho} : \Gamma \to G_0(R)$.*

*Proof.* By Lemma 3.7, it suffices to see that each $\rho_n : \Gamma_0 \to G_0(R) \to G_0/G_n(R)$ is continuous, i.e. has finite image. For this we argue by induction on $n$.

For $n = 1$, the group $G_0/G_1(R) = \overline{\mathfrak{g}}_t^+(R)$ is an abelian torsion group. As $\Gamma_0$ is finitely generated, it follows that any map $\varphi : \Gamma_0 \to \overline{\mathfrak{g}}_t^+(R)$ has finite image. Thus $\ker \varphi$ is a finite index subgroup, which is open by the assumption $\Gamma = \widehat{\Gamma}_0$. This gives the case of $\rho_1$.

For the induction step, we consider the short exact sequence obtained from Proposition 3.6

$$0 \to \overline{\mathfrak{g}}_t^+(R) \to G_0/G_{n+1}(R) \to G_0/G_n(R) \to 1.$$

By induction hypothesis, the image $\mathrm{im}(\rho_n)$ of $\Gamma_0$ on the right is finite, so $H := \ker(\rho_n)$ is a finite index subgroup of $\Gamma_0$. In particular, $H$ is again finitely generated. The restriction of $\rho_{n+1}$ to $H$ factors through $\overline{\mathfrak{g}}_t^+(R)$, and thus has finite index kernel by the induction start. This shows that $\ker \rho_{n+1}$ contains a finite index subgroup, hence is open. $\qquad\square$

Hence $\mathscr{H}om(\Gamma, G_0) = \mathscr{H}om(\Gamma_0, G_0)$ is represented by a Zariski-closed subspace of the space $(G_0)^r$ parametrising the images of the $g_i$, cut out by the relations between the $g_i$.
Step 3: The general case. Consider now any finite index subgroup $H \subseteq \Gamma_0$ and let $F_H \subseteq \mathscr{H}om(\Gamma, G)$ be the open subfunctor defined as the preimage of $\mathscr{H}om(H, G_0) \subseteq \mathscr{H}om(H, G)$ under the restriction map $\mathrm{res}_H : \mathscr{H}om(\Gamma, G) \to \mathscr{H}om(H, G)$. Then by continuity, we have an equality of sheaves $\mathscr{H}om(\Gamma, G) = \varinjlim_{H < \Gamma_0} F_H$ where $H$ ranges through the finite index subgroups of $\Gamma_0$. It thus suffices to prove that each $F_H$ is representable by a semi-normal rigid space; the transition maps in the colimit are then clearly open immersions. For this let $s = (s_1, \ldots, s_l)$ be a finite set of generators of $H$, and for each $k = 1, \ldots, l$ let $w_{s_k}(g)$ be the group-theoretic word expressing $s_k$ in terms of the generators $g = (g_1, \ldots, g_r)$ of $\Gamma_0$. Let $E \subseteq G^r$ be the Zariski-closed subspace cut out by the relations between the $g_i$. Then

$$
\begin{array}{ccccc}
E & \hookrightarrow & G^r & \xrightarrow{\ w_s\ } & G^l \\
{\scriptstyle \mathrm{ev}(g)}\uparrow & & & & \uparrow{\scriptstyle \mathrm{ev}(s)} \\
F_H & & \xrightarrow{\ \mathrm{res}_H\ } & & \mathscr{H}om(H, G_0)
\end{array}
$$

is a Cartesian diagram. It follows that $F_H$ is represented by the fibre product in rigid spaces. Let us denote the latter by $X_H$. It remains to pass from $X_H$ to its semi-normalisation:

**Lemma 8.18** ([KL16, Remark 3.7.3][SW20, p.78]). *The inclusion*

$$\{semi\text{-}normal\ rigid\ spaces\ over\ K\} \hookrightarrow \{rigid\ spaces\ over\ K\}$$

*admits a right-adjoint, the semi-normalisation functor* $-^{\mathrm{sn}}$. *For any rigid space* $X$, *the co-unit map induces an isomorphism* $X^{\mathrm{sn}\diamond} \to X^{\diamond}$.

*Proof.* After replacing $X$ with its reduced subspaces $X^{\mathrm{red}}$, it suffices to construct this on reduced rigid spaces. Any reduced commutative ring $A$ has a semi-normalisation $A^{\mathrm{sn}}$ with the desired universal property by [Swa80, Theorem 4.1], and the map $A \to A^{\mathrm{sn}}$ is subintegral, hence finite since $A$ is excellent. Thus $A^{\mathrm{sn}}$ is again of topologically finite type over $K$. This glues since seminormality is stable under rational localisation [KL16, Proposition 3.7.2]. $\square$

It follows that the semi-normal rigid space $X_H^{\mathrm{sn}}$ represents $\mathscr{H}om(\pi, G)$. $\square$

Returning to the $p$-adic Simpson correspondence, we wish to apply Proposition 8.16 to the étale fundamental group $\Gamma = \pi_1^{\text{ét}}(X)$ of a smooth proper rigid space $X$. In this situation, the algebraic conditions imposed on $\Gamma$ apply in decent generality:

**Corollary 8.19.** *Let* $X$ *be the rigid analytification of a smooth proper* $K$-*variety. Then the* $v$-*sheaf* $\mathscr{H}om(\pi_1^{\text{ét}}(X), G)$ *is represented by a semi-normal rigid space.*

*Proof.* By the Lefschetz principle, we can assume that $X$ has a model $X_0$ over a subfield $K_0 \subseteq K$ such that there is an isomorphism of fields $K_0 \cong \mathbb{C}$. By invariance of $\pi_1^{\text{ét}}$ under extension of scalars, $\pi_1^{\text{ét}}(X)$ is then the profinite completion of the topological fundamental group of the compact Kähler variety $X_0(\mathbb{C})$, which is finitely generated. $\square$

**Question 8.20.** Let $X$ be any (smooth) proper rigid space over $K$. Is the étale fundamental group $\pi_1^{\text{ét}}(X)$ topologically finitely generated? If yes, then Corollary 8.19 holds for such $X$.

8.7. **The Hitchin morphism for representations.** Finally, we explain how for the analytification $X$ of a connected smooth proper variety over a complete algebraically closed field $K$ over $\mathbb{Q}_p$ with fixed base point $x \in X(K)$, the Hitchin morphism $\widetilde{\mathcal{H}}$ attaches to any continuous representation $\pi := \pi_1^{\text{ét}}(X, x) \to \mathrm{GL}_n(K)$ a set of "Hitchin–Hodge–Tate weights" in a way that is compatible in families. The key to this is the following relation between $G$-representations of $\pi_1^{\text{ét}}(X, x)$ and $v$-$G$-bundles:

**Definition 8.21.** Let $G$ be any rigid group. There is a natural morphism of $v$-stacks

$$\alpha : \mathscr{H}om_{\mathrm{cts}}(\pi, G) \to \mathscr{B}un_{G,v}$$

defined as follows (cf. [Heu22b, Theorem 5.2] for $G = \mathrm{GL}_n$): Let $\widetilde{X} \to X$ be the pro-finite-étale universal cover from Remark 4.4.2, which is a $\pi$-torsor in a natural way. For any $Y \in \mathrm{Perf}_K$, consider the projection $q : \widetilde{X} \times Y \to X \times Y$ as a $\pi \times Y$-torsor relatively over the base $Y$. Then $\alpha(Y)$ sends any continuous homomorphism $\rho : \pi \times Y \to G$ over $Y$ in the sense of Definition 8.15 to the $G$-torsor on $(X \times Y)_v$ defined by pushout of $q$ along $\rho$.

**Theorem 8.22.** *Let* $G$ *be a rigid group over* $K$ *satisfying the assumptions of Lemma 8.9, for example* $G$ *could be reductive, or commutative. Then* $\widetilde{\mathcal{H}} \circ \alpha$ *defines a canonical morphism of rigid analytic spaces*

$$\widetilde{H} : \mathscr{H}om(\pi_1^{\text{ét}}(X, x), G) \to \mathcal{A}_G$$

*from the continuous* $G$-*representation variety of* $\pi_1^{\text{ét}}(X, x)$ *to the Hitchin base. Moreover:*

*(1)* $\widetilde{H}$ *is functorial in* $G$ *and* $X \to \mathrm{Spa}(K)$.
*(2) For any continuous representations* $\rho_1, \rho_2 : \pi_1^{\text{ét}}(X, x) \to G(Y)$ *over a perfectoid space* $Y$ *for which there is a finite index subgroup* $\Gamma \subseteq \pi_1^{\text{ét}}(X, x)$ *such that the restrictions* $\rho_{1|\Gamma} \sim \rho_{2|\Gamma}$ *become conjugated over a* $v$-*cover* $Y' \to Y$, *we have* $\widetilde{H}(\rho_1) = \widetilde{H}(\rho_2)$.
*(3) If* $G$ *is commutative, then* $\widetilde{H}$ *is a group homomorphism.*

(4) *If $G = \mathrm{GL}_n$, then for any short exact sequence of representations $\rho_1 \to \rho \to \rho_2$, we have $\widetilde{H}(\rho) = \widetilde{H}(\rho_1) \cdot \widetilde{H}(\rho_2)$ where $\cdot$ is the product on $\mathcal{A}_n$ defined in Remark 8.6.*

*Proof.* The composition $\widetilde{\mathcal{H}} \circ \alpha : \mathscr{H}om(\pi_1^{\text{ét}}(X, x), G) \to \mathcal{A}_G$ is a morphism of $v$-sheaves over $K$. Both sides are represented by semi-normal rigid spaces by Corollary 8.19 and Lemma 8.9. As diamondification from semi-normal rigid spaces to $v$-sheaves over $K$ is fully faithful [KL16, Theorem 8.2.3][Sch, Proposition 10.2.3], this defines a unique morphism of rigid spaces.

Parts 1 and 4 are then immediate from Proposition 8.13. Part 2 follows from functoriality in $X$: After pullback to the finite étale cover $X' \to X$ corresponding to $\Gamma$, and any choice of lift $x'$ of $x$ to $X'$, the representations define the same element in $\mathscr{H}om(\pi_1^{\text{ét}}(X', x'), G)$. Since the natural map $\mathcal{A}_{G,X}(R) \to \mathcal{A}_{G,X'}(R')$ is injective, it follows that $\widetilde{H}(\rho_1) = \widetilde{H}(\rho_2)$.

For part 3, it suffices by part 2 to prove this on an open subgroup $G_0 \subseteq G$ on which exp is defined. As $K[\mathfrak{g}]^G = K[\mathfrak{g}]$, we have $\mathcal{A}_{G_0} = \mathcal{A}_G = \mathbb{G}_a \otimes \mathfrak{g} \otimes \widetilde{\Omega}$, and the result follows by functoriality from the morphism $\exp : G_0 \to \mathfrak{g} \otimes \mathbb{G}_a$ and the case of $\mathbb{G}_a$.                $\square$

We believe that $\widetilde{H}$ encodes deep arithmetic information in a geometric way. Indeed, from known instances of the $p$-adic Simpson correspondence, one can extract the following:

**Example 8.23.**    (1) For $G = \mathbb{G}_m$, it follows from §8.5 that $\widetilde{H}$ is the homomorphism from [Heu24a, Theorem 1.2], hence fits into an exact sequence of rigid group varieties

$$0 \to \mathbf{Pic}_{X,\text{ét}}^{\text{tt}} \to \mathscr{H}om_{\text{cts}}(\pi, \mathbb{G}_m) \xrightarrow{\widetilde{H}} H^0(X, \widetilde{\Omega}) \otimes_K \mathbb{G}_a \to 0.$$

In particular, $\widetilde{H}$ is then a torsor under $\mathbf{Pic}_{X,\text{ét}}^{\text{tt}}$, the topologically torsion Picard variety. For curves, the first map is a geometric incarnation of the Weil pairing of the Jacobian.
(2) If a representation $\rho : \pi \to \mathrm{GL}_n(K)$ has abelian image, then $\widetilde{H}(\rho)$ can be described in a way that is closely related to the abeloid $p$-adic Simpson correspondence of [HMW23]: In this case, $K^n$ can be decomposed into simultaneous eigenspaces. Let $\lambda_i : \pi \to K^\times$ be the corresponding characters and set $a_i := \widetilde{H}(\lambda_i) \in H^0(X, \widetilde{\Omega})$, then it follows from Theorem 8.22.(4) that $\widetilde{H}(\rho)$ is given by the coefficients of $(X - a_1) \cdots (X - a_n)$.
(3) If $X$ has a model $X_0$ over a discretely valued subfield of $K$, then any representation $\rho : \pi \to \mathrm{GL}_n(\mathbb{Q}_p)$ associated to a $\mathbb{Q}_p$-local system $\mathbb{L}$ over $X_0$ lands in the fibre over 0 by Galois equivariance in Theorem 8.22.(1), giving a different perspective on why the Higgs field associated to $\mathbb{L}$ by the $p$-adic Simpson functor of Liu–Zhu [LZ17] is nilpotent.
(4) Similarly, all the representations in the image of Deninger–Werner's functor [DW20] are also in the fibre over 0, because the associated Higgs fields vanish. This includes all representations with finite image, for which this also follows from Theorem 8.22.2.

If $X$ is a smooth proper curve of genus $g \geq 2$, we think it is plausible that Example 8.23.1 generalises to $\mathrm{GL}_n$ for $n \geq 1$, namely that $\widetilde{H}$ could generically on $\mathcal{A}_G$ be a torsor under a certain rigid group related to the spectral curve. We will pursue this further in future work.

8.8. **Relation to the complex Corlette–Simpson correspondence.** In complex geometry, the Corlette–Simpson correspondence for a smooth projective variety $X$ over $\mathbb{C}$ is an equivalence of categories between finite dimensional complex representations of the topological fundamental group $\pi_1(X)$ on the one hand ("Betti side"), and semi-stable Higgs bundles on $X$ with vanishing Chern classes on the other ("Dolbeault side") [Sim94]. It can be regarded as an analogue of the Hodge decomposition for non-abelian coefficients.

More generally, the Corlette–Simpson correspondence can be generalised to an equivalence of $G$-representations and $G$-Higgs bundles for any reductive group $G$ over $\mathbb{C}$ [Sim92]. However, we are not aware of an extension to general complex Lie groups. From this perspective, it is arguably somewhat surprising that our result holds in the given generality. This is one reason

why in the introduction we were cautious and imposed a condition that $G$ is reductive in the globalisation of Theorem 1.2 to proper $X$.

Regarding moduli spaces, the Corlette–Simpson correspondence for $\mathrm{GL}_n$ induces for any smooth projective variety $X$ over $\mathbb{C}$ a canonical and functorial homeomorphism

$$(31) \qquad \mathcal{M}_{\mathrm{B}}(\mathbb{C}) \xrightarrow{\sim} \mathcal{M}_{\mathrm{Dol}}(\mathbb{C})$$

between the Betti moduli space of complex representations of the topological fundamental group $\pi_1(X)$ of dimension $n$ on the one hand, and the Dolbeault moduli space of semi-stable Higgs bundles on $X$ of rank $n$ with vanishing Chern classes on the other [Sim94]. However, even though both sides have natural complex analytic structures, the correspondence does not respect these: In fact, the analogue of $\widetilde{H}$ in this setting would be the composition

$$\mathcal{M}_{\mathrm{B}}(\mathbb{C}) \xrightarrow{\sim} \mathcal{M}_{\mathrm{Dol}}(\mathbb{C}) \xrightarrow{\mathcal{H}} \mathcal{A}_n(\mathbb{C})$$

of the Corlette–Simpson correspondence with the Hitchin morphism. But this fails to be complex analytic already for line bundles on curves [Sim92, Example on p21]. In particular, the above homeomorphism cannot be upgraded to an isomorphism between moduli stacks.

From this perspective, we find it very surprising that $\widetilde{H}$ exists as a morphism of rigid analytic spaces: This seems to be a stronger statement than what is possible in the complex theory. It thus appears that the correspondence over $K$ is less canonical as it requires the choice of lift of $X$, but on the other hand preserves more structure.

8.9. **Relation to Sen theory.** In order to provide more context for the Hitchin–Hodge–Tate map, we now briefly sketch how the above construction is related to Hodge–Tate–Sen weights:

Let $L$ be a $p$-adic local field and let $L_\infty | L$ be the extension obtained by adjoining all $p$-power roots of unity. Let $C$ be the completed algebraic closure of $L$. Classical Sen theory associates to any semi-linear representation of the Galois group $G_L$ of $L$ on a finite dimensional $C$-vector space $W$ an $L_\infty$-vector space $E = D_{\mathrm{Sen}}(W)$ such that $E \otimes_{L_\infty} C = W$ together with an $L_\infty$-linear operator $\theta : E \to E$ (see [Sen81]). We call $(E, \theta)$ a Sen module, and the Hodge–Tate–Sen weights of $W$ are defined as the generalised eigenvalues of $\theta$.

The construction of $(E, \theta)$ is very similar to the small $p$-adic Simpson correspondence, and one can turn this analogy into a more precise comparison: Consider the adic space $X = \mathrm{Spa}(L)$, then we can make a very similar construction for $X$ (the "arithmetic setting") as we did for smooth rigid spaces over $C$ in the last section (the "geometric setting"): The pro-finite-étale cover $\widetilde{X} \to X$ of the last section is for $X = \mathrm{Spa}(L)$ simply the cover $\mathrm{Spa}(C) \to \mathrm{Spa}(L)$. Like in Definition 8.21, one obtains by descent along this cover an equivalence

$$\left\{ \begin{array}{c} \text{semi-linear representations of } G_L \\ \text{on fin. dim. } C\text{-vector spaces} \end{array} \right\} \xrightarrow{\sim} \left\{ v\text{-vector bundles on } \mathrm{Spa}(L) \right\}.$$

On the other hand, a result of Tate in Galois cohomology [Tat66, Theorem 1] can be interpreted as saying that for $\nu : X_v \to X_{\text{ét}}$, we have $R^1 \nu_* \mathcal{O} = \mathcal{O}$ and $R^2 \nu_* \mathcal{O} = 0$. From the point of view taken in this article, it would therefore be sensible to define $\widetilde{\Omega}_X := R^1 \nu_* \mathcal{O} = \mathcal{O}$, so that according to Definition 2.10, a Higgs bundle on $X_{\text{ét}}$ for this notion of "differentials' would be a vector bundle $E$ on $X_{\text{ét}}$ together with an endomorphism of $E$. In particular, from such a Higgs bundle, we get a Sen module by base-changing the global sections to $L_\infty$.

Going through the construction of the Sen module $D_{\mathrm{Sen}}(W)$, one now sees that one can interpret this as being obtained from a "local $p$-adic Simpson correspondence on the space $X = \mathrm{Spa}(L)$": The coefficients $L_\infty$ appear since one needs to go up the cyclotomic tower $L_\infty | L$ to make the $v$-vector bundle small, as with the toric tower in the geometric setting.

Carrying out the analogous constructions of the last section in the arithmetic setting now yields the map that sends a semilinear $G_L$-representation $W$ to the characteristic polynomial of its Sen operator $\theta$, which is an equivalent datum to the Hodge–Tate–Sen weights. This is

the precise sense in which the Hitchin–Hodge–Tate morphism $\widetilde{H}$ is conceptually analogous in the geometric setting to sending a $G_L$-representation to its Hodge–Tate–Sen weights.

## APPENDIX A. RECOLLECTIONS ON NON-ABELIAN CONTINUOUS COHOMOLOGY

In this appendix, we recall some generalities on non-abelian group cohomology sets.

**Definition A.1.** Let $\Gamma$ and $A$ be not necessarily abelian topological groups, written multiplicatively. Assume that we have a continuous left-action of $\Gamma$ on $A$, i.e. a group homomorphism $\Gamma \to \mathrm{Aut}(A)$ such that the induced morphism $\Gamma \times A \to A$ is continuous. For $g \in \Gamma$, $a \in A$, we write $ga$ for its image under this map. Then the set of continuous 1-cocycles is

$$\mathcal{Z}^1_{\mathrm{cts}}(\Gamma, A) := \{\text{continuous maps } c : \Gamma \to A \mid c(gh) = c(g) \cdot gc(h)\}$$

The continuous group cohomology set $H^1_{\mathrm{cts}}(\Gamma, A) := \mathcal{Z}^1_{\mathrm{cts}}(\Gamma, A)/\sim$ is then defined as the quotient by the following equivalence relation:

$$f \sim f' \Leftrightarrow \exists a \in A : \quad a^{-1} \cdot f(g) \cdot ga = f'(g) \text{ for all } g \in \Gamma.$$

For any continuous 1-cocycle $c$, we denote its cohomology class by $[c]$.

**Definition A.2.** Let $B$ be a topological group with continuous $\Gamma$-action and let $A \subseteq B$ be a normal subgroup that is preserved by the action of $\Gamma$. Let $b \in \mathcal{Z}^1_{\mathrm{cts}}(\Gamma, B)$ be any continuous 1-cocycle. Then we denote by $_bA$ the $\Gamma$-module with underlying group $A$ equipped with the $b$-twisted action, given for $g \in \Gamma$, $a \in A$ by $g *_b a := b(g) \cdot ga \cdot b(g)^{-1}$, where the right hand side is calculated inside $B$.

**Proposition A.3** ([Ser94], §5.5)**.** *Let $\Gamma$ be a topological group, and let $0 \to A \to B \to C \to 0$ be a short exact sequence of (not necessarily abelian) topological groups with continuous left-$\Gamma$-actions such that all maps in the sequence are $\Gamma$-equivariant. Then:*

(1) *There is a short exact sequence of pointed sets*

$$0 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to H^1_{\mathrm{cts}}(\Gamma, A) \to H^1_{\mathrm{cts}}(\Gamma, B) \to H^1_{\mathrm{cts}}(\Gamma, C).$$

(2) *Let $b \in \mathcal{Z}^1_{\mathrm{cts}}(\Gamma, B)$, then the subset of elements of $H^1_{\mathrm{cts}}(\Gamma, B)$ with the same image as $b$ in $H^1_{\mathrm{cts}}(\Gamma, C)$ is in natural bijection with the quotient of $H^1_{\mathrm{cts}}(\Gamma, {}_bA)$ by $C^\Gamma$.*

(3) *Assume that $A$, $B$, $C$ are discrete, that $A$ is abelian and let $c \in \mathcal{Z}^1_{\mathrm{cts}}(\Gamma, C)$, then there is a class $\Delta(c) \in H^2_{\mathrm{cts}}(\Gamma, {}_cA)$ that vanishes if and only if $[c]$ is in the image of $H^1_{\mathrm{cts}}(\Gamma, B) \to H^1_{\mathrm{cts}}(\Gamma, C)$. The class $\Delta(c)$ is canonical and functorial both in $\Gamma$ and in the sequence.*

We note that it is assumed in [Ser94] that $\Gamma$ is profinite and $A$, $B$, $C$ are discrete, but one verifies from all explicit constructions that the arguments go through in the given generality.

One natural way that continuous group cohomology arises in practice is that it can be used to describe Čech-cohomology for Galois covers with group $\Gamma$ for non-abelian sheaves, by a non-abelian version of the Cartan–Leray spectral sequence:

**Proposition A.4** ([Heu22b], Proposition 2.8)**.** *Let $q : X \to Y$ be a morphism of diamonds over $K$ that is Galois for the action of a locally profinite group $\Gamma$ on $X$. Let $\mathcal{F}$ be a sheaf of (not necessarily abelian) topological groups on $Y_v$ with the property that for $i = 1, 2$ we have*

$$\mathcal{F}(X \times \Gamma^i) = \mathrm{Map}_{\mathrm{cts}}(\Gamma^i, \mathcal{F}(X)),$$

*for example this holds if $\mathcal{F}$ is represented by a rigid group over $K$ that admits a locally closed embedding as a rigid space into some affine space $\mathbb{A}^n$ (e.g. any linear algebraic group, or rigid open subgroup thereof). Then there is a left-exact sequence of pointed sets*

$$0 \to H^1_{\mathrm{cts}}(\Gamma, \mathcal{F}(X)) \to H^1_v(Y, \mathcal{F}) \to H^1_v(X, \mathcal{F})^\Gamma.$$

*Proof.* The only part that is not proved in the reference is that $\mathcal{F}$ satisfies the displayed condition if it embeds into $\mathbb{A}^n$. It suffices to prove this for $i = 1$ and $X$ in the basis of affinoid perfectoid spaces. By the known case of $\mathcal{F} = \mathcal{O}$, we then have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{cts}}(\Gamma, \mathcal{F}(X)) & \hookrightarrow & \mathrm{Map}_{\mathrm{cts}}(\Gamma, \mathcal{O}^n(X)) \\
\wedge \vdots & & \cong \uparrow \\
\mathrm{Mor}_K(\Gamma \times X, \mathcal{F}) & \hookrightarrow & \mathrm{Mor}_K(\Gamma \times X, \mathbb{A}^n).
\end{array}
$$

By evaluating at all points $g \in \Gamma$, we get the dotted arrow in the diagram: Here the continuity holds because the assumptions ensure that $\mathcal{F}(X) \subseteq \mathcal{O}^n(X)$ inherits the subspace topology. Also due to the locally closed assumption, we can check on points whether a map $\Gamma \times X \to \mathbb{A}^n$ factors through $\mathcal{F}$. This shows that the morphism on the left is surjective. $\square$

When the Galois group is $\cong \mathbb{Z}_p^d$, we have the following, see the proof of [Sch13a, Lemma 5.5]:

**Lemma A.5.** *Let* $\Delta = \mathbb{Z}_p^d$ *be topologically generated by elements* $\gamma_1, \ldots, \gamma_d$. *Let* $R$ *be a* $p$-*adically complete* $\mathbb{Z}_p$-*algebra and let* $M$ *be a* $p$-*adically complete* $R$-*module with a continuous* $R$-*linear* $\Delta$-*action. Then* $R\Gamma_{\mathrm{cts}}(\Delta, M)$ *is computed by the complex*

$$
\mathcal{C}^*_{\mathrm{grp}} := \left[ M \to M \otimes_R R^d \to M \otimes_R \wedge^2 R^d \to \cdots \to M \otimes_R \wedge^d R^d \right]
$$

*whose* $k$-*th transition map is given by* $m \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_{j=1}^d (\gamma_j - 1) m \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_j$.

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BEN HEUER, INSTITUT FÜR MATHEMATIK, JOHANN WOLFGANG GOETHE-UNIVERSITÄT, ROBERT-MAYER-
STR. 6-8, 60325 FRANKFURT AM MAIN, GERMANY; `heuer@math.uni-frankfurt.de`