G-TORSORS ON PERFECTOID SPACES

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ABSTRACT. For any rigid analytic group variety G over a non-archimedean field K over $\mathbb{Q}_p$, we study G-torsors on adic spaces over K in the v-topology. Our main result is that on perfectoid spaces, G-torsors in the étale and v-topology are equivalent. This generalises the known cases of $G = \mathbb{G}_a$ and $G = \mathrm{GL}_n$ due to Scholze and Kedlaya–Liu.

On a general adic space X over K, where there can be more v-topological G-torsors than étale ones, we show that for any open subgroup $U \subseteq G$, any G-torsor on X_v admits a reduction of structure group to U étale-locally on X. This has applications in the context of the p-adic Simpson correspondence: For example, we use it to show that on any adic space, generalised $\mathbb{Q}_p$-representations are equivalent to v-vector bundles.

1. INTRODUCTION

Let G be a smooth group scheme over a field K, and let X be a K-scheme. Then by a well-known Theorem of Grothendieck [Gro68, Théorème 11.7, Remark 11.8], the functor

$$\{G\text{-torsors on } X_{\text{ét}}\} \xrightarrow{\sim} \{G\text{-torsors on } X_{\text{fppf}}\}$$

is an equivalence of categories. This article studies a similar functor in p-adic geometry:

Let K be a non-archimedean field over $\mathbb{Q}_p$ and let X be an adic space over K, considered as a diamond in the sense of Scholze [Sch22, §15]. Then X has an étale topology and a much finer v-topology [Sch22, §14]. Let G be any rigid analytic group variety over K, not necessarily commutative, considered as a diamond. In this article, we study the functor

$$\{G\text{-torsors on } X_{\text{ét}}\} \hookrightarrow \{G\text{-torsors on } X_v\},$$

where a G-torsor is a sheaf with a G-action that is locally isomorphic to G (see Section 3.3). Let $\nu : X_v \to X_{\text{ét}}$ be the natural morphism of sites, then this functor is fully faithful under the mild technical assumption that $\nu_*\mathbb{G}_a = \mathbb{G}_a$ (e.g. by results of Scholze and Kedlaya–Liu, this holds when X is a perfectoid space or a semi-normal rigid space), see Proposition 3.16. The question whether the functor is essentially surjective can be phrased in terms of ν by asking whether $R^1\nu_*G$ vanishes.

In this p-adic situation, the analogy to Grothendieck's Theorem works best when X is a perfectoid space: Indeed, in this case, torsors under $G = \mathbb{G}_a$ on $X_{\text{ét}}$ and X_v agree, namely Scholze proves that $R\nu_*\mathbb{G}_a = \mathbb{G}_a$ [Sch22, Proposition 8.8]. By a Theorem of Kedlaya–Liu, also vector bundles on $X_{\text{ét}}$ and X_v agree [KL16, Theorem 3.5.8] (cf. [SW20, Lemma 17.1.8]), which is the case of $G = \mathrm{GL}_n$. This implies the case of linear algebraic G by the Tannakian formalism [SW20, §19.5]. Our main result is the following generalisation of all of these statements:

Theorem 1.1. *Let X be a perfectoid space and let G be a rigid group over K. The functor*

$$\{G\text{-torsors on } X_{\text{ét}}\} \xrightarrow{\sim} \{G\text{-torsors on } X_v\}$$

is an equivalence of categories. If G is commutative, then we more generally have $R\nu_*G = G$.

Our method is different to that of the aforementioned works, and also gives a new proof of Kedlaya–Liu's Theorem. One interesting new case is $G = \mathrm{GL}_n(\mathcal{O}^+)$, which says that the categories of finite locally free $\mathcal{O}^+$-modules on $X_{\text{ét}}$ and X_v agree.

As another application, recall that for any locally Noetherian adic space X over $\mathbb{Q}_p$ (for example a rigid space), we have a hierarchy of topologies

$$X_v \to X_{\text{qproét}} \to X_{\text{proét}} \to X_{\text{ét}}.$$

Here $X_{\text{qproét}}$ is the quasi-pro-étale site, which makes sense without the assumption that X is locally Noetherian. Since the three topologies on the left are locally perfectoid, we deduce:

Corollary 1.2. *Let X be any adic space over K, then the categories of G-torsors on X_v and $X_{\text{qproét}}$ (and $X_{\text{proét}}$ if X is locally Noetherian) are equivalent.*

1.1. **G-torsors on rigid spaces.** Let now X be a rigid space, then it is known that there are in general many more v-topological G-torsors than there are étale ones: For example, if K is a complete algebraically closed extension of $\mathbb{Q}_p$ and X is smooth over K, then already for $G = \mathbb{G}_a$, a key step in Scholze's construction of the Hodge–Tate spectral sequence is a canonical isomorphism on $X_{\text{ét}}$

$$R^1\nu_*\mathbb{G}_a = \Omega_X(-1)$$

where $\Omega_X(-1)$ is a Tate twist of the Kähler differentials on X [Sch13b, Proposition 3.23].

Also for $G = \mathrm{GL}_n$, it is known that étale vector bundles and v-vector bundles on X are not the same, e.g. see [Heu22a] for the case of $G = \mathbb{G}_m$. The difference is closely related to the p-adic Simpson correspondence: As we explain in Section 2, there is an equivalence

$$\{\text{finite locally free } \mathcal{O}^+\text{-modules on } X_v\} \xrightarrow{\sim} \{\text{generalised representations on } X\}$$

where following Faltings [Fal05], generalised representations are compatible systems $(V_n)_{n\in\mathbb{N}}$ of finite locally free $\mathcal{O}^+/p^n$-modules V_n on $X_{\text{ét}}$. The equivalence is similar in spirit to the equivalence between lisse $\mathbb{Q}_l$-sheaves on a scheme and locally free $\mathbb{Q}_l$-modules on the pro-étale site in the sense of Bhatt–Scholze [BS15, §1]. The proof hinges on the following:

Proposition 1.3. *We have an isomorphism $R\nu_*(\mathcal{O}^+/p^n) = \mathcal{O}^+/p^n$ (already before passing to the almost category) and $R^1\nu_*\mathrm{GL}_n(\mathcal{O}^+/p^n) = 1$. In particular, we have an equivalence*

$$\nu^* : \left\{ \begin{array}{c} \textit{finite locally free} \\ \mathcal{O}^+/p^n\textit{-modules on } X_{\text{ét}} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{finite locally free} \\ \mathcal{O}^+/p^n\textit{-modules on } X_v \end{array} \right\}.$$

One reason why we are interested in v-topological G-torsors under general rigid groups G is for generalisations of the p-adic Simpson correspondence to more general non-abelian coefficients, as explored in [Heu22b]: These relate v-topological G-bundles to G-Higgs bundles. The relevance of Theorem 1.1 in this context is that it shows that v-topological G-bundles are "locally small", namely they admit reductions of structure groups to small open subgroups.

1.2. **Reduction of structure group.** The technical heart of this article is the study of sheaves F that "commute with tilde-limits":

Definition 1.4. A sheaf F on the big étale site of sousperfectoid spaces over K is said to satisfy the **approximation property** if for any affinoid perfectoid tilde-limit $X \sim \varprojlim_{i\in I} X_i$ of affinoid spaces X_i such that $\varinjlim_{i\in I} \mathcal{O}(X_i) \to \mathcal{O}(X)$ has dense image, $F(X) = \varinjlim_{i\in I} F(X_i)$.

Examples of such sheaves include $\mathcal{O}^+/p^n$ and thus also $\mathrm{GL}_n(\mathcal{O}^+/p^n)$. We show:

Theorem 1.5. *Let F be a sheaf of groups satisfying the approximation property. Then F is already a v-sheaf and $R^1\nu_*F = 1$. If F is a sheaf of abelian groups, then $R\nu_*F = F$.*

The relevance to our study of G-torsors is then the following:

Proposition 1.6. *Let G be any rigid group. Let $U \subseteq G$ be any rigid open subgroup, not necessarily normal. Then G/U satisfies the approximation property.*

For example, both $\mathcal{O}^+/p^n$ and $\mathrm{GL}_n(\mathcal{O}^+/p^n)$ are of this form. We use the Proposition to show that any G-torsor on X admits a reduction of structure group to U on an étale cover:

Theorem 1.7. *Let G be a rigid group over K and let $U \subseteq G$ be a rigid open subgroup. Let X be a sousperfectoid space over K and let $\nu : X_v \to X_{\text{ét}}$ be the natural morphism of sites. Then the natural map*

$$R^1\nu_*U \to R^1\nu_*G$$

is surjective. If G is commutative, we more generally have $R^k\nu_*U = R^k\nu_*G$ for all $k \geq 1$.

For non-commutative G, the map $R^1\nu_*U \to R^1\nu_*G$ is not in general an isomorphism.

The idea for the proof of Theorem 1.1 is now that by the theory of p-adic Lie groups, there is a large supply of open subgroups of G. If G is commutative, then these are isomorphic to open subgroups of the Lie algebra of G via the exponential, and one can deduce the result from the case of $\mathbb{G}_a$. In general, the exponential does not respect the group structure, but the relation to the Lie algebra suffices to trivialise G-torsors étale-locally by inductive lifting.

Conceptually speaking, the main aim of this work is to launch a systematic study of G-torsors on adic spaces with a view towards generalisations and reformulations of the p-adic Simpson correspondence. Apart from the above results, we therefore prove some further foundational results on G-torsors. We continue our study in [Heu22b], where based on the results of this article, we give an explicit description of the sheaf $R^1\nu_*G$ on smooth rigid spaces, and use this to construct analytic moduli spaces of G-torsors on rigid spaces.

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Setup and Notation

Throughout let p be a prime and K a non-archimedean field of residue characteristic p. We fix a subring $K^+ \subseteq K$ of integral elements, which we often drop from notation. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathfrak{m}_K$ its maximal ideal. We also denote this by $\mathfrak{m}$ if K is clear from context. For any $\epsilon > 0$ for which there exists an element $a \in K$ with $|a| = |p|^\epsilon$, we denote by $p^\epsilon\mathfrak{m}_K$ the ideal of $\mathcal{O}_K$ which is the image of $\mathfrak{m}_K \subseteq \mathcal{O}_K$ under the morphism $a\cdot : \mathcal{O}_K \to \mathcal{O}_K$ of multiplication by a. This only depends on ϵ: Indeed, it is explicitly given by $p^\epsilon\mathfrak{m}_K = \{x \in K \text{ s.t. } |x| < |p|^\epsilon\}$. We note that the right hand side makes sense for any $\epsilon > 0$, so by a mild abuse of notation we can use this as a definition for $p^\epsilon\mathfrak{m}_K$ for any $\epsilon > 0$.

By an adic space over K we mean an adic space over $\mathrm{Spa}(K, K^+)$ in the sense of Huber. All adic spaces occurring in this article will be adic spaces over K, in particular they will automatically be Tate. For any adic space X, we use the étale site $X_{\text{ét}}$ in the sense of Kedlaya–Liu [KL15, Definition 8.2.16]. By a rigid space over K we mean an adic space locally of topologically finite type over $\mathrm{Spa}(K, K^+)$. One example we use several times is the closed ball $\mathbb{B}^d = \mathrm{Spa}(K\langle T_1, \dots, T_d\rangle)$. We use perfectoid spaces in the sense of [Sch12], as well as Scholze's category of diamonds [Sch22]. Most adic spaces we consider throughout will be sousperfectoid in the sense of [SW20, §6.3][HK]: examples of such are smooth rigid spaces, perfectoid spaces, and products thereof.

Throughout, we will want to work with adic spaces which are locally given by affinoid adic spaces $\mathrm{Spa}(A, A^+)$ such that for any finite étale morphism of Huber pairs $(A, A^+) \to (B, B^+)$, the pair (B, B^+) is again sheafy. In order to express this, we follow Kedlaya–Liu and work with pre-adic spaces in the sense of [KL15, Definition 8.2.3]. Let us say that an adic space X is étale-sheafy if for any pre-adic space Y with an étale morphism of pre-adic spaces $Y \to X$, also Y is an adic space. For example, when X is sousperfectoid, any pre-adic space $Y \to X$ is even a sousperfectoid adic space, hence sousperfectoid spaces are étale-sheafy.

For any adic space X over K, we consider the associated locally spatial diamond $X^\diamondsuit$ in the sense of [Sch22, §15]. As we have fixed a structure map to K, we may regard $X^\diamondsuit$ as a sheaf on the v-site $\mathrm{Perf}_{K,v}$ of perfectoid spaces over K. Then diamondification identifies the étale sites [Sch22, Lemma 15.6] of X and $X^\diamondsuit$. We may therefore often switch back and forth between X and its associated diamond $X^\diamondsuit$. However, there is one subtlety: On the diamantine side, we have a structure sheaf $\mathcal{O}$ on $X^\diamondsuit$ induced from that on $\mathrm{Perf}_{K,v}$. We caution that we currently do not in general know whether the equivalence $X^\diamondsuit_{\text{ét}} = X_{\text{ét}}$ identifies the structure sheaves on both sides for sousperfectoid X, but this is true when X is a perfectoid space or a semi-normal rigid space ([Sch22, Theorem 8.7], [SW20, Proposition 10.2.3]).

We denote by Sous_K the category of sousperfectoid spaces over K. We make this into a big étale site $\mathrm{Sous}_{K,\text{ét}}$ by equipping it with the étale topology. Second, we can also make sense of v-sheaves on Sous_K: We define a morphism $X \to Y$ in Sous_K to be a v-cover if the associated morphism $X^\diamondsuit \to Y^\diamondsuit$ is. This does not make Sous_K into a site because the fibre product $X^\diamondsuit \times_{Y^\diamondsuit} X^\diamondsuit$ may not be represented by an object of Sous_K. But this fibre product is still a diamond, hence it is covered by a perfectoid space in Sous_K, which will suffice to define what it means to be a v-sheaf on Sous_K (see Definition 2.13.2). We denote by $\mathrm{Dmd}_{K,v}$ the larger category of diamonds over K equipped with the v-topology. As any diamond admits a v-cover by a perfectoid space, the categories of v-sheaves on $\mathrm{Perf}_{K,v}$, on Sous_K and on $\mathrm{Dmd}_{K,v}$ are all equivalent.

2. Generalised representations on rigid spaces

Before we discuss G-torsors for general rigid groups G, we study in this section the simpler case of v-vector bundles, that is, GL_n-torsors on X_v. While this simplifies the setting, the overall line of argument is already the same as in the general case. We begin by giving an alternative description of v-vector bundles that is of interest in the p-adic Simpson correspondence: The equivalence between v-vector bundles and Faltings' generalised representations.

2.1. Generalised representations.

We start by adapting generalised representations as defined by Faltings [Fal05, §2 and Theorem 3] from the algebraic setting of log schemes to an analytic setting: Throughout this section, let K be any non-archimedean extension of $\mathbb{Q}_p$ and let X be any étale-sheafy adic space over K. We recall that "étale-sheafy" was defined in the previous section and means that for any étale morphism of pre-adic spaces $Y \to X$ in the sense of [KL15, Definition 8.2.16], the pre-adic space Y is an adic space.

Definition 2.1. (1) A **generalised representation** on X is a system $(M_k)_{k\in\mathbb{N}}$ of finite locally free $\mathcal{O}^+/p^k$ modules M_k on $X_{\text{ét}}$ with isomorphisms $M_{k+1}/p^k \xrightarrow{\sim} M_k$ for all k.

(2) The isogeny category is the localisation of the category of generalised representations on X at multiplication by p. A **generalised $\mathbb{Q}_p$-representation** on X is a system of generalised representations $(M_{U_i})_{U_i\in\mathfrak{U}}$ on some étale cover $\mathfrak{U}$ of X with isomorphisms $M_{U_i|U_i\times_X U_j} \to M_{U_j|U_j\times_X U_i}$ in the isogeny category that satisfy the cocycle condition.

Remark 2.2. The name "generalised representation" stems from the following observation: Fix any base-point $x \in X(\overline{K})$ and consider the étale fundamental group $\pi_1(X) := \pi_1^{\text{ét}}(X, x)$, then to any semilinear continuous representation $\rho : \pi_1(X) \to \mathrm{GL}(E)$ on a finite free $\mathcal{O}_K$-module E, we can associate a generalised representation: Namely, by continuity, the reduction $\rho_k : \pi_1(X) \to \mathrm{GL}(E/p^k)$ factors through a finite quotient of $\pi_1(X)$, corresponding to a finite étale cover $X' \to X$. We can then regard ρ_k as a descent datum for a finite locally free $\mathcal{O}^+/p^k$-module M_k along $X' \to X$, by defining M_k on any $U \in X_{\text{ét}}$ to be

$$M_k(U) := \{x \in E(U \times_X X')/p^k \mid \gamma^* x = \rho^{-1}(\gamma)x \text{ for all } \gamma \in \pi_1(X)\}.$$

The system $(M_k)_{k\in\mathbb{N}}$ then defines a generalised representation in the above sense.

The main goal of this short section is to prove the following equivalent characterisation:

Proposition 2.3. *Let X be any étale-sheafy adic space over a non-archimedean field K over $\mathbb{Q}_p$ (e.g. X can be any rigid spaces, or sousperfectoid).*

(1) The morphism of sites $\nu : X_v \to X_{\text{ét}}$ induces a natural equivalence of categories

$$\{\textit{finite locally free } \mathcal{O}^+\textit{-modules on } X_v\} \to \{\textit{generalised representations on } X\}$$
$$V \mapsto (\nu_*(V/p^k))_{k\in\mathbb{N}}.$$

Moreover, these are equivalent to finite locally free $\mathcal{O}^+$-modules on $X_{\text{qproét}}$, and if X is locally Noetherian also to finite locally free $\mathcal{O}^+$-modules on $X_{\text{proét}}$.

(2) By localising at multiplication by p, this defines an equivalence of categories

$$\{\textit{finite locally free } \mathcal{O}\textit{-modules on } X_v\} \to \{\textit{generalised } \mathbb{Q}_p\textit{-representations on } X.\}$$

Moreover, these are equivalent to finite locally free $\mathcal{O}$-modules on $X_{\text{qproét}}$, and if X is locally Noetherian also to finite locally free $\mathcal{O}$-modules on $X_{\text{proét}}$.

Definition 2.4. We also call a finite locally free $\mathcal{O}$-module on X_v a **v-vector bundle**. We then call finite locally free $\mathcal{O}$-modules on $X_{\text{ét}}$ "étale vector bundles" to clarify the topology.

From this perspective, one could reverse-engineer Faltings' definition of generalised $\mathbb{Q}_p$-representations by saying that they describe v-vector bundles purely in terms of $X_{\text{ét}}$.

Remark 2.5. The idea that generalised representations on a rigid space X are vector bundles on $X_{\text{proét}}$ is mentioned or implicit in other works: In the arithmetic setting of smooth rigid spaces over discretely valued fields, it is hinted at by Liu–Zhu [LZ17] (see Remark 2.6). In more general analytic settings, the idea appears in the works of Würthen [Wü23, footnote on p.3] and Mann–Werner [MW22], and a related result appears in the work of Morrow–Tsuji (cf [MT21, Theorem 5.7]) in a good reduction setting. In more algebraic geometric settings, related statements have been shown by Xu [Xu17, §3.29] and Yang–Zuo [YZ21, Lemma 2.10].

Proposition 2.3 now provides a result in the full generality of adic spaces over K, free from any choice of integral model, or additional assumptions on X or K. We note that this analytic setting introduces some additional subtleties due to the difference between $\mathcal{O}^+/p^k$-modules on $X_{\text{ét}}$ and modules on the reduction mod p^k of some given formal model.

The step to finite locally free $\mathcal{O}^+$-modules in the v-topology is newer, although we build on work of [MW22]. This step is relevant because $X_{\text{proét}}$ is only defined for locally Noetherian X, while the v-topology is available more generally, which is useful even for rigid spaces e.g. in the relative setting studied in [Heu22b].

The proof will take up the entire section as we will also discuss some related topics to prepare the more general version for G-torsors. The basic idea is to compare locally free $\mathcal{O}^+/p$-modules for various topologies. For this we can relax the setup of this section:

Let K be any non-archimedean field, not necessarily over $\mathbb{Q}_p$, and let $\varpi \in \mathcal{O}_K$ be any pseudo-uniformiser of K. The following Proposition is the technical heart of this section:

Proposition 2.6. *Let X be an adic space over K which is étale-sheafy. Then for $m = 0, 1$ the map*

$$H^m_{\text{ét}}(X, \mathrm{GL}_r(\mathcal{O}^+/\varpi)) \xrightarrow{\sim} H^m_v(X, \mathrm{GL}_r(\mathcal{O}^+/\varpi)) \qquad (1)$$

is an isomorphism for all $r \in \mathbb{N}$. The analogous statement holds for $\mathrm{GL}_r(\mathcal{O}^+/\varpi\mathfrak{m})$.

In particular, the functor

$$\nu^* : \left\{ \begin{array}{c} \textit{finite locally free} \\ \mathcal{O}^+/\varpi\textit{-modules on } X_{\text{ét}} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{finite locally free} \\ \mathcal{O}^+/\varpi\textit{-modules on } X_v \end{array} \right\}$$

is an equivalence of categories, where $\nu : X_v \to X_{\text{ét}}$ is the natural morphism.

Proof. To see the isomorphism (1) for $m = 0$, it suffices to see $\nu_*\mathrm{GL}_r(\mathcal{O}^+/\varpi) = \mathrm{GL}_r(\mathcal{O}^+/\varpi)$.

Second, to see that the functor ν^* is fully faithful, it suffices to see $\nu_*(\mathcal{O}^+/\varpi) = \mathcal{O}^+/\varpi$, i.e. $H^0_{\text{ét}}(X, \mathcal{O}^+/\varpi) = H^0_v(X, \mathcal{O}^+/\varpi)$ is an (honest, not just almost) isomorphism: Indeed, for any two finite locally free $\mathcal{O}^+/\varpi$-modules E_1 and E_2 on $X_{\text{ét}}$, we have $\mathscr{H}om(\nu^*E_1, \nu^*E_2) =$

$\nu^*\mathscr{H}om(E_1,E_2)$, and locally where the locally free $\mathcal{O}^+/\varpi$-module $\mathscr{H}om(E_1,E_2)$ becomes trivial, the statement then follows.

Third, to see essential surjectivity, it suffices to see that $R^1\nu_*\mathrm{GL}_r(\mathcal{O}^+/\varpi)=1$. By passing to isomorphism classes, this will also imply the isomorphism (1) for $m=1$.

All of these three statements, and their analogues for $\mathcal{O}^+/\varpi\mathfrak{m}$ will be proved by Proposition 2.14 below. Namely, they follow by approximation arguments, which we now axiomatise and discuss in detail in the following subsection.

2.2. **Approximation of non-abelian sheaf cohomology.** For the proof of Proposition 2.6, we will use ideas from [Sch13a, §3], [Sch22, Proposition 14.7], [MW22, §2], [Heu21b, Theorem 2.18]. We shall axiomatise the argument as we will apply it several times in the following section. Throughout this subsection, K is any non-archimedean field. We begin by recalling some properties of affinoid tilde-limits of adic spaces:

Definition 2.7 ([Hub96, (2.4.1)][SW13, §2.4])**.** Let $(X_i)_{i\in I}$ be a cofiltered inverse system of affinoid adic spaces over K and let X be an affinoid adic space with compatible maps $X\to X_i$ for all $i\in I$. We write

$$X\sim\varprojlim_{i\in I}X_i$$

if $|X|=\varprojlim|X_i|$ and if there is a cover of X by affinoid opens U for which the induced map $\varinjlim_{U_i}\mathcal{O}(U_i)\to\mathcal{O}(U)$ has dense image, where the direct limit runs through all $i\in I$ and all affinoid opens $U_i\subseteq X_i$ through which $U\to X_i$ factors.

Lemma 2.8. *In the situation of Definition 2.7, we have $X^\diamondsuit=\varprojlim_{i\in I}X_i^\diamondsuit$ for the associated diamonds. For the associated quasi-compact quasi-separated étale sites, we have:*

$$X_{\text{ét,qcqs}}=2\text{-}\varinjlim_{i\in I}X_{i,\text{ét,qcqs}}$$

Proof. The first part is [SW13, Proposition 2.4.5], the second [Sch22, Proposition 11.23]. $\square$

Definition 2.9. In the situation of Definition 2.7, we write

$$X\approx\varprojlim_{i\in I}X_i$$

if already $\varinjlim_i\mathcal{O}(X_i)\to\mathcal{O}(X)$ has dense image.

One reason why this stronger notion is useful is the following elementary observation:

Lemma 2.10. *Let I and J be directed sets and let $(X_{i,j})_{i,j\in I\times J}$ be an inverse system of affinoid adic spaces over K, indexed over $I\times J$. Suppose that we have affinoid tilde-limits $X_i\approx\varprojlim_{j\in J}X_{i,j}$ for all i as well as $X\approx\varprojlim_{i\in I}X_i$. Then $X\approx\varprojlim_{i,j\in I\times J}X_{i,j}$.*

Proof. The condition on topological spaces is clear, the one on global sections follows by sequential approximation of elements. $\square$

The crucial technical property of $\mathrm{GL}_n(\mathcal{O}^+/\varpi)$ that we use to prove Proposition 2.6 is:

Lemma 2.11. *Let $X\approx\varprojlim_{i\in I}X_i$ be an affinoid perfectoid tilde-limit of affinoid adic spaces over K. Then*

$$H^0_{\text{ét}}(X,\mathrm{GL}_n(\mathcal{O}^+/\varpi))=\varinjlim_{i\in I}H^0_{\text{ét}}(X_i,\mathrm{GL}_n(\mathcal{O}^+/\varpi)).$$

The analogous statement for $\mathrm{GL}_n(\mathcal{O}^+/\varpi\mathfrak{m})$ also holds.

Proof. Consider the Cartesian diagram of sheaves of sets

$$\begin{tikzcd}
\mathrm{GL}_n(\mathcal{O}^+/\varpi) \arrow[r] \arrow[d, "\det"] & M_n(\mathcal{O}^+/\varpi) \arrow[d, "\det"] \\
(\mathcal{O}^+/\varpi)^\times \arrow[r] & \mathcal{O}^+/\varpi.
\end{tikzcd}$$

The statement holds for the bottom row by [Heu21a, Lemma 3.10]. It thus also holds for $M_n(\mathcal{O}^+/\varpi)$, and thus for the top left since H^0 preserves finite limits.

The statement for $\mathcal{O}^+/\varpi\mathfrak{m} = \varinjlim_{\epsilon\to 0} \mathcal{O}^+/\varpi^{1+\epsilon}$ follows by taking colimits. □

For later applications, we more generally consider sheaves satisfying an analogous property in a more general setting:

Setup 2.12. Throughout the rest of this subsection, let S be an adic space over K and let $\mathcal{C}$ be a full subcategory of adic spaces over S satisfying the following conditions:

(1) $\mathcal{C}$ contains every perfectoid space X over S as well as $X \times \mathbb{B}^d$ for every $d \in \mathbb{N}$.
(2) For any $X \in \mathcal{C}$ and any étale morphism of pre-adic spaces $Y \to X$ in the sense of [KL15, Definition 8.2.16], the pre-adic space Y is an adic space (i.e. X is étale-sheafy) and Y is contained in $\mathcal{C}$.

We endow $\mathcal{C}$ with the étale topology. The latter requirement ensures that this is well-defined.

For example, $\mathcal{C}$ could be the big étale site $\mathrm{Sous}_{S,\text{ét}}$ of sousperfectoid spaces over S, which satisfies axioms (1) and (2) by [SW20, Proposition 6.3.3]. In particular, for $S = \mathrm{Spa}(K)$ it could be $\mathrm{Sous}_{K,\text{ét}}$.

Definition 2.13. Let $\mathcal{C}$ be a site of adic spaces as in Setup 2.12. Let F be a sheaf on $\mathcal{C}$.

(1) We say that F satisfies the **approximation property** on $\mathcal{C}$ if for any affinoid perfectoid tilde-limit $X \approx \varprojlim_{i\in I} X_i$ of affinoid adic spaces in $\mathcal{C}$, we have
$$F(X) = \varinjlim_{i\in I} F(X_i).$$
(2) Diamondification ([Sch22, §15]) defines a natural morphism of sites $\lambda : \mathrm{Dmd}_{K,v} \to \mathcal{C}$. We call F a ***v*-sheaf on** $\mathcal{C}$ if the natural map $F \to \lambda_*\lambda^*F$ is an isomorphism. Since any object of $\mathrm{Dmd}_{K,v}$ admits a cover by a perfectoid space over S, hence an object of $\mathcal{C}$, this means explicitly that for any $X, Y, Z \in \mathcal{C}$ and any v-covers $Y^\diamondsuit \to X^\diamondsuit$ and $Z^\diamondsuit \to Y^\diamondsuit \times_{X^\diamondsuit} Y^\diamondsuit$,
$$F(X) \to F(Y) \rightrightarrows F(Z)$$
is an equaliser diagram.

The goal of this subsection is to show:

Proposition 2.14. *Let $\mathcal{C}$ be a site of adic spaces as in Setup 2.12. Let F be a sheaf of sets on $\mathcal{C}$ satisfying the approximation property in the sense of Definition 2.13.1. Then:*

(1) F is already a v-sheaf.
(2) If F is a sheaf of groups, then for any X in $\mathcal{C}$, we have $R^1\nu_*F = 1$ for $\nu : X_v \to X_{\text{ét}}$.
(3) If F is a sheaf of abelian groups, then $R^m\nu_*F = 1$ for any $m \geq 1$.

The application we have in mind for the purpose of this section is to show:

Corollary 2.15. *Let X be any sousperfectoid space, then the map*
$$H^n_{\text{ét}}(X, \mathcal{O}^+/\varpi) \to H^n_v(X, \mathcal{O}^+/\varpi)$$
is an (honest, not just almost) isomorphism for all $n \geq 0$.

This is already non-trivial for $n = 0$ and X perfectoid, where almost acyclicity on affinoid perfectoid spaces ([Sch12, Proposition 7.13] for the étale topology, [Sch22, Proposition 8.8] for the v-topology) a priori only implies an almost isomorphism. For rigid X, it is related to [Sch13a, Lemma 4.2], which implies the analogous result for $X_{\text{proét}}$. For strictly totally disconnected X, the Corollary recovers [MW22, Proposition 2.13] by a similar proof.

Proof of Proposition 2.14. We start with a few technical Lemmas about the situation:

Lemma 2.16. *Let F be a sheaf of (not necessarily abelian) groups on $\mathcal{C}$ satisfying the approximation property. Then for $m = 0, 1$ and any $X \approx \varprojlim_{i\in I} X_i$ as in Definition 2.13,*

$$H^m_{\text{ét}}(X, F) = \varinjlim_{i\in I} H^m_{\text{ét}}(X_i, F).$$

If F is abelian, this holds for any $m \geq 0$.

Proof. For $m = 0$ this holds by definition. For $m \geq 1$ it follows by a Čech argument: Following [Heu21a, Definition 3.6], let us call a morphism of affinoid adic spaces standard-étale if it is a successive composition of finite étale maps and rational open immersions. Such maps form a basis for the étale topology, hence every class in $H^m_{\text{ét}}(X, F)$ is trivialised by a standard-étale cover X' of X. By Lemma 2.8, this cover arises via pullback from some $X'_i \to X_i$. For $j \geq i$ let $X'_j := X_j \times_{X_i} X'_i$, then by [Heu21a, Lemma 3.13], we have a tilde-limit relation $X' \approx \varprojlim X'_j$, so this fulfils the assumptions of the lemma and we therefore have $H^0_{\text{ét}}(X', F) = \varinjlim H^0_{\text{ét}}(X'_j, F)$. The statement for H^m now follows by comparing Čech cohomology for the covers $X' \to X$ and $X'_j \to X_j$ for $j \geq i$. □

Lemma 2.17. *Any morphism $f : Y \to X$ of affinoid perfectoid spaces over K arises as the tilde-limit $Y \approx \varprojlim_{i\in I} Y_i$ of a cofiltered inverse system of sousperfectoid spaces $Y_i \subseteq \mathbb{B}^d \times X \to X$ that are rational open in unit balls over X. If moreover X is strictly totally disconnected and f is a v-cover, then each morphism $Y_i \to X$ admits a splitting.*

Proof. This fact is used in the proof of [Sch22, Proposition 14.7]. For a proof, see [Heu21a, Proposition 3.17], and [Heu21b, Lemma 2.23] for the last sentence. □

Lemma 2.18. *Let X be any affinoid adic space over K. Then:*

1. *There is an inverse system $(X_i \to X)_{i\in I}$ of finite étale Galois covers with affinoid perfectoid tilde-limit $X_\infty \approx \varprojlim X_i$.*
2. *There is an inverse system $(X_i \to X)_{i\in I}$ of surjective étale morphisms with strictly totally disconnected tilde-limit $\widetilde{X} \approx \varprojlim X_i$.*

Proof. Part 1 is [SW13, Lemma 10.1.6-7] (see also [Sch22, Lemma 15.3, Proposition 15.4]) and goes back to Colmez [Col02, §4].

For part 2, we first recall that by [Sch22, Lemma 7.18], the perfectoid space X_∞ admits an affinoid pro-étale cover by a strictly totally disconnected space $\widetilde{X}$. By [Sch22, Proposition 6.5], this means that there is a cofiltered inverse system of étale maps from affinoid perfectoid spaces $(X_{\infty,l} \to X_\infty)_{l\in L}$ such that $\widetilde{X} \approx \varprojlim_{l\in L} X_{\infty,l}$. Using Lemma 2.8 and Lemma 2.10, these combine to give the desired system of étale morphisms over X. □

Lemma 2.19. *Let $Y \approx \varprojlim_{i\in I} Y_i \to X$ be one of the affinoid perfectoid tilde-limits of Lemma 2.17 or Lemma 2.18. Then for any $m \in \mathbb{N}$, the m-fold fibre product $Y^{\times m}_{/X}$ of Y with itself over X exists in the category of uniform adic spaces and is again affinoid perfectoid. Moreover, we still have*

$$Y^{\times m}_{/X} \approx \varprojlim_{i\in I} Y^{\times m}_{i/X}.$$

Proof. In the case of Lemma 2.17 where $Y \to X$ is a morphism of affinoid perfectoid spaces, this can be seen exactly as in [BGH+22, Lemma 2.8], by approximation of simple tensors.

In the case of Lemma 2.18, we can see inductively that the fibre product exists because for any $m \geq 2$, the projection $Y^{\times m}_{/X} \to Y^{\times(m-1)}_{/X}$ is pro-étale over an affinoid perfectoid space, hence itself affinoid perfectoid. Moreover, for any $m \geq 2$, we have

$$Y^{\times m}_{/X} = Y \times_X Y^{\times(m-1)}_{/X} \approx \varprojlim_{i\in I} Y \times_X Y^{\times(m-1)}_{i/X}$$

as this is true for affinoid pro-étale maps of perfectoid spaces. Second, for any fixed i, we have

$$Y \times_X Y_{i/X}^{\times(m-1)} \approx \varprojlim_{j\in I} Y_j \times_X Y_{i/X}^{\times(m-1)}$$

by [Heu21a, Lemma 3.13]. We can therefore invoke Lemma 2.10 to deduce that

$$Y_{/X}^{\times m} \approx \varprojlim_{i,j\in I\times J} Y_j \times_X Y_{i/X}^{\times(m-1)}.$$

Finally, the diagonal embedding $I \subseteq I \times I$ identifies I with a cofinal subset of $I \times I$, hence the cofiltered inverse systems $(Y_j \times_X Y_{i/X}^{\times(m-1)})_{i,j\in I\times J}$ and $(Y_i \times_X Y_{i/X}^{\times(m-1)})_{i\in I}$ are isomorphic. □

We can now give the proof of Proposition 2.14: For the reader interested in an exposition of the argument in the special case of $F = \mathcal{O}^+/p^n$, we also refer to [Zav21, Appendix C].

For strictly totally disconnected X, part 1 is proved in [MW22, Lemma 2.12]. We repeat the argument here as it ties in well with the general case: Let $\lambda : \mathrm{Dmd}_{K,v} \to \mathcal{C}$ be the natural morphism of sites and let $F_v := \lambda_*\lambda^*F$ be the v-sheafification of F on $\mathcal{C}$ from Definition 2.13.

Step 1: $F \to \lambda_*\lambda^*F$ is injective for strictly totally disconnected X. Let $\beta_1, \beta_2 \in F(X)$ be such that the images of β_1, β_2 under the map $F(X) \to F_v(X)$ agree. Then there is a v-cover $Y \to X$ by a perfectoid space such that the images of β_1 and β_2 under the map $F(X) \to F(Y)$ agree. Applying the approximation property to the tilde-limit $Y \approx \varprojlim Y_i$ from Lemma 2.17, we see that $F(Y) = \varinjlim F(Y_i)$, so there is some cover $Y_i \to X$ such that the images of β_1 and β_2 already agree under $F(Y) \to F(Y_i)$. But Lemma 2.17 also says that the map $Y_i \to X$ has a section, so it follows that $\beta_1 = \beta_2$.

Step 2: $F \to \lambda_*\lambda^*F$ is injective for any X in $\mathcal{C}$. Let $\widetilde{X} \approx \varprojlim X_i$ be the strictly totally disconnected pro-étale cover of X from Lemma 2.18. Using that F is an étale sheaf, then the approximation property, and finally step 1, we find that the following map is injective:

$$F(X) \hookrightarrow \varinjlim F(X_i) = F(\widetilde{X}) \hookrightarrow F_v(\widetilde{X}).$$

As this factors through $F_v(X)$, this shows that $F(X) \to F_v(X)$ is injective.

Step 3: $F \to \lambda_*\lambda^*F$ is surjective for strictly totally disconnected X. Let $\alpha \in F_v(X)$, then there is an affinoid perfectoid v-cover $Y \to X$ such that α appears in $F(Y)$. By step 2 applied to $F(Y \times_X Y) \hookrightarrow F_v(Y \times_X Y)$, it thus lies in $\check{H}^0(Y \to X, F)$.

By the same approximation $Y \approx \varprojlim Y_i$ as in step 1, as well as Lemma 2.19 and the approximation property, we have $\check{H}^0(Y \to X, F) = \varinjlim \check{H}^0(Y_i \to X, F)$. It follows that for i large enough, α already lies in the image of $\check{H}^0(Y_i \to X, F) \to F_v(X)$.

We are not quite done yet because we do not require $Y_i \to X$ to be a cover in $\mathcal{C}$. To resolve this, observe that we can endow $\mathcal{C}$ with a "smooth" topology $\mathcal{C}_{\mathrm{sm}}$ for which we define covers to be the surjective morphisms $U \to V$ that are locally given by a composition $U \to V \times \mathbb{B}^d \to V$ where the first map is étale and the second map is the projection. This indeed defines a site, and this "smooth" topology clearly refines the étale one. We conclude that $\lambda : \mathrm{Dmd}_{K,v} \to \mathcal{C}$ factors through a morphism of sites $\mu : \mathcal{C}_{\mathrm{sm}} \to \mathcal{C}$. However, since X is strictly totally disconnected, any cover of X in $\mathcal{C}_{\mathrm{sm}}$ admits an étale refinement. For this reason, we see directly from the explicit definition of the sheafification in terms of the iteratively formed colimit of the 0-th Čech cohomology over all covers (see e.g. [dJ+23, 00W1]) that $\mu^*\mathcal{F}(X) = \mathcal{F}(X)$. Since $\check{H}^0(Y_i \to X, F) \to F_v(X)$ factors through $\check{H}^0(Y_i \to X, \mu^*F) = \mu^*\mathcal{F}(X)$, this shows that α lies in the image of $F(X) \to F_v(X)$.

Step 4: $F \to \lambda_*\lambda^* F$ is an isomorphism for any adic space X in $\mathcal{C}$. Like in step 2, we consider the cover $\widetilde{X} \to X$ from Lemma 2.18.2. Let us compute $\check{H}^0(\widetilde{X} \to X, F)$. By step 1 and step 3, we have

$$F(\widetilde{X}) = F_v(\widetilde{X}).$$

Second, $\widetilde{X} \times_X \widetilde{X}$ is a perfectoid space by Lemma 2.19, hence in $\mathcal{C}$, so by step 2 the map

$$F(\widetilde{X} \times_X \widetilde{X}) \hookrightarrow F_v(\widetilde{X} \times_X \widetilde{X})$$

is injective. Unravelling the definition of $\check{H}^0$, this combines to show that

$$F_v(X) = \check{H}^0(\widetilde{X} \to X, F_v) = \check{H}^0(\widetilde{X} \to X, F).$$

But we have $F(\widetilde{X}) = \varinjlim F(X_i)$ and by Lemma 2.19, we moreover have $F(\widetilde{X} \times_X \widetilde{X}) = \varinjlim F(X_i \times_X X_i)$. This shows that

$$\check{H}^0(\widetilde{X} \to X, F) = \varinjlim \check{H}^0(X_i \to X, F) = F(X).$$

In combination, this shows that $F_v(X) = F(X)$, as we wanted to see.

We now prove the vanishing of $R^m\nu_*F$ for $m \geq 1$ by an intertwined induction on $m \geq 1$: the induction step is first carried out for strictly totally disconnected X, then for general X.

Step 5: induction step to prove $R^m\nu_*F = 1$ for strictly totally disconnected X. As X is strictly totally disconnected, it has trivial étale cohomology, so we need to show that $H^m_v(X, F) = 0$. Let $\alpha \in H^m_v(X, F)$. By locality of cohomology, there is an affinoid perfectoid v-cover $Y \to X$ such that α becomes trivial in $H^m_v(Y, F)$.

For the non-abelian case and $m = 1$, we now use the short exact sequence of pointed sets

$$\check{H}^1(Y \to X, F) \to H^1_v(X, F) \to H^1_v(Y, F).$$

For the abelian case and $m \geq 1$, we more generally have the Čech-to-sheaf spectral sequence

$$\check{H}^k(Y \to X, H^j_v(-, F)) \Rightarrow H^{k+j}_v(X, F)$$

for $m = k + j$. In either case, we use the inverse system from Lemma 2.17 and invoke Lemma 2.19 and Lemma 2.16: By induction hypothesis we have for $j < m$ that

$$\check{H}^k(Y \to X, H^j_v(-, F)) = \check{H}^k(Y \to X, H^j_{\text{ét}}(-, F)) = \varinjlim \check{H}^k(Y_i \to X, H^j_{\text{ét}}(-, F)).$$

As $Y_i \to X$ is split, the last term vanishes for $k > 0$. Hence only the term for $k = 0$, $j = m$ contributes. This means that $H^m_v(X, F) \to H^m_v(Y, F)$ has trivial kernel. Hence $\alpha = 0$.

Step 6: induction step to prove $R^m\nu_*F = 1$ for general X. Finally, we assume by induction hypothesis that $R^j\nu_*F = 1$ for any $1 \leq j < m$ and any X, as well as for $j = m$ if X is totally disconnected, and deduce that $R^m\nu_*F = 1$. For this we again use the cover $\widetilde{X} \to X$ from Lemma 2.18.2. As in step 5, we consider the short exact sequence of pointed sets, respectively the Čech-to-sheaf spectral sequence for abelian F

$$\check{H}^k(\widetilde{X} \to X, H^j_v(-, F)) \Rightarrow H^{k+j}_v(X, F)$$

for $k + j = m$. By induction hypothesis, we have

$$\check{H}^k(\widetilde{X} \to X, H^j_v(-, F)) = \varinjlim_i \check{H}^k(X_i \to X, H^j_{\text{ét}}(-, F))$$

for all $k > 0$ and $j < m$, by Lemma 2.19 and Lemma 2.16. For $j = m$, $k = 0$, this equation also holds since by step 5 we have $H^m_v(\widetilde{X}, F) = 0$ and $\varinjlim_i H^m_{\text{ét}}(X_i, F) = H^m_{\text{ét}}(\widetilde{X}, F) = 0$. By comparing to the Čech-to-sheaf spectral sequence for the étale cover $X_i \to X$,

$$\check{H}^k(X_i \to X, H^j_{\text{ét}}(-, F)) \Rightarrow H^{k+j}_{\text{ét}}(X, F),$$

we deduce that $H^m_{\text{ét}}(X, F) = H^m_v(X, F)$, as we wanted to see. □

This finishes the proof of Proposition 2.6. □

Before we continue, we record a Lemma which we will use later. Recall that the v-sheaf property of Definition 2.13 means that we can extend F uniquely to a sheaf on $\mathrm{Dmd}_{K,v}$.

Lemma 2.20. *Let F be a sheaf of sets on $\mathcal{C}$ satisfying the approximation property in the sense of Definition 2.13. Let $(X_i)_{i\in I}$ be a cofiltered inverse system of locally spatial diamonds over K with qcqs transition maps. Let $X := \varprojlim_{i\in I} X_i$, this is a locally spatial diamond by* [Sch22, Lemma 11.22]. *Then*

$$F(X) = \varinjlim_{i\in I} F(X_i).$$

Proof. We begin by following the proof of [Sch22, Lemma 11.22], providing some additional details for the reader's convenience: Since the statement is local on X_0 for any fixed $0 \in I$, we can first reduce to the case that all X_i and hence also X are qcqs. By the "refined argument" in the proof of [Sch22, Lemma 11.22], we can then find a cofiltered inverse system of strictly totally disconnected spaces $(\widetilde{X}_i)_{i\in I}$ with compatible quasi-pro-étale surjections $\widetilde{X}_i \to X_i$: Let us provide some details on this claim. Using [AR94, Lemma 1.6] and arguing like in the proof of [AR94, Corollary 1.7], it suffices to consider the case that I is a chain, i.e. that I is the set of ordinals $< \lambda$ for some ordinal λ. We can then argue by transfinite induction, as follows: Given any limit ordinal $\mu < \lambda$, if we have found $\widetilde{X}_i \to X_i$ for all $i < \mu$, then

$$\varprojlim_{i<\mu} \widetilde{X}_i \to \varprojlim_{i<\mu} X_i$$

is a quasi-pro-étale cover, hence so is its base-change

$$X'_\mu := \varprojlim_{i<\mu} \widetilde{X}_i \times_{\varprojlim_{i<\mu} X_i} X_\mu \to X_\mu.$$

By the statement of [Sch22, Lemma 11.22], X'_μ is a diamond, and we can define $\widetilde{X}_\mu \to X'_\mu$ to be any strictly totally disconnected quasi-pro-étale cover. Then $\widetilde{X}_\mu \to X_\mu$ has all desired properties.

By [Sch22, Proposition 6.5], there exists an affinoid perfectoid tilde-limit

$$\widetilde{X} \approx \varprojlim_{i\in I} \widetilde{X}_i$$

which is a quasi-pro-étale cover $\widetilde{X} \to X$. Since quasi-pro-étale covers of strictly totally disconnected spaces are again strictly totally disconnected by [Sch22, Lemma 7.19], it follows that $\widetilde{X}_i \times_{X_i} \widetilde{X}_i \to X_i$ is still affinoid perfectoid. Since limits in the category of diamonds commute with fibre products, we see again by [Sch22, Proposition 6.5] that

$$\widetilde{X} \times_X \widetilde{X} \approx \varprojlim_{i\in I} \widetilde{X}_i \times_{X_i} \widetilde{X}_i.$$

We now use that F is a v-sheaf. Combined with the approximation property, this implies

$$F(X) = \check{H}^0(\widetilde{X} \to X, F) = \varinjlim_{i\in I} \check{H}^0(\widetilde{X}_i \to X, F) = \varinjlim_{i\in I} F(X_i)$$

as we wanted to see. $\square$

Remark 2.21. For any sousperfectoid space X over K and any sheaf F on $X_{\text{ét}}$, the restriction to $\mathrm{Sous}_{X,\text{ét}}$ of the pullback ν^*F along $\nu : X_v \to X_{\text{ét}}$ satisfies the approximation property by [Sch22, Proposition 14.9]. However, not any sheaf on $\mathrm{Sous}_{X,\text{ét}}$ satisfying the approximation property is of this form: For example, for $X = \mathrm{Spa}(K, \mathcal{O}_K)$ and $F = \mathcal{O}^+/p$ on $X_{\text{ét}}$, let C be a complete algebraically closed field extension of K whose residue field is transcendental over that of K. Let $\overline{K} \subseteq C$ be the algebraic closure of K and consider $Y := \mathrm{Spa}(C)$ in X_v. Then one sees directly from the definition of ν^* that $\nu^*F(C) = \mathcal{O}_{\overline{K}}/p$ whereas $\mathcal{O}^+/p(C) = \mathcal{O}_C/p$.

2.3. **Generalised representations on perfectoid spaces.** With these preparations, we can now also prove an integral version of the following result of Kedlaya–Liu already mentioned in the introduction (see also [SW20, Lemma 17.1.8]):

Theorem 2.22 ([KL16, Theorem 3.5.8])**.** *Let X be a perfectoid space. Then the categories of vector bundles on X_{an}, on $X_{\text{ét}}$, on $X_{\text{proét}}$, on $X_{\text{qproét}}$ and X_v, respectively, are all equivalent.*

Our integral version is the following:

Theorem 2.23. *Let X be a perfectoid space. Then the categories of finite locally free $\mathcal{O}^+$-modules on $X_{\text{ét}}$, on $X_{\text{proét}}$, on $X_{\text{qproét}}$ and X_v, respectively, are all equivalent.*

Remark 2.24. We suspect that they might not be equivalent to finite locally free $\mathcal{O}^+$-modules on X_{an}, or even to finite projective $\mathcal{O}^+(X)$-modules if X is affinoid perfectoid.

Remark 2.25. It follows that finite locally free $\mathcal{O}^+$-module on strictly totally disconnected spaces are trivial. Already for $\mathrm{Spa}(\mathbb{C}_p)$, this becomes wrong in the almost category: For $c \in \mathbb{R}\backslash\mathbb{Q}$, the module $p^c\mathfrak{m}$ becomes $\overset{a}{=} \mathcal{O}^+$ on the v-cover $\mathrm{Spa}(C) \to \mathrm{Spa}(\mathbb{C}_p)$ where C is any extension whose value group contains c. We thank Lucas Mann for pointing this out to us.

We will later prove a much more general version in Theorem 4.28, the Proposition being the case of $G = \mathrm{GL}_n(\mathcal{O}^+)$. But for greater clarity in the much simpler case of $G = \mathrm{GL}_n(\mathcal{O}^+)$, we first give them in this case and then later explain how to generalise.

Recall that we use perfectoid spaces in the sense of [Sch12], so X is assumed to live over a perfectoid field (K, K^+), and we denote by ϖ any pseudo-uniformiser of K.

We now first observe:

Lemma 2.26. *Let X be an affinoid perfectoid space.*

(1) Let V be a ϖ-torsionfree $\mathcal{O}^+$-module on X_v such that $V = \varprojlim_k V/\varpi^k$. Assume that there is $r \in \mathbb{N}$ such that $V/\varpi\mathfrak{m} \cong \mathcal{O}^{+r}/\varpi\mathfrak{m}$ as $\mathcal{O}^+$-modules. Then $V \cong \mathcal{O}^{+r}$.

(2) Let V be a finite locally free $\mathcal{O}^+/\varpi$-module. If $V/\mathfrak{m}$ is free over $\mathcal{O}^+/\mathfrak{m}$, then V is free.

Proof. Since V is ϖ-torsionfree, we have for any $k > 0$ a short exact sequence on X_v

$$0 \to \mathfrak{m}V/\varpi\mathfrak{m} \xrightarrow{\varpi^k} V/\varpi^{k+1}\mathfrak{m} \to V/\varpi^k\mathfrak{m} \to 0.$$

For part 1, assume that $V/\varpi^k\mathfrak{m}$ is free of rank r, and let $v_1, \dots, v_r$ be any basis of $V/\varpi^k\mathfrak{m}(X)$ as a finite free $\mathcal{O}^+/\varpi^k\mathfrak{m}(X)$-module. Consider the long exact sequence

$$0 \to \mathfrak{m}V/\varpi\mathfrak{m}(X) \to V/\varpi^{k+1}\mathfrak{m}(X) \to V/\varpi^k\mathfrak{m}(X) \to H^1_v(X, \mathfrak{m}V/\varpi\mathfrak{m}).$$

The last term vanishes: Indeed, we have $\mathfrak{m}V/\varpi\mathfrak{m} \cong \mathfrak{m}\mathcal{O}^{+r}/\varpi\mathfrak{m} = \mathfrak{m} \otimes_{K^+} \mathcal{O}^{+r}/\varpi\mathfrak{m}$ by assumption and moreover $H^1_v(X, \mathcal{O}^+/\varpi\mathfrak{m}) \overset{a}{=} 0$ by almost acyclicity, hence

$$H^1_v(X, \mathfrak{m}V/\varpi\mathfrak{m}) \cong H^1_v(X, \mathcal{O}^{+r}/\varpi\mathfrak{m}) \otimes_{K^+} \mathfrak{m} = 0$$

vanishes "without almost", because any almost zero module vanishes after tensoring with $\mathfrak{m}$. This shows that we can lift the basis vectors to sections $v'_1, \dots, v'_r \in V/\varpi^{k+1}\mathfrak{m}(X)$. Starting with $k = 1$, we can thus inductively define a map

$$\phi_{k+1} : \mathcal{O}^{+r}/\varpi^{k+1}\mathfrak{m} \to V/\varpi^{k+1}\mathfrak{m}$$

which reduces mod $\varpi^k\mathfrak{m}$ to ϕ_k. By the 5-Lemma, ϕ_{k+1} is an isomorphism. In the limit, due to the completeness assumption on V, we obtain an isomorphism $\phi = \varprojlim_k \phi_k : \mathcal{O}^{+r} \to V$.

To deduce part 2, observe that we can by the same argument lift any basis of $V/\mathfrak{m}$ to sections of $V/\varpi\mathfrak{m}$: Indeed, the lifting obstruction again lies in $H^1_v(X, \mathfrak{m}V/\varpi\mathfrak{m}) = 0$. This basis defines a map $\phi : \mathcal{O}^{+r}/\varpi\mathfrak{m} \to V/\varpi\mathfrak{m}$ that is an isomorphism mod $\mathfrak{m}$. If we know a priori that $V/\varpi\mathfrak{m}$ is finite locally free, we can consider $\det \phi \in \wedge^r(V/\varpi\mathfrak{m})(X)$ which is an invertible section mod $\mathfrak{m}$, thus invertible. This shows that ϕ is an isomorphism. $\square$

Proof of Theorem 2.23. We clearly have a chain of fully faithful functors, so it suffices to prove that any finite locally free $\mathcal{O}^+$-module V on X_v is already free étale-locally. For this we may assume that X is affinoid perfectoid. Then by Proposition 2.6 there is an affinoid perfectoid étale cover $X' \to X$ on which $V/\varpi\mathfrak{m}$ becomes trivial. By Lemma 2.26, already V is trivial on X'. $\square$

Corollary 2.27. *Let $X = \mathrm{Spa}(K, K^+)$ where K is a perfectoid field, then any locally free $\mathcal{O}^+$-module on X_v is trivial.*

Proof. Any étale cover of X is a disjoint union of maps $\mathrm{Spa}(L, L^+) \to \mathrm{Spa}(K, K^+)$ where $L|K$ is finite Galois and $L^+|K^+$ is the integral closure, hence faithfully flat. Any v-vector bundle on X thus defines a finite projective K^+-module. This is free as K^+ is a valuation ring. $\square$

As any adic space over K has a pro-étale perfectoid cover by Lemma 2.18, we deduce:

Corollary 2.28. *Let X be any diamond, then the categories of finite locally free $\mathcal{O}^+$-modules on $X_{\mathrm{qpro\acute{e}t}}$ and X_v (and $X_{\mathrm{pro\acute{e}t}}$ if X is locally Noetherian and $\mathrm{char}\, K = 0$) are equivalent.*

We now return to the proof of Proposition 2.3. For the second part, we also need the following, which we will also generalise later in Proposition 4.8, by a different proof:

Lemma 2.29. *Let X be an adic space over K. Let V be a v-vector bundle on X. Then étale-locally on X, there is a finite v-locally free $\mathcal{O}^+$-module V^+ such that $V^+[\frac{1}{p}] = V$.*

Proof. We may assume that X is affinoid. By Lemma 2.18, there is then a pro-finite-étale affinoid perfectoid cover $X_\infty \to X$ that is Galois for some profinite group N. The pullback of V to X_∞ is étale-locally free by Theorem 2.22. By Lemma 2.8 we can replace X by some étale cover to assume that the pullback of V to X_∞ is trivial.

It follows that V is associated to a descent datum on the trivial vector bundle on X_∞, thus defines a class in the first term of the Cartan–Leray sequence (see [Heu22a, Proposition 2.8])

$$0 \to H^1_{\mathrm{cts}}(N, \mathrm{GL}_n(\mathcal{O}(X_\infty))) \to H^1_v(X, \mathrm{GL}_n(\mathcal{O})) \to H^1_v(X_\infty, \mathrm{GL}_n(\mathcal{O})).$$

Let $\rho : N \to \mathrm{GL}_n(\mathcal{O}(X_\infty))$ be any continuous 1-cocycle representing V. Let N_0 be the inverse image of the open subspace $\mathrm{GL}_n(\mathcal{O}^+(X_\infty)) \subseteq \mathrm{GL}_n(\mathcal{O}(X_\infty))$. Then by continuity of ρ, the subspace $N_0 \subseteq N$ is an open neighbourhood of the identity in N, so N_0 contains an open subgroup $N_1 \subseteq N$. Since N/N_1 is finite, this corresponds to a finite étale cover $f : X' \to X$, and the pullback of V to X' is defined by the 1-cocycle $\rho : N_1 \to \mathrm{GL}_n(\mathcal{O}^+(X_\infty))$. By functoriality of the Cartan–Leray sequence this defines an element in $H^1_v(X', \mathrm{GL}_n(\mathcal{O}^+(X_\infty)))$ whose image in $H^1_v(X', \mathrm{GL}_n(\mathcal{O}))$ corresponds to the isomorphism class of f^*V. $\square$

We can now prove the equivalence of v-vector bundles and generalised representations:

Proof of Proposition 2.3. The equivalence between the categories of finite locally free modules in various topologies is Corollary 2.28. We now first consider the functor in part 1: By Proposition 2.6, the $\mathcal{O}^+/p^n$-module V/p^n is already locally free in the étale topology, so that $\nu_*(V/p^n)$ is a locally free $\mathcal{O}^+/p^n$-module on $X_{\mathrm{\acute{e}t}}$. Thus the functor is well-defined.

It is fully faithful by Proposition 2.6 and because any system of compatible morphisms $V/p^n \to W/p^n$ on X_v in the limit induces a unique $\mathcal{O}^+$-linear morphism $V \to W$.

It remains to see that the functor is essentially surjective. Let $(M_n)_{n\in\mathbb{N}}$ be a generalised representation and consider the v-sheaf $V := \varprojlim_{n\in\mathbb{N}} \nu^* M_n$. As the v-topology is replete in the sense of [BS15, §3.1] (see [Heu21b, Lemma 2.6]), we have $V/p^n = M_n$. To see that V is finite locally free, let $X' \to X$ be an étale cover that trivialises M_1. Choose a pro-étale cover by an affinoid perfectoid $X'' \to X'$. Then by Lemma 2.26, the fact that V is p-adically complete and V/p is free on X'' implies that V is a finite free $\mathcal{O}^+$-module on X''. This shows that V is a finite locally free $\mathcal{O}^+$-module on X_v.

It thus remains to prove Proposition 2.3.2 about generalised $\mathbb{Q}_p$-representations. It is clear from Theorem 2.23 that the data in the definition of generalised $\mathbb{Q}_p$-representations translates into gluing data for vector bundles on $X_{\text{proét}}$. We thus have a fully faithful functor

$$\{\text{generalised } \mathbb{Q}_p\text{-representations on } X_{\text{ét}}\} \to \{\text{finite locally free } \mathcal{O}\text{-modules on } X_{\text{proét}}\}.$$

This is essentially surjective if any finite locally free $\mathcal{O}$-modules on $X_{\text{proét}}$ comes from a finite locally free $\mathcal{O}^+$-module on an étale cover, which is guaranteed by Lemma 2.29. $\square$

3. G-torsors for rigid groups G on adic spaces

We now pass from vector bundles to torsors under any rigid analytic group G, the p-adic analogue of a complex Lie group. We start by discussing some background on rigid analytic group varieties, since non-commutative rigid groups are not so common in the literature. For this reason, and to provide a reference for our sequel articles, we discuss slightly more than is strictly necessary to prove the main result of this article.

From now on, we assume that K is a perfectoid field over $\mathbb{Q}_p$. The characteristic 0 assumption is necessary in this context to obtain a p-adic exponential.

Definition 3.1. By a rigid analytic group variety, or just **rigid group**, we mean a group object G in the category of adic spaces locally of topologically finite type over $\mathrm{Spa}(K, K^+)$.

We refer to [Far19, §1.2-1.3] for some background on rigid groups, some of which we recall below. Rigid groups have been studied primarily in the commutative case, e.g. in [Lüt95], but we do not assume G to be commutative. We therefore write the group operation multiplicatively as $m : G \times G \to G$, and write $1 \in G$ for the identity. As before, we freely identify G with its associated v-sheaf over K. This is harmless in characteristic 0 due to:

Lemma 3.2 ([Far19, Proposition 1])**.** *Any rigid group G is smooth. Moreover, there is a rigid open subspace $1 \in U \subseteq G$ for which there is an isomorphism $U \xrightarrow{\sim} \mathbb{B}^d$ of rigid spaces.*

Example 3.3.
(1) For any algebraic group G over $\mathrm{Spec}(K)$, we get an associated rigid analytic group by analytification. Again, we often identify G with its analytification as well as with the associated v-sheaf. If G is affine, then the latter can explicitly be described in terms of the algebraic group as being the sheaf $G(\mathcal{O})$ on Perf_K sending $(R, R^+) \mapsto G(R)$. We are particularly interested in the case $G = \mathrm{GL}_n$. If G is not affine, the description of the v-sheaf is still true after sheafification.

Two examples that we use frequently throughout are $G = \mathbb{G}_a$, which is the rigid affine line with its additive structure and represents $\mathcal{O}$, as well as $G = \mathbb{G}_m$, which is the rigid affine line punctured at the origin with its multiplicative structure and represents $\mathcal{O}^\times$.

More generally, for any finite dimensional vector space W over K, we have the rigid group $W \otimes_K \mathbb{G}_a$. We call any rigid group of this form a **rigid vector group.**

(2) Let $\mathcal{G}$ be a smooth formal group scheme over $\mathrm{Spf}(K^+)$, i.e. a group object in the category of formal schemes locally of topologically finite presentation over $\mathrm{Spf}(K^+)$ that is smooth over $\mathrm{Spf}(K^+)$. Then the adic generic fibre $\mathcal{G}^{\mathrm{ad}}_\eta \to \mathrm{Spa}(K, K^+)$ in the sense of [SW13, §2.2] is naturally a rigid group. We say that a rigid group G over K has **good reduction** if it arises in this way, i.e. if there is a smooth formal group scheme $\mathcal{G}$ over $\mathrm{Spf}(K^+)$ whose adic generic fibre is isomorphic as a rigid group to G.

(3) Assume that G is an algebraic group over K that extends to a smooth algebraic group G_{K^+} over K^+. For instance, by a Theorem of Chevalley–Demazure, such a model always exists if G is a split connected reductive group, namely the Chevalley model [SGA3, XXV Corollaire 1.2]. Consider the p-adic completion $\mathcal{G}$ of G_{K^+}. Then the adic generic fibre of $\mathcal{G}$ is an open rigid subgroup $G^+ \subseteq G$ that has good reduction.

For $G = \mathbb{G}_a$ with its canonical extension to K^+, this construction yields the closed unit ball $\mathbb{G}_a^+ \subseteq \mathbb{G}_a$ which represents the v-sheaf $\mathcal{O}^+$. For $G = \mathrm{GL}_n$, it recovers the rigid open subgroup $\mathrm{GL}_n(\mathcal{O}^+) \subseteq \mathrm{GL}_n(\mathcal{O})$ of integral matrices from the last section.

3.1. **The correspondence between rigid groups and Lie algebras.** For any rigid group G over (K, K^+), we denote by

$$\mathfrak{g} := \operatorname{Lie} G := \ker(G(K[X]/X^2) \to G(K))$$

the **Lie algebra** of G, defined exactly like for algebraic groups (see also [Far19, §1.2]). Its underlying K-vector space is the tangent space of G at the identity, so $\dim \mathfrak{g} = \dim G$. Similarly as for algebraic groups, we can also regard $\mathfrak{g}$ as a rigid vector group over K, explicitly given by $\operatorname{Lie} G \otimes_K \mathbb{G}_a$. As usual, we also consider this as a v-sheaf on Perf_K, explicitly given for any perfectoid K-algebra (R, R^+) by $\mathfrak{g}(R) = \ker(G(R[X]/X^2) \to G(R))$. We shall therefore from now on write $\mathfrak{g}(K)$ when we mean the underlying K-vector space.

One application of the v-sheaf perspective is that it immediately shows that we have a rigid analytic **adjoint action**

$$\operatorname{ad} : G \to \operatorname{GL}(\mathfrak{g}),$$

a homomorphism of rigid groups sending $g \in G(R)$ to the R-linear automorphism of $\mathfrak{g}(R)$ induced on tangent spaces by the conjugation map $G \times_K R \to G \times_K R$ that sends $h \mapsto g^{-1}hg$. On tangent spaces, this induces a map $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ which defines the Lie bracket on $\mathfrak{g}$.

With these definitions, there is a p-adic analogue of the Lie group-Lie algebra correspondence in complex geometry, the main difference to the complex case being that one needs to account for the fact that in the p-adic setting there are many more open subgroups. While we do not know of a place in the literature where this is discussed in the present setting of rigid analytic groups, the construction is of course essentially classical as it follows immediately from the theory of p-adic Lie groups, as developed in [Bou72][Ser06][Sch11]:

Theorem 3.4 ([Ser06, II]). *Sending $G \mapsto \operatorname{Lie}(G)$ defines a functor*

$$\operatorname{Lie} : \{\textit{rigid groups over } K\} \to \{\textit{fin. dim. Lie algebras over } K\}.$$

This becomes an equivalence of categories after localising the left hand side at the class of homomorphisms of rigid groups that are open immersions.

Moreover, for any homomorphism $f : G \to H$ of rigid groups for which the morphism $\operatorname{Lie}(f) : \operatorname{Lie}(G) \to \operatorname{Lie}(H)$ is an isomorphism, there are rigid open subgroups $G_0 \subseteq G$ and $H_0 \subseteq H$ such that f restricts to an isomorphism of rigid groups $G_0 \to H_0$.

Proof. Due to Lemma 3.2, there are open neighbourhoods of $1 \in G$ and $1 \in H$ inside of which rigid open subdiscs around 1 with morphisms of rigid spaces between them are equivalent to open subdiscs of $G(K)$ and $H(K)$ centered at 1 and analytic maps between them in the sense of p-adic Lie groups as defined in [Ser06, II, 2]. In particular, the category on the left becomes after localisation equivalent to Serre's "analytic group chunks".

Again by Lemma 3.2, the completion $\widehat{G}$ of G at 1 is isomorphic as a formal scheme to $\operatorname{Spf}(K[[X_1, \dots, X_d]])$ and inherits the structure of a formal group law. This defines a functor

$$R : \{d\text{-dim. rigid groups over } K\} \to \{d\text{-dim. formal group laws over } K\}.$$

On the other hand, sending a formal group law H over K to the tangent space $\mathfrak{h}$ at the origin defines an equivalence of categories [Ser06, II, 5, §6]

$$S : \{d\text{-dim. formal group laws over } K\} \to \{d\text{-dim. Lie algebras over } K\}.$$

The composition is easily seen to coincide with Lie.

It thus suffices to prove the results for the functor R: That R is essentially surjective follows from [Ser06, II, 4, §8, Theorem 1] (or [Sch11, Proposition 17.6]). It becomes full after the localisation by [Ser06, II, 5, §7, Theorem 1]. It becomes faithful after the localisation because any morphism from a connected rigid group is determined on any open subgroup by Zariski-density.

The last sentence follows from [Ser06, II, 5, §7, Corollary 1.2]. For an alternative proof of this statement, see also [Far19, §1, Lemme 1, 2]. ∎

3.2. **The p-adic exponential of a rigid group G.** For the rest of this section, we fix any rigid group G over (K, K^+). We now give a brief account of the p-adic exponential in the non-commutative setting of rigid groups: Exactly like for Theorem 3.4, this is essentially a translation of a classical result from the theory of p-adic Lie groups into the setting of rigid groups:

Proposition 3.5. *Let G be a rigid group over K and $\mathfrak{g}$ its Lie algebra. Then there is an open $\mathcal{O}_K$-linear subgroup $\mathfrak{g}^\circ \subseteq \mathfrak{g}$ for which there is a unique open immersion of rigid spaces*

$$\exp : \mathfrak{g}^\circ \to G$$

with $\exp(0) = 1$ *that induces the identity* $\mathfrak{g} \to \mathfrak{g}$ *on tangent spaces and makes the diagram*

$$\begin{tikzcd} \mathfrak{g}^\circ \times \mathfrak{g}^\circ \arrow[r, "\exp"] \arrow[d, "\mathrm{BCH}"'] & G \times G \arrow[d, "m"] \\ \mathfrak{g}^\circ \arrow[r, "\exp"] & G \end{tikzcd}$$

commute, where BCH *is the Baker–Campbell–Hausdorff formula*

$$\mathrm{BCH}(x, y) = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}[x - y, [x, y]] + \ldots$$

It satisfies the following properties:

(1) For any open subgroup $\mathfrak{g}_1 \subseteq \mathfrak{g}^\circ$, the map $\exp : \mathfrak{g}_1 \to G$ is an isomorphism of rigid spaces onto an open subgroup of G (but not necessarily a homomorphism).
(2) $\exp$ is functorial in G (i.e. after shrinking $\mathfrak{g}^\circ$, the obvious diagram commutes).

Definition 3.6. In particular, part 1 says that $\exp : \mathfrak{g}^\circ \to G$ is an isomorphism onto an open subgroup $G^\circ \subseteq G$. We denote the inverse by $\log : G^\circ \to \mathfrak{g}^\circ$.

Remark 3.7. (1) If G is commutative, then $\mathrm{BCH}(x, y) = x + y$ and the diagram says that exp is an isomorphism of rigid groups. This case is discussed in [Far19, §1.5].
(2) There is in general no canonical way to choose the subgroup $\mathfrak{g}^\circ$: Already if $G = \mathrm{GL}(W)$ for some finite dimensional K-vector space W, we need a basis of W to get $\mathfrak{g}^\circ$. In the following, we shall therefore fix a choice of $\mathfrak{g}^\circ$ and thus G°, but this choice will be harmless as we will always be free to replace them by open subgroups. More canonically, one could consider the filtered system of all open subgroups on which exp is defined.
(3) For $G = \mathrm{GL}_n$, the exponential can be explicitly described by the usual formulas: We have $\mathfrak{g} = M_n$ and for any uniform Huber pair (R, R^+) over (K, K^+), the p-adic exponential and logarithm series define continuous homomorphisms

$$\begin{aligned} \exp\colon \mathfrak{g}^\circ := p^{\alpha_0}\mathfrak{m}M_n(R^+) &\to 1 + p^{\alpha_0}\mathfrak{m}M_n(R^+), & x &\mapsto \textstyle\sum_{n=0}^\infty \frac{x^n}{n!} \\ \log\colon \quad 1 + \mathfrak{m}M_n(R^+) &\to M_n(R), & 1 + x &\mapsto -\textstyle\sum_{n=1}^\infty \frac{(-x)^n}{n} \end{aligned}$$

which are mutually inverse when restricted to the domain and codomain of the exponential, where we set $\alpha_0 := 1/(p-1)$ if $p > 2$ and $\alpha_0 = 1/4$ otherwise. More generally, this describes exp for linear algebraic groups, or more generally any rigid group G that admits an injective homomorphism $G \hookrightarrow \mathrm{GL}_n$: The maps exp and log restrict to the closed subgroups $\mathfrak{g}^\circ := \mathfrak{g} \cap p^{\alpha_0}\mathfrak{m}M_n(\mathcal{O}^+)$ and $G^\circ := G \cap (1 + \mathfrak{p}^{\alpha_0}\mathfrak{m}M_n(\mathcal{O}^+))$.

Proof. This is developed in [Bou72, II] and [Sch11, III, Corollary 18.19] for p-adic Lie groups instead of rigid groups, but since p-adic Lie groups are locally analytic, one can obtain from this a morphism of rigid groups on a neighbourhood of the identity. We sketch the proof:

We use the functor S already mentioned in the proof of Theorem 3.4. Its inverse is given by sending a Lie algebra $\mathfrak{h}$ to the formal group law $F_{\mathfrak{h}} := U(\mathfrak{h})^*$, where $U(\mathfrak{h})^*$ is the K-linear dual of the universal enveloping algebra with its natural co-algebra structure.

Applying this construction to $\mathfrak{g}$, the Theorem of Poincaré–Birkhoff–Witt identifies $U(\mathfrak{g})^*$ with the completed symmetric algebra $K[[\mathfrak{g}^*]]$ of $\mathfrak{g}^*$ [Sch11, Proposition 18.3]: This defines an isomorphism of formal schemes $\psi : F_{\mathfrak{g}} \to \widehat{\mathbb{G}}_a \otimes \mathfrak{g}$. Then BCH expresses the formal group law on $\widehat{\mathbb{G}}_a \otimes \mathfrak{g}$ transported from $F_{\mathfrak{g}}$ via ψ, see [Sch11, Corollary 18.15].

The crucial calculation is now that BCH converges on an open ball in $\mathfrak{g}$: Namely, choose any K-basis of $\mathfrak{g}$ and for $k \geq 0$ let $\mathfrak{g}_k^\circ$ be the $p^k\mathfrak{m}\mathbb{G}_a^+$-span of the basis vectors, considered as an open ball. Then by [Sch11, Proposition 17.6], BCH defines for $k \gg 0$ a function

$$\mathrm{BCH} : \mathfrak{g}_k^\circ \times \mathfrak{g}_k^\circ \to \mathfrak{g}_k^\circ$$

that endows $\mathfrak{g}_k^\circ$ with the structure of a rigid group with Lie algebra $\mathfrak{g}$. Let us call this $\mathfrak{g}_{k,\mathrm{BCH}}^\circ$.

By the above equivalence of categories S, there is now a natural isomorphism between the formal groups associated to $\mathfrak{g}_{k,\mathrm{BCH}}^\circ$ and G that is the identity on tangent spaces. By Theorem 3.4 and its proof, this is analytic in a neighbourhood of the identity, i.e. for $k \gg 0$ there is an open subgroup $G^\circ \subseteq G$ for which there is an isomorphism of rigid groups $\exp : \mathfrak{g}_{k,\mathrm{BCH}}^\circ \xrightarrow{\sim} G^\circ$. It is clear from the construction that this is functorial. □

Corollary 3.8. *Any rigid group G has a neighbourhood basis of the identity $(G_k)_{k\in\mathbb{N}}$ that consists of open subgroups $G_k \subseteq G$ whose underlying rigid space is isomorphic to the closed ball $\mathbb{B}^d$ of dimension $d = \dim G$. In particular, each G_k has good reduction.*

Proof. This is true for $\mathfrak{g}$ which is isomorphic as a rigid group to $\mathbb{G}_a^d$. By Proposition 3.5.1 it then also holds for G. The group G_k has good reduction because taking $\mathcal{O}^+$ of $m : G_k \times G_k \to G_k$ defines a formal group scheme structure on the unit ball over $\mathrm{Spf}(K^+)$. □

One way to make exp explicit is to use the following consequence of Theorem 3.4:

Corollary 3.9. *For any rigid group G, there exist $n \geq 0$ and an open subgroup $G^\circ \subseteq G$ that admits a homomorphism of rigid groups $G^\circ \hookrightarrow \mathrm{GL}_n$ that is a locally closed immersion.*

Proof. By Ado's Theorem, there is a faithful representation $\mathrm{Lie}\, G \to \mathrm{Lie}(\mathrm{GL}_n)$ for some n. By Theorem 3.4, it follows that this comes from a morphism $\rho : G^\circ \to \mathrm{GL}_n$ for some open subgroup $G^\circ \subseteq G$. It follows from Proposition 3.5.2 that ρ is Zariski-closed. □

In particular, one can then describe exp on G_k using the explicit formulas in Remark 3.7.

We now record some more properties of exp and log that we do not need in this article, but that fit naturally in the discussion and that we will use in [Heu22b] in order to compute $R^1\nu_*G$.

Lemma 3.10. *Let (R, R^+) be any (K, K^+)-Banach algebra.*

(1) If $A, B \in \mathfrak{g}^\circ(R)$ satisfy $[A, B] = 0$, then $\exp(A)$ and $\exp(B)$ commute in $G(R)$, and

$$\exp(A + B) = \exp(A)\exp(B).$$

(2) If $g, h \in G^\circ(R)$ satisfy $gh = hg$, then $[\log(g), \log(h)] = 0$ and

$$\log(gh) = \log(g) + \log(h).$$

(3) Let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \mathfrak{g}^\circ$ and let $G_1 \subseteq G_2 \subseteq G$ be the images under exp. If $g \in G(R)$ is such that $\mathrm{ad}(g)$ sends $\mathfrak{g}_1$ into $\mathfrak{g}_2$, then conjugation by g sends G_1 into G_2 and for any $A \in \mathfrak{g}_1(R)$ we have

$$\exp(\mathrm{ad}(g)(A)) = g^{-1}\exp(A)g.$$

Proof. For the first part, if $[A, B] = 0$, then $\mathrm{BCH}(A, B) = A + B$, so this follows from the diagram in Proposition 3.5.

For part 3, we can reduce to the universal situation and replace R by a reduced affinoid K-algebra. Then we can check the statement on K-points. Here it follows from functoriality of exp applied to the conjugation $G \to G$, $h \mapsto g^{-1}hg$.

For part 2, we deduce from part 1 that $\log([p^n]g) = p^n \log(g)$. Since it suffices to prove that $p^n \log(g)$ and $p^n \log(h)$ commute, we may therefore shrink G°, and may thus assume by functoriality of exp that the restriction of ad to G° fits into a commutative diagram

$$\begin{tikzcd} G^\circ \arrow[r, "\mathrm{ad}"] \arrow[d, "\log"] & \mathrm{GL}(\mathfrak{g})^\circ \arrow[d, "\log"] \\ \mathfrak{g}^\circ \arrow[r, "\mathrm{ad}"] & \mathrm{End}(\mathfrak{g})^\circ. \end{tikzcd}$$

By part 3, we know that $\mathrm{ad}(g)(\log h) = \log(g^{-1}hg) = \log h$. It follows that

$$[\log(g), \log(h)] = \log(\mathrm{ad}(g))(\log h) = -\sum_{n=1}^{\infty} \tfrac{(1-\mathrm{ad}(g))^n}{n}(\log h) = 0,$$

where the first equality comes from the diagram since the bottom row defines $[-,-]$.

The displayed formula now follows from applying log to the one from 1. $\square$

For $G = \mathrm{GL}_n$ with notation as in Remark 3.7, we slightly more generally have:

Lemma 3.11. *Let A, $B \in p^{\alpha_0}\mathfrak{m}M_n(R^+)$ and $N \in M_n(R)$. The following are equivalent:*

(1) $AN = NB$
(2) $\exp(A)N = N\exp(B)$

Proof. (1) $\Rightarrow$ (2) is clear from the formula for exp. To see (2) $\Rightarrow$ (1), we can check the equality on points, which reduces us to the case of $(R, R^+) = (K, K^+)$.

If N is invertible, then $N^{-1}\exp(A)N = \exp(B) \in 1 + p^{\alpha_0}\mathfrak{m}M_n(R^+)$ and the statement follows from applying log. In general, (2) implies that $\exp(B)$ preserves $\ker N$, which is equivalent to B preserving $\ker N$. In particular, either defines an operator on $\mathrm{coim}(N)$. Similarly $\exp(A)$ preserves $\mathrm{im}(N)$, which is equivalent to A preserving $\mathrm{im}(N)$. It follows that we can reduce to the map $N : \mathrm{coim}(N) \to \mathrm{im}(N)$, which is an isomorphism. $\square$

3.3. **G-torsors in the étale and v-topology.** The theme of this section is that we pass from vector bundles to G-torsors. There are several ways to define these, and we shall mainly use the following. For now, we can allow X to be any étale-sheafy adic space over a non-archimedean field K over $\mathbb{Q}_p$.

Definition 3.12. Consider X_τ for τ one of ét or v. Then by a **G-torsor** P on X_τ we mean a **cohomological G-torsor** on the site X_τ where G is regarded as a sheaf by sending $Y \in X_\tau$ to the morphisms of adic spaces $Y \to G$ over K. Explicitly, P is a sheaf on X_τ with a right action $m : P \times G \to P$ by G such that τ-locally on X, there is a G-equivariant isomorphism $G \xrightarrow{\sim} P$. The morphisms of G-bundles are the G-equivariant morphisms of sheaves on X_τ.

We shall also refer to G-torsors on $X_{\text{ét}}$ as **étale G-torsors**, and to G-torsors on X_v as **v-G-torsors**. In either case, we say that the torsor is trivial if it is globally on X_τ isomorphic to G. It is clear from this definition that G-torsors on X_τ up to isomorphism are classified by the non-abelian sheaf cohomology set $H^1_\tau(X, G)$, with the distinguished point corresponding to the trivial torsor.

We can make the analogous definition when X is instead a locally spatial diamond with X_τ being either $X_{\text{ét}}$ or X_v, and where G is the sheaf on X_τ represented by $G^\diamondsuit$ in $\mathrm{Dmd}_{K,v}$.

Remark 3.13. For $G = \mathrm{GL}_n$, we recall that there is a natural functor from GL_n-torsors on X_τ to τ-vector bundles (i.e. finite locally free $\mathcal{O}$-modules on X_τ) of rank n, that is essentially surjective but not fully faithful: The morphisms of GL_n-torsors are precisely the isomorphisms of vector bundles. In particular, the difference is harmless as long as we are only concerned with isomorphisms, for example in the context of moduli stacks.

As usual, there is also a geometric perspective on G-torsors:

Definition 3.14. Let X be an étale-sheafy adic space over K and let τ be one of ét or v. Then a **geometric G-torsor** on X_τ is a morphism $E \to X$ of sheaves on X_τ with a right G-action $E \times G \to E$ over X such that there is a cover $X' \to X$ in X_τ with a G-equivariant isomorphism $X' \times G \xrightarrow{\sim} X' \times_X G$ over X'. The morphisms of geometric G-torsors are the G-equivariant morphisms of sheaves on X_τ.

We can make the analogous definition when X is instead a locally spatial diamond where X_τ is either $X_{\text{ét}}$ or X_v, and G is seen as the sheaf on X_τ represented by $G^\diamondsuit$ in $\mathrm{Dmd}_{K,v}$.

Remark 3.15. (1) We caution that both in Definition 3.12 and Definition 3.14, depending on the space X, it can in general make a difference whether we consider G-torsors on $X_{\text{ét}}$ or G-torsors on $X^\diamondsuit_{\text{ét}}$, because in general the structure sheaves are not identified via the equivalence $X_{\text{ét}} = X^\diamondsuit_{\text{ét}}$. For example, for the rigid space $X = \mathrm{Spa}(K[X]/X^2)$, the automorphisms of the trivial $\mathbb{G}_a$-torsor on $X_{\text{ét}}$ are given by $\mathcal{O}_{X_{\text{ét}}}(X) = K[X]/X^2$, whereas the automorphisms of the trivial $\mathbb{G}_a$-torsor on $X^\diamondsuit_{\text{ét}}$ are given by $\mathcal{O}_{X^\diamondsuit_{\text{ét}}}(X^\diamondsuit) = K$ because perfectoid spaces don't see the non-reduced structure.

(2) While it may be true for an adic space X that geometric G-torsor on $X_{\text{ét}}$ are represented by adic spaces, it is definitely not the case that geometric G-torsors on X_v are always represented by adic spaces. However, we will see in Corollary 4.29 that they are still always diamonds, which is not immediately obvious from the definition. In particular, when X is a locally spatial diamond and $\tau = v$, one gets an equivalent definition if one takes $E \to X$ to be a morphism of diamonds instead of v-sheaves.

Proposition 3.16. *(1) Let X be an étale-sheafy adic space over K, or let X be a locally spatial diamond. The functor sending a geometric G-torsor $E \to X$ to its sheaf of sections $E \leftarrow X$ defines an equivalence of categories*

$$\{\textit{geometric } G\textit{-torsors on } X_\tau\} \xrightarrow{\sim} \{\textit{cohomological } G\textit{-torsors on } X_\tau\}.$$

Any morphism in either category is an isomorphism.

(2) When X is a locally spatial diamond, there is a natural fully faithful functor

$$\{G\textit{-torsors on } X_{\text{ét}}\} \hookrightarrow \{G\textit{-torsors on } X_v\}$$

given by sending a G-torsor on $X_{\text{ét}}$ to the v-sheaf of sections of its associated geometric G-torsor. On isomorphism classes, this functor is given by the natural map

$$H^1_{\text{ét}}(X, G) \to H^1_v(X, G).$$

(3) When X is an étale-sheafy adic space, diamondification defines a functor

$$\{G\textit{-torsors on } X_{\text{ét}}\} \to \{G^\diamondsuit\textit{-torsors on } X^\diamondsuit_{\text{ét}}\}.$$

(4) Let X be an étale-sheafy adic space such that $\nu : X_v \to X_{\text{ét}}$ satisfies $\nu_*\mathcal{O} = \mathcal{O}$, for example this holds when X is perfectoid or a semi-normal rigid space. Then the functor from (3) is an equivalence. In particular, we then have a fully faithful functor

$$\{G\textit{-torsors on } X_{\text{ét}}\} \hookrightarrow \{G\textit{-torsors on } X_v\}.$$

As one can already see from the case of $G = \mathbb{G}_m$ discussed in [Heu22a], the functor from Proposition 3.16.(4) is in general far from being essentially surjective when X is a rigid space.

Proof. (1) It is clear that both cohomological G-torsors on X_τ and geometric G-torsors on X_τ satisfy τ-descent. It therefore suffices to see that the endomorphisms of the trivial object are identified via the functor: For the trivial cohomological G-torsor, it is clear that the G-equivariant morphisms $G \to G$ over X correspond to $g \in G(X)$ via the map $h \mapsto gh$. Similarly, for the trivial geometric G-torsor $X \times G$, any G-equivariant morphism $\phi : X \times G \to X \times G$ of τ-sheaves over X is uniquely determined by the map $X \xrightarrow{\mathrm{id},1} X \times G \xrightarrow{\phi} X \times G \xrightarrow{\pi_2} G$, hence the endomorphisms of the trivial geometric G-torsor are also identified with $G(X)$.

(2) It is clear from (1) that the functor is well-defined. The claim that this is fully faithful can be checked locally, so we can again reduce to the trivial torsor. Here it follows from the fact that for $\mu : X_v^\diamondsuit \to X_{\text{ét}}^\diamondsuit$, we have $\mu_*G = G$ by definition.
(3) This is clear from interpreting both sides as geometric G-torsors.
(4) The condition $\nu_*\mathcal{O} = \mathcal{O}$ guarantees that for any $Y \in X_{\text{ét}}$, the morphisms $Y \to G$ of adic spaces over K are in bijection with the morphisms $Y^\diamondsuit \to G^\diamondsuit$ of diamonds over K. The statement then follows from the identification $X_{\text{ét}} = X_{\text{ét}}^\diamondsuit$. □

Remark 3.17. If G is a linear algebraic group, there is a third perspective on G-torsors: the Tannakian point of view. We sketch it here and refer to [SW20, Appendix to Lecture 19] for details: A **Tannakian G-torsor** on a sousperfectoid space X is an exact tensor functor

$$P : \mathrm{Rep}_G \to \mathrm{Bun}_{X,\text{ét}}$$

where Rep_G is the category of algebraic representations $G \to \mathrm{GL}(V)$ of G considered as an algebraic group scheme, and $\mathrm{Bun}_{X,\text{ét}}$ is the category of étale vector bundles on X. By [SW20, Theorem 19.5.1], the categories of G-torsors and Tannakian G-torsors on $X_{\text{ét}}$ are equivalent via the functor that sends a G-torsor P to the Tannakian G-torsor $V \mapsto P \times^G V$.

One can deduce using Kedlaya Liu's Theorem 2.22 that on a perfectoid space X, the categories of G-torsors on $X_{\text{ét}}$ and G-torsors on X_v are equivalent for linear algebraic G.

However, we do not know if the case of $G^+ := G(\mathcal{O}^+)$ for linear algebraic groups over K^+ could be deduced in a similar way from that of $\mathrm{GL}_n(\mathcal{O}^+)$ discussed in Section 2: The Tannakian approach works via algebraisation, but for G^+-torsors it seems less clear when these come from G^+-torsors on $\mathrm{Spec}(R^+)$, already for $G = \mathrm{GL}_n$.

4. Reduction of structure group to open subgroups

In the past section, we have seen using the exponential that any rigid group G has a neighbourhood basis of open subgroups. In this section, we study when G-torsors in the v-topology admit a reduction of structure group to open subgroups. We then deduce the Main Theorem.

4.1. Approximation for quotient sheaves.

The key technical result of this section is a generalisation of the approximation property Lemma 2.11 for $\mathrm{GL}_n(\mathcal{O}^+/\varpi)$ to all rigid groups and all rigid open subgroups:

Proposition 4.1. *Let G be a rigid group and let $U \subseteq G$ be any rigid open subgroup (not necessarily normal). Then the sheaf of cosets G/U on $\mathrm{Sous}_{K,\text{ét}}$ satisfies the approximation property of Definition 2.13: For any affinoid perfectoid tilde limit $X \approx \varprojlim_{i\in I} X_i$, we have*

$$G/U(X) = \varinjlim_{i\in I} G/U(X_i).$$

In particular, G/U is already a v-sheaf on Sous_K.

Proof. For any $i \in I$ and any $W_i \to X_i$ in $X_{i,\text{ét},\text{qcqs}}$, let $W := W_i \times_{X_i} X$ and $W_j := W_i \times_{X_i} X_j$ for $j \geq i$ be the pullbacks. Then we obtain a natural pullback map

$$\varinjlim_{j\geq i} G(W_j)/U(W_j) \to G(W)/U(W).$$

Using the identification $X_{\text{ét},\text{qcqs}} = 2\text{-}\varinjlim_{i\in I} X_{i,\text{ét},\text{qcqs}}$ from Lemma 2.8, we can for varying i and W_i regard this as a morphism of presheaves on $X_{\text{ét},\text{qcqs}}$. We consider its sheafification: If we denote by $\mu_i : X_{\text{ét},\text{qcqs}} \to X_{i,\text{ét},\text{qcqs}}$ the projections, then this can be described as the natural map

$$\phi : \varinjlim_{i\in I} \mu_i^*(G/U) \to G/U.$$

By [dJ+23, 09YN], the left hand side satisfies

$$\varinjlim_{i\in I} \mu_i^*(G/U)(X) = \varinjlim_{i\in I} G/U(X_i).$$

Hence it suffices to prove that ϕ is an isomorphism.

Claim 4.2. *The map ϕ is injective.*

Proof. If $g_1, g_2 \in G(X_i)$ have the same image in $G(X)/U(X)$, then $\delta = g_1^{-1} \cdot g_2 \in G(X_i)$ lands in $U(X) \subseteq G(X)$. To deduce that δ lands in $U(X_j)$ for some $j \geq i$, we no longer need the group structure on G and use the following argument that we learnt from [Sch12, Lemma 6.13.(iv)]:

Claim 4.3. *Let H be an affinoid adic space over K and $V \subseteq H$ a rational subspace. Let $X_i \to H$ be a morphism such that $X \to X_i \to H$ factors through V. Then $X_j \to X_i \to H$ factors through V for $j \gg i$.*

Proof. For $j \geq i$, let W_j be the pullback of V along $X_j \to H$, then $W_j \subseteq X_j$ is still rational open, hence affinoid. The assumptions imply that $X \to X_j$ factors through W_j and induce a homeomorphism $|X| \to \varprojlim_{j\geq i} |W_j|$. We now use that $(W_j)_{j\geq i}$ is an inverse system of affinoid adic spaces, hence $(|W_j|)_{j\geq i}$ is an inverse system of spectral spaces with spectral transition maps. Since $\varprojlim_{j\geq i} |X_j|\backslash|W_j|$ is empty, it follows from [dJ$^+$23, Tag 0A2W] that $X_j\backslash W_j$ is empty for some $j \geq i$. Hence $X_j = W_j$ and thus $X_j \to H$ factors through V. □

We apply the claim locally on X: Let H be any affinoid open subspace of G and cover $U \cap H$ by rational opens V. As X_i is quasi-compact, finitely many such opens cover the image of $X_i \to U$. Replacing X and the X_i by the respective pullback of $V \subseteq H$, we may assume that δ lies in $V(X)$. The claim shows that already $\delta \in U(X_j)$ for some $j \geq i$. □

It thus remains to prove that ϕ is surjective. For this we again localise on G: Let $\mathrm{Spa}(A, A^+) = V \subseteq G$ be any affinoid open such that (A, A^+) is of topologically finite type over (K, K^+). By [Hub94, Lemma 3.3.(ii)], this means that there is a subring of definition $A_0 \subseteq A^+$ of topologically finite type (hence automatically of topologically finite presentation) over K^+ such that A^+ is the integral closure of A_0 in A. Write $X = \mathrm{Spa}(R, R^+)$ and $X_i = \mathrm{Spa}(R_i, R_i^+)$, then any point $x \in G(X)$ that factors through V defines a map $f : A_0 \to R^+$. By [Heu21a, Lemma 3.10], the assumption $X \approx \varprojlim X_i$ implies $R^+/p^n = \varinjlim R_i^+/p^n$ for all $n \in \mathbb{N}$. Since A_0/p^n is of finite presentation over K^+/p, the reduction of f mod p^n factors through a morphism

$$A_0/p^n \to R_i^+/p^n.$$

for some $i \in I$. We now use:

Lemma 4.4. *Let A be a p-adically complete flat K^+-algebra such that $A[\frac{1}{p}]$ is a smooth affinoid K-algebra. Let us assume for simplicity that $\Omega^1_{A[\frac{1}{p}]|K}$ is finite free. Then there is $t \geq 0$ such that for any p-adically complete K^+-algebra R and any homomorphism of K^+-algebras*

$$f_t : A/p^n \to R/p^n$$

for some $n > t$, there is a homomorphism of K^+-algebras $f : A \to R$ with $f \equiv f_t \bmod p^{n-t}$.

Proof. This is a statement about torsion in the cotangent complex: By [Ill71, III.2.2.4], there is for any $s \leq n \leq 2s$ in $\mathbb{N}$ and any morphism $f_n : A \to R/p^n$ an obstruction class

$$o_{n,s} \in \mathrm{Ext}^1_{n,s} := \mathrm{Ext}^1_{R/p^n}(L_{A/p^n|K^+/p^n} \otimes^{\mathbb{L}}_{A/p^n} R/p^n, R/p^s)$$

that vanishes if and only f_n lifts to a morphism $A \to R/p^{n+s}$.

Claim 4.5. *There is $t \in \mathbb{N}$ independent of s, n such that $\mathrm{Ext}^1_{n,s}$ is p^t-torsion.*

Proof. We first note that the inclusion $A \subseteq A[\frac{1}{p}]^\circ$ has bounded p-torsion cokernel. Similarly for $K^+ \to \mathcal{O}_K := K^\circ$. We may therefore without loss of generality replace A by a ring of integral elements that is of topologically finite presentation over $\mathcal{O}_K$ and R by $R\hat{\otimes}_{K^+}\mathcal{O}_K$, so that we may assume that $K^+ = \mathcal{O}_K$. In this setting, we can use the analytic cotangent complex for formal schemes and rigid spaces introduced by Gabber–Ramero [GR03, §7]: This is a pseudo-coherent complex $L^{\mathrm{an}}_{A|\mathcal{O}_K}$ of A-modules such that $H^0(L^{\mathrm{an}}_{A|\mathcal{O}_K}) = \Omega_{A|\mathcal{O}_K}$ and

$$L^{\mathrm{an}}_{A|\mathcal{O}_K} \otimes^{\mathbb{L}}_A A/p^k = L_{A|\mathcal{O}_K} \otimes^{\mathbb{L}}_A A/p^k$$

for all $k \in \mathbb{N}$. Moreover,

$$L^{\mathrm{an}}_{A|\mathcal{O}_K}[\tfrac{1}{p}] = L^{\mathrm{an}}_{A[\frac{1}{p}]|K} = \Omega_{A[\frac{1}{p}]|K}$$

due to the assumption that $A[\frac{1}{p}]|K$ is smooth. Let Ω^+ be any finite free A-sublattice of $\Omega_{A[\frac{1}{p}]|K}$ that contains the image of $\Omega_{A|\mathcal{O}_K} \to \Omega_{A[\frac{1}{p}]|K}$. Then it follows that the cone $C :=$ cone(h) of the canonical map $h : L^{\mathrm{an}}_{A|\mathcal{O}_K} \to \Omega_{A|\mathcal{O}_K} \to \Omega^+$ is exact after inverting p. Thus by the same argument as in the proof of [Sch12, Proposition 6.10.(iii)], it follows that for all $i \leq 0$, the cohomology $H^i(C)$ is killed by p^m for some m depending on i. As C is bounded above, we may therefore choose t such that $H^i(C)$ is killed by p^t for all $i \geq -2$.

We now first apply $\otimes^{\mathbb{L}}_A R/p^n$ and then $\mathrm{Ext}^1_{R/p^s}(-, R/p^s)$ to the distinguished triangle

$$C \to L^{\mathrm{an}}_{A|\mathcal{O}_K} \to \Omega^+$$

and obtain an exact sequence

$$\mathrm{Ext}^1(C \otimes^{\mathbb{L}}_A R/p^n, R/p^s) \to \mathrm{Ext}^1_{n,s} \to \mathrm{Ext}^1(\Omega^+ \otimes_A R/p^n, R/p^s).$$

The last term vanishes because Ω^+ is a finite free A-module by construction. The first term is p^t-torsion because the second argument of Ext^1 is concentrated in degree 0, so only the first two terms of C contribute, and the truncation $\tau_{\geq -2}C$ is killed by p^t. □

Let now s, n be such that $t < s \leq n < 2s$. It is clear from short exact sequences that

$$\mathrm{Ext}^1_{n,s} \xrightarrow{p^t} \mathrm{Ext}^1_{n-t,s}$$

sends the obstruction class $o_{n,s}$ to $o_{n-t,s}$. By the claim, this morphism is zero. It follows that we may lift the reduction $f_{n-t} : A \xrightarrow{f_n} R/p^n \to R/p^{n-t}$ to a morphism $A \to R/p^{n+s-t}$. As $s > t$, we see inductively that we obtain a lift of f_{n-t} to the complete $\mathcal{O}_K$-algebra R. □

Since we may after localising assume that $\Omega_{A|K}$ is finite free, the Lemma applies and setting $k = n-t$, we see that there is a sequence of indices $(i_k)_{k\in\mathbb{N}}$ and morphisms $f_k : A_0 \to R^+_{i_k}$ such that $f_k \equiv f \bmod p^k$. Since the ring extension $A_0 \to A^+$ is integral with bounded cokernel, it is clear that f_k extends to $f_k : A^+ \to R^+_{i_k}$, and that after re-indexing we may still assume that $f_k \equiv f \bmod p^k$. After inverting p, the f_k now define a sequence of points in $G(X_{i_k})$. Let $x_k \in G(X)$ be the images of these points under $G(X_{i_k}) \to G(X)$.

Claim 4.6. *For $k \gg 0$, the difference $\delta_k := x^{-1}x_k$ lies in $U(X)$.*

This will prove that

$$\varinjlim_{i\in I} G(X_i) \to G(X)/U(X)$$

is surjective (generalising [Heu21a, Lemma 3.10], which is the case of $G = \mathcal{O}^+$, $U = \varpi\mathcal{O}^+$). This will in turn finish the proof of the approximation property. That G/U is a v-sheaf then follows from Proposition 2.14.

Proof. Let $Z_k \subseteq X$ be the preimage of $U \subseteq G$ under the map $\delta_k : X \xrightarrow{\delta_k} G$. It is clear that Z_k is open. It therefore remains to prove that each $z \in X$ is contained in Z_k for some k. From this the claim follows by compactness of X.

Let $z \in X$ be any point, and let $\mathrm{Spa}(C, C^+) \to X$ be any morphism where (C, C^+) is a perfectoid field such that z is in the image. The points x and x_k both define maps $\mathrm{Spa}(C, C^+) \to X \to V$ that correspond to two homomorphisms of K^+-algebras

$$A^+ \to R^+ \to C^+.$$

By construction, these agree modulo p^k. We now consider the affinoid rigid space $V_C := V \times_{\mathrm{Spa}(K)} \mathrm{Spa}(C)$ over $\mathrm{Spa}(C)$ and the topological space $V_C(C)$. Let $W_k \subseteq V_C(C)$ be the subspace of points whose associated morphisms $A^+\hat{\otimes}_{K^+}C^+ \to C^+$ agree with x mod p^k.

Claim 4.7. *Any open neighbourhood of $x \in V_C(C)$ contains one of the W_k.*

Proof. We reduce to the case of $\mathbb{B}^d$, where the statement is clear: For this reduction, embed $V \subseteq \mathbb{B}^d$ into some d-dimensional ball corresponding to a morphism $K^+\langle T_1, \dots, T_n\rangle \to A^+$ with bounded p-torsion cokernel, then also the compositions

$$K^+\langle T_1, \dots, T_n\rangle \to A^+ \to A^+/p^k \to C^+/p^k$$

agree mod p^k for any two points in W_k. The claim follows since $V_C(C) \subseteq \mathbb{B}^d(C)$ carries the subspace topology. $\square$

For the base-change $U_C \subseteq G_C$ of $U \subseteq G$ to C, this now means that the open neighbourhood $(x \cdot U_C(C)) \cap V_C(C)$ of x contains W_k for $k \gg 0$. Thus $x_k \in W_k \subseteq x \cdot U_C(C)$, showing $\delta_k(z) = x^{-1}x_k \in U_C(C)$, whence $z \in Z_k$. $\square$

This finishes the proof of Proposition 4.1. $\square$

Proposition 4.8. *Let G be a rigid group over K and let $U \subseteq G$ be a rigid open subgroup. Let X be a sousperfectoid adic space over K and let $\nu : X_v \to X_{\text{ét}}$ be the natural morphism of sites. Then the natural map*

$$R^1\nu_*U \to R^1\nu_*G$$

is surjective. In other words, any G-torsor E on X_v admits a reduction of structure group to U étale-locally on X, i.e. there is an étale cover $Y \to X$ and a U-torsor F on Y_v such that $F \times^U G = E \times_X Y$ as G-torsors. If G is commutative, then we more generally have

$$R^k\nu_*U = R^k\nu_*G$$

and $R^k\nu_*(G/U) = 1$ for all $k \geq 1$. In particular, in this case F is unique up to isomorphism.

Proof. We first observe that if $U \subseteq G$ is normal, then we can consider the short exact sequence of sheaves of pointed sets

$$R^1\nu_*U \to R^1\nu_*G \to R^1\nu_*(G/U)$$

and the result follows from the fact that by Proposition 4.1, we may apply Proposition 2.14.2 to G/U and conclude that $R^1\nu_*(G/U) = 1$. If G is commutative, we more generally have $R^k\nu_*(G/U) = 1$ by Proposition 2.14.3, which gives the result in the commutative case.

In general, G/U is only a sheaf of cosets, but we can still make sense of the change-of-fibre

$$\overline{E} := E \times^G G/U := G\backslash(E \times G/U)$$

where the last term is the quotient of $E \times G/U$ with respect to the left-G-action via $g \cdot (e, x) := (eg^{-1}, gx)$. Then $\overline{E}$ is a fibre bundle on X_v with structure group G and fibres G/U. For this we can show:

Lemma 4.9. *Let X be a locally spatial diamond and let E be a G-torsor on X_v. Then the following are equivalent:*

(1) E admits a reduction of structure group to U.
(2) $\overline{E}$ admits a section $s \in \overline{E}(X)$.

When $U \subseteq G$ is a normal subgroup, these are furthermore equivalent to

(3) $\overline{E}$ is isomorphic to $X \times G/U$.

Proof. To see the implication (1) $\Rightarrow$ (2), let F be a U-torsor such that $E = F \times^U G$. Then

$$\overline{E} = F \times^U G \times^G G/U = F \times^U G/U = U\backslash(F \times G/U).$$

We claim that the composition

$$t : F \to E \to \overline{E}$$

is constant in the sense that it factors through an X-point $s : X \to \overline{E}$. To see this, it suffices to see that there is a basis of X_v of spaces Z such that the image of $t(Z)$ is a single element. But the above description of $\overline{E}$ shows that for any Z on which $F \simeq U$ is trivial, t becomes isomorphic to

$$U \to G \to G/U,$$

which has image 1. Hence t is locally constant, and therefore constant, as we wanted to see.

To see (2) ⇒ (1), let $F \subseteq E$ be the subsheaf defined as the fibre product

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "s"] & \overline{E} \end{tikzcd}$$

in v-sheaves on X_v. We claim that F is a reduction of structure group of E to U. To see this, we first observe that the right-action of U on E leaves the reduction map $E \to \overline{E}$ invariant. It follows that the right G-action on E restricts to a right U-action on F.

Let now $f : Y \to X$ be a v-cover by a strictly totally disconnected space on which E admits a trivialisation $\gamma : Y \times_X E = Y \times G$. This induces an isomorphism $\overline{\gamma} : Y \times_X \overline{E} = Y \times G/U$ that we can use to regard f^*s as a section of $G/U(Y)$. Using that $G/U(Y) = G(Y)/U(Y)$ since Y is strictly totally disconnected, we can now compose γ with an element in $G(Y)$ to ensure that $\overline{\gamma}(s) = 1$.

Then via γ, the pullback of the above diagram along $Y \to X$ is isomorphic to

$$\begin{tikzcd} Y \times_X F \arrow[r] \arrow[d] & Y \times G \arrow[d] \\ Y \arrow[r, "(\mathrm{id},1)"] & Y \times G/U. \end{tikzcd}$$

It follows that we have a U-equivariant isomorphism

$$Y \times_X F \xrightarrow{\sim} Y \times U$$

as the right hand side is also the fibre product. Thus F is a v-topological U-torsor on Y.

Since the natural map $F \to E$ is U-equivariant by construction, it now follows from the fact that any morphism between G-torsors is an equivalence (Proposition 3.16.(1)) that the map $F \times^U G \xrightarrow{\sim} E$ is an isomorphism, hence F is a reduction of structure group of E. □

It thus suffice to prove:

Lemma 4.10. *Let X be a sousperfectoid adic space and let E be a v-G-torsor on X. Then there is an étale cover $Y \to X$ over which $\overline{E}$ admits a section.*

Proof. We may without loss of generality replace U by a smaller rigid open subgroup. By Corollary 3.8, we may therefore assume that the adic space underlying U is isomorphic to the unit ball $\mathbb{B}^d$. In this case, we can argue similarly as in Proposition 2.14:

We first deal with the case that X is strictly totally disconnected. Let $Y \to X$ be a v-cover by an affinoid perfectoid space on which there is an isomorphism $\alpha : G \times Y \xrightarrow{\sim} E \times_X Y$. This induces a class $\psi \in G(Y \times_X Y)$ such that $\overline{E}$ can be described on any $Z \in X_v$ as the equaliser

$$\overline{E}(Z) \longrightarrow G/U(Y \times_X Z) \overset{\psi \cdot \pi_1^*}{\underset{\pi_2^*}{\rightrightarrows}} G/U(Y \times_X Y \times_X Z).$$

Approximating $Y \to X$ by sousperfectoid spaces $Y_i \hookrightarrow \mathbb{B}^n \times X$ as in Lemma 2.17, and using that $Y^{\times 2}_{/X} \approx \varprojlim_{i \in I} Y^{\times 2}_{i/X}$ by Lemma 2.19, it follows by the approximation property for G/U from Proposition 4.1 that

$$\overline{E}(Y) = \varinjlim_{i \in I} \overline{E}(Y_i).$$

This shows that the composition

$$Y \xrightarrow{1,\mathrm{id}} G/U \times Y \xrightarrow{\alpha} \overline{E} \times_X Y \to \overline{E}$$

factors through a morphism $y : Y_i \to \overline{E}$ for some i. Since $Y_i \to X$ is split, we thus obtain a section $X \to \overline{E}$. This proves the Lemma for strictly totally disconnected X.

The general case follows from this by a similar approximation argument: Let X be any affinoid sousperfectoid space and let $Y \to X$ be the quasi-pro-étale cover $Y \to X$ by a strictly totally disconnected space from Lemma 2.18, i.e. $Y \approx \varprojlim Y_i$ is a cofiltered tilde-limit of étale surjective maps $Y_i \to Y$. By the first part, there is an isomorphism $\alpha : G/U \times Y \xrightarrow{\sim} \overline{E} \times_X Y$. Again by the approximation property, we have $\overline{E}(Y) = \varinjlim \overline{E}(Y_i)$. ☐

This finishes the proof of Proposition 4.8 ☐

As a further application, the strategy of this section also gives the following useful results:

Proposition 4.11. *Let G be a rigid group and let $U \subseteq G$ be any open subgoup. Let X be a locally spatial diamond and let E be a G-torsor on X_v. Let $Y = \varprojlim_{i\in I} Y_i$ be a limit of locally spatial diamonds with qcqs transition maps and compatible maps $Y_i \to X$. If $E \times_X Y$ has a reduction of structure group to U, then there is $i \in I$ such that $E \times_X Y_i$ has a reduction of structure group to U.*

Proof. By Lemma 4.9, it suffices to prove that we have

$$\overline{E}(Y) = \varinjlim_i \overline{E}(Y_i).$$

To see this, let $\widetilde{X} \to X$ be any v-cover by a perfectoid space on which E becomes trivial and consider the fibre products $\widetilde{Y} := Y \times_X \widetilde{X}$ and $\widetilde{Y}_i := Y_i \times_X \widetilde{X}$ in the category of locally spatial diamonds (these exist by [Sch22, Corollary 11.29]). Then since fibre products commute with limits, we have $\widetilde{Y} = \varprojlim \widetilde{Y}_i$. By Proposition 4.1 and Lemma 2.20, it follows that we have

$$G/U(\widetilde{Y}) = \varinjlim_i G/U(\widetilde{Y}_i),$$

and similarly for $\widetilde{Y} \times_Y \widetilde{Y}$. Since $\overline{E} \simeq G/U$ over $\widetilde{X}$, and hence also over each $\widetilde{Y}_i$ and $\widetilde{Y}$, it follows from this that

$$\overline{E}(Y) = \check{H}^0(\widetilde{Y} \to Y, \overline{E}) = \varinjlim_i \check{H}^0(\widetilde{Y}_i \to Y_i, \overline{E}) = \varinjlim_i \overline{E}(Y_i),$$

as we wanted to see. ☐

Corollary 4.12. *Let X be any locally spatial diamond. Let S be any adic space over K. Let E be a G-torsor on $(X \times S)_v$. Let $\eta : \mathrm{Spa}(C, C^+) \to S$ be any morphism where C is a perfectoid field and assume that the pullback of E to $X \times \eta$ admits a reduction of structure group to U. Then there is an étale map $S' \to S$ whose open image contains $\mathrm{im}(\eta)$ such that E admits a reduction of structure group to U over $X \times S'$.*

Proof. We first assume that S is strictly totally disconnected, then by Lemma 2.17 we can choose an approximation of η of the form $\mathrm{Spa}(C, C^+) \approx \varprojlim S_i \to S$ where each $S_i \to S$ is split. Then $X \times \eta = \varprojlim X \times S_i$ in locally spatial diamonds, hence Proposition 4.11 shows that V admits a reduction of structure group to U on some $X \times S_i$. Since $S_i \to S$ is smooth, its image is an open in S, which is again strictly totally disconnected. Hence the map is split over its open image, so we obtain an open of S with the desired property.

We deduce the general case by considering the pro-étale strictly totally disconnected cover $\widetilde{S} \approx \varprojlim_i S_i \to S$ of Lemma 2.18: The map η then factors through a map $\widetilde{\eta} : \mathrm{Spa}(C, C^+) \to \widetilde{S}$. By the first part, there is an open $W \subseteq \widetilde{S}$ such that E admits a reduction of structure group to U over $X \times W$. After shrinking W, we may without loss of generality assume that it comes via pullback from some open $W_i \subseteq S_i$. Applying Proposition 4.11 once more, this shows that E becomes trivial over $X \times W_i$ for some open $W_i \subseteq S_i$. $\square$

4.2. **p-adic integral subgroups for rigid groups of good reduction.** In this subsection, we take a closer look at the structure of rigid open subgroups in the special case that G has good reduction. Throughout this subsection we fix a rigid group G of good reduction as well as a smooth formal model $\mathcal{G}$ of G over $\mathrm{Spf}(K^+)$. The Lie algebra $\mathfrak{g}^+$ of $\mathcal{G}$ is a finite projective K^+-module, hence free, and we can consider it as a canonical open K^+-lattice $\mathfrak{g}^+ \subseteq \mathfrak{g}$. As before, we also consider these as v-sheaves on K_v and write explicitly $\mathfrak{g}^+(K)$ and $\mathfrak{g}(K)$ for the underlying modules.

Lemma 4.13. *The sheaf G on $\mathrm{Sous}_{K,\text{ét}}$ is the analytic sheafification of the functor*

$$(R, R^+) \mapsto \mathcal{G}(R^+)$$

Proof. For any formal affine open $\mathrm{Spf}\, A \subseteq \mathcal{G}$, the adic generic fibre is by definition given by $(\mathrm{Spf}\, A)^{\mathrm{ad}}_\eta = \mathrm{Spa}(A[\frac{1}{p}], A^+)$ where A^+ is the integral closure of A in $A[\frac{1}{p}]$. Hence the points $G(R, R^+)$ correspond to the homomorphisms of K^+-algebras $A \to R^+$. $\square$

We can use this to give a natural generalisation of the sheaves $\mathrm{GL}_n(\mathcal{O}^+/p^k\mathfrak{m})$:

Definition 4.14. For any $0 \le k \in \log|K|$, consider the sheaf $\overline{G}_k$ on the big étale site of sousperfectoid spaces $\mathrm{Sous}_{K,\text{ét}}$ defined by étale sheafification of the presheaf on $\mathrm{Sous}_{K,\text{ét}}$

$$Y \mapsto \mathcal{G}(\mathcal{O}^+/p^k\mathfrak{m}\mathcal{O}^+(Y)).$$

We denote by $G_k \subseteq G$ the kernel of the morphism $G \to \overline{G}_k$.

For $k = 0$, the restriction to Perf_K of $\overline{G}_k$ is the sheaf $\overline{G}^\diamond$ studied in [Heu21b, §5].

Lemma 4.15. *The following sequence on $\mathrm{Sous}_{K,\text{ét}}$ is short exact:*

$$0 \to G_k \to G \to \overline{G}_k \to 0.$$

Proof. Left-exactness is clear by definition. It is right exact because $\mathcal{G}$ is formally smooth over K^+, so for any $k > 0$ and any affinoid (S, S^+) in $\mathrm{Sous}_{K,\text{ét}}$, the map $\mathcal{G}(S^+) \to \mathcal{G}(S^+/p^k\mathfrak{m})$ is surjective. For $k = 0$, we observe that after passing to an analytic cover, any $x \in \mathcal{G}(S^+/\mathfrak{m})$ lifts to $\mathcal{G}(S^+/p^\epsilon)$ for some $\epsilon > 0$ as $\mathcal{G}$ is locally of topologically finite presentation. $\square$

We can describe G_k in more classical terms as follows: Let $\widehat{\mathcal{G}}$ be the completion of $\mathcal{G}$ at the origin. This is a formal Lie group: Any local choice of generators of the sheaf of ideals defining the unit section in G induces an isomorphism of formal schemes

$$\widehat{\mathcal{G}} \cong \mathrm{Spf}(K^+[[T_1, \dots, T_d]]),$$

see [Mes72, p25-26]. In particular, on global sections, $\widehat{\mathcal{G}}$ defines a formal group law over K^+.

Lemma 4.16. *(1) G_0 is represented by the adic generic fibre $\widehat{\mathcal{G}}^{\mathrm{ad}}_\eta$ of $\widehat{\mathcal{G}}$.*

(2) The natural morphism $G_k \to G_0$ is isomorphic to the adic generic fibre of the morphism of affine formal schemes defined on global sections by

$$K^+[[T_1, \dots, T_d]] \to K^+[[T'_1, \dots, T'_d]], \quad T_i \mapsto p^k T'_i.$$

In particular, $G_k \subseteq G$ is represented by a normal rigid open subgroup whose underlying rigid space is isomorphic to an open ball.

Proof. Let $\mathrm{Spf}(A) \subseteq \mathcal{G}$ be any open neighbourhood of the unit section, and let $I \subseteq A$ be the ideal of the unit section. Then the functor $\widehat{\mathcal{G}}^{\mathrm{ad}}_\eta$ is the sheafification of the functor that sends (R, R^+) to the continuous homomorphisms $\varphi : A \to \varprojlim A/I^n = K^+[[T_1, \dots, T_d]] \to R^+$. By continuity, the images of the T_i land in the topologically nilpotent elements $\mathfrak{m}R^+$. Thus φ reduces mod $\mathfrak{m}$ to the unit section $A \to K^+/\mathfrak{m} \to R^+/\mathfrak{m}$. Conversely, if $A \to R^+$ factors mod $\mathfrak{m}$ through the unit section, it already does so mod p^ϵ for some $\epsilon > 0$ since A is of topologically finite presentation. Thus for all $n \in \mathbb{N}$, the map $A \to R^+ \to R^+/p^n$ factors through A/I^m for some m, and in the limit we get the desired map.

For part 2, we similarly observe that if a map $A \to R^+$ factors through $K^+[[T_1', \dots, T_d']]$, then it factors through the unit section after reducing mod $p^k\mathfrak{m}$. Conversely, if it factors through the unit section mod $p^k\mathfrak{m}$, then the induced morphism $K^+[[T_1, \dots, T_d]] \to R^+$ from the first part must send T_i into $p^k\mathfrak{m}R^+$ and thus factors through the displayed map.

That G_k is normal can be proved on the level of sheaves, where it follows from the short exact sequence in Lemma 4.15. □

Combined with our work in the previous sections, this shows:

Proposition 4.17. *The sheaf $\overline{G}_k$ on $\mathrm{Sous}_{K,\text{ét}}$ is already a v-sheaf. Moreover,*

$$R^1\nu_*\overline{G}_k = 1.$$

If G is commutative, we more generally have $R\nu_*\overline{G}_k = \overline{G}_k$.

Proof. By Lemma 4.15 and Lemma 4.16, we know that $\overline{G}_k = G/G_k$ is the quotient of the rigid group G by the normal open subgroup G_k. By Proposition 4.1, it follows that $\overline{G}_k$ is a v-sheaf with the desired properties. □

This allows us to pass from adic spaces to diamonds in the following: The rigid groups G and G_k represent sheaves on the site $\mathrm{Dmd}_{K,v}$ of diamonds over K with the v-topology. Due to Proposition 4.17 and Lemma 4.15, we may also extend $\overline{G}_k$ to a v-sheaf on the site $\mathrm{Dmd}_{K,v}$ with the v-topology by setting $\overline{G}_k := G/G_k$.

The system $(G_k)_{k\in\mathbb{N}}$ forms a neighbourhood basis of open subgroups and gives us a notion of completeness of G, generalising the isomorphism $\mathrm{GL}_n(\mathcal{O}^+) = \varprojlim_k \mathrm{GL}_n(\mathcal{O}^+/p^k\mathfrak{m})$:

Lemma 4.18. *The natural map $G \to \varprojlim_{k\in\mathbb{N}} \overline{G}_k$ is an isomorphism of v-sheaves.*

Proof. It is clear from Lemma 4.16 that $\varprojlim_{k\in\mathbb{N}} G_k = \cap_{k\in\mathbb{N}} G_k = 1$, so the map is injective. To see that it is surjective, suppose we are given a compatible system of elements $x_k \in G/G_k(R, R^+)$ for some perfectoid (R, R^+). Using that the v-site is replete, we can inductively find a v-cover $(R, R^+) \to (S, S^+)$ such that the x_k are in the image of $\mathcal{G}(S^+/p^k\mathfrak{m}) \to G/G_k(S, S^+)$ and form a compatible system of elements in $\mathcal{G}(S^+/p^k\mathfrak{m})$. On the level of formal schemes, this defines an element $x \in \mathcal{G}(S^+)$ whose image in $G(S, S^+)$ is a preimage of $(x_k)_{k\in\mathbb{N}}$. □

Remark 4.19. Due to the Lemma, one could now generalise Faltings' notion of generalised representations by defining "generalised G-representations" to be compatible systems of $\overline{G}_k$-torsors on the étale site. Due to Proposition 4.17 and Lemma 4.18, one then sees exactly like in Section 2 that these are equivalent to G-torsors on X_v.

We can now identify the subgroups that appeared in the context of the exponential with the integral subgroups G_k of this section in the context of good reduction:

Definition 4.20. For any $0 \le k \in \log|K|$, we denote by $\mathfrak{g}_k^+ \subseteq \mathfrak{g}$ the subsheaf $p^k\mathfrak{m}\mathfrak{g}^+$, represented by an open rigid subgroup of $\mathfrak{g}$. We then set $\overline{\mathfrak{g}}_k^+ := \mathfrak{g}^+/\mathfrak{g}_k^+$ on $\mathrm{Dmd}_{K,v}$, the v-site of diamonds. Since $\mathfrak{g}^+$ is free, this is isomorphic as an $\mathcal{O}^+$-module to $(\mathcal{O}^+/p^k\mathfrak{m}\mathcal{O}^+)^d$ where $d = \dim G$.

Lemma 4.21. *There is $\alpha \ge 1/(p-1)$ such that for any $\alpha < k \in \log|K|$, the exponential of G is defined on $\mathfrak{g}_k^+$ and restricts to an isomorphism of rigid spaces*

$$\exp : \mathfrak{g}_k^+ \xrightarrow{\sim} G_k.$$

Proof. By Proposition 3.5, the exponential $\mathfrak{g}^\circ \to G$ induces the identity on Lie algebras. It therefore corresponds to an isomorphism from $\mathfrak{g}_0^+$ to G_0 in the localised category of Theorem 3.4. Its morphism of Lie algebras being the identity means that the associated morphism of formal Lie groups over K is therefore of the form $F : K[[T_1,\dots,T_n]] \to K[[T_1,\dots,T_n]]$ with $F(T_i) = T_i +$ (terms of higher degree). As F converges on some disc centred at 0, this shows that after replacing T_i by T_i' on both sides via $T_i \mapsto p^k T_i'$ for some $k \in \log|K|$, $k \gg 0$, it restricts to $F : K^+[[T_1',\dots,T_n']] \to K^+[[T_1',\dots,T_n']]$. Passing to the adic generic fibre, we see from Lemma 4.16.2 that the left hand side becomes $\mathfrak{g}_k^+$ and the right hand side becomes G_k. Explicitly, we now define α to be the infimum of all $k > 1/(p-1)$ with this property. □

Definition 4.22. Recall that $\alpha_0 := 1/(p-1)$ if $p > 2$ and $\alpha_0 = 1/4$ otherwise. We denote by α the infimum of all $k \geq \alpha_0$ for which Lemma 4.21 holds. As before, we denote by log the inverse of exp.

Lemma 4.23. *For any $k > \alpha$, the adjoint action of G_k preserves $\mathfrak{g}_k^+$.*

Proof. This follows from Lemma 4.21 and Lemma 3.10. □

Lemma 4.24. *For any $\alpha < r < s \in \mathbb{Q}$ with $s \leq 2r - \alpha_0 \in \mathbb{Q}$, the exponential induces an isomorphism of abelian sheaves on $\mathrm{Dmd}_{K,v}$*

$$\exp : \mathfrak{g}_r^+/\mathfrak{g}_s^+ \xrightarrow{\sim} G_r/G_s.$$

In fact, we already have an isomorphism $\exp : \mathfrak{g}_r^+(X)/\mathfrak{g}_s^+(X) \xrightarrow{\sim} G_r(X)/G_s(X)$ for any $X \in \mathrm{Sous}_K$. We thus get a short exact sequence on $\mathrm{Dmd}_{K,v}$ (in fact, already on $\mathrm{Sous}_{K,\text{ét}}$)

$$0 \to \overline{\mathfrak{g}}_{s-r}^+ \xrightarrow{\exp} \overline{G}_s \to \overline{G}_r \to 1.$$

Remark 4.25. For $G = \mathrm{GL}_n$, this is the natural isomorphism

$$M_n(p^r\mathfrak{m}\mathcal{O}^+/p^s\mathfrak{m}) \xrightarrow{\sim} 1 + p^r\mathfrak{m}M_n(\mathcal{O}^+)/1 + p^s\mathfrak{m}M_n(\mathcal{O}^+), \quad x \mapsto 1 + x$$

which coincides with the exponential because the conditions on r and s ensure that

$$x^2/2! + x^3/3! + \dots \in p^s\mathcal{O}^+ \text{ for any } x \in p^r\mathcal{O}^+$$

by the usual estimate $v_p(x^n/n!) > nv_p(x) - \frac{n-1}{p-1}$. Similarly for linear algebraic groups.

Proof. By the commutative diagram in Proposition 3.5, BCH defines on $\mathfrak{g}_r^+$ a rigid group structure isomorphic to G_r. It suffice to prove that this agrees with the additive group structure on the quotient $\mathfrak{g}_r^+/\mathfrak{g}_s^+$. This is guaranteed by the following Lemma: □

Lemma 4.26. *For any $r > \alpha$, let $x, y \in \mathfrak{g}_r^+$. Then $\mathrm{BCH}(x,y)$ converges in $\mathfrak{g}_r^+$ and*

$$\mathrm{BCH}(x,y) \equiv x + y \bmod \mathfrak{g}_s^+$$

for any $r < s < 2r - \alpha_0$.

Proof. This follows from the estimates in [Sch11, (26) and Proposition 17.6]: We have $\mathrm{BCH} = \sum_{n\geq 1} H_n$ where $H_n(X,Y)$ is homogeneous of degree n, and $\|H_n\| \leq |p|^{-(n-1)\alpha_0}$. Let us define a valuation v_p on $\mathfrak{g}$ via $v_p(z) := \sup\{\log_{|p|}|a| : a \in K \text{ s.t. } z \in a\cdot\mathfrak{g}^+\}$ for $z \in \mathfrak{g}$. Then since $x, y \in \mathfrak{g}_r^+$ satisfy $v_p(x), v_p(y) > r$, it follows that for $n \geq 2$, we have

$$v_p(H_n(x,y)) > nr - (n-1)\alpha_0 = n(r - \alpha_0) + \alpha_0 \geq 2r - \alpha_0 \geq s.$$

Thus

$$\mathrm{BCH}(x,y) = x + y + H_2(x,y) + \dots \in x + y + p^s\mathfrak{m}\mathfrak{g}^+$$

is of the desired form. □

4.3. **G-torsors on perfectoid spaces.** Given our technical preparations, we can now generalise the results of Section 2. First, by a generalisation of Lemma 2.26, small G-torsors on affinoid perfectoid Z are trivial:

Lemma 4.27. *Let X be any diamond over K with $H^1_v(X, \mathfrak{m}\mathcal{O}^+/p^\epsilon\mathfrak{m}) = 0$ for any $\epsilon > 0$, for example any affinoid perfectoid space, and let $k > \alpha$. Then a G-torsor V on X_v is trivial if and only if the associated $\overline{G}_k$-torsor V_k on X_v is trivial. In particular,*

$$H^1_v(X, G_k) = 1.$$

Proof. We set $r := k > \alpha > \alpha_0$ and let $s \in \mathbb{R}$ be such that $r < s < 2r - \alpha_0$. We then apply $H^1_v(X, -)$ to the short exact sequence of v-sheaves in Lemma 4.24. Since $\overline{\mathfrak{g}}^+_{s-r} = (\mathfrak{m}\mathcal{O}^+/p^{s-r}\mathfrak{m})^d$, it follows from the assumption that $H^1_v(X, \overline{\mathfrak{g}}^+_{s-r}) = 0$. This shows that if V_k is trivial, then so is V_s. Replacing k by s and arguing inductively, this shows that V_s is trivial for any $s > k$. The same long exact sequence shows inductively that any element in $\overline{G}_k(X)$ can be lifted to $\overline{G}_s(X)$ for any $s > k$.

We then use that we have a Milnor exact sequence of pointed sets

$$1 \to R^1 \varprojlim_s \overline{G}_s(X) \to H^1_v(X, G) \to \varprojlim_s H^1_v(X, \overline{G}_s).$$

Since we have just seen that $(\overline{G}_s(X))_{s\in\mathbb{N}}$ is a Mittag-Leffler system, the first term vanishes. This shows that an element of $H^1_v(X, G)$ vanishes if and only if its image in $\varprojlim_s H^1_v(X, \overline{G}_s)$ vanishes, if and only if its image in $H^1_v(X, \overline{G}_k)$ vanishes. This shows the first part.

By Lemma 4.18, we can also deduce that the map $G(X) \to \overline{G}_k(X)$ is surjective. The last sentence then follows from the long exact sequence of $H^0_v(X, -)$ on Lemma 4.15. $\square$

We can now prove the main theorem of this article, generalising Kedlaya–Liu's Theorem 2.22 and Theorem 2.23, which were the cases of $G = \mathrm{GL}_n$ and $G = \mathrm{GL}_n(\mathcal{O}^+)$:

Theorem 4.28. *Let X be a perfectoid space and let G be any rigid group over K. Then the categories of G-torsors on $X_{\text{ét}}$ and G-torsors on X_v are equivalent.*

Proof. By Corollary 3.8, there exists a rigid open subgroup $U \subseteq G$ which has good reduction. By Proposition 4.8, the map $R^1\nu_*U \to R^1\nu_*G$ is surjective, so it suffices to prove that $R^1\nu_*U = 1$. We have thus reduced to the case that G has good reduction. By the same argument, we may then further replace G by the open subgroup G_k for $k > \alpha$. Then by Lemma 4.27, we have $R^1\nu_*G_k = 1$. $\square$

Corollary 4.29. *Let X be any adic space over K. Then the categories of G-torsors on $X_{\text{qproét}}$ and X_v are equivalent. Moreover, any geometric G-torsor on X_v is a diamond.*

Proof. The first part follows since by Lemma 2.18, the space X admits a quasi-pro-étale cover by a perfectoid space. For the second part, let E be a geometric G-torsor on X_v. The statement is étale-local on X, so we may assume that X has a pro-finite-étale perfectoid Galois cover $X' \to X$ such that $E \times_X X' \cong G \times X'$. Then the projection $G \times X' \to E$ is a pro-finite-étale cover of E by a sousperfectoid space. Hence E is a diamond. $\square$

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