Brief outline

[Unedited and Uncorrected. BVRao]

Cauchy-Riemann; [135-151]

holomorphic functions; more on curves and vector calculus; normal integral, Dini theorem; Uniform of convergence of integrals depending on parameter; continuity, change of order of integration, change of order of integration and differentiation; Heat equation [152-161]

Uniform convergence; continuity, integration; differentiation [162-168]

Home Assignment

1 (169-172) 2 (173-175) 3 (176-177)

4 (178-180) 5 (181-184) 6 (185-186)

7 (187-188) 8 (189-190) 9 (191-193)

10 (194-197)

Midsem 198 semestral (199-200).

Prelude:

Last semester we understood some aspects of the set R of real numbers —- rational numbers, irrational numbers, sequences, their convergence, series, absolute convergence, products of series and so on. We understood some aspects of functions defined on R taking values in R —- continuity, derivatives, power series, integration of bounded functions over bounded intervals, integration when the function is not bounded or when the interval is not bounded and so on.

As a by-product of the analysis we saw several interesting facts. Functions which have enough derivatives can be expanded in powers of x (Taylor expansion), Fundamental theorem on power series says they define continuous functions which can be differentiated term by term just as we do for polynomials, continuous function on a closed bounded interval can be uniformly approximated by a polynomial as close as desired (Weierstrass); $n!$ can be explained in simpler terms but using complicated numbers like π and e (Stirling); π can be expressed as a (infinite) product of simple ratios of integers (Walli) and so on.

We devised methods to compute simple integrals —- fundamental theorem of calculus allowed us to recognize differentiation and integration as inverse operations in a precise sense. This helped us to convert product rule of differentiation into 'integration by parts', chain rule of differentiation into 'method of substitution'. We devised methods to compute complicated integrals like $\int\limits_0^\infty \frac{\sin x}{x} dx$.

At the same time, you should keep in mind, that there are several problems that we have not discussed. For example, given a bounded interval, what exactly are the conditions for a bounded function to be Riemann integrable? What reasonable conditions are needed so that we can differentiate a series of functions term by term? And so on. Questions like the first one are theoretical in character. However questions like the second are of immense practical use. We shall discuss some of those later in this course.

It would be a good idea for you to do a quick review of what we did so far.

Euclidean spaces:

This semester we shall learn about functions of several variables. First we make clear to ourselves what we mean by 'several variables'. We shall consider the set

$$R^d = \{(x_1, \cdots, x_d) : \quad x_i \in R \quad 1 \le i \le d\}.$$

That is, the set of d-tuples of real numbers. Here $d \ge 1$ is an integer. Of course, the case $d = 1$ corresponds to the set of one-tuples, thus it is really no different from the set of real numbers. These spaces are called Euclidean spaces. R^d is called d-dimensional Euclidean space.

We have picturized R as a line; plotted all real numbers on the line —- after arbitrarily marking zero and marking one to its right (then nothing else is in our hands, other numbers have specific places on the line). Every number corresponds to a point on the line and every point on the line corresponds to a real number. In a similar fashion, we can picturize R^2 and you have already done so in high-school (actually we also did when we drew graphs of functions and calculated areas, but we did not stress this aspect). Here it is.

The paper or board is the picture corresponding to R^2. You draw two perpendicular lines: the horizontal line is called x-axis and vertical line is called y-axis. Their point of intersection is taken as the pair $(0, 0)$. Now you think of the two lines as copies of the real line. Plot all numbers on the horizontal line after fixing the place for 1. Similarly on the y-axis, plot all numbers. Just as we have fixed the right side of zero as positive numbers, we fix (just a convention, after all we have to follow something or the other) numbers 'above' zero to be positive on the y-axis. We follow the same units in both axes (it is pleasing).

The pair $(4, 3)$ is plotted on the paper as follows: Start from $(0, 0)$ move on the x axis to 4, then move three units up, mark this point as $(4, 3)$. Similarly, every pair $(a, b) \in R^2$, whether the numbers a and b are positive or not, is identified with a point on the paper. Conversely, every point on the paper corresponds to a point in R^2.

We could not only picturize, but also 'draw' R and R^2. You can not draw R^3 but can imagine as follows. Think of three lines from where you stand: Two lines on the floor they are x-axis and y-axis and the third line is yourself.

You are standing at $(0,0,0)$. If you want to plot the point $(4,7,3)$ go four units right on the floor and from there go seven units to the front and from there go up three units above, that is the point $(4,7,3)$. Of course if the z coordinate is negative, the point is on the other side of the floor!

The reason for this detailed discussion is that you should start making mental pictures for the cases $d = 2, 3$. We consider $d > 1$. You can read our analysis by thinking $d = 1$ too and it remains true. But we said $d > 1$ because we shall be explaining all the concepts using what we already know about real numbers, namely, the case $d = 1$.

We would like to now understand sequences. A sequence is a function defined on natural numbers with values in R^d. Instead of thinking of it as a function f we think of a sequence by its values at one, at two and so on. Think of them as the first term, second term etc of the sequence. We write the sequence , as earlier, as (x^n) or $(x^n : n \geq 1)$ or $(x^n)_{n\geq 1}$. We are using n as super fix, rather than suffix. This is because we used suffix to denote coordinates. Thus $x^n = (x_1^n, x_2^n, \cdots, x_d^n)$.

Incidentally, there is nothing new in the concept of function. In fact we discussed functions from a set A to a set B. But just keep in mind the following. If $f(x) = \pm\sqrt{x}$ on the interval $(0,1)$ then it is *not* a function. However $g(x) = +\sqrt{x}$ and $h(x) = -\sqrt{x}$ are functions. of course, we made a convention that $\sqrt{x}$ means $+\sqrt{x}$ (just as, 2 means +2).

Returning to sequences and paralleling the earlier development, we wish to say that a sequence converges to a point $x \in R^d$ if the terms of the sequence are getting closer to the point x. Thus we first need to understand what is meant by 'close'.

Norm and Distance:

We define norm on R^d by

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}; \quad x = (x_1, x_2, \cdots, x_d).$$

Thus, square the coordinates and add them up and then take squareroot. Of course, if $d = 1$ this turns out to be just the familiar modulus. Thus, sometimes we would write $|x|$ instead of $||x||$. We think of norm as the distance of the point from the origin 0, the point with all coordinates zero. The reason for this definition comes from Pythagoras theorem. Imagine the case

$d = 2$. If you have a point (x, y), you can make the right angled triangle $(0, 0); (x, 0); (x, y)$ and apply Pythagoras theorem.

As you have noticed, we denoted point in R^2 by (x, y) and not (x_1, x_2). This is how we are used to in school and so we continue. But this has the disadvantage that unless you are alert, you may think that $x = (x_1, x_2)$, $y = (y_1, y_2)$ are two points rather than understanding that x y are real numbers and the pair (x, y) is the point we are talking about.

In R we felt that the distance from 4 to 10 is same as the distance from 0 to $10 - 4 = 6$. Same philosophy we adapt here too. For two points $u, v \in R^d$, the distance between them is

$$d(u, v) = ||u - v|| = \sqrt{\sum_1^d (u_j - v_j)^2}.$$

Squaring is complicated operation where as linear operations are simple to understand and manipulate. It is pleasing to note that the norm is indeed driven by a linear operation. Define

$$u \cdot v = \sum_j u_j v_j; \quad u, v \in R^d.$$

This is called inner product between u and v, sometimes also denoted $\langle u, v\rangle$. It is linear in each argument when the other argument is fixed. Now $||u||$ is nothing but $u \cdot u$. You are probably familiar with these concepts from your linear algebra course. In fact we do need all that material as we go along. We have already used the vector space structure when we used $u - v$ above.

Here are some properties of norm and inner product.
Theorem 1:
(**A**) Norm has the following properties. Here $u, v, w \in R^d$ and $\alpha \in R$.
(i) $||u||$ is a real number; $||u|| \geq 0$; $||u|| = 0$ iff $u = 0$.
(ii) $|u \cdot v| \leq ||u||||v||$ (Cauchy-Schwarz inequality)
(iii) $||\alpha u|| = |\alpha|||u||$ and $||u + v|| \leq ||u|| + ||v||$

(**B**) Distance satisfies the following
$d(u, v) \geq 0$ (positive);
$d(u, v) = d(v, u)$ (symmetric);
$d(u, v) = 0 \leftrightarrow u = v$:
$d(u, v) \leq d(u, w) + d(w, v)$ (triangle inequality)
$d(u + w, v + w) = d(u, v)$ (translation invariant)

$d(\alpha u, \alpha v) = \alpha d(u, v)$ if $\alpha > 0$.

All the properties can be easily verified. We only recall proof of (ii). If $u_1, \cdots, u_k$ and $v_1, \cdots, v_k$ are non-negative real numbers then the following quadratic in λ is always non-negative,

$$\sum (u_j - \lambda v_j)^2 = (\sum v_j^2)\lambda^2 - (2 \sum u_j v_j)\lambda + (\sum u_j^2) \geq 0.$$

The fact that its discriminant is nonpositive is precisely the C-S inequality.

To get familiarity with distance, let us get a feel for the following. Given a point u what are all the points which are at a distance smaller than one from u? In case of real line, we have already noted that this distance is same as $|u - v|$. Thus given a number u, the set of points that are at a distance smaller than 1 is just the interval $(u - 1, u + 1)$.

In case $u \in R^2$, your high school familiarity tells you that this set consists precisely points in the interior of the circle with radius 1 and centered at u. In case $u \in R^3$ it is the set of all points in the interior of the sphere with radius one and centered at u. In a sense, this is the meaning of circle and sphere.

convergence of sequences:

Returning to sequences, we say that a sequence (x^n) of points in R^d converges to a point x in R^d, in case the points of the sequence are getting closer and closer to the point x. More precisely, given $\epsilon > 0$, there is a N such that $d(x^n, x) < \epsilon$ for all $n \geq N$. We denote this by $x^n \to x$.

It is useful to explain this notion in terms of sequences of real numbers familiar to us.

Theorem: Let (x^n) be a sequence in R^d and $a \in R^d$. Then the following are equivalent.

(i) $x^n \to a$.

(ii) given $\epsilon > 0$, there is N such that $|x_j^n - a_j| < \epsilon$ for each j; $1 \leq j \leq d$ and for each $n \geq N$.

(iii) For each j, $x_j^n \to a_j$.

Most of you could guess the proof, I suggest you practice writing proof.

This will immediately tell us several things. If $u^n \to u$ and $v^n \to v$, then $(u^n + v^n) \to (u + v)$. Also if we have real numbers $\alpha^n \to \alpha$, then $\alpha^n u^n \to \alpha u$.

In other words the vector space operations are respected by this notion of convergence.

Thus for example the sequence $(1/n, 1 + 2^{-n}) \to (0, 1)$.

The sequence $(e^{-n} \sin n, e^{-n} \cos n) \to (0, 0)$. You plot this sequence and see that it spirals around the origin, getting closer to $(0, 0)$. Intuitively speaking, it is not heading in any fixed direction.

continuous functions:

A function $f : R^d \to R$ is continuous at a point a if for points close to a functional values are close to $f(a)$. More, precisely, given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. A function is continuous if it is continuous at every point.

The concept of continuity makes sense, and is useful, for functions not necessarily defined on all of R. suppose f is defined on a set $S \subset R^d$ and $a \in S$. We say that f is continuous at a if the following happens: given $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \in S$.

Just as in the case of real line, we have the following result.

Theorem: Let f be defined on $S \subset R^d$ and $a \in S$. the following are equivalent.

(i) f is continuous at a.

(ii) $f(x_n) \to f(a)$ whenever (x_n) is a sequence of points in S and $x_n \to a$.

You should be careful, superficial appearance may be deceptive. Consider the function

$$f(x, y) = \frac{xy}{x^2 + y^2}; \quad (x, y) \neq (0, 0); \quad f(0, 0) = 0.$$

If you fix any number x, say, 4 or π or zero, whatever, then $y \mapsto f(\pi, y)$ is a function of one variable. Similarly when you fix a value of y, say, $\sqrt{2}$, then $x \mapsto f(x, \sqrt{2})$ is a function of one variable. In the present example these are all continuous functions. However the function f is not a continuous function on R^2. Reason: calculate $f(1/n, 1/n)$ and see.

Does the intermediate value theorem hold? Yes. Suppose f is a function defined on all of R^d. Let $f(a) = 4$ and $f(b) = 20$. Is there a point c such that $f(c) = 15$? Yes. Join a and b by straight line. In other words consider the set $\{\lambda a + (1-\lambda)b : 0 \le \lambda \le 1\}$. This is not part of real line, but none-the-less looks like a line segment and there must be a point on this line with value of f as desired. In fact, if we define $g(\lambda) = f(\lambda a + (1-\lambda)b)$, then g is a continuous function on the interval $[0,1]$ and from what you have learnt earlier, there must be a number λ with $g(\lambda) = 15$ giving what we wanted.

But what happens if the function is not defined on all of R? If you can draw paths joining points, then the result should be true. Yes. We take this occasion to develop some set theoretic terminology.

Open, Closed, Connected sets:

For a point $a \in R^d$ and a real number $r \ge 0$, we put

$$B(a,r) = \{x \in R^d : ||x-a|| < r\}. \quad \overline{B}(a,r) = \{x \in R^d : ||x-a|| \le r\}.$$

These are called the open ball and closed ball respectively, with center a and radius r. As already seen earlier, in R this amounts to the intervals $(a-r, a+r)$ and $[a-r, a+r]$ respectively. In case of R^2, $B(a,r)$ is precisely points inside the circle with centre a radius r. And $\overline{B}(a,r)$ is precisely set of points inside as well as points on the circle.

A set $V \subset R^d$ is open if whenever $a \in V$, some space around a is also in the set V. More precisely, there is an $r > 0$ such that $B(a,r) \subset V$. For example in R, the set of rational numbers, the set $[0,1]$ are not open where as the set $(0,1)$ is open.

Clearly union of any number of open sets is open. In fact if a point a is in the union, then it must be in one of those sets and then some ball around is contained in that set and hence in the union. Also intersection of finitely many open sets is open. In fact if a is in the intersection then it is in all of them so some ball around a is contained in each of them, take minimum of those finitely many radii to see that this ball around a is contained in the intersection.

Each of the sets $(-1/n, +1/n)$ are open sets and their intersection is just the singleton $\{0\}$ which is not an open set.

Let $S \subset R^d$ and $a \in R^d$. We say that a is a limit point of the set S if every $B(a,r)$ contains a point of S other than a. We say that a set $C \subset R^d$ is closed if all limit points of S are in S. In other words the set is 'closed under limits'.

Just as in case of R we can show that point a is a limit point of S iff there is a sequence (x_n) such that each $x_n \neq a$ and $x_n \to a$.

A set is closed iff its complement is open. Let us prove this. Let A be the set. suppose A is open. If $a \in A$ then there is $r > 0$ such that $B(a, r) \subset A$. In other words, it has no point of A^c. Thus if you take a point of A it can not be a limit point of A^c. As a result all limit pouts of A^c are already in A^c. Thus A^c is closed.

Conversely, let us assume that A^c is closed. To show that A is open, fix $a \in A$. If we can not show $r > 0$ such that $B(a, r) \subset A$, it means that every $B(a, r)$ contains points of A^c. This is precisely the statement that a is a limit point of A^c, see definition. But then A^c does not contain its limit point a, contradicting that A^c is closed.

This shows that closed sets are precisely complements of open sets. There are sets which are neither closed nor open. For example consider the set $[0, 1)$.

A set $S \subset R^d$ is connected if whenever you cut it into two pieces, there is a point which is on the boundary of both pieces. More precisely, if $S = A \cup B$ where $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ then there is a point $x \in S$ such that x is a limit point of A and also x is a limit point of B.

For example very interval contained in R is connected and actually these are the only connected sets in R. Recall, interval means whenever there are two numbers in the set, everything in between is also in the set.

Returning to intermediate value property, here is the precise result.

Let $S \subset R^d$ be a connected set and $f : S \to R$ be continuous. Then f has intermediate value property.

Indeed, let $\alpha < \gamma < \beta$ be three numbers and α, β are both in the range of f. let us define

$$A = \{x \in S : f(x) \leq \gamma\}, \quad B = \{x \in S : f(x) \geq \gamma\}$$

Since α and β are in the range of f it is clear that these sets are non-empty. Also $S = A \cup B$.

The set A contains all its limit points. Indeed let $x_n \in A$ for all n and $x_n \to x$. Then $f(x_n) \leq \gamma$ for all n and $f(x_n) \to f(x)$. Hence $f(x) \leq \gamma$. Thus

$x \in A$. Similarly B contains all its limit points.

Thus there is no point of S which is a limit point of both A and B. In case $A \cap B = \emptyset$ we have a contradiction for the connectedness of S. Thus there is a point common to both A and B. Clearly value of f at such a point equals γ. This proves the intermediate value property.

The argument that we gave earlier using paths is also interesting and let us show the following interesting fact. Suppose that V is a connected open set. Then any two points of V can be connected by a path which lies entirely in V. We have seen several open sets, U-shaped; star shaped, open sets with holes and so on where it was tricky to join points by a path. The fact that we can do at all is a miracle and definitely needs proof. Also, this involves absolutely no complicated maths. If you omit 'open' it is no longer true in general. Of course if you omit 'connected' it is never true.

Well, what do we mean by path? A path is a continuous function $\gamma : [0,1] \to R^d$. This is called path joining the points $\gamma(0)$ and $\gamma(1)$. The path lies entirely in a set if all values of γ belong to that set.

Returning to our problem, let V be an open connected set. Let us agree temporarily to say $x \sim y$ if we can join x and y by a path entirely lying in V. Fix any point $a \in V$ you like. Let

$$A = \{x \in V : x \sim a\}; \quad B = \{x \in V : x \not\sim a\}.$$

Clearly A and B are disjoint. Also A is non-empty. Indeed, using the fact that V is open take $r > 0$ with $B(a,r) \subset V$. Every point in this ball can be joined to a by straight line which lies entirely in V. Further $A \cup B = V$.

We show A is open. Let $x \in A$. Take $r > 0$ such that $B(x,r) \subset V$, possible because V is open. Every point of this ball can be joined to a because you join to x and then draw a straight line from x to the point in the ball. In other words, this entire ball is contained in A. Thus A is open. In particular, no point of A can be limit point of B.

We show that B is open. Let $x \in B$. Take $r > 0$ with $B(x,r) \subset V$. If any point of this ball can be joined to a, then we can join that end point to x by straight line contradicting the fact that x can not be joined to a. In other words this entire ball is contained in B. Thus B is open. In particular, no point of B can be limit point of A.

If both the sets A, B are non-empty, then connectedness of V is contradicted. Since A is already known to be non-empty, we conclude that B must be empty. This proves our result.

There are many interesting properties and facts about these three concepts; open, closed, connected sets. We can not afford to spend much time on these matters; even if we do, some of you may find it rather abstract. We pick them up when needed.

Differentiation:

The idea of derivative is to understand the rate at which a function is changing; or to find the best linear function that approximates 'near' a given point or to understand velocity etc.

Let $f : R^2 \to R$ be given and $a = (a_1, a_2) \in R^2$.

Remembering our expertise on R, we can try to get functions of one variable using f. This is easily done. We can restrict the function to the horizontal line and study. Thus put,

$$g(x) = f(x, a_2).$$

This is a function of one variable. If it has a derivative at the point a_1, it is reasonable to feel that this is the rate of change at a when you consider the horizontal line, that is, the direction of x-axis. If so, then we look at

$$\lim_{h \to 0} \frac{f(a_1 + h, a_2) - f(a_1, a_2)}{h} = f_x(a_1, a_2).$$

When this limit exists, it is called partial derivative of f w.r.t. x at the point a. This has several notations

$$f_x(a); \quad \frac{\partial}{\partial x} f(a); \quad \frac{\partial f}{\partial x}(a); \quad f_1(a); \quad D_x f(a).$$

Or, one could try to understand the function along the y-axis; in other words, see if the function $y \mapsto f(a_1, y)$ has a derivative at a_2. It is reasonable to feel that this is the rate of change at a when you consider the vertical line, that is the direction of y-axis. If so, then we consider

$$\lim_{k \to 0} \frac{f(a_1, a_2 + k) - f(a_1, a_2)}{k} = f_y(a_1, a_2).$$

This is called (when it exists) partial derivative of f w.r.t. y at the point a. This has several notations

$$f_y(a); \quad \frac{\partial}{\partial y} f(a); \quad \frac{\partial f}{\partial y}(a); \quad f_2(a); \quad D_y f(a).$$

Interestingly, even if these partial derivatives, vertical and horizontal, boh exist at a point, the function need not be continuous. Try the example given earlier $xy/(x^2 + y^2)$. Moreover, there are several other directions, lines passing through the given point and we can talk about the rate of change along all those lines. Situation appears unmanageable.

Let us see what we are lead to if we take the other attitude: what is the best linear fit?

Using our expertise with functions of one variable, we have defined partial derivatives; rate of change in the two directions: horizontal and vertical or equivalently in the x-direction and y-direction respectively.

There are several peculiarities that take place. We shall just try to understand some so that they serve as warning. However, the main focus of our course is to develop smooth calculus and not really to spend time on pathologies that can occur. They are also interesting, but not part of our course.

Conditions on f_1, f_2 for continuity of f:

Let f be a function defined on an open set $\Omega \subset R^2$ and $f : \Omega \to R$ and $a = (a_1, a_2) \in \Omega$. Then

$$D_x f(a) = \lim_{h \to 0} \frac{f(a_1 + h, a_2) - f(a_1, a_2)}{h}$$

and

$$D_y f(a) = \lim_{k \to 0} \frac{f(a_1, a_2 + k) - f(a_1, a_2)}{k}$$

whenever these limits exist. However existence of these derivatives does not even imply that the function is continuous at the point a. For example the function

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0); \quad f(0, 0) = 0.$$

has partial derivatives at the point $(0, 0)$ but is not continuous at that point.

Here is one condition when the existence of these partial derivatives implies continuity. Suppose that the partial derivatives are bounded in Ω. Then the functioc is continuous in Ω.

Theorem: if the partial derivatives are bounded in Ω then f is continuous in Ω.

Obviously, you see that, as in the one dimensional case, continuity is a local property. Recall, though unnecessary, this means the following: if f

and g are defined in Ω and both agree in $B(a,r)$ for some $r>0$, then continuity of f at a is equivalent to continuity of g at a. In particular, if $a \in \Omega$ and if there is a number $r>0$ such that f is continuous in $B(a,r)$ then f is continuous at the point a. Thus if the partial derivatives are bounded in $B(a,r)$ for some $r>0$, then f is continuous at a.

The proof of the theorem is simple. Take $a \in R$ and $\epsilon > 0$. First fix $M>0$ a bound for $|f_x|$ and $|f_y|$. Fix a small rectangle or square, say, $Q = (a_1-\delta, a_1+\delta) \times (a_2-\delta, a_2+\delta) \subset \Omega$ with $\delta < \epsilon/2M$. We show that

$$x \in Q \Rightarrow |f(x)-f(a)| < \epsilon.$$

we can write $x=(x_1,x_2)$ where $x_1 = a_1+h_1$ and $x_2 = a_2+h_2$ with $|h_1|<\delta$ and $|h_2|<\delta$

$$|f(x)-f(a)| \le |f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2)| + |f(a_1+h_1, a_2) - f(a_1, a_2)|.$$

Note that all the points that appear on the right side are in the rectangle Q and hence so are the lines joining the first two points (vertical line) and the last two points (horizontal line). The mean value theorem that we learnt for functions of one variable tells us that the first term on right side equals $|h_2||f_y(P_1)|$ for some point P_1 on the vertical line. But this is smaller than $(\epsilon/2M)M = \epsilon/2$. similarly the second term is also smaller than $\epsilon/2$. This completes the proof of continuity.

equality of mixed derivatives:

It is also possible that f has partial derivatives everywhere giving us the functions f_x and f_y everywhere. It is also quite possible that they again have partial derivatives

$$f_{xx} = (f_x)_x, \quad f_{xy} = (f_x)_y, \quad f_{yx} = (f_y)_x, \quad f_{yy} = (f_y)_y.$$

One would like to have $f_{xy} = f_{yx}$, that is the order in which we differentiate should not matter. But it is a peculiarity that this may not happen. For example, if you take the function

$$f(x,y) = \frac{x^3y - y^3x}{x^2+y^2}, \quad (x,y) \neq (0,0); \quad f(0,0) = 0.$$

Then

$$f_1(x,y) = \frac{3x^2y - y^3}{x^2+y^2} - \frac{(x^3y - y^3x)2x}{(x^2+y^2)^2}, \quad (x,y) \neq (0,0);$$

and

$$f_1(0,0) = 0.$$

$$f_2(x,y) = \frac{x^3 - 3y^2x}{x^2+y^2} - \frac{(x^3y - y^3x)2y}{(x^2+y^2)^2}, \quad (x,y) \neq (0,0);$$

and

$$f_2(0,0) = 0.$$

$$f_{12}(0,0) = \lim_{k\to 0} \frac{f_1(0,k) - f_1(0,0)}{k} = -1.$$

while

$$f_{21}(0,0) = \lim_{h\to 0} \frac{f_2(h,0) - f_2(0,0)}{h} = +1.$$

This is rather unpleasant.

However if f_{12} and f_{21} are continuous, then they are equal.

Before you loose track where we are heading, let me repeat the following. the concept of derivative is rather delicate, if you are not careful. Many unpleasant things happen. We are seeing those and along the way try to remedy the situation; if you assume suitable hypothesis then it does not happen. For every ill, if you use separate hypothesis to rectify it then life becomes difficult. We soon get out of all these warning signals and assume *one* nice hypothesis that allows good things to happen.

Returning to the problem we are discussing, we shall now show that if f_{12} and f_{21} are continuous then they are equal. So let us take a point $a \in \Omega$. Need to show $f_{12}(a) = f_{21}(a)$. You fix any $\epsilon > 0$. We show that

$$|f_{12}(a) - f_{21}(a)| < \epsilon \qquad (\diamondsuit).$$

You agree that this would suffice.

Using continuity of the 'mixed' derivatives, fix a rectangle, or square, $Q = (a - \delta, a + \delta) \subset \Omega$ such that

$$x \in Q \Rightarrow |f_{12}(a) - f_{12}(x)| < \epsilon/2; \quad |f_{21}(a) - f_{21}(x)| < \epsilon/2. \quad (\heartsuit)$$

Here we have used an abbreviation:

$$(a - \delta, a + \delta) = (a_1 - \delta, a_1 + \delta) \times (a_2 - \delta, a_2 + \delta).$$

Let us fix $0 < h < \delta$ and consider the quantity

$$\Delta = f(a_1 + h, a_2 + h) - f(a_1 + h, a_2) - f(a_1, a_2 + h) + f(a_1, a_2).$$

The plan is to show

$$(\exists\, P_1, P_2 \in Q) \quad f_{12}(P_1) = \Delta = f_{21}(P_2). \qquad (\dagger)$$

If this is done then ($\heartsuit$) with $x = P_1$ in the first inequality and $x = P_2$ in the second inequality will give us ($\diamondsuit$) as required.

Towards proof of ($\dagger$), consider the functions of one variable,

$$\varphi(x) = f(x, a_2 + h) - f(x, a_2); \qquad \Psi(y) = f(a_1 + h, y) - f(a_1, y).$$

Apply mean value theorem to get $0 < \theta_1 < 1$ and $0 < \eta_2 < 1$ such that

$$\varphi(a_1 + h) - \varphi(a_1) = h\varphi'(a_1 + \theta_1 h); \quad \Psi(a_2 + h) - \Psi(a_2) = h\Psi'(a_2 + \eta_2 h).$$

Observe

$$\varphi(a_1 + h) - \varphi(a_1) \;=\; \Delta \;=\; \Psi(a_2 + h) - \Psi(a_2).$$

and also that

$$\varphi'(x) = f_1(x, a_2 + h) - f_1(x, a_2); \qquad \Psi'(y) = f_2(a_1 + h, y) - f_2(a_1, y).$$

Thus

$$h[f_1(a_1 + \theta_1 h, a_2 + h) - f_1(a_1 + \theta_1 h, a_2)] = \Delta,$$

and

$$h[f_2(a_1 + h, a_2 + \eta_2 h) - f_2(a_1, a_2 + \eta_2 h)] = \Delta.$$

Now applying mean value theorem to the left sides of the two equations above we get $0 < \theta_2 < 1$ and $0 < \eta_1 < 1$ such that

$$h^2 f_{12}(a_1 + \theta_1 h, a_2 + \theta_2 h) = \Delta = h^2 f_{21}(a_1 + \eta_1 h, a_2 + \eta_2 h).$$

Take

$$P_1 = (a_1 + \theta_1 h, a_2 + \theta_2 h), \quad P_2 = (a_1 + \eta_1 h, a_2 + \eta_2 h),$$

to complete the proof of ($\dagger$).

differentiability:

As discussed last time, let us turn to the problem of understanding a function near a point — either in the sense of geometry like drawing a tangent plane at that point or in the sense of approximating by simpler functions.

Let $f : \Omega \to R$ where $\Omega \subset R^2$ is an open set and $a \in \Omega$. What is the simplest function that approximates f near a. First we need to make it precise. we are looking for a function $\varphi(x)$ on Ω so that $f(x) - \varphi(x) \to 0$ as x approaches a, that is as $||x - a|| \to 0$. The answer is simple, take the function whose value at every point is the number $f(a)$. Then this is the simplest function we can think of and it satisfies the requirement.

Next, as in the case of R, let us demand that $\varphi(a) = f(a)$ and $f(x) - \varphi(x) \to 0$ faster than $||x - a||$. So what is meant by faster. well even the ratio $|f(x) - \varphi(x)|/||x - a|| \to 0$ as $||x - a|| \to 0$. But we are now allowed linear functions. What are linear functions? Just like in the case of R, they are of the form $\varphi(x) = a_1x_1 + a_2x_2 + b$ for some numbers a_1, a_2, b. This can be succinctly expressed as $\varphi(x) = \alpha \cdot x + \beta$ where $\alpha \in R^2$ and dot is the inner product.

If that happens, then

$$f(a) = \varphi(a) = \alpha \cdot a + \beta; \quad \text{that is} \quad \beta = f(a) - \alpha \cdot a.$$

Thus the function is $\varphi(x) = \alpha \cdot x + f(a) - \alpha \cdot a$. Of course, we still do not low what is the vector α. It should satisfy, $|f(x) - \varphi(x)|/||x - a|| \to 0$ as $||x - a|| \to 0$.

Let us agree to say that the function f is differentiable at the point a in case there is a vector $\alpha \in R^2$ such that

$$\lim_{||x-a||\to 0} \frac{f(x) - f(a) - \alpha \cdot (x - a)}{||x - a||} \to 0. \qquad (\bullet)$$

In such a case, we can refer to the vector α as the derivative of f at a.

Let us see what could be the vector α. Taking the sequence $x_n = (a_1 + \frac{1}{n}, a_2)$ we see that $\alpha_1 = f_1(a)$. Similarly, taking $x_n = (a_1, a_2 + \frac{1}{n})$ we see $\alpha_2 = f_2(a)$.

Thus if the function is differentiable then

$$\alpha = (f_1(a), f_2(a)).$$

But is this enough, that is, if the partial derivatives exist then will $(\bullet)$ hold? Not always.

In fact, existence of partial derivatives need not even imply that the function is continuous. We shall return to this in a minute, but let us also see the geometric picture of the derivative.

Just as the graph $\{(x, y) : y = mx + c\}$ of a linear function $\varphi(x) = mx + c$ is a straight line (never mind, we have missed y-axis by loosely representing line as above), the graph $\{(x, y, z) : z = ax + by + c\}$ of linear function $\varphi(x, y) = ax + by + c$ is a plane. Just as graphs of functions on R to R are called curves in the plane R^2, graphs of functions from R^2 to R are called surfaces in R^3.

How do you imagine surfaces. Imagine the ground to be the plane and think of a tent that has height $f(x, y)$ at the point (x, y) on the ground. You can imagine it as a tent or bowl or inverted bowl etc, whatever you are comfortable with. of course specific functions have specific shapes.

A tangent plane to the surface $z = f(x)$ at the point $a = (a_1, a_2) \in \Omega$ is the graph of a map $\varphi(x) = \alpha \cdot x + \beta$ that passes through the point $(a, f(a)) \in R^3$ on the surface but makes 'stronger' contact with the surface than the constant function $\Psi(x) \equiv f(a)$. What does this mean? We mean, the ratio $[f(x) - \varphi(x)]/||x - a|| \to 0$ as $x \to a$; not simply $f(x) - \varphi(x) \to 0$.

This concept of tangent plane again leads to the same conclusion as earlier, namely solving $(\bullet)$ for vector α and you end up with the same answer as above.

continuity of f_1, f_2 implies differentiability:

We saw that existence of partial derivatives does not imply $(\bullet)$ holds. However if the partial derivatives are continuous then the function is differentiable and at any point a, the vector $(f_1(a), f_2(a))$ is indeed the derivative.

Theorem: Let $\Omega \subset R^2$ be open set and $f : \Omega \to R$ be such that the partial derivatives f_1 and f_2 are continuous. Then f is differentiable and derivative of f at a point $a \in \Omega$ is the vector $(f_1(a), f_2(a))$.

Fix $a \in \Omega$ and denote $\alpha = (f_1(a), f_2(a))$. Need to show

$$\lim_{||x-a|| \to 0} \frac{f(x) - f(a) - \alpha \cdot (x-a)}{||x-a||} \to 0.$$

Fix $\epsilon > 0$. Using continuity of f_1 and f_2 choose $\delta > 0$ so that $B(a, \delta) \subset \Omega$ and

$$||x-a|| < \delta \Rightarrow |f_1(x) - f_1(a)| < \epsilon/2; \quad |f_2(x) - f_2(a)| < \epsilon/2.$$

Let us take any point $x = (x_1, x_2) \in B(a, \delta)$. Then using the mean value theorem (for functions of one variable) get points P_1 and P_2 on the appropriate horizontal and vertical line segments such that

$$f(x) - f(a) = [f(x_1, x_2) - f(a_1, x_2)] + [f(a_1, x_2) - f(a_1, a_2)]$$

$$= (x_1 - a_1) f_1(P_1) + (x_2, a_2) f_2(P_2) = (x-a) \cdot v$$

where v is the vector $v = (f_1(P_1), f_2(P_2)$. Thus

$$|f(x) - f(a) - \alpha \cdot (x-a)| = |(x-a) \cdot (v - \alpha)|$$

$$\leq ||x-a|| \, ||v - \alpha|| \leq ||x-a|| \sqrt{2\epsilon^2/4} \leq \epsilon ||x-a||.$$

Thus for $x \in B(a, \delta)$ we have

$$\left| \frac{f(x) - f(a) - \alpha \cdot (x-a)}{||x-a||} \right| < \epsilon.$$

proving the result.

differentiability $\not\Rightarrow$ f_1, f_2 conitnuous:

We have shown that if the partial derivatives f_1, f_2 are continuous then f is differentiable. However differentiability is not equivalent to the continuity of partial derivatives.

Let us consider the function,

$$f(x,y) = (x^2 + y^2) \sin \frac{1}{x^2 + y^2}, \quad (x,y) \neq (0,0); \qquad f(0,0) = 0.$$

Then

$$f_1(0,0) = \lim_{h \to 0} \frac{f(h,0) - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h^2} = 0.$$

When $(x, y) \neq (0, 0)$

$$f_1(x, y) = 2x \sin \frac{1}{x^2 + y^2} - \frac{2x}{x^2 + y^2} \cos \frac{1}{x^2 + y^2}.$$

You can see that f_1 is not continuous at $(0, 0)$. Indeed the first term in f_1 converges to zero. Argue that the second term is unbounded. However, f is differentiable at $(0, 0)$ simply because

$$\frac{f(x, y)}{||(x, y)||} \to 0.$$

showing that f is differentiable and the derivative is the vector $(0, 0)$. you can show that f_2 is also not continuous at $(0, 0)$.

bringing some order:

I hope you do understand how confusing is the situation. so to avoid all bad things, we are going to assume that the functions we deal with are all C^1 functions. A function is C^1 if its first partial derivatives are continuous functions. Recall, then the function is differentiable and derivative at point a is the vector $(f_1(a), f_2(a))$. Derivative is denoted by $\nabla f(a)$., read it as gradient or grad or nabla.

Let f be a C^1 function on $\Omega \subset R^2$ and $a \in \Omega$. Denoting points in R^2 by $x = (x_1, x_2)$; the hyperplane $x_3 = f(a) + \nabla f(a) \cdot (x - a)$ is the tangent plane to the surface $x_3 = f(x_1, x_2)$ at the point a.

We shall study properties of the derivatives. But before we do this, let us get back to a statement regarding partial derivatives. We did recognise the possibility of rate of change, not only in the x and y directions, but also in any direction at the point. Let us take-up this idea now.

What is meant by direction? it is simply a unit vector u. It is the direction pointed by that vector. More precisely, join origin to u and consider the full line — extended both ways, but remember the direction is from origin to u. In common parlance, it is customary to say the line extended in both directions. we would not use it, because it would be confusing. There are no two directions, there is only one direction, namely *that pointed by* u. When one uses the phrase, the line extended in 'both directions', one is only referring to the act of drawing the line to 'both sides' — join origin to u and extend beyond u and also beyond zero.

Now, let f be a C^1 function on $\Omega \subset R^2$ and $a \in \Omega$. If we take the unit vector $e_1 = (1,0)$ then the quantity

$$\lim_{t \to 0} \frac{f(a + te_1) - f(a)}{t}$$

gives precisely the partial derivative $f_1(a)$. similarly, if we take the unit vector $e_2 = (0,1)$, then

$$\lim_{t \to 0} \frac{f(a + te_2) - f(a)}{t}$$

gives precisely $f_2(a)$. We use the same idea to define the directional derivative in the direction of u;

$$D_u f(a) = \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}.$$

You agree that this does indeed give rate of change of the function in the direction of u. You already know that

$$f_1(a) = \nabla f(a) \cdot e_1; \qquad f_2(a) = \nabla f(a) \cdot e_2.$$

It is natural to expect that $D_u f(a) = \nabla f(a) \cdot u$. This is indeed so. We see this soon. But note that in the new notation, f_1 and f_2 become $D_{e_1} f$ and $D_{e_2} f$ respectively.

It is also natural to think of direction as an angle θ, $0 \leq \theta \leq 2\pi$. of course angles zero and 2π correspond to the same direction. Thus x direction corresponds to $\theta = 0$, y direction corresponds to $\theta = \pi/2$. The negative x direction corresponds to $\theta = \pi$ and the negative y direction corresponds to $\theta = 3\pi/2$.

The reason we did not use angles is because, in higher dimensions you need several angles. Of course unit vector still specifies a direction. all the concepts and results that we discussed have analogues in R^d as well.

A chain rule:

let $f : \Omega \to R$ be a C^1 function. Here $\Omega \subset R^2$ is an open set. Suppose φ_1 and φ_2 are two real C^1 functions defined on an interval (a, b) such that for every t, the point $(\varphi_1(t), \varphi_2(t)) \in \Omega$. Then it makes sense to define the composed function.

$$F(t) = f(\varphi_1(t), \varphi_2(t)) : (a, b) \to R.$$

Thus the function is a real valued function defined on an interval. Thus given a real number t, it produces a real number $F(t)$. However, to calculate the number $F(t)$ we pass through R^2. It is natural to expect that this is again a differentiable function. What is the formula for the derivative at a point?

Theorem (Chain rule): F is C^1 and

$$F'(t) = f_1(\varphi_1(t), \varphi_2(t))\ \varphi_1'(t) + f_2(\varphi_1(t), \varphi_2(t))\ \varphi_2'(t).$$

The result can be restated in several ways. For example

$$F'(t) = \nabla f \cdot (\varphi_1', \varphi_2').$$

where ∇f is evaluated at the point $(\varphi_1(t), \varphi_2(t))$ and the derivatives φ_1' and φ_2 are evaluated at the point t.

If we denote the map $\Phi(t) : (a, b) \to R^2$ by $\Phi(t) = (\varphi_1(t), \varphi_2(t))$ and make the convention $\Phi'(t) = (\varphi_1'(t), \varphi_2'(t))$ then the formula takes the pleasing form,

$$F'(t) = \nabla f(\Phi(t)) \cdot \Phi'(t).$$

The reason it is pleasing is that it is the same formula we learnt in one dimension. There is nothing new really, is it not?

Here is another way of stating, which uses a different suggestive notation. The maps φ_1 and φ_2 are denoted by $x(t)$ and $y(t)$ respectively. This is because they denote the x and y coordinates when we proceed to compose with f. Points in R^2 are denoted as (x, y) and the function f is $f(x, y)$. With this notation,

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

We have used ∂f to remind that f is a fucntion of more than one variable and we have differentiated w.r.t. one of those variables. On the other hand we used dF, dx, dy because these are functions of only one variable and derivative is taken w.r.t. that variable.

Proof is not difficult. But let us see why we are interested in this.

Consider a function f defined on a region $\Omega \subset R^2$ and a point $a \in \Omega$. let us fix a unit vector $u \in R^2$. We want to see if the limit

$$\lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}$$

exists and equals $\nabla f \cdot u$. Remember the points $a + tu$ are on the line in the direction of u at the point a and this is the derivative at the point a in the direction u.

This is an immediate consequence of the chain rule. Take Φ to be the function $\Phi(t) = a + tu$ defined on the interval $(-1, 1)$. Do not forget to notice that a, u are vectors but t is a real number. The differentiability of the composed function at $t = 0$ is precisely the limit required above and chain rule says that it exists and equals the claimed quantity. You need to only note that $\Phi'(t) = u$.

The earlier observation combined with Cauchy-Schwarz inequality leads to an interesting interpretation of the gradient: it gives the direction in which the derivative is the largest in modulus, assuming that the gradient is non-zero vector. This is trivial because

$$|\nabla f \cdot u| \leq ||\nabla f||\ ||u||.$$

equality above holds when and only when u is a multiple of ∇f, assuming that ∇f is non-zero. If ∇f is zero vector then for any u the quantity above is zero. Since u is a unit vector, it should be the vector: normalised ∇f.

The chain rule has another interesting consequence, the mean value theorem. Suppose $\Omega \subset R^2$ open $a, b \in \Omega$ and the line joining the points a and b is contained in Ω. Then there is some point θ on this line such that

$$f(b) - f(a) = \nabla f(\theta) \cdot (b - a). \qquad (\dagger)$$

This is again easy. Consider the composition as above with

$$\phi(t) = tb + (1 - t)a.$$

Then $F(1) = b$ and $F(0) = a$ and it is a continuous function on the interval $[0,1]$ which is differentiable at every point inside the interval. Hence usual mean value theorem applies to give a number $\eta \in (0,1)$ such that $F(1) - F(0) = F'(\eta)$. apply chain rule to see this is same as (†) with $\theta = \Phi(\eta)$.

The chain rule also leads to Taylor expansion but since it needs higher derivatives we return to this later. Let us now prove the chain rule. Fix a t_0. Need to show

$$F'(t_0) = \nabla f(\Phi(t_0)) \cdot \Phi'(t).$$

That is,

$$\frac{f(\Phi(t)) - f(\Phi(t_0))}{t - t_0} - \nabla f(\Phi(t_0)) \cdot \Phi'(t_0) \to 0.; \quad \text{as} \quad t \to t_0.$$

Let $\epsilon > 0$ be any fixed number. We show $\delta > 0$ such that

$$|t - t_0| < \delta \Rightarrow \left| \frac{f(\Phi(t)) - f(\Phi(t_0))}{t - t_0} - \nabla f(\Phi(t_0)) \cdot \Phi'(t_0) \right| < \epsilon. \quad (\dagger)$$

We reduce it two separate simple problems. We show $\delta_1 > 0$ such that

$$|t - t_0| < \delta_1 \Rightarrow \left| \frac{f(\Phi(t)) - f(\Phi(t_0))}{t - t_0} - \nabla f(\Phi(t_0)) \cdot \left[\frac{\Phi(t) - \Phi(t_0)}{t - t_0} \right] \right| < \epsilon/2. \qquad (\spadesuit)$$

We show a $\delta_2 > 0$ such that

$$|t - t_0| < \delta_2 \Rightarrow \left| \nabla f(\Phi(t_0)) \cdot \left[\frac{\Phi(t) - \Phi(t_0)}{t - t_0} \right] - \nabla f(\Phi(t_0)) \cdot \Phi'(t_0)] \right| < \epsilon/2. \qquad (\clubsuit).$$

Then for $|t - t_0| < \min\{\delta_1, \delta_2\}$ both inequalities hold; adding them we get (†) as required.

($\clubsuit$) is simple. In fact the long expression, by Cauchy-Schwarz inequality, is at most

$$||\nabla f(\Phi(t_0))|| \quad \left\| \frac{\Phi(t) - \Phi(t_0)}{t - t_0} - \Phi'(t_0)] \right\|$$

Denote $1 + ||\nabla f(\Phi(t_0))|| = c$. By using definition of derivative (for usual one variable functions), choose $\delta_2 > 0$ so that for $|t - t_0| < \delta_2$

$$\left| \frac{\varphi_1(t) - \varphi_1(t_0)}{t - t_0} - \varphi_1'(t_0)] \right| < \frac{\epsilon}{2c}; \quad \left| \frac{\varphi_2(t) - \varphi_2(t_0)}{t - t_0} - \varphi_2'(t_0)] \right| < \frac{\epsilon}{2c}.$$

(I have used mean value theorem here while explaining in the class and you suggested definition of derivative is enough.) Of course, the choice of δ_2

implies that $(t_0 - \delta_2, t_0 + \delta_2) \subset (a, b)$. if you are not able to see, just you can as well take smaller value of δ_2 that satisfies this condition.

Then definition of norm and definition of Φ' will give you ($\clubsuit$).

To achieve ($\spadesuit$) you do exactly similar thing. Denote $\Phi(t_0) = a$. using definition of derivative (that we have learnt now for functions of two variables), first choose η so that for $||x - a|| < \eta$ we have

$$\left| \frac{f(x) - f(a) - \nabla f(a) \cdot (x - a)}{||x - a||} \right| < \epsilon/2. \qquad (*)$$

Just to remind you, here the dot in the neumerator is scalar product. By using continuity of Φ, that is, continuity of φ_1 and φ_2; get $\delta_1 > 0$ so that when $|t - t_0| < \delta_1$ then $||\Phi(t) - \Phi(a)|| < \eta$.

Let us see what happens to the right side expression of ($\spadesuit$) when $|t - t_0| < \delta_1$. Fix such a t. In case $\Phi(t) = \Phi(t_0)$ then that expression is zero and nothing for us to do to see the required inequality. Other wise denoting $\Phi(t) = x$, that expression equals

$$\left| \frac{f(x) - f(a) - \nabla f(a) \cdot (x - a)}{||x - a||} \right| \times \frac{||x - a||}{|t - t_0|.}$$

Here the first quantity is smaller than $\epsilon/2$ by $(*)$. What about the second term?

By MVT, of the last semester, applied to the functions φ_1 and φ_2, there are points P_1 and P_2 in the interval $(t_0 - \delta_1, t_0 + \delta_1)$ so that the second term in the above display in nothing but norm of the vector $\langle \varphi_1'(P_1), \varphi_2'(P_2) \rangle$. If $c/2$ is a bound for these derivatives of φ_1, φ_2 over this interval then this norm is smaller than c. Thus the he second term is smaller that c and hence the above expression is smaller than $c\epsilon/2$.

So it appears that a better choice of δ_1 would do, namely, choose your η so that for $||x - a|| < \eta$ we have

$$\left| \frac{f(x) - f(a) - \nabla f(a) \cdot (x - a)}{||x - a||} \right| < \epsilon/2c, \qquad (**)$$

instead of $(*)$ and then chose δ_1 for this η.

Yes, if you now go back and choose δ_1 for this η the proof seems to work perfectly. But this argument is faulty because c depended on δ_1 (see where we got into this c) and δ_1 now depends on c. So actually nothing is achieved. A wise thinking, that involves looking ahead before you take your step, would help. Here is the precise argument to show that a δ_1 can be chosen to satisfy ($\spadesuit$).

First choose $\delta' > 0$ so that

$$[t_0 - \delta', t_0 + \delta'] \subset (a, b).$$

Since φ_1' and φ_2' are continuous functions, they are bounded on this interval and let $c/2$ be a bound for these functions. Choose η so that $(**)$ holds. Choose δ_1 exactly as earlier for this η. If necessary, make it smaller so that $\delta_1 < \delta'$. This choice would do to show ($\spadesuit$).

I have given the thought process in choosing δ_1 as part of the proof. At the end, if you are confused, ignore the thought process and the faulty argument we went through. Go to the para where we choose η earlier, replace it by the above para and then proceed with the argument. You must convince yourself that ($\spadesuit$) is achieved. You must also convince ourself that the proof is actually very simple and this was precisely what was done last semester too for the chin rule; a two step procedure.

Thus we have completed proof of the chain rule. You must understand what we achieved. We have a given formula to calculate the derivative the function F from R to R. Then did we not do such things already last semester? No. Here to get the value of the function you pass through R^2.

You can generalise to passing through R^n. This means, you have C^1 functions $\varphi_1, \cdots, \varphi_n$ on an interval (a, b) and you have a C^1 function $f : \Omega \to R$. Here $\Omega \subset R^n$ is an open set and it includes the vector $(\varphi_1(t), \cdots, \varphi_n(t))$ for very $t \in (a, b)$. Then it makes sense to define $F : R \to R$ by

$$F(t) = f(\varphi_1(t), \cdots, \varphi_n(t)); \quad a < t < b.$$

Then F is C^1 and

$$F'(t) = \sum f_i(\varphi_1(t), \cdots, \varphi_n(t))\varphi_i'(t) = \sum \frac{\partial f}{\partial x_i}(\varphi_1(t), \cdots, \varphi_n(t))\varphi_i'(t).$$

$$= \nabla f(\Phi(t)) \cdot \Phi'(t) = f'(\Phi(t)) \cdot \Phi'(t).$$

where

$$\Phi(t) = (\varphi_1(t), \cdots, \varphi_n(t)); \quad \Phi'(t) = (\varphi_1'(t), \cdots, \varphi_n'(t)).$$

Several new problems arise. Suppose φ_1 and φ_2 are C^1 functions defined on Ω_1 taking values in R. The only difference is that they are not defined on an interval contained in R. Then you can think of $\Phi(x) = (\varphi_1(x), \varphi_2(x))$ on as a function on Ω_1 with values in R^2, as earlier. Suppose its values fall in the open set $\Omega \subset R^2$ on which we have real valued C^1 function f. Then it makes sense to talk about the composition:

$$F(x) = f(\Phi(x)); \qquad x \in \Omega_1.$$

is this differentiable? Is

$$F'(x) = f'(\Phi(x)) \cdot \Phi'(x).$$

It should be correct, the only problem is that we do not know the meaning of $\Phi'(t)$ because Φ is a function on R^2 to R^2. We need to assign meaning to derivative of function defined on R^m and taking values in R^n.

Let us pause for a moment and see what happened so far. First we had functions from R to R and we defined derivative at a point and it is a number. Then we had function from R^n to R and we defined its derivative at a point and it is a vector. In the previous theorem we had Φ from R to R^n and we defined Φ'. This was facilitated by the fact that such a function Φ is made up of n real valued functions, namely, $\varphi_i(t)$ equals the i-th coordinate of $\Phi(t)$. It turned out that $\Phi'(t)$ is also a vector.

There is again some chaos, sometimes we have functions $R \mapsto R$, sometimes $R^2 \mapsto R$ and sometimes $R \mapsto R^2$. Now we have functions $R^2 \mapsto R^2$. Sometimes derivatives are numbers, sometimes they are vectors. What are they? Some order has to be brought in and a clear understanding has to be achieved.

Let us go back and see what was the purpose of derivative. We wanted to make best linear approximation of f at a point a in its domain. That is, we wanted $g(x) = L(x)+\beta$ such that $g(a) = f(a)$ and $||f(x)-g(x)||/||x-a|| \to 0$ as $x \to a$. Thus $f(x) - g(x)$ approaches zero faster than x approaches a. Of course the only non-trivial thing is $L(x)$ because $\beta = f(a) - L(a)$ since we want (or know) $\varphi(a) = f(a)$.

(a)

If $f : R \to R$, then the derivative at a denoted by the number c has the property that it determines g. More precisely the linear transformation from

R to R is the map $L(x) = cx$.

(**b**)

If $f : R^2 \to R$, then the derivative at $a \in R^2$ denoted by the vector $c = \nabla f(a)$ has the property that it determines g. More precisely the linear transformation from R^2 to R is the map $L(x) = c \cdot x$. Here is another way of stating the same thing. Even though everything, points in R^2 as well as derivatives are vectors, let us give them proper dress so that we can recognize them easily. Think of R^2 as space of column vectors. That is

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let us think of the derivative c as row vector. Then the linear transformation we are talking about is nothing but $L(x) = cx$. This makes perfect sense because c is 1×2 vector and x is 2×1 vector and so cx is 1×1 vector, or a number. Thus the derivative $\nabla f(a)$ is actually the row vector. But it is to be thought of as a linear transformation of R^2 to R.

(**c**)

If $f : R \to R^2$, then the derivative at $a \in R$ denoted by the vector c has the property that it determines g. Remember that if $f_1(t)$ is the first coordinate of $f(t)$ and $f_2(t)$ is the second coordinate of $f(t)$ then $f(t) = (f_1(t), f_2(t))$. But just now we decided to think of R^2 as column vectors. Thus

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

More precisely the linear transformation from R to R^2 is the map $L(x) = cx$, that is,

$$x \in R^1 \mapsto cx = \begin{pmatrix} f_1'(a) \\ f_2'(a) \end{pmatrix} x \in R^2.$$

In other words the derivative of f is the column vector

$$f'(a) = \begin{pmatrix} f_1'(a) \\ f_2'(a) \end{pmatrix}.$$

again to avoid confusion regarding column and row vectors, let us think of f as the linear transformation $L(x) = cx$. This makes sense c is 2×1 vector and x is 1×1 and cx makes sense and is 2×1 vector, in other words, an element of R^2. This is the linear transformation from R to R^2.

Before proceeding further, we should note one thing here. We have not defined derivative of $f : R \to R^2$ via best linear approximations. We defined outright $f'(a) = \langle f_1'(a), f_2'(a)\rangle$. But one can easily show that

$$\frac{||f(t) - f(a) - f'(a)(t-a)||}{|t-a|} \to 0 \quad \text{as} \quad t \to a.$$

This only depends on the fact that a sequence of vectors converges to zero iff coordinate-wise it so happens.

This brings some order into things. Derivatives are linear operators, no need to confuse whether it is row vector or column vector or a number and so on. Elements of R^n are column vectors: x and so row vectors r define linear maps on them to $R : x \mapsto rx$. again in the notation there is nothing to tell you whether a symbol is a row vector or column vector.

Taking clue from the above, suppose we have a map $f : R^m \to R^n$ and $a \in R^m$. The best linear approximation of f at a determines the derivative. We say that a linear transformation $L(x) : R^m \to R^n$ is derivative of f at a if the map $\varphi(x) = L(x) + f(a) - L(a)$ has the property

$$\frac{||f(x) - \varphi(x)||}{||x-a||} \to 0 \quad \text{as} \quad x \to a.$$

In other words

$$\frac{||f(x) - f(a) - L(x-a)||}{||x-a||} \to 0 \quad \text{as } ||x-a|| \to 0.$$

Since linear transformations from R^m to R^n are given by $n \times m$ matrices A via $L(x) = Ax$ we can reformulate the idea of derivative as follows. A $n \times m$ matrix A is derivative at the point a if

$$\frac{||f(x) - f(a) - A(x-a)||}{||x-a||} \to 0 \quad \text{as } ||x-a|| \to 0.$$

Of course there is another way to define the derivative taking a clue from the adhoc definition we employed earlier. remember if $f : R \to R^2$ is a C^1 map we defined $f'(a) = (f_1'(a), f_2'(a))$. If

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

then we can put

$$f'(a) = \begin{pmatrix} \nabla f_1(a) \\ \nabla f_2(a) \\ \vdots \\ \nabla f_n(a) \end{pmatrix}$$

Note that it has n rows each an m-vector. Thus it is $n \times m$ matrix.

We shall, as earlier, restrict our attention to only C^1 functions and proceed. But first we need to familiarize ourselves with maps from R^m to R^n. So you can forget about derivatives for a while.

functions from R^m to R^n:

Elements of R^n are column vectors v. But it takes space to write column vectors. One way is to think of your symbols v as row vectors and elements of R^n as their transpose: v^t. Again it is a burden in reading and so we shall not do this either. We just do as we have been doing all along. Do not scratch unless it itches. When this distinction is specifically needed, then we shall use it. Otherwise we enjoy being careless, but remember elements of R^n are column vectors.

Let $\Omega \subset R^m$. Suppose we have n real valued functions on Ω. Then we can cook up a R^n valued function on Ω as

$$f(x) = (f_1(x), f_2(x), \cdots, f_n(x)) \quad x \in \Omega.$$

Conversely, every function on Ω taking values in R^n is obtained like this. More precisely, suppose $f : \Omega \to R^n$ is given to us, then there are n uniquely determined real functions $f_1, f_2, \cdots, f_n$ on Ω such that

$$f(x) = (f_1(x), f_2(x), \cdots, f_n(x)).$$

This is obvious, $f_i(x)$ must be the i-th coordinate of the point $f(x)$ and this choice does it.

Let us say that f is continuous if each f_i is so. remember we defined continuity for real valued functions. Of course, the natural way to define is to say that $f(x)$ should be close to $f(a)$ if x is close to a. Yes, both ideas are same mathematically.

Theorem: let $f = (f_1, f_2, \cdots, f_n) : \Omega \to R^n$. Let $a \in \Omega$. Then the following are equivalent.

(i) Each of the n real valued functions $f_1, f_2, \cdots, f_n$ are continuous on Ω.
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$x \in \Omega, ||x - a|| < \delta \Rightarrow ||f(x) - f(a)|| < \epsilon.$$

(iii) If $\{x^i : i \geq 1\}$ is a sequence of points in Ω and $x^i \to a$ then $f(x^i) \to f(a)$.

We have proved it in class. Try to do so without writing.

The ideas you learnt last semester are powerful to give you theorems about continuous functions on R^m also. Here is one such. A subset $S \subset R^m$ is bounded if there is a number c such that $||x|| \leq c$ for all $x \in S$.

Theorem: Let $S \subset R^m$ be a closed bounded set and f be a continuous function $f : S \to R$. Then f is bounded, that is, there is a number M such that $|f(x)| \leq M$ for all $x \in S$. This is also same as saying that the range of the function f is a bounded subset of R.

There is no new idea. Let us execute it for $S \subset R^2$. Since the set S is bounded get a square $R_0 = [a, b] \times [a, b]$ which includes S. You only need to note that if $||x|| \leq c$ then $x \in [-c, c] \times [-c, c]$. We prove the theorem by contradiction.

suppose that the function is not bounded. Divide the square into four parts by cutting each side at the mid point. Then f must be unbounded on part of S contained in one of these smaller squares. Take one such square, $R_1 = [a_1, b_1] \times [c_1, d_1]$. Just be careful, do not be under the impression that this square is like $[c, d] \times [c, d]$ just because the earlier square was $[a, b] \times [a, b]$. That was in your hands, this is not. Do the same to R_1 and get R_2 and so on.

Thus you get a sequence of squares R_n such that length of each side of R_n is half length of previous square side. By cantor intersection theorem, we get a point (a_1, a_2) common to all these squares. Indeed all the sides $[a_n, b_n]$ have a point x in common and all sides $[c_n, d_n]$ have a point y in common. Since each square R_n contains points of S, we see that (x, y) is a limit point of S and must be in S because S is closed.

But then continuity of f tells that there is a $\delta > 0$ such that $x \in S, ||x - a|| < \delta$ implies $||f(x) - f(a)|| < 1$, thus $||f(x)|| \leq ||f(a)) + 1$. In particular, f is bounded by this number at all points of S in this disc. Now one of your R_n must be contained in this disc, because their lengths are converging to zero. This contradicts the fact that f is not bounded on the part of S contained in R_n.

This completes the proof.

Let us now consider $S \subset R^m$ and $f : S \to R^n$. Say that f is bounded if there is number M such that $||f(x)|| \leq M$ for all $x \in S$. If $F = (f_1, f_2, \cdots, f_n)$ then f is bounded iff each f_n is so. This is because, if f is bounded by M then each f_i is also bounded by M and if each f_i is bounded by M then f is bounded by $M\sqrt{n}$.

Say that $f : R^m \to R^n$ is a C^1 function if each f_i is so, Note that f_i is C^1 map means that each of its m partial derivatives are continuous functions.

let $\Omega \subset R^m$ be an open set and $F : \Omega \to R^n$ be C^1 map. Let $a \in \Omega$ and let A be the $n \times m$ matrix whose i-th row consists of $\nabla f_i(a)$. In other words, (i, j)-th entry of A is $D_j f_i(a)$; the partial derivative of the i-th function f_i w.r.t. the j-th coordinate.

Theorem:

$$\frac{||f(x) - f(a) - A(x-a)||}{||x-a||} \to 0 \quad \text{as} \quad x \to a. \qquad (\diamondsuit)$$

In other words, A confirms to our intuition as a suitable candidate for being the derivative of f at A. Indeed we *define* A to be the derivative of f at a. Of course, what we mean is that the linear map $x \mapsto Ax$ of R^m to R^n is the derivative at the point a. Of course, you can also think of the matrix A itself as the derivative. Then you need to keep track of the confusion that some times derivatives are numbers, sometimes they are vectors and yet other times they are matrices.

proof of the above theorem is simple, no work is needed. Note that the quantity in $(\diamondsuit)$ is simply norm of a vector. Carefully decipher the notation, to see that the i-th entry of the above vector is nothing but

$$\frac{f_i(x) - f_i(a) - \nabla f_i(a) \cdot (x-a)}{||x-a||}$$

and by definition of ∇f_i this quantity does converge to zero as $||x-a|| \to 0$. Hence $(\diamondsuit)$ is verified.

Can there be another matrix A_1 satisfying ($\diamondsuit$)? If so, fix $r > 0$ so that $B(a, r) \subset \Omega$. Then for $h \in B(0, r)$

$$\frac{||Ah - A_1h||}{||h||} \leq \frac{||f(a+h) - f(a) - Ah||}{||h||} + \frac{||f(a+h) - f(a) - A_1h||}{||h||}$$

$$\longrightarrow 0.$$

If you take any non-zero vector $v \in R^m$ then for all large integers i, we see $v/i \in B(0, r)$ so that

$$\frac{||A(v/i) - A_1(v/i)||}{||v/i||} \to 0, \quad \text{as} \quad i \to \infty.$$

In other words $Av - A_1v = 0$. This is true for every vector $v \in R^m$ and hence the two matrices/linear transformations are same.

Thus our definition of derivative is a good definition. Of, course you can say that an f is differentiable if ($\diamondsuit$) holds for some linear transformation A; without assuming that f is C^1 to begin with. Then also you can show that such a transformation is unique, by the same argument s above. However, the derivative may exist but the function f may not be C^1. We have already seen such examples earlier. It is to avoid some pathologies we started restricting to C^1 maps.

We denote the derivative of f at a by $Df(a)$ or $(D_jf_i(a))_{ij}$ or the more compact notation $f'(a)$.

examples:

1. Fix a vector $u \in R^n$ and consider the map $f(x) = u$ for all $x \in R^m$. This is a C^1 map and $f'(a) \equiv 0$. That is, it is the zero linear transformation or it is the matrix with all entries zero.

2. Fix one $n \times n$ matrix A and consider $f(x) = Ax + u$. Then for every a, we have $f'(a) = A$.

3. $f(x, y) = x + y$ from R^2 to R. Then $f'(a) = (1, 1)$.

   More generally, if we take the map from R^n to R given by $f(x) = \sum x_i$, then it is C^1 and $f'(a)$ is the vector with all entries equal to one.

   Or if you take the map R^5 to R^2 given by

   $$f(x_1, x_2, \cdots, x_5) = (x_1 + x_2 + x_3.x_4 + x_5).$$

then it is C^1 and

$$f'(a) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Or define f from R^{2n} to R^n by $f(x,y) = x + y$. Note that here we are denoting point of R^{2n} by $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. then $f'(a,b) = (\mathbf{I}, \mathbf{I})$where I is the $n \times n$ identity matrix. thus the matrix $(\mathbf{I}, \mathbf{I})$ has n rows and $2n$ columns.

4. Define from R^2 to R; $f(x,y) = x.y$. Then

$$f'(a,b) = (b,a).$$

For example $f'(3,4)$ is the linear transformation given by $(4,3)$; in other words $L(x,y) = 4x + 3y$.

Consider $f(x,y) = \sum\limits_{1}^{n} x_i y_i$ from R^{2n} to R^n. Again see how we denoted points of R^{2n}, not as $(x_i : 1 \leq i \leq 2n)$ but as $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. Then

$$f'(a_1, \cdots, a_n, b_1, \cdots, b_n) = (b_1, \cdots, b_n, a_1, \cdots, a_n).$$

5. Let us consider a symmetric 2×2 matrix A and consider $f(x) = x^t A x$ a function from R^2 to R. Here x^t is the transpose of the column vector x. In other words with usual notation of (x,y) for points of R^2

$$f(x,y) = \alpha x^2 + 2\beta xy + \gamma y^2 \qquad A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

Then $f'(a,b) = (2\alpha a + 2\beta b, 2\beta a + 2\gamma b)$. Reverting back to the notation $x = (x_1, x_2)$ we see $f'(a) = 2Aa$. This is pleasing, just like derivative of x^2 at a being $2a$.

You can take a symmetric $n \times n$ matrix A and define $f(x) = x^t A x$ for $x \in R^n$. Then the same argument as above shows you

$$f'(x) = 2Ax, \quad x \in R^n.$$

6. You can think of more complicated functions. For example you can consider 2×2 matrix as a point in R^4.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (a_{11}, a_{12}, a_{21}, a_{22}) \in R^4.$$

You can think of matrix multiplication as a map from R^8 to R^4

$$f(a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= (a_{11}b_{11} + a_{12}b_{21}, a_{11}b_{12} + a_{12}b_{22}, ---, ---).$$

Those are not difficult to handle, but we need to wait till we gain experience and and not get confused easily.

Now let us return to the main problem with which we started, namely, the chain rule.

Let $\Omega \subset R^m$ and $\Omega_1 \subset R^n$ be open sets.

$$\Phi : \Omega \to R^n; \quad f : \Omega_1 \to R^k$$

be C^1 maps. Assume that $\Phi(x) \in \Omega_1$ for all $x \in \Omega$. Define the composition

$$F(x) = f(\Phi(x)), \qquad x \in \Omega.$$

Theorem: F is a C^1 map from $\Omega \to R^k$. Further

$$F'(a) = f'(\Phi(a))\Phi'(a).$$

Note that f' is $k \times n$ matrix and Φ' is $n \times m$ matrix and so the product is $k \times m$ matrix and defines linear transformation from R^m to R^k.

proof is exactly like the earlier situation.

Let $a \in \Omega$ be fixed. let $b = \Phi(a)$. Let $f'(b) = B$ and $\Phi'(a) = A$. Fic $\epsilon > 0$. We need to show $\delta > 0$ so that

$$||x - a|| < \delta \Rightarrow \frac{||F(x) - F(a) - BA(x - a)||}{||x - a||} < \epsilon.$$

that is

$$||x - a|| < \delta \Rightarrow \frac{||f(\Phi(x)) - f(\phi(a)) - BA(x - a)||}{||x - a||} < \epsilon.$$

We show $\delta_1 > 0$ so that

$$||x - a|| < \delta_1 \Rightarrow \frac{||f(\Phi(x)) - f(\phi(a)) - B[\Phi(x) - \Phi(a)]||}{||x - a||} < \epsilon/2.$$

We show $\delta_2 > 0$ so that

$$||x-a|| < \delta_2 \Rightarrow \frac{||B[\Phi(x) - \phi(a)] - BA(x-a)||}{||x-a||} < \epsilon/2.$$

If $||x-a|| < \delta = \min\{\delta_1.\delta - 2\}$ then both the inequalities hold and adding them gives the desired inequality.

<u>To get δ_2:</u>

First note that given any $k \times n$ matrix B, there is a number c so that $||Bx|| < c||x||$ for any $x \in R^n$. In fact let

$$M = \max\{|b_{i,j} : i \leq k, j \leq n\}$$

then,

$$Bx = (\sum b_{1j}x_j, \sum b_{2j}x_j, \cdots \sum b_{kj}x_j)$$

Since

$$(\sum b_{ij}x_j)^2 \leq (\sum |b_{ij}||x_j|)^2 \leq M^2 k||x||^2.$$

Here we have used the following fact: if you square the sum, you get square terms and cross products, but $2\alpha\beta \leq \alpha^2 + \beta^2$. Thus cross products are again bounded by square terms. This is true for each of the k coordinates of Bx. Hence

$$||Bx||^2 \leq k^2 M^2 ||x||^2.$$

Thus $c = kM$ would do.

Returning to our problem, fix $c > 0$ as above. Using differentiability of ϕ get $\delta_2 > 0$ so that

$$||x-a|| < \delta_2 \Rightarrow \frac{||\Phi(x) - \Phi(a) - A(x-a)||}{||x-a||} < \epsilon/(2c).$$

This will satisfy requirement of δ_2.

<u>To get δ_1:</u>

First fix an $r > 0$ so that the closed ball around a of radius r is contained in Ω. Since $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_n)$ is a C^1 function, we see that $\nabla\Phi_i$ is bounded on the closed ball. Now fix M so that

$$||x-a|| \leq r \Rightarrow ||\nabla\Phi_i(x)|| \leq M/n, \quad i = 1, 2, \cdots n.$$

Using differentiability of f get $\eta > 0$ so that

$$||y - b|| < \eta \Rightarrow \frac{||f(y) - f(b) - B(y - b)||}{||y - b||} < \epsilon/(2M).$$

Npw choose $\delta_1 > 0$ so that $||x - a|| < \delta_1$ implies $||\Phi(x) - b|| < \eta$. This is just by continuity of Φ. By reducing if necessary, we shall assume $\delta_1 < r$.

Let now x be such that $||x - a|| < \delta_1$. Need to show

$$\frac{||f(\Phi(x)) - f(\phi(a)) - B[\Phi(x) - \Phi(a)]||}{||x - a||} < \epsilon/2.$$

if $\Phi(x) = \Phi(a)$ there is nothing to be done. Otherwise, denote the point $\Phi(x)$ by y the above expression equals

$$\frac{||f(y) - f(b) - B(y - b)||}{||y - b||} \, \frac{||\Phi(x) - \Phi(a)||}{||x - a||}$$

By choice of δ_1, we conclude that $||\phi(x) - b|| < \eta$ so that choice of η tells us that the first term above is at most $\epsilon/(2M)$. The second term is norm of

$$\left(\frac{\Phi_i(x) - \Phi_i(a)}{||x - a||} : 1 \leq i \leq k \right).$$

By the mean values theorem, there are points P_i such that this vector is

$$\left(\frac{\nabla \Phi_i(P_i) \cdot (x - a)}{||x - a||} 1 \leq i \leq k \right).$$

Note that we can apply the mean slue theorem, the points x and a are all in a disc which is contained in ω and hence the lines joining are also contained in Ω. Since $\delta_1 < r$, choice of M tells us, with Cauchy-Schwarz that each entry of the vector above is at most M/n and hence its norm is at most M. so the product is at most $\epsilon/2$.

This completes the proof of chain rule.

Taylor:

The chain rule will now be applied to derive Taylor formula for function of several variables. This will be *exactly* same as the one you learnt last semester. There is absolutely no change. First let us recall the Taylor we know.

Let f be a function on an open interval I which is n times continuously differentiable. Let $a, b \in I$, Then

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b f^{(n)}(t)\frac{(b-t)^{n-1}}{(n-1)!}dt.$$

If we have functions defined on R^2 and a and b are points in R^2, then the integral term above is a little tricky. so let us reformulate the above equation. We change the variable of integration to

$$t = ub + (1-u)a.$$

The beauty is that as t goes from a to b, the variable u goes from zero to one. It is beautiful because the range of integration no longer depends on the points a and b.

Note that $dt = (b-a)du$ and $(b-t) = (b-a)(1-u)$. Thus

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_0^1 f^{(n)}(ub+(1-u)a)(b-a)^n\frac{(1-u)^{n-1}}{(n-1)!}du.$$

Let us use convenient notation. We use D for derivative.

Thus $[D]f$ means the function f'. And $[D]f(a)$ means the earlier function evaluated at the point a, namely the number $f'(a)$.

$[4D]f$ means the function $4f'$. And $[4D]f(a)$ means the earlier function evaluated at the point a, namely the number $4f'(a)$.

$[(b-a)D]f$ means the function $(b-a)f'$. And $[(b-a)D]f(13)$ means the earlier function evaluated at the point 13, namely the number $(b-a)f'(13)$.

In particular, $[(b-a)D]f(a)$ means the number $(b-a)f'(a)$.

Symbols can be operated again and again. For example $[(b-a)D]^5 f$ means the function $(b-a)^5 D^5 f = (b-a)^5 f^{(5)}$. Recall $f^{(5)}$ is the fifth derivative of the function f. Thus $[(b-a)D]^5 f(13)$ means you need to evaluate this function at 13, thus you get the number $(b-a)^5 f^{(5)}(13)$.

In general if we say $[(b-a)D]^k f$ means it is the function $(b-a)^k f^{(k)}$; namely, The k-th derivative multiplied by the number $(b-a)^k$.
Thus $[(b-a)D]^k f(a)$ means the number $(b-a)^k f^{(k)}(a)$.

We can rewrite Taylor as follows:

$$f(b) \quad = \quad \sum_{k=0}^{n-1} \frac{[(b-a)D]^k f(a)}{k!} \quad +$$

$$\int_0^1 [(b-a)D]^n f(ub+(1-u)a) \frac{(1-u)^{n-1}}{(n-1)!} du. \quad (\spadesuit)$$

The Taylor formula we shall prove is the following.

Theorem: Let $\Omega \subset R^2$ be an open set. Let $a, b \in \Omega$. Assume that the line joining these points is contained in Ω. Let f be a real valued C^n function defined on Ω. Then

$$f(b) \quad = \quad \sum_{k=0}^{n-1} \frac{[(b-a)\cdot D]^k f(a)}{k!} \quad +$$

$$\int_0^1 [(b-a)\cdot D]^n f(ub+(1-u)a) \frac{(1-u)^{n-1}}{(n-1)!} du. \quad (\clubsuit)$$

You see that this formula is exactly same as the earlier one. We only need to explain the notation. you will see that proof of the theorem is itself not difficult. In fact it is trivial from what you already know in the one variable case and the chain rule.

A function f is C^1 if the partial derivatives f_1 and f_2 are continuous functions on Ω. such a statement means that the partial derivatives at each point exist and the function so obtained on Ω is continuous. We met this notation earlier before defining the concept of derivative.

We say that f is C^2 if these functions f_1 and f_2 are also C^1. In other words, $f_{11}, f_{12}, f_{21}, f_{22}$ are continuous functions. Remember that when this happens we already know that the functions f_{12} and f_{21} are same. Thus there are three second order derivatives.

In general, proceeding to define by induction, we say that f is C^n, if f_1 and f_2 are C^{n-1}. Let us use an earlier notation. $D_1 g$ stands for g_1 and $D_2 g$ stands for g_2, the partial derivatives.

With this notation, $f \in C^n$ is same as saying the following: whenever you take a sequence $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ consisting of ones and twos, the function

$$D_{\epsilon_n} D_{\epsilon_{n-1}} \cdots D_2 D_1 f$$

should exist and is continuous.

Using equality of mixed derivatives that we proved, it is a joy to show the following. When the above happens, the function so obtained depends not on the exact sequence, but only on the number of ones and twos in the sequence!

To continue with the notation, we use D for the vector (D_1, D_2). Of course, you should not get worried because, so far, vectors are something like $(4, 3)$ and so on consisting of numbers. Here this vector we have introduced has symbols. Let it be. We shall do more symbolic operations.

Observe $(b - a) \in R^2$, is a vector. By $(b - a) \cdot D$ we mean inner product between these vectors. Again, you should not get worried. We only mean that $(b - a) \cdot D$ stands for the symbol

$$(b_1 - a_1)D_1 + (b_2 - a_2)D_2.$$

Of course, what is the meaning of this symbol? Just as the symbol D_1 applied to a function f gives you a new function $D_1 f = f_1$ so is this symbol.

$$[(b - a) \cdot D]f = [(b_1 - a_1)D_1 + (b_2 - a_2)D_2]f$$

$$= (b_1 - a_1)f_1 + (b_2 - a_2)f_2.$$

Thus if you want to calculate this function at a point, say, $(4, 3)$ you get

$$[(b - a) \cdot D]f(4, 3) == (b_1 - a_1)f_1(4, 3) + (b_2 - a_2)f_2(4, 3).$$

More symbolic things would follow. When we say $[(b - a) \cdot D]^2$ we understand exactly the same thing as in the one dimensional case above.

$$[(b - a) \cdot D]^2 == [(b_1 - a_1)D_1 + (b_2 - a_2)D_2]^2$$

$$= (b_1 - a_1)^2 D_1^2 + 2(b_1 - a_1)(b_2 - a_2) D_2 D_1 + (b_2 - a_2)^2 D_2^2.$$

Thus

$$[(b - a) \cdot D]^2 f = (b_1 - a_1)^2 f_{11} + 2(b_1 - a_1)(b_2 - a_2) f_{12} + (b_2 - a_2)^2 f_{22}.$$

More specifically, if you want to evaluate the function at any point you can do so

$$[(b - a) \cdot D]^2 f(a) =$$

$$(b_1 - a_1)^2 f_{11}(a) + 2(b_1 - a_1)(b_2 - a_2) f_{12}(a) + (b_2 - a_2)^2 f_{22}(a).$$

If you have got a feeling, then

$$[(b - a) \cdot D]^k = \sum_{j=0}^{k} \binom{k}{j} (b_1 - a_1)^j (b_2 - a_2)^{k-j} D_2^{k-j} D_1^j.$$

Or

$$[(b - a) \cdot D]^k f = \sum_{j=0}^{k} \binom{k}{j} (b_1 - a_1)^j (b_2 - a_2)^{k-j} D f_{1^j 2^{k-j}}.$$

For example $f_{1^3 2^4}$ means $f_{1112222}$. Still more specifically, if you want to calculate the function above at the point a,

$$[(b - a) \cdot D]^k f(a) = \sum_{j=0}^{k} \binom{k}{j} (b_1 - a_1)^j (b_2 - a_2)^{k-j} D f_{1^j 2^{k-j}}(a).$$

Now let us go back to the simple chain rule and calculate higher order derivatives of composed function in a special case. The special case is the following. We have an open set $\Omega \subset R^2$ and a real valued c^2 function on Ω and two points $a, b \in \Omega$. We have an open interval $I \subset R$. Define

$$\Phi(u) = ub + (1 - u)a.$$

We consider the real valued function $F(u) = f(\Phi(u))$ defined on I. By earlier chain rule

$$F'(u) = f'(\Phi(u)) \cdot \Phi'(u) = [(b - a) \cdot D] f(\Phi(u)). \qquad (*)$$

Recall that to compare with earlier notation, here we have

$$\Phi(u) = (\varphi_1(u), \varphi_2(u))$$

$$\varphi_1(u) = ub_1 + (1 - u)a_1; \quad \varphi_2(u) = ub_2 + (1 - u)a_2.$$

$$\Phi' = (b_1 - a_1, b_2 - a_2)$$

Thus $(*)$ is a consequence of earlier chain rule. Now let us consider this function F' as a sum of two composed function.

$$F'(u) = f_1(\Phi(u))(b_1 - a_1) + f_2(\Phi(u))(b_2 - a_2).$$

Exactly the same argument applied to f_1 and f_2 in place of $f$gives

$$[f_1(\Phi(u)]' = f_{11}(\Phi(u))(b_1 - a_1) + f_{12}(b_2 - a_2).$$

$$[f_2(\Phi(u)]' = f_{21}(\Phi(u))(b_1 - a_1) + f_{22}(b_2 - a_2).$$

Substituting above and simplifying we get

$$F''(u) = [(b - a) \cdot D]^2 f(u). \qquad (**)$$

If you have understood the argument of arriving at $(**)$ from $(*)$, you should have no problem in arriving at the following, assuming that f is C^3.

$$F^{(3)}(u) = [(b - a) \cdot D]^3 f(\Phi(u)). \qquad (***)$$

Thus, by induction, one has

$$F^{(k)}(u) = [(b - a) \cdot D]^k f(\Phi(u)).$$

Proof of Taylor:

Finally, we return to proof of (♣). Since the line joining a and b is contained in Ω and Omega is open, you can easily see that there is a an $\epsilon > 0$ so that

$$\Phi(u) = ub + (1 - u)a$$

defined on $(-\epsilon, 1 + \epsilon)$ has range contained in Ω. Now consider the function

$$F(u) = f(\Phi(u))$$

on this interval. all the above argument shows that when f is C^n then so is F and gives a formula to calculate its derivatives.

Expand $F(1)$ around zero. This means apply the usual one variable Taylor, namely (♠), to F with $b = 1$ and $a = 0$.

$$f(1) = \sum_{k=0}^{n-1} \frac{[D]^k F(0)}{k!} + \int_0^1 [D]^n F(u) \frac{(1 - u)^{n-1}}{(n - 1)!} du.$$

That is,

$$F(1) = \sum_{k=0}^{n-1} \frac{F^{(k)}(0)}{k!} + \int_0^1 F^{(n)}(u) \frac{(1-u)^{n-1}}{(n-1)!} du.$$

Observe

$$F(1) = f(b); \quad F^{(k)}(0) = [(b-a) \cdot D]^k f(a).$$

This completes proof.

Again, this can be stated differently

$$f(b) = \sum_{k=0}^{n-1} \frac{[(b-a) \cdot D]^k f(a)}{k!} + \frac{[(b-a) \cdot D]^n f(\theta)}{n!};$$

where θ is a point on the line joining a to b. This follows from the version proved above by noting that the integral is between $c/n!$ and $C/n!$ where c and C are bounds for $[(b-a) \cdot D]^n f(\theta)$ as θ runs on the line.

It is also customary to state the Taylor with x instead of b

$$f(x) \quad = \quad \sum_{k=0}^{n-1} \frac{[(x-a) \cdot D]^k f(a)}{k!} \quad +$$

$$\int_0^1 [(x-a) \cdot D]^n f(ux + (1-u)a) \frac{(1-u)^{n-1}}{(n-1)!} du.$$

But there is a subtle scope for confusion in understanding this and so I did not state it this way.

Exact the same formula holds in dimensions more than two as well. Now $(b-a)$ and D are of that dimension. But formula is the same and so is the proof.

Though we do not have much use of this formula, we shall see one important consequence of this formula. But it is satisfying to realise that the ideas of last semester are indeed very powerful to let the same formulae to hold even in higher dimensions.

Extrema:

Let $\Omega \subset R^2$ be open and f be a real valued C^1 function on Ω. Let $a \in \Omega$. We say a is a point of local maximum if there is an $r > 0$ such that $f(x) \leq f(a)$ for all $x \in B(a, r)$. call it strict local maximum if the inequality is strict for every x in the ball different from a.

Similarly we say that a is a point of local minimum if there is an $r > 0$ such that $f(x) \geq f(a)$ for all $x \in B(a, r)$. A point which is a local minimum or local maximum is called a local extremum.

Let a be a local maximum. If $a = (a_1, a_2)$ then clearly, $x \mapsto f(x, a_2)$ has local maximum at $x = a_1$ and $y \mapsto f(a_1, y)$ has local maximum at $y = a_2$. Thus

$$f_1(a) = 0 = f_2(a).$$

Same holds even if a is a local minimum.

Just like in one dimensions here too, the above equations would not guarantee either a maximum or minimum. For example $f(x, y) = x^3$ has both derivatives at $(0, 0)$ but f has neither a max nor min at that point. This is not surprising.

However, now something spectacular may also happen. The point may be maximum in several directions and minimum in several other directions at that point! For example,

$$f(x, y) = x^2 - y^2$$

has derivatives zero at the point $(0, 0)$. There are several lines passing through the origin such that if you restrict f to that line it has a max at this point and for several other lines it is a point of minimum.

In one dimensions the strict positivity of the second derivative at a would ensure that a is a local minimum. Same is true here too.

Assume that f is C^2. Recall there are now four (actually three distinct) second derivatives. Let us temporarily, only for this section, denote by f'' the following 2×2 matrix.

$$f''(a) = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}.$$

Just to draw your attention to the fact that the matrix is symmetric, we have written, in the first entry of second row, f_{12} instead of the expected $f_{21} = f_{12}$.

Let A be a 2×2 symmetric matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

Let us say that A is positive definite or simply positive if $v^t A v > 0$ for every non-zero vector v. That is

$$a_{11}v_1^2 + 2a_{12}v_1v_2 + a_{22}v_2^2 > 0.$$

We write this as $A \gg 0$. We shall need matrix theory, but no deep results. Here is a fact to which we shall return later.

$$A \gg 0 \quad \leftrightarrow \quad a_{11} > 0; \;\; a_{11}a_{22} - a_{12}^2 > 0.$$

Here is a criterion for local minimum.

Theorem: Let f be a C^2 function on an open set $\Omega \subset R^2$ and $a \in \Omega$. If $f'(a) = 0$ and $f'' \gg 0$ the a is a point of strict local minimum.

Proof is simple. Consider the two functions: f_{11} and $f_{11}f_{22} - f_{12}^2$. These are continuous and are strictly positive at the point a. So fix $r > 0$ so that they are strictly positive for all $x \in B(a, r)$. Now take any point b in this ball, use Taylor

$$f(b) = f(a) + [(b-a)\cdot D]f(a) + \frac{[(b-a)\cdot D]^2 f(\theta)}{2!};$$

where θ is a point on the line joining a to b. But whatever it be, the positive definiteness of f'' says that the last term on the right side is positive. The second term is zero. Thus $f(b) > f(a)$. This completes the proof.

Similarly, we have a criterion for local minimum too. Say that A is negative definite if for all non-zero vectors $v^t A v < 0$. We write $A \ll 0$.

$$A \ll 0 \quad \leftrightarrow \quad a_{11} < 0; \;\; a_{11}a_{22} - a_{12}^2 > 0.$$

The same proof as earlier shows that If $f'' \ll 0$ at a point a, then it is so in a ball around the point a.

functions without formula:

So far what we have been doing is just imitation of the development of calculus we learnt last semester. of course, this statement does not mean it is a trivial job. It has taken us rather too far. We can now differentiate functions defined on any Euclidean space (R^k) and taking values in any Euclidean space (R^n).

If you think about it, the achievement is really spectacular. Imagine, $n \times n$ matrix is nothing but a point in n^2-dimensional euclidean space. Since determinant is a continuous function of the entries of the matrix (it is a polynomial), the set of non-singular matrices is an open set (determinant non zero). Thus you can define a function on this open set of n^2 dimensional space to itself, namely, the matrix inverse map. We know how to differentiate this function!

This is all fine. But is there anything we can do now about functions of one variables (we learnt last semester) which we could not do last semester? Yes, there are several things that we can do now even for functions of one variable. Just because this course is functions of several variables, you should not be under any wrong impression.

Let us start with a simple example. Consider the function $y = \varphi(x)$ defined by the formula $x^2 + y^2 - 1 = 0$. Of course you might ask if there is such a function at all. This is a happy situation, you can explicitly solve the equation. There are two functions defined on the open interval $(-1, = 1)$ by the formula

$$\varphi(x) = +\sqrt{1 - x^2}; \qquad \psi(x) = -\sqrt{1 - x^2}.$$

Just a word about notation: I do not have to put + sign in describing φ because by convention a square root is always taken positive. However we did so to draw your attention.

$$\varphi'(x) = -\frac{x}{\sqrt{1 - x^2}}.$$

Here is another way of arriving at the answer. Consider the function

$$f(x, y) = x^2 + y^2 - 1$$

on R^2 and $F(x) = f(x, \varphi(x))$. Then we know that $F \equiv 0$ and hence $F'(x) \equiv 0$. But by chain rule

$$F'(x) = (1, \varphi'(x)) \cdot (2x, 2y) = 2x + 2y\varphi'(x) \equiv 0.$$

This gives

$$\varphi'(x) = -\frac{x}{y} = -\frac{x}{\sqrt{1 - x^2}}.$$

Of course what is the purpose of using the chain rule when you can get explicit formula for your function and calculate using expertise of last semester. As far as this example is concerned this is just another way of doing it. But sometimes this may be the only way of doing it! This happens especially when you are not lucky to get a formula in your hand.

Let us now consider another example.

Consider the function $y = \varphi(x)$ on the interval $(1, \ \infty)$ defined by the formula

$$y = \log(x + y).$$

is there such a function? Yes, consider the equation

$$x = e^y - y.$$

The right side as a function of y starts off at 1 when $y = 0$; derivative being positive it strictly increases towards infinity as y becomes large. Thus it assumes all values between one and infinity exactly once as y travels from zero to infinity. Thus given any number $x > 1$ there is exactly one $y > 0$ satisfying the above equation. This is the number $\varphi(x)$.

Is this function differentiable? We do not have explicit formula. suppose it is differentiable. What is φ'? Again as above consider the function

$$f(x, y) = y - \log(x + y)$$

so that

$$F(x) = f(x, \varphi(x)) \equiv 0.$$

Thus

$$F'(x) = (1, \varphi'(x)) \cdot (\frac{1}{x+y}, 1 - \frac{1}{x+y}) = 0.$$

giving

$$\varphi'(x) = \frac{1}{x+y} \ \frac{x+y}{x+y-1} = \frac{1}{x+y-1}.$$

Thus without any explicit formula for our function, we have been able to differntiate.

Of course, you can still avoid two variable calculus. You can say, consider

$$\psi(y) = e^y - y : (0, \infty) \to (1, \infty)$$

Thus $\varphi(x)$ is precisely inverse of this map and we know

$$\varphi'(x) = \frac{1}{\psi'(\varphi(x))} = \frac{1}{e^y - 1} = \frac{1}{x + y - 1}.$$

But many times even such a recognition is not possible. Then we need to use the method outlined above.

Such functions are called 'implicit functions'. That is, functions that *are there already* in the relation you want to be satisfied. They may have explicit formula or may not.

But as soon as you know that you have a differentiable function $y = \varphi(x)$ satisfying the relation $f(x, y) \equiv 0$ then the above argument tells us that

$$\varphi'(x) = -\frac{f_1(x, \varphi(x))}{f_2(x, \varphi(x))}.$$

Thus we need to understand the problem: given a relation to be satisfied between x and y is there a function $\varphi(x)$ so that when you take $y = \varphi(x)$ then (x, y) satisfies your relation. Is such a function differentiable? This is precisely the question answered by the 'implicit function theorem'.

Imagine starting with the relation $x^2 + y^2 + 1 = 0$, so that,

$$f(x, y) = x^2 + y^2 + 1$$

and the above formula gives

$$\varphi'(x) = -\frac{x}{y}.$$

But this is non-sense because there is no function at all!

So let us take a point (x_0, y_0) satisfying the given relation. You can see from the formula for the derivative of $y = \varphi$, we need $f_2(x_0, y_0) \neq 0)$. Surprisingly this condition is enough.

Theorem (Implicit function theorem)

Let $f : \Omega \subset R^2 \to R$ be a C^1 function. Suppose $(a, b) \in \Omega$ Suppose $f(a, b) = 0$ and $f_2(a, b)] \neq 0$. Then

(i) there is a rectangle $Q = (a - \delta, a + \delta) \times (b - \eta, b + \eta) \subset \Omega$ and a unique function φ defined on the interval $(a - \delta, a + \delta)$ whose graph is contained in the rectangle Q and such that $f(x, \varphi(x)) \equiv 0$.

(ii) This function is differentiable.
(iii)

$$\varphi'(x) = -\frac{f_1(x, \varphi(x))}{f_2(x, \varphi(x))}.$$

We shall prove the implicit function theorem. Recall

Theorem (Implicit function theorem)

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(ii) This function is differentiable.

(iii)

$$\varphi'(x) = -\frac{f_1(x,\varphi(x))}{f_2(x,\varphi(x))}.$$

Let us assume without loss of generality that $f_2(a,b) = 2m > 0$. Since f is C^1, we can fix a rectangle $S = (a-\delta_1, a+\delta_1) \times (b-\eta, b+\eta)$ such that

$$[a-\delta_1, a+\delta_1] \times [b-\eta, b+\eta] \subset \Omega \text{ and } f_2(x,y) > m \text{ for } (x,y) \in S.$$

Here is the idea. Since $f_2 > 0$ on S, for each fixed $x \in (a-\delta_1, a+\delta_1)$, the function $y \mapsto f(x,y)$ is strictly increasing on $[b-\eta, b+\eta]$. If only we can ensure that it is of opposite sign at the end points, there will be exactly only number y in between where $f(x,y)$ takes the value zero and that number will be our $\varphi(x)$. this plan is executed by taking a suitable δ smaller than δ_1. This choice is made after seeing (or getting an idea of) the values $f(x,y)$ at the end points $y = b-\eta$ and $y = b+\eta$.

Choose $M > 0$ so that $|f_1(x,y)| < M$ and $|f(x,y)| < M$ on S. This is possible because these are continuous functions on the closed rectangle.

Let us take one x, $a-\delta_1 < x < a+\delta_1$. Using $f(a,b) = 0$ we have

$$f(x,b+\eta) = f(x,b+\eta) - f(x,b) + f(x,b) - f(a,b) = f_2(P_2)\eta + f_1(P_1)(x-a)$$

by Mean value theorem (one variable), where P_2 and P_1 are points on appropriate lines, but within the rectangle we have. Thus

$$f_1(P_1) < M, a-\delta_1 < x < a \Rightarrow f_1(P_1)(x-a) \geq -M(a-x)$$

whereas

$$f_1(P_1) > -M, a < x < a+\delta_1 \Rightarrow f_1(P_1)(x-a) \geq -M(x-a)$$

Thus in any case

$$f(x, b+\eta) \geq m\eta - M|x-a|.$$

Let us choose $\delta > 0$ smaller than δ_1 and also smaller than $m\eta/M$. Then,

$$|x-a| < \delta \Rightarrow M|x-a| < m\eta \Rightarrow f(x, b+\eta) > 0.$$

Similarly, let us take an x, $a-\delta_1 < x < a+\delta_1$. Using $f(a,b) = 0$ we have exactly as above,

$$f(x, b-\eta) = f(x, b-\eta) - f(x,b) + f(x,b) - f(a,b) = f_2(P_2)(-\eta) + f_1(P_1)(x-a)$$

so that

$$f_1(P_1) > -M, a-\delta_1 < x < a \Rightarrow f_1(P_1)(x-a) \leq -M(a-x)$$

whereas

$$f_1(P_1) < -M, a < x < a+\delta_1 \Rightarrow f_1(P_1)(x-a) \leq -M(x-a)$$

Thus in any case

$$f(x, b-\eta) \leq -m\eta + M|x-a|.$$

$$|x-a| < \delta < m\eta/M \Rightarrow M|x-a| < m\eta \Rightarrow f(x, b-\eta) < 0.$$

Thus with this choice of δ we have achieved what we wanted. Thus we have a function φ whose value at $x \in (a-\delta, a+\delta)$ is the unique $y \in (b-\eta, b+\eta)$ with $f(x,y) = 0$. As a result we have $f(x, \varphi(x)) \equiv 0$.

shall now show φ is a continuous function on the interval $(a-\delta, a+\delta)$. Let us take two points x and $x+h$ in this interval. Then by the mean value theorem (two variable)

$$0 = f(x+h, \varphi(x+h)) - f(x, \varphi(x)) = f'(P) \cdot (h, \varphi(x+h) - \varphi(x)).$$

that is,

$$f_1(P)h + f_2(P)[\varphi(x+h) - \varphi(x)] = 0.$$

Thus

$$|\varphi(x+h) - \varphi(x)| \leq \frac{M}{2m}|h|.$$

This shows uniform continuity of the function φ.

We shall now show that φ is differentiable. The same calculation above gives

$$\frac{\varphi(x+h) - \varphi(x)}{h} = -\frac{f_1(P)}{f_2(P)}.$$

Of course the point P above depends on h, though not visible in the notation. It is on the line joining $(x+h, \varphi(x+h))$ and $(x, \varphi(x))$. As $h \to 0$ we see that this point converges to $(x, \varphi(x))$, we already know that φ is continuous. Thus

$$\lim_{h \to 0} \frac{\varphi(x+h) - \varphi(x)}{h} = -\frac{f_1(x, \varphi(x))}{f_2(x, \varphi(x))}.$$

Remember that $f_2 > 0$ for all points under consideration.

Of course, the formula above shows that φ is C^1 as well.

Uniqueness is clear in this rectangle $(a-\delta, a+\delta) \times (b-\eta, b+\eta)$ because there is only one point y for each x satisfying $f(x,y) = 0$.

This completes proof of the theorem.

Sometimes this is also stated as follows. One does not say unique function with graph contained in the rectangle. One states as follows: there is an interval $(a-\delta, a+\delta)$ and a unique function φ on this interval with

(i) $\varphi(a) = b$ and $f(x, \varphi(x)) = 0$ for all x in this interval.
Further this function satisfies conditions (ii) and (iii) above.

Uniform continuity:

We are trying to understand whether the new knowledge gained about functions of several variables tells us some new things about functions of one variable which we could not have seen last semester. Yes, sometimes a function of one variable may itself arise from functions of two variables. Before proceeding further, we need to understand uniform continuity.

Suppose that $S \subset R^2$ is a closed bounded set and $f : C \to R$ be a continuous function. Then f is uniformly continuous. That is, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$a, b \in S; \quad ||a - b|| < \delta \Rightarrow |f(a) - f(b)| < \epsilon.$$

Recall that continuity is also defined exactly the same way, but first we defined continuity at a point a and f is continuous if it is continuous at every point a. Thus for a continuous function, given a point a and $\epsilon > 0$, there is a $\delta > 0$ which may depend not only on ϵ but also on the point a so that

$$b \in S; \quad ||a - b|| < \delta \Rightarrow |f(a) - f(b)| < \epsilon.$$

Thus uniform continuity demands that given $\epsilon > 0$ we should be able to find one $\delta > 0$ that works for all points a.

We have seen continuous functions that are not uniformly continuous in case of R. similar examples can be constructed in R^2 and so on. The above theorem was proved for R and the same proof works even in R^2. here it is.

Let $\epsilon > 0$ be given. Suppose that we can not find $\delta > 0$ as stated. Then for each $n \geq 1$, we can find two points $a_n, b_n \in S$ so that $||a_n - b_n|| < 1/n$ but $|f(a) - f(b)| \geq \epsilon$. By taking subsequence if necessary, we can assume that the sequences (a_n) and (b_n) converge. But then they converge to the same point $v \in S$. But then continuity of f leads to $\lim\limits_n |f(a_n) - f(b_n)| = 0$ whereas each of these quantities are at least ϵ..

Maximum: $\max\limits_y f(x, y)$

Let $f(x, y)$ be a continuous function on a rectangle $S = [a, b] \times [c, d]$. Put

$$\varphi(x) = \max\{f(x, y) : c \leq y \leq d\}; \quad x \in [a, b].$$

then φ is a continuous function.

In fact given $\epsilon > 0$, using uniform continuity of f, select $\delta > 0$ such that

$$P, Q \in S; \quad ||P - Q|| < \delta \Rightarrow |f(P) - f(Q)| < \epsilon/2.$$

We claim that

$$x_1, x_2 \in [a, b]; \quad |x_1 - x_2| < \delta \Rightarrow |\varphi(x_1) - \varphi(x_2)| < \epsilon.$$

Indeed, for any y, choosing $P = (x_1, y)$ and $Q = (x_2, y)$ we see

$$f(x_2, y) - \epsilon/2 \leq f(x_1, y) \leq f(x_2, y) + \epsilon/2.$$

Taking supremum through out we get

$$\varphi(x_2) - \epsilon/2 \leq \varphi(x_1) \leq \varphi(x_2) + \epsilon/2.$$

That is

$$|\varphi(x_1) - \varphi(x_2)| < \epsilon.$$

This result can be interpreted as follows. We know that supremum of a finite number of continuous functions is continuous. We also now that this need not be true for a sequence of continuous functions. Here we have a family of continuous functions, namely one function $x \mapsto f(x,y)$ for each $y \in [c,d]$. We are asserting that supremum of this bunch of functions is again a continuous function.

integral of a function of two variables:

Let f be a continuous function on $S = [a,b] \times [c,d]$. Put

$$\varphi(x) = \int_c^d f(x,y)dy.$$

Then φ is a continuous function on $[a,b]$.

Given $\epsilon > 0$, choose $\delta > 0$ as above, using uniform continuity, so that

$$||P - Q|| < \delta \Rightarrow |f(P) - f(Q)| < \epsilon/(d-c).$$

If $|x_1 - x_2| < \delta$ then,

$$|\int_c^d f(x_1,y)dy - \int_c^d f(x_2,y)dy| \le \int_c^d |f(x_1,y) - f(x_2,y)|dy < \epsilon.$$

This can be interpreted as follows. Sum of two continuous functions is continuous. But sum of a sequence of continuous functions need not be continuous. What we are saying here is the following. We have a bunch of continuous functions $x \mapsto f(x,y)$ one for each y. Their 'continuous sum' which you think of integral, is continuous.

differentiation under integral:

Let f be a continuous function on $S = [a,b] \times [c,d]$. Suppose that f_1 is a continuous function on S. Let

$$\varphi(x) = \int_c^d f(x,y)dy.$$

Then φ is C^1 and

$$\varphi'(x) = \int_c^d f_1(x,y)dy.$$

In symbols

$$\frac{d\varphi}{dx} = \int_c^d \frac{\partial f}{\partial x}(x,y)dy.$$

or
$$\frac{d}{dx}\int_c^d f(x,y)dy = \int_c^d \frac{\partial f}{\partial x}(x,y)dy.$$
This has the following interpretation. We know that sum of a finite number of differentiable functions is again differentiable and derivative of sum equals sum of derivative. We are here saying continuous analogue of this statement. Derivative of integral equals integral of derivative. This is also referred to, as the symbols suggest, interchange of the order of derivative and integration.

proof is not difficult. Since f_1 is uniformly continuous, given $\epsilon > 0$, choose $\delta > 0$ as earlier using $\epsilon/(d-c)$. Now take any x. Let $|h| < \delta$.
$$\left|\frac{\varphi(x+h)-\varphi(x)}{h} - \int_c^d f_1(x,y)dy\right|$$
$$= \left|\int_c^d [\frac{f(x+h,y)-f(x,y)}{h} - f_1(x,y)]dy\right|$$
But by mean value theorem (one variable) the fraction in the integrand equals $f_1(P)$ for some P. Since P is on the line joining (x,y) and $(x+h,y)$ and since $|h| < \delta$; we conclude that the integrand is at most $\epsilon/(d-c)$. This is true for each y to complete the proof.

Change of order of integration:

Let f be a continuous fundtion on $S = [a,b]\times[c,d]$. Then
$$\int_c^d \left[\int_a^b f(x,y)dx\right]dy = \int_a^b \left[\int_c^d f(x,y)dy\right]dx.$$

Note that both sides make sense because we already know that after one integration the function you get is again continuous function (of the remaining variable) and hence can be integrated.

These integrals are called repeated integrals. Thus if you are doing 'repeated integration' of a continuous function on a closed bounded rectangle, then the order does not matter.

A natural question should occur to you. Can we not imitate integration we learnt and adapt it to functions of two variables? For example divide the rectangle into smaller ones, calculate the upper and lower sums and so on? Yes, can and will be done. These are called double integrals.

To prove the theorem, put

$$\Phi(u,y) = \int_a^u f(x,y)dx; \quad a \le u \le b; \quad c \le y \le d$$

Fundamental theorem of calculus (one variable, anyway, this is the only one we have at this stage) tells us that (for each fixed y) Φ is differentiable w.r.t. x and

$$\frac{\partial}{\partial u}\Phi(u,y) = f(u,y).$$

In other words Φ and Φ_1 are continuous functions on S.

Using the earlier theorem we conclude that

$$\frac{d}{du}\int_c^d \Phi(u,y)dy = \int_c^d f(u,y)dy.$$

Integrating both sides from a to b we get

$$\int_c^d \Phi(b,y)dy - \int_c^d \Phi(a,y)dy = \int_a^b \int_c^d f(u,y)dydu$$

Noting $\Phi(a,y) = 0$ and substituting $\Phi(b,y)$ we get the stated equation.

All this is fine, but the most important cases are when the integrand is unbounded or when the range of integration is infinite — in other words improper integrals. In such cases the above results would not apply. However, we can use these special cases to prove more general theorems.

But why are we doing all this? Well, these results enable us to understand some functions better, enable us to evaluate some integrals which can not be evaluated by usual methods etc. For example let us consider the function

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt; \quad x \in (0,\infty).$$

We know that this is a well defined function, that is, for every $x > 0$ the integral does give us a number. It is pertinent to ask if this is a differentiable function and if so whether

$$\Gamma'(x) = \int_0^\infty e^{-t}t^{x-1}\log t \;\; dt; \quad x \in (0,\infty).$$

What about second derivative? There are several other interesting integrals that appear in practice.

At some stage, to understand things better we have specialised to functions of two variables. We should keep in mind that this is done only to facilitate understanding and these theorems hold for any number of variables.

There are several directions in which our course can proceed.

We can discuss analogues of the above theorems for improper integrals and understand their use.

We have mentioned about double integral. We can discuss its development parallel to what we have learnt last semester and its relation with repeated integrals we talked about. This brings us to discover change of variables formula. After all, just as last semester we had areas in mind, we can now ask abut finding volumes.

We can use our knowledge to understand geometry, nature of curves and surfaces.

Amidst all these we should not forget one important thing: We knew for one variable C^1 function f, if $f'(a) \neq 0$ then there is a small interval $I = (a - \delta, a + \delta)$ in which the function is one to one, takes you to another interval J and you can define inverse function g on this interval J and $g'(y) = 1/f'(x)$ if $f(x) = y$. What is its analogue for functions of two variables?

And so on!

unbounded intervals/functions:

We discussed functions of one variable obtained from two variable functions by performing integration etc. We also showed that derivative of integral equals integral of derivative (loosely speaking). But the entire action took place over bounded intervals. Most of the integrals that we need in practice are over unbounded intervals. Even if it is over a bounded interval, the function is usually unbounded. These are what are (unfortunately) called improper integrals.

Instead of discussing general theory we illustrate with an example. Consider

$$f(x,t) = e^{-t}t^{x-1}, \qquad (x,t) \in (0,\infty) \times (0,\infty).$$
$$\Gamma(x) = \int_0^\infty f(x,t)dt, \qquad x \in (0,\infty).$$

We have seen last semester that this integral is finite. We shall now show that this is a continuous function of x on the interval $(0,\infty)$. For this it suffices to show that the two functions

$$\varphi(x) = \int_1^\infty f(x,t)dt; \qquad \psi(x) = \int_0^1 f(x,t)dt$$

are continuous functions.

continuity of φ:

Fix $0 < a < b < \infty$. Enough to show that φ is continuous on the interval $[a,b]$. Indeed to show continuity at a point α, use the fact that it is continuous on the interval $[\alpha/2,\ 2\alpha]$.

If you fix any number $c > 1$ then the function

$$\varphi_c(x) = \int_1^c f(x,t)dt$$

is a continuous from what we have discussed earlier. We show that $\varphi_c \to \varphi$ uniformly over $[a,b]$ as $c \to \infty$.

Let $\epsilon > 0$ be given. Choose $C > 1$ so that

$$\int_C^\infty f(b,t)dt < \epsilon.$$

This is possible because $\varphi_c(b) \to \varphi(b)$. Now for $x \in [a, b]$, the fact $t \geq 1$ implies $f(x, t) \leq f(b, t)$. Thus for any $c > C$ we have

$$|\varphi_c(x) - \varphi(x)| \leq \int_C^\infty f(x,t)dt \leq \int_C^\infty f(b,t)dt < \epsilon.$$

<u>continuity of ψ:</u>

Again we fix $0 < a < b < \infty$ and show ψ is continuous on $[a, b]$.
If you fix any number $0 < c < 1$ then the function

$$\psi_c(x) = \int_c^1 f(x,t)dt$$

is a continuous from what we have discussed earlier. We show that $\psi_c \to \psi$ uniformly over $[a, b]$ as $c \to 0$.

Let $\epsilon > 0$ be given. Choose C so that

$$\int_C^\infty f(a,t)dt < \epsilon.$$

This is possible because $\psi_c(a) \to \psi(a)$. Now if we take any $x \in [a, b]$ note that $t \leq 1$ tells us that $f(x, t) \leq f(a, t)$. Thus for any $0 < c < C$ we have

$$|\psi_c(x) - \psi(x)| \leq \int_0^C f(x,t)dt \leq \int_0^C f(a,t)dt < \epsilon.$$

This completes the proof that the Gamma function is a continuous function.

The function $\Gamma(x)$ is a differentiable and

$$\Gamma'(x) = \int_0^\infty e^{-t}\, t^{x-1}\, \log t\, dt.$$

In other words, you can differentiate under the integral sign.

First we show that this function is a continuous function and the argument is similar to the above by getting bounds. One needs to go a little below a or a little above b to compensate for the log factor. Then one shows differentiability under the integral sign by imitating argument similar to the one we used in power series: tail difference quotients are small and over bounded interval you can differentiate under integral sign.

I will not execute this. It is trivial if you have understood the idea and it needs maturity and clarity in thought. It appears highly non-trivial otherwise. So let us wait for some time. First you assimilate what is done (and understand why it is done!). These are important matters.

Inverse function theorem:

Recall that the inverse function theorem in one variable is the following.

Let $\Omega \subset R$ be an open interval, $f : \Omega \to R$ be C^1, and $a \in \Omega$. Suppose that $f'(a) \neq 0$. Then there is an open interval $U \subset \Omega$ and an open interval V such that the following hold:
(i) $a \in U$ and $b = f(a) \in V$.
(ii) f is one-one on U onto V.
(iii) The inverse map $g : V \to U$ is differentiable and for $y = f(x) \in V$ we have $g'(y) = 1/[f'(x)]$.

What is its analogue for functions of two variables? Interestingly, the exact same theorem is true. here it is.

Theorem: Let $\Omega \subset R^2$ be open, $f : \Omega \to R^2$ be C^1, and $a \in \Omega$. Suppose that $f'(a)$ (which is a 2×2 matrix) is non-singular. Then there is an open set $U \subset \Omega$ and an open set V such that the following hold:
(i) $a \in U$ and $b = f(a) \in V$.
(ii) f is one-one on U onto V.
(iii) The inverse map $g : V \to U$ is differentiable and for $y = f(x) \in V$ we have $g'(y) = [f'(x)]^{-1}$.

Let us understand the theorem. For every $x \in \Omega$, f associates a point of R^2, let its coordinates be denoted by $f_1(x)$ and $f_2(x)$. Thus $f(x) = (f_1(x), f_2(x))$. Then function f is C^1 means the two real valued functions f_1 and f_2 have continuous partial derivatives. The derivative at a point x is the linear transformation whose matrix representation is

$$f'(x) = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix} = \begin{pmatrix} D_1 f_1(x) & D_2 f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) \end{pmatrix}.$$

Why is this theorem non-trivial in R^2. After all, in R we said the following. Assume without loss of generality $f'(a) > 0$. Then in a small interval $I = (a - \delta, a + \delta)$ around a, we have $f'(x) > 0$ and $[a - \delta, a + \delta] \subset \Omega$. denote $y_1 = f(a - \delta)$ and $y_2 = f(a + \delta)$. Thus f is strictly increasing on $[a - \delta, a + \delta]$

and transforms it to $[y_1, y_2]$. It is then clear that f transforms the interval I onto $J = (y_1, y_2)$. The fact that it is increasing makes matters simple. Unfortunately in R^2 such an argument is no longer possible.

On R, if $f' \neq 0$ then the fact that f' is continuous tells us that either $f' > 0$ always or $f' < 0$ always. Thus on all of the interval Ω the function is one-one. Again this depends on the fact that f is strictly increasing or strictly decreasing. However such a conclusion can not be drawn in R^2.

Consider

$$f : R^2 \to R^2; \quad f(x, y) = (e^x \cos y, e^x \sin y).$$

Just be careful, though we denote points, in general, by $x = (x_1, x_2)$; in specific examples we do not follow this. This is done not to confuse you, but to make you comfortable. You are used to (x, y) for points of R^2 and it is better to keep it that way.

For the function above

$$f'(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

which is non-singular, in fact its determinant is e^{2x}. Thus at every point in R^2 the derivative is an invertible linear transformation. Yet this function is not one-one.

Its range is all of R^2 except the point $(0, 0)$. Each value in the range is assumed infinitely many times. Each horizontal strip $R \times [k2\pi, (k+1)2\pi)$ is mapped to $R^2 - \{(0, 0)\}$. Any horizontal strip of width 2π does this. For each fixed x the values of f trace the circle of radius e^x around the origin. Of course, in this case given any point you can clearly plot an disc around that point on which f is one-one and invertible.

Before starting proof of the theorem, we make three observations about the landscape in R^2. These are true in all R^n, but that is for later.

1. Let us use the word 'compact' to denote closed bounded sets. here is a fact.

If $K \subset R^2$ is compact and $f : K \to R^2$, then its range $f(K)$ is again compact.

This is easy to see because from what we have already proved about real valued functions, the first coordinates of points in $f(K)$ form a bounded set and so is the set of second coordinates of points in $f(K)$. This is enough to conclude that $f(K)$ is a bounded set. to show that it is closed, let $y_n \to y$ and $f(x_n) = y_n$. We exhibit x so that $f(x) = y$. This will show that the limit y is in $f(K)$ showing that $f(K)$ is closed. Of course if x_n converges to x, then K being closed we se $x \in K$ and continuity of f tells us that

$$f(x) = \lim f(x_n) = \lim y_n = y.$$

If x_n does not converge, take a subsequence of (x_n) that converges and argue with its limit. Note that K being compact every sequence in K has a limit point and every sequence has a convergent subsequence.

2. Let K be a compact set and $z \notin K$. Then there is an $r > 0$ such that the ball $B(z, r) \subset K^c$. recall ball $B(z, r)$ means the set of points x such that $||x - z|| < r$.

This is easy because our earlier characterisation: a set is closed iff its complement is open. Thus z is in the open set K^c and hence a ball around z is contained in this open set.

3. Let $U \subset R^2$ be an open set and $f : U \to R^2$ be continuous. then for any open set $V \subset R^2$ we have $f^{-1}(V)$ is an open subset of U.

This again clear because if $x \in f^{-1}(V)$, then $f(x) \in V$. Since V is open there is $\epsilon > 0$ so that $B(f(x), \epsilon) \subset V$. Continuity of f gives a $\delta > 0$ so that $B(x, \delta) \subset f^{-1}(V)$.

We shall prove the theorem assuming that $f'(a)$ is the identity matrix. That is,

$$f'(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (\spadesuit)$$

This will make things neat, otherwise we need to hang on to maximum of its entries etc making some of the estimates ugly. the general case then follows easily as we see later. We start drawing some simple consequences of out hypotheses.

4. Non-singularity is expressed in terms of the determinant, namely

$$D_1 f_1(x) D_2 f_2(x) - D_2 f_1(x) D_1 f_2(x)$$

which is a continuous function of x we see that if it is non-zero at a, then it is non-zero in a ball around a.

5. There is a ball around a such that for x in that ball

$$|D_j f_i(x) - D_j f_i(a)| < 1/4.$$

This is follows from the continuity of the partial derivatives.

6. There is a ball around a such that for all points $x \neq a$ in that ball, $f(x) \neq f(a)$.

If this is false, we get $x_n \neq a$, $x_n \in B(a, 1/n)$, $f(x_n) = f(a)$. Since $f'(a) = I$ we conclude

$$\frac{||f(x_n) - f(a) - I(x_n - a)||}{||x_n - a||} \to 0.$$

But $f(x_n) = f(a)$ tells that this ratio is always one, leading to a contradiction.

7. Take an open ball $B(a, r)$ such that the closed ball

$$\overline{B(a,r)} = \{x : ||x - a|| \leq r\} \subset \Omega$$

and **4,5,6** hold in this ball. we claim that

$$x, \tilde{x} \in B(a,r) \Rightarrow ||f(x) - f(\tilde{x})|| \geq \frac{1}{2}||x - \tilde{x}||. \qquad (\clubsuit)$$

Note that this will in particular shows that f is one-to-one in this ball. In fact norm being continuous function,the inequality remains true in the closed ball and that f is actually one-to-one in the closed ball.

This inequality is shown as follows. Consider the function $g(x) = f(x) - x$ in the closed ball. The mean value theorem applied to the first and second coordinate functions

$$g_1(x) = f_1(x) - x_1; \qquad g_2(x) = f_2(x) - x_2$$

of g will give the following (use **5.**)

$$|[f_1(x) - x_1] - [f(\tilde{x}_1) - \tilde{x}_1]| \leq \sqrt{\frac{2}{16}}\ ||x - \tilde{x}||$$

and

$$|[f_2(x) - x_2] - [f(\tilde{x}_2) - \tilde{x}_2]| \leq \sqrt{\frac{2}{16}}\ ||x - \tilde{x}||$$

so that

$$||[f(x) - x] - [f(\tilde{x}) - \tilde{x}]|| \leq \sqrt{\frac{4}{16}}\ ||x - \tilde{x}|| = \frac{1}{2}||x - \tilde{x}||.$$

Thus using triangle inequality,

$$\begin{aligned}||x-\tilde{x}|| &\le ||[f(\tilde{x})-\tilde{x}]-[f(x)-x]|| \quad + \quad ||f(x)-f(\tilde{x})|| \\ &\le \frac{1}{2}||x-\tilde{x}|| + ||f(x)-f(\tilde{x})||\end{aligned}$$

proving (♣).

8. Let us denote $f(a) = b$.

The set $K = \{x : ||x-a|| = r\}$ is clearly closed (norm is continuous) and bounded so that $f(K)$ is compact by **1.** and $b \notin f(K)$ by **7.** Hence by **2.** there is an $\eta > 0$ so $B(b, 2\eta) \cap f(K) = \emptyset$.

Let $V = B(b, \eta)$. Note that

$$y \in V \Rightarrow ||y - f(x)|| > ||y - f(a)|| \quad \forall x \in K \qquad (\bullet).$$

This is because $||y-b|| < \eta$ whereas $||y - f(x)|| \ge \eta$ for $x \in K$.

We now claim that for $y \in V$ there is a unique $x \in B(a, r)$ such that $f(x) = y$. That there can not be two points x and $\tilde{x}$ satisfying the condition follows from (♣). we only need to show the existence of a point.

As you realise, in the one dimensional case the strictly increasing nature and the intermediate value theorem for continuous functions settled the matter. Here we do not have it. But what you can not see easily will be shown by linear algebra as follows.

So take one $y \in V$ and define the function

$$\varphi(x) = ||f(x) - y||^2 = [f_1(x) - y_1]^2 + [f_2(x) - y_2]^2$$

on the compact set $\overline{B(a,r)}$. This real valued function assumes a minimum at some point. Also this is assumed in the open ball $B(a, r)$ because $(\bullet)$ shows that $\varphi(x) > \varphi(a)$ for all $x \in K$. Since the minimum is attained at a point, say x^* in an open set (not on the boundary) we conclude that $\nabla\varphi(x^*) = 0$. That is,

$$\begin{aligned}[f_1(x^*) - y_1]D_1f_1(x^*) + [f_2(x^*) - y_2]D_1f_2(x^*) = 0 \\ [f_1(x) - y_1]D_2f_1(x^*) + [f_2(x) - y_2]D_2f_2(x^*) = 0.\end{aligned}$$

If we denote

$$f_1(x^*) - y_1 = v_1; \quad f_2(x^*) - y_2 = v_2; \quad v = (v_1, v_2)$$

$$A = \begin{pmatrix} D_1 f_1(x^*) & D_2 f_1(x^*) \\ D_1 f_2(x^*) & D_2 f_2(x^*) \end{pmatrix}$$

then in matrix notation, we have

$$vA = 0.$$

But A is non-singular by **4.** Thus $v = 0$ which means $f(x^*) = y$.

9. Let $U = f^{-1}(V) \cap B(a, r)$. Then U is open by **3.** From **8.** we see that $f : U \to V$ is one-one and onto. Thus it has inverse $g : V \to U$.

A restatement of (♣) is

$$||g(y) - g(\tilde{y})|| \leq 2||y - \tilde{y}||; \qquad y, \tilde{y} \in V.$$

This shows that g is continuous.

10. Fix $y \in V$ and let $g(y) = x$, that is, $f(x) = y$. We show that g is differentiable at y and $g'(y) = [f'(x)]^{-1} = A^{-1}$, say. Take $y_n \in V$, $y_n \neq y$ for all n. Need to show

$$\frac{g(y_n) - g(y) - A^{-1}(y_n - y)}{||y_n - y||} \to 0. \qquad (*)$$

Since A is an invertible matrix, in order to show that $v_n \to 0$ for a sequence of vectors (v_n) one could as well show $Av_n \to 0$. Denoting $g(y_n) = x_n$, that is, $f(x_n) = y_n$ we need to show

$$\frac{A(x_n - x) - (f(x_n) - f(x))}{||f(x_n) - f(x)||} \to 0.$$

That is, need to show

$$\frac{(f(x_n) - f(x)) - A(x_n - x)}{x_n - x} \quad \frac{x_n - x}{||f(x_n) - f(x)||} \to 0.$$

Because of (♣) the second term is bounded by 2, the first term converges to zero because $A = f'(x)$.

This completes proof of the inverse function theorem.

inverse function theorem:

We proved the inverse function theorem assuming that $f'(a) = I$. We shall now deduce the general case. We start with some auxiliary observations.

Let A be a 2×2 matrix and consider the linear transformation of R^2 to itself, $\varphi(x) = Ax$. This is a continuous map, simply because each coordinate of $\varphi(x)$ is a linear combinations of coordinates of x. If A is non-singular, then it has an inverse A^{-1} and the map $x \mapsto A^{-1}x$ is likewise continuous.

When A is non-singular, then for every closed set S, the set

$$A(S) = \{Ax : x \in S\}$$

is closed. Indeed, if $y_n \in A(S)$ and $y_n \to y$, then $A^{-1}(y_n) \in S$ and $A^{-1}(y_n) \to A^{-1}(y)$ and S being closed we conclude $A^{-1}(y) \in S$ which shows that $y = A(A^{-1}y) \in A(S)$. Since closed sets are precisely complements of open sets, we conclude that for any open set V, the set $A(V)$ is open.

Returning to the inverse function theorem, let $f : \Omega \to R^2$ be C^1 function with $f'(a) = A$, non-singular. Then define the function $g(x) = A^{-1}f(x)$ on Ω. This is again a C^1 function and $g'(a) = A^{-1}A = I$. Thus there is are open sets U, V such that $a \in U$, $b = g(a) \in V$, g is one-to-one on U onto V, the inverse map $h : V \to U$ is C^1, and $h'(y) = [g'(x)]^{-1}$ for $y \in V$ and $g(x) = y$.

the set $W = A(V)$ is an open set. Easy to see that $f : U \to W$ is one-one on U onto W, indeed $f(x) = Ag(x)$. The inverse map $\xi : W \to U$ is given by $\xi(z) = h(A^{-1}z)$ and is hence composition of C^1 maps. Thus it is again C^1. Chain rule now verifies the formula for the derivative of the inverse map ξ.

Integration:

We shall now proceed to imitate the concept of lower sums and upper sums and the concept of integral for function of two variables. Of course, this is not just for the sake of imitation. Just as finding areas motivated us towards integration last semester, finding volumes is one of the reasons for integrating functions of two variables, We do not spend time but you should understand why we are doing all this.

Basically, we would like to start with a bounded function defined on a bounded region; partition the region into small sets; for each set T in the partition wish to calculate $a(T)M(T)$ where $a(T)$ is the area of T and $M(T)$ is the supremum of the function in that set T. Sum of all these will then give us the upper sum. Similarly we define the lower sum.

There is one problem in implementing above plan. How do we know the area of T in the calculation above? We follow the maxim: solve the simplest problem first. From high school we are familiar with rectangles and their areas. so we decide to partition into rectangles. But if your original region is not a rectangle, then you can not partition it into rectangles.

Thus we consider, as a first step, bounded functions defined on rectangle and also consider partitions into rectangles with sides parallel to the axes.

Till further announcement, rectangle always means usual rectangle with sides parallel to the two standard axes.

Let $S = [a, b] \times [c, d]$ be a (closed) rectangle and $f : S \to R$ be a bounded function. Recall a partition of the interval $[a, b]$ is a finite sequence of points

$$a = a_0 < a_1 < a_2 < \cdots < a_k = b$$

or equivalently the intervals

$$[a_0, a_1], \quad [a_1, a_2], \quad \cdots \quad , [a_{k-1}, a_k].$$

If we also have a partition of $[c, d]$

$$c = c_0 < c_1 < c_2 < \cdots < c_l = d$$

then we define product partition of the rectangle S as the collection of the non-overlapping rectangles,

$$S_{ij} = [a_i, a_{i+1}] \times [b_j, b_{j+1}]; \quad 0 \leq i \leq k-1; 0 \leq j \leq l-1.$$

Such partitions of S are called product partitions. It is denoted simply as $\Pi = \{S_{ij}\}$.

Let us agree on meanings to some phrases. if we have a rectangle $T = [\alpha, \beta] \times [\gamma, \delta]$ we say that points

$$\{(x, y) : \alpha < x < \beta;\ \gamma < y < \delta\}$$

are interior points of the rectangle T. In other words it is precisely the set $(\alpha, \beta) \times (\gamma, \delta)$. The remaining points of T are called boundary points of T. In other words boundary consists of

$$\{\alpha\} \times [\gamma, \delta] \ \cup \ \{\beta\} \times [\gamma, \delta] \ \cup \ [\alpha, \beta] \times \{\gamma\} \ \cup \ [\alpha, \beta] \times \{\delta\}.$$

Two rectangles are non-overlapping if any point common to both of them is boundary point of both the rectangles. Thus, as sets they may not be disjoint, but they have no common interior points. Any two different rectangles above are non-overlapping.

For a rectangle T as above, its area is $(\beta - \alpha) \times (\delta - \gamma)$. This is also denoted by $|T|$.

Let us now consider a product partition as above of S. Denote

$$M_{ij} = \sup\{f(x) : \ x \in S_{ij}\}, \quad m_{ij} = \inf\{f(x) : \ x \in S_{ij}\}.$$

$$U(f, \Pi) \ = \ \sum_{i,j} M_{ij}|S_{ij}| \quad L(f, \Pi) \ = \ \sum_{i,j} m_{ij}|S_{ij}|.$$

$U(f, \Pi)$ is called the upper sum for the partition Π and $L(f, \Pi)$ is called the lower sum.

Say that f is **integrable** if the supremum of all lower sums equals infimum of the upper sums (over all product partitions). We denote this common value by

$$\int_S f; \quad \int_S f(x, y)d(x, y), \qquad \int_{[a,b]\times[c,d]} f(x, y)d(x, y).$$

this is called the integral or double integral of f.

Here are standard facts whose proofs are exactly same (?) as in the one dimensional case.

Theorem: $L(f, \Pi) \leq U(f, \Pi)$.

Recall that a partition η of the interval $[a, b]$ is finer than a partition π if every point that appears in π also appears in η. In other words η is obtained by adding more (possibly zero) points to π. Thus a finer partition cuts the interval into more pieces or into finer pieces. A product partition Π_2 is finer than Π_1 if the corresponding partitions on each side are finer for Π_2.

Theorem: If Π_2 is finer than Π_1, then

$$L(f, \Pi_1) \leq L(f, \Pi_2); \qquad U(f, \Pi_1) \geq U(f, \Pi_2).$$

In other words as the partition becomes finer the upper sums reduce while the lower sums increase.

Since we are considering only product partitions the theorem above was stated for product partitions, but it is true for any partitions, one finer than the other.

Theorem: For any partitions Π_1 and Π_2, $L(f, \Pi_1) \leq U(f, \Pi_2)$.

Theorem: f is integrable iff for any $\epsilon > 0$, there is a product partition Π such that $U(f, \Pi) - L(f, \Pi) < \epsilon$.

Proof: f is integrable iff sup of lower sums equals inf of upper sums. this is same as saying that for every $\epsilon > 0$, there is a lower sum and an upper sum which are ϵ-close. Since lower sums increase and upper sums decrease, this is same as saying that there is one partition for which the lower and upper sums differ by at most ϵ.

Theorem: Every continuous function is integrable.

The proof is along the expected lines. Given $\epsilon > 0$, use uniform continuity to get a product partition so that within each rectangle of the partition the values of the function differ by at most $\epsilon / Aa(S)$. Remember $a(S) = (b - a)(d - c)$, area of S,

We have been considering product partitions. We can consider any partition into rectangles, in fact, any reasonable partition. But to see that this also leads to the same answer, we have to wait. Right now, we can at least observe one thing. Any partition into rectangles parallel to the axes can also be considered, we arrive at the same answer.

Let us for this para, by a partition mean partition into rectangles with sides parallel to the axes. Superficially it appears if you allow all possible, not necessarily product, partitions you have many possibilities. But the crucial fact is the following.

Theorem: given any partition (rectangles, sides parallel to the axes) Π there is a finer partition Π_1 which is product partition.

In fact you only have to consider all the corners of all the rectangles of the partition Π, take their first coordinates as the partition of the side $[a, b]$; take all the second coordinates of all the corners as the partition of $[c, d]$ and consider Π_1 to be the product of these partitions of the two sides.

Let A be the set of all lower sums corresponding to only product partitions and B be the set of all lower sums corresponding to all partitions (into rectangles with sides parallel to the axes). Thus $A \subset B$, so that $\sup A \leq \sup B$.

Since finer partitions (whether product or not) have larger lower sums, the theorem above tells that for every number in B there is something larger than that in A. Thus $\sup B \leq \sup A$. This shows that A and B have the same sup.

Thus whether you make lower sums with product partitions or you make lower sums with all possible partitions (rectangles with sides parallel to the axes) you get the same sup. Similar remark holds for the upper sums. As a consequence if you define integrability using these partitions you get nothing new.

Theorem: If f and g are (bounded) integrable then so is $f + g$ and cF where c is any number. Further,

$$\int (f + g) = \int f + \int g; \quad \int cf = c \int f.$$

First observe the following. If f and g are two functions (real valued) on a set T with supremums $M(f)$ and $M(g)$, then for any point $x \in T$

$$(f + g)(x) = f(x) + g(x) \leq M(f) + M(g).$$

Thus $M(f + g) \leq M(f) + M(g)$. Similar remark applies for the infimums. This leads to the followng. For any partition Π

$$L(f) + L(g) \leq L(f + g) \leq U(f + g) \leq U(f) + U(g).$$

Since f and g are integrable, take partitions Π_n such that

$$L(f, \Pi_n) \to \int f; \quad U(f, \Pi_n) \to \int f;$$

$$L(g, \Pi_n) \to \int g, \ U(g, \Pi_n) \to \int g.$$

Then the earlier display shows that

$$L(f+g, \Pi_n) \to \int f + \int g; \quad U(f+g, \Pi_n) \to \int f \int g.$$

Tis shows that $f+g$ is integrable and $\int (f+g) = \int f + \int g$.

Similar and simpler argument shows $\int cf = c \int f$.

Let S be union of non-overlapping rectangles $S_1, \cdots S_k$ all having sides parallel to the axes. Then

Theorem: f is integrable on S iff it is integrable on each S_i and then

$$\int_S f = \sum \int_{S_i} f.$$

Strictly speaking, we should not say f is integrable on S_i, we should say restriction of f to S_i is integrable on S_i. Similarly, we should be writing

$$\int_{S_i} f_i$$

where f_i is restriction of f to S_i. But we shall not be so strict.

Proof: If f is integrable on S then you can take a partition Π such that $U(f, S, \Pi) - L(f, S, \Pi) < \epsilon$. By taking a larger partition, if necessary, we assume that each S_i is union of sets in the partition. If this is not already so, you only need to put-in the x coordinates of all the corner points of all the S_i in the partition of $[a, b]$ on the x-axis, and similarly put-in the y coordinates of all the corner points of all the rectangles S_i to get a partition of $[c, d]$. clearly for each i, those sets in Π that are contained in S_i is a product partition Π_i of S_i and

$$U(f, S_i, \Pi_i) - L(f, S_i, \Pi_i) \leq U(f, S, \Pi) - L(f, S, \Pi) \leq \epsilon$$

showing that f is integrable on each S_i.

Conversely, if f is integrable on each S_i then you take product partition of η_i of S_i for each i so that

$$U(f, S_i, \eta_i) - L(f, S_i, \eta_i) < \epsilon/k.$$

Putting all the x coordinates of all the corners of all the partition rectangles together we get a partition of $[a, b]$ and and putting together all the

y-coordinates of all the corners of all the partition rectangles together we get a partition of $[c, d]$ and thus we get product partition Π of S. If we restrict this Π to S_i we get a product partition Π_i which is finer than the η_i we started with on S_i. Thus

$$U(f, S, \Pi) - L(f, S, \Pi) = \sum_1^k [U(f, S_i, \Pi_i) \leq \epsilon.$$

This shows that f is integrable on S.

In fact this last display can be strengthened into

$$U(f, S, \Pi) = \sum U(f, S_i, \Pi_i); \quad L(f, S, \Pi) = \sum L(f, S_i, \Pi_i)$$

Small sets:

In case of functions of one variable, we did not answer the question: which functions (bounded function on a bounded interval) are integrable. This was because most of the functions that we come across at the elementary level have finitely many discontinuities and we have shown that such functions are integrable. This was enough for life to go on.

Unfortunately, in higher dimensions, we do need to tackle this problem head on. The reason is the following. After all, we need to integrate functions which are not necessarily defined on rectangles, even if it is defined on rectangles, then the rectangle need not have sides parallel to the axes. Thus partitions that we have been considering will be not enough. Of course, we would not complicate too much. If f is afunction given on a bounded set Ω, we just enclose Ω in a rectangle S, extend f to all of Ω by defining its value to be zero for points of $S - \Omega$ to be zero.

But then even if we started with a nice continuous function on Ω, this extension is rarely continuous on S and we need some assurance that we have not destroyed continuity too much and this extended function on S is indeed integrable. If moreover this value does not depend on which rectangle you choose to enclose Ω, then we are justified to regard integral of this extended function on the rectangle S as integral of f on Ω.

A set is small if it can be fit into any bag, no matter how small is the bag. More precisely, a set $A \subset R^2$ is small if given any $\epsilon > 0$, we can get finitely many or countable many rectangles $S_1, S_2, \cdots$ such that

$$\sum a(S_i) < \epsilon; \qquad A \subset \cup_i S_i.$$

It does not matter whether we take closed rectangles or open rectangles in the above definition. The reason is the following. If you can do with open rectangles $(a_i, b_i) \times (c_i, d_i)$ then you can, without changing areas consider the closed rectangles $[a_i, b_i] \times [c_i, d_i]$. Conversely, if you can do with closed rectangles, such a simple minded argument of removing boundary will not work because those open rectangles may not cover all of the set A. But this is achieved as follows. Let $\epsilon > 0$ be given. Get closed rectangles with total area smaller than $\epsilon/2$ covering A. Increase each of the rectangles a little bit so that the area of i-th rectangle is increased by only $\epsilon/2^{i+2}$ an make it closed. These will do.

Clearly every single point set is small. Even a countable set is a small set because you can use $\epsilon/2^i$ argument. In fact the same $\epsilon/2^i$ allows you to show that union of countably many small sets is small. Also clear is that subset of a small set is small. It is a nice exercise to show that this concept agrees with your intuition by showing the following. A rectangle $[a, b] \times [c, d]$ with $a < b$, $c < d$ is indeed not small where as its boundary is small.

Here is then the relevance of small sets to our problem.

Theorem: Let f be a bounded function on a bounded rectangle $S = [a, b] \times [c, d]$. then f is integrable iff its set of discontinuity points form a small set.

We shall prove this theorem. But we first need some preliminaries.

Oscillation:

so we have to finally understand discontinuity points. Of course we did discuss a little about left limits, right limits and discontinuity points last semester. But now we need to get a quantitative feeling for discontinuity.

Let $f : S \subset R^2 \to R$ and $a \in S$. We want to understand how far away from continuity is f at the point a. for $\delta > 0$, let us define

$$O(a, \delta) = \sup\{f(x) : x \in [a - \delta, a + \delta]\} - \inf\{f(x) : x \in [a - \delta, a + \delta]\}.$$

Here we have used a notation, $[a - \delta, a + \delta]$ is not an interval but is the rectangle $[a_1 - \delta, a_1 + \delta] \times [a_2 - \delta, a_2 + \delta]$. This is suggestive if understood carefully, otherwise it would be confusing.

In case the function is continuous at a then definition of continuity tells us that, given any $\epsilon > 0$ we can choose a $\delta > 0$ so that $O(a, \delta) < 2\epsilon$. Observe that $O(a, \delta)$ decreases as $\delta \downarrow 0$. Let us define

$$O(a) = \lim_{\delta \downarrow 0} O(\delta).$$

This is called oscillation of the function at the point a. Here is its importance.

Theorem: f is continuous at a iff $O(a) = 0$.

In fact if the function is continuous we have already seen that given $\epsilon > 0$ there is $\delta > 0$ such that $O(a, \delta) < 2\epsilon$ and this holds for all smaller δ too. This shows that $O(a) = 0$. Conversely, if $O(a) = 0$ then given $\epsilon > 0$, there is a $\delta > 0$ so that $O(a, \delta) < \epsilon$. In particular $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ showing continuity of f at a.

Theorem: For any $\epsilon > 0$, the set $\{a \in S : O(a) \geq \epsilon\}$ is closed in S. That is, if $a_n \to a$ and all these points a_n and a are in S and if each a_n is in this set then so is a.

Note that if $O(a) < \epsilon$ then there is $\delta > 0$ so that $O(a, \delta) < \epsilon$. But then for every point $b \in (a - \delta, a + \delta)$ we have small square around b contained in $(a - \delta, a + \delta)$ which shows that $O(b) < \epsilon$. In other words none of the a_n are in this square ahowing a_n does not converge to a; a contradiction.

In defining oscillation we have used squares. It is alright to use recatngles, but then $O(\delta)$ will be indexed by two numbers $O(\delta_1, \delta_2)$, lengths of the two sides. You should take limit as both $\delta_1 \to 0$ and $\delta_2 \to 0$. You get nothing new, simply because every such rectangle contains a square.

You can also take instead of square around a, a disc around $B(a, \delta) = \{x : ||x - a|| \leq \delta\}$ and define $O(a, \delta)$. Again since every square contains a disc and conversely, you get nothing new.

As you may have noticed, we have been taking closed square or closed disc. But you can take open squares or open discs too. Since closed squares contain open squares and open square contains a closed square, we get nothing new.

compact sets:

A closed bounded subset $K \subset R^2$ is simply called a compact set. Compact sets have the following nice property: if some one puts into many bags, we can fit it in finitely many of those bags.

Theorem: Let K be a compact set and you have a collection of open rectangles which cover K. This means that every point of K is in one of these rectangles. Then you can select finitely many of the given rectangles which also cover K.

The argument is standard and was seen several times. First put the set K in a big closed bounded rectangle S. Let $\mathcal{U}$ be a collection of open rectangles covering K for which the conclusion is false. Cut the rectangle into four parts at the mid points of the sides, one of these parts can not be covered by finitely many sets from $\mathcal{U}$. Take one such part S_1. Cut this into four pieces as above and pick a part S_2 so that the part of K in S_2 can not be covered by finitely many sets from $\mathcal{U}$. Thus we get a sequence of closed rectangles (S_n) which are decreasing.n fact lengths of sides are hlf length of the previous one; the part of K in S_n can not be covered by finitely many sets from $\mathcal{U}$. Cantor intersection theorem gives you a point a common to all the rectangles which will be in S because S is closed. This point is in some open rectangkle T of the family $\mathcal{U}$
and hence some $S_n \subset T$ contradicting the choice of S_n.

back to integration:

suppose that $f : S \to R$ bounded function on a rectangle $S = [a, b] \times [c, d]$. If $O(a) < \epsilon$ for all $a \in S$, then there is a product partition Π such that $U(\Pi) - L)\Pi) < \epsilon a(S)$.

Proof. For every point $a \in S$ there is a open rectangle T_a such that $O(f, T_a) < \epsilon$. In stead of showing δ we are showing the rectangle T_a. These open rectangles cover S and use finitely many of them to cover. Take their intersection with S to obtain finitely many rectangles$T_1, T_2, \cdots, T_k$ which cover S and in each of them the sup minus inf of the fumction is $< \epsilon$.. Now do the usual thing. Take first coordinates of the corners of these rectangles and the second coordinates of the corners to obtain partition of the sides of S Take product partition. Within each rectangle for the partition, we have sup minus inf of the function is $< \epsilon$. Thus for this partition Π, we have $U(\Pi) - \text{Ł}(\Pi) < \epsilon a(S)$.

We shall now prove the theorem on integrability. Let f be a closed bounded rectangle and $f : S \to R$ be bounded, say $|f| \leq M$. Let us assume that the set D of points where f is not continuous is a small set. Fix $\epsilon > 0$. We wish to exhibit a partition so that $U - L < \epsilon$. The idea is the following. The set $D_\epsilon = \{x : O(x) \geq \epsilon/2a(S).\}$ is a closed and bounded set and hence is compact. Put each of these points in a small open rectangle of area $< \epsilon/2Ma(S)$. Take finitely many of these which cover D_ϵ. Let this part of S be denoted S_1. This finite collection already gives a partition of S_1. On the remaining part $S_2 = S - S_1$ the oscillation is small and we can make $U - L$ smaller than $\epsilon/2$ by proper choice of partition, using previous result. The previous theorem is for a rectangle and our S_2 is unlikely to be a rectangle. We need to carefully argue and this we shall do later.

This is simple and will be precisely executed later.

integrability:

we shall now prove the following result:

Q is a rectangle $[a, b] \times [c, d]$ and $f : Q \to R$ is a bounded function whose set of discontinuity points is a small set.
Then f is integrable.

Let $\epsilon > 0$. We shall exhibit Π such that $U(\Pi) - L(\Pi) < \epsilon$. Tis is done in a few simple steps. fix $M > 1$ such that $|f(p) \leq M$ for all points $p \in Q$. Let $\alpha = (b - a)(d - c)$, area of the rectangle Q.

First consider the set

$$D = \left\{ p \in Q : O(p) \geq \frac{\epsilon}{4M\alpha}. \right\}$$

we know that D is a closed bounded set, that is, compact. Since it is a subset of a small set, it is also small, we can cover it by finitely many open rectangles $\mathcal{S}_0$ whose total area is at most $\epsilon/4M$.

Take each open rectangle in $\mathcal{S}_0$, consider the corresponding closed rectangle and denote this family by $\mathcal{S}$. This is a finite family of closed rectangles with total area still at most $\epsilon/4M$. remember that each point of D is in the interior of one of these rectangles. we assume that all our rectangles are contained in Q, otherwise take the closed rectangle and intersect with Q. It is again a closed rectangle.

Consider the usual product partition starting from $\mathcal{S}$. That is, take all the corners of rectangles of $\mathcal{S}$, their x-coordinates will give a partition η_1 of $[a, b]$ and their y-coordinates gives a partition η_2 of $[c, d]$ and let $\eta = \eta_1 \times \eta_2$ be the product partition. Clearly, every rectangle in $\mathcal{S}$ is union of rectangles from η.

This partition has two types of rectangles. Type 1: part of a rectangle of $\mathcal{S}$. Type 2; rectangle that does not overlap with any rectangle of $\mathcal{S}$. Whatever partition Π we produce later on, it will be finer than the present partition η. Thus every rectangle $R \in \Pi$ will be contained in some rectangle of η. Let us say a rectangle of Π is type I if it is contained in a type 1 rectangle of η, otherwise Type II. Thus if we denote

$$A = \sum \ \{ \ [\sup_R f - \inf_R f] \ a(R) : \ \ R \in \Pi, \ \ \text{Type I}\}$$

$$B = \sum \ \{ \ [\sup_R f - \inf_R f] \ a(R) : \ R \in \Pi, \ \ \text{Type II}\}$$

$$U(\Pi) - L(\Pi) = A + B.$$

let us observe one thing. No matter what our future partition Π is, we have $A < \epsilon/2$. Because in each rectangle of type I, we can bound the sup minus inf by $2M$. Thus the sum corresponding to the type I rectangles is at most $2M$ times total area of those rectangles. But their total area is at most $\epsilon/4M$. Thus

$$A \leq 2M\frac{\epsilon}{4M} = \frac{\epsilon}{2}.$$

Let us consider a rectangle T of η of type II. At each point of T we have oscillation at most $\epsilon/4M\alpha$. Observe a subtle point. Each point of D is in the interior of one of the rectangles of $\mathcal{S}$. Thus if you take a rectangle of type II, then even at the boundary points of this rectangle we have oscillation smaller than $\epsilon/4M\alpha$. (if it were larger, the point would be in the interior of one of those rectangles etc). Thus by one of the previous theorems there is a partition of T, say $\pi(T)$ such that

$$U(\pi(T)) - L(\pi(T)) < \frac{\epsilon}{4M\alpha}a(T)$$

. Get such a partition for each T of type II. Consider the product partition made up of all the rectangles of all these $\pi(T)$ as T ranges over type II rectangles and all rectangles in η. Remember this means the following. Take all x-coordinates of all corners of all these rectangles mentioned and similarly the y-coordinates and consider the product partition Π.

Clearly this Π is finer than η. From the work we did above, we only need to show that that $B < \epsilon/2$. again keep a subtle point in mind. if you take the 'trace' of this grand partition on T above, you may not get back $\pi(T)$ because the other corners also influence this grand partition. Think about it. But of course the trace of this Π on T will be finer than $\pi(T)$ and hence by the property of upper and lower sums we have

$$\sum \ \{ \ [\sup_R f - \inf_R f] \ a(R) : \ R \in \Pi, \ \ R \subset T\}$$

$$\leq \ U(\pi(T)) - L(\pi(T)) \ \leq \ \frac{\epsilon}{4M\alpha}a(T).$$

as a consequence

$$B = \sum \{\ [\sup_R f - \inf_R f]\ a(R) :\ R \in \Pi,\ \ \text{Type II}\}$$

$$= \sum_{T \text{ type II}} \ \sum \{\ [\sup_R f - \inf_R f]\ a(R) :\ R \in \Pi,\ \ R \subset T\}$$

$$\leq \sum_{T \text{ type II}} \ U(\pi(T)) - L(\pi(T)) \ \leq \sum_{T \text{ type II}} \ \frac{\epsilon}{4M\alpha} a(T).$$

$$\leq \ \frac{\epsilon}{4M\alpha}\alpha \ \leq \ \frac{\epsilon}{2}.$$

This completes the proof.

Let f and g be two (bounded) functions on a (bounded) rectangle Q and assume that $D = \{(x,y) : f(x,y) \neq g(x,y)\}$ is contained in a small compact set. Then f is integrable iff g is integrable. More over when they are integrable, their integrals are same.

Proof is very simple. Incidentally the hypothesis implies that D itself is small. But the compact containment tells us that given any $\epsilon > 0$ you can cover the set by finitely many rectangles of small total area.

Suppose that f is integrable. Let $\epsilon > 0$ be given. Let $|f| < M$ and $|g| < M$. Take any partition Π of Q such that $U(\Pi, f) - L(\Pi, f) < \epsilon$. cover D by finitely many open rectangles of total area smaller than $\epsilon/(2M|Q|)$, where $|Q|$ is area of the rectangle Q. Take a product partition η finer than these finitely many rectangles and Π. When you calculate U or L then the summand that participates in the sum is same for both f and g in all rectangles except those that are involved in covering D. But for each of the rectangles T involved in covering D the summand is at most $2M|T|$ and hence their sum is at most $\epsilon/2$. Thus $U(f)$ and $U(g)$ differ by at most ϵ. Same argument shows that $L(f)$ and $L(g)$ also differ by at most ϵ. Thus in particular, $U(g) - L(g) < 4\epsilon$. Since $\epsilon > 0$ is arbitrary, this shows that g is integrable.

This also shows that integrals are same too. (why?)

definition of integral:

So far we have been discussing a very very special case of integration. We are dealing with integral for bounded functions defined on a bounded rectangle (with sides parallel to the usual axes). We have defined upper and lower sums by taking product partitions with sides parallel to the axes. These are the simplest.

The whole thing appears very very unsatisfactory. However when we complete the discussion you see that you can take your function on any kind of (reasonable) bounded region, not necessarily rectangle. you can also take any kind of reasonable partition of the region. I am using the adjective reasonable because we need to definitely put some conditions, but the condition would not be serious; in the sense you have to work hard to find cases not satisfying the conditions we put!

By the way, do keep in mind, we already know that we can take partitions into rectangles with sides parallel to the axes, not necessarily product partitions.. You get the same sup of all lower sums and you get the same inf of all upper sums.

So now let us implement the idea described before introducing small sets. Take a bounded set S and bounded real function f on S. Take any rectangle $Q \supset S$. This is possible because S is bounded. Define g on Q by $g(p) = f(p)$ if the point $p \in S$ and for points $p \in Q - S$ put $g(p) = 0$.

Say that f *is integrable on* S if g is integrable on Q and in that case declare value of the integral

$$\int_S f = \int_Q g.$$

The first question to be addressed is whether this definition depends on the Q taken. if you take a bigger rectangle $Q' \supset Q$, then you see that it makes no difference. You can express $Q' - Q$ as union of non-overlapping rectangles, one of them being Q. (First rigorously prove that if a rectangle is contained in another then the corresponding sides are contained in one another and proceed).

We already had a theorem: A function is integrable on a rectangle which is union of non-overlapping rectangles iff it is integrable on each of these rectangles and then integral is sum of the integrals. apply this theorem to see integrability of f as well as the value of integral remains same.

Now if you take two different rectangles $Q \supset S$ and $Q' \supset S$, possibly one not contained in the other, you can arrive at the same answer by taking another bigger rectangle which includes both Q and Q' and comparing both with that rectangle.

The second thing to be attended to is whether it gives the same answer as earlier in case the set S is already a rectangle. This is easily settled because you can take the original rectangle itself as the Q containing S.

The third question is whether many functions are integrable and whether integral has properties that we would like, linearity etc. We shall discuss this now.

Even though we made the definition for functions defined on an arbitrary set S, we shall be interested in concrete sets. For example f be a continuous function on the disc $\{(x, y) : x^2 + y^2 \leq 1\}$, is it integrable? Or f is a continuous function on a closed triangle or quadrilateral which may not be a rectangle. is it integrable? all these questions are answered rather easily. if you take any rectangle which contains the disc or triangle or whatever and define g on the rectangle as suggested, namely, define zero for the new points, then it is easy to see that this function has all its discontinuity points contained the boundary of the disc/triangle/quadrilateral. Thus the only thing one needs to verify is that these sets are small.

The set $D = \{(x, y) : 0 \leq x = y \leq 1\}$ is a small set. Indeed, the set

$$\bigcup_{1}^{n} \; [(k-1)/n, \; k/n] \; \times \; [(k-1)/n, \; k/n]$$

is a union of rectangles, contains D and sum of areas of these rectangles is $1/n$. Thus D is small. This is prototype of proof.

Let $\varphi : [0, 1] \to R$ be a continuous function. Then its graph

$$G \;\; = \;\; \{(x, y) : \;\; 0 \leq x \leq 1; \;\; \varphi(x) = y\}$$

and

$$H \;\; = \;\; \{(x, y) : \;\; 0 \leq y \leq 1; \;\; \varphi(y) = x\}$$

are small sets. This is seen as follows. let $\epsilon > 0$ be given. get $n \geq 1$ such that

$$|x_1 - x_2| \leq 1/n \Rightarrow |\varphi(x) - \varphi(y)| < \epsilon/2$$

Denote $y_k = \varphi(k/n)$. Then

$$G \subset \bigcup_{1}^{n} \; [(k-1)/n, \; k/n] \; \times \; [y_k - \epsilon/2, \; y_k + \epsilon/2].$$

a finite union of rectangles whose total area is at most ϵ. Similar argument applies for H.

You can see that the interval could be any closed bounded interval, not necessarily $[0, 1]$. Either you can repeat this proof or use that a finite union of small sets is small.

In particular you get that boundaries of rectangles, discs, triangles, quadrilaterals are all small. There are open sets whose boundaries are not small, but you need to work a little hard. There are also simple open sets whose boundaries are not small. Here simple means their boundaries are given as image of a continuous function on the unit interval (a simple closed curve). However to construct such things you need to work very vey hard. Thus the open sets you come across have boundaries small.

Here are simple facts that follow from properties of integrals on rectangles and properties of small sets.

(1) if f_1 and f_2 are integrable on S so is $f_1 + f_2$ and

$$\int_S (f_1 + f_2) = \int_S f_1 + \int_S f_2.$$

$39f$ is integrable and

$$\int_S (39f) = 39 \int_S f.$$

(2) Let S be a bounded open set whose boundary is small. suppose that f is a bounded continuous function on S. Then f is integrable.

If you extend f to a rectangle, then the set of discontinuities are small.

Let us say that a set S has *area* in case the function $f \equiv 1$ (defined on S) is integrable. In that case we put

$$a(S) = \int_S 1.$$

We denote area by $|S|$ also.

(3) If V is a bounded open set with small boundary, then it has area. simply because the function 1 on V is continuous and (2) above takes care.

Such sets arise often and let us give a name. an open set is *good* if it is bounded and its boundary is small.

(4) if V is a good open set and if $\overline{V} = V \cup \partial V$ then both have areas and $|V| = |\overline{V}|$.
this is because they differ on a small set and an earlier theorem takes care.

(5) if $f \leq g$ are both integrable on S then

$$\int_S f \leq \int_S g.$$

(6) Let V be good open set and let $m \leq f \leq M$ on V, then

$$m\ a(S) \leq \int_S f \leq M\ a(S).$$

This follows from the above. We already knew f is integrable.

(7) Let S_1 and S_2 be disjoint sets and f_1 and f_2 defined on S_1, S_2 respectively are integral. Define

$$S = S_1 \cup S_2; \quad f = f_1 \text{ on } S_1; = f_2 \text{ on } S_2.$$

Then f is integrable on S and

$$\int_S f = \int_{S_1} f_1 + \int_{S_2} f_2.$$

This follows by taking large rectangle that includes both S_1 and S_2 and applying known results for integrals on rectangles. Observe that you are already told that f_1 and f_2 are integrable and that S_1 and S_2 are disjoint.

(8) V is a good open set and f a bounded continuous function on $\overline{V}$. Then f is integrable on V (this means restriction of f to V is integrable on V) and f is integrable on $\overline{V}$ and

$$\int_V f = \int_{\overline{V}} f$$

This is because when you put both in a rectangle, they differ on a small set. Go by the rule book. Take large rectangle Q that includes $\overline{V}$. when you calculate $\int\limits_V f$ you take g on Q to be zero outside V. When you calculate $\int\limits_{\overline{V}} f$ you take g' to be zero outside $\overline{V}$.

(9) Let V_1 and V_2 be good open sets, then so is their union and

$$|V_1 \cup V_2| = |V_1| \cup |V_2|.$$

This follows from (7).

Thus area adds up for disjoint open sets.

Let V be an open set with small boundary. A finite collection of open sets Π is said to be a *good partition* of V, if they are disjoint open subsets of V each having a small boundary and their closures cover V. By closure of an open set W we mean $W \cup \partial W$.

if you feel uncomfortable with this definition you can consider $\overline{V}$ to start with. Then the family $\{\overline{W} : W \in \Pi\}$ is indeed a partition of $\overline{V}$. Theses sets are non-overlapping with union equal to $\overline{V}$.

We shall now show that you can take any nice open set and any good partition of it to calculate integrals. This removes the unnatural conditions of taking partitions with rectangles sides parallel to the axes.

We need a definition. For a bounded set A, diameter of A is defined by

$$d(A) = \sup\{d(p, q) : p \in A, q \in A\}.$$

here $d(p, q)$ is the distance between the two points p and q, that is $||p - q||$. if you take a disc with diameter a then the diameter of the set consisting of the disc, as defined above, is indeed a, verify this.

For a collection Π of sets,

$$||\Pi|| = \sup\{d(A) : A \in \Pi\}.$$

(10). Let V be a good open set. f be a continuous function on $\overline{V}$. Given $\epsilon > 0$, there is an $\delta > 0$ such that for any good partition Π of V with $||\Pi| < \delta$

$$|U(\Pi, f) - \int_V f| < \epsilon; \quad |L(\Pi, f) - \int_V f| < \epsilon.$$

In other words even if you take finer and finer partitions with sets you like and calculate upper or lower sums you will still get the integral we got. But you need to take good partitions. After all, you need to calculate the sup of the function and multiply by the area, so you need areas for your sets.

Otherwise you can not calculate the sums.

if you feel uncomfortable with the function being given on $\overline{V}$ and integrals being talked about are on V, you can take integrals also on $\overline{V}$. an earlier theorem tells you both are same.

Proof is simple. Using uniform continuity of the continuous function f on the compact set $\overline{V}$ take $\delta > 0$ so that

$$p, q \in \overline{V}; \;\; d(p,q) < \delta \rightarrow |f(p) - f(q) < \frac{\epsilon}{2|V|}.$$

Let now be Π be any good partition. We have enough theorems above to justify each of the following equalities.

$$L(\Pi, f) = \sum_{T \in \Pi} m_T |T| \leq \sum_{T \in \Pi} \int_T f = \int_{\cup \Pi} f = \int_V f$$

Last equality is from the fact that $\cup\Pi$ and V differ by a small set, namely at most union of all the ∂T for $T \in \Pi$ put together. similarly

$$U(\Pi, f) = \sum_{T \in \Pi} M_T |T| \geq \sum_{T \in \Pi} \int_T f = \int_{\cup \Pi} f = \int_V f$$

finally

$$U - L \leq \frac{\epsilon}{2|V|}|V| = \epsilon/2.$$

This completes the proof.

Thus you can use some rectangles with sides not necessarily parallel to the axes, some sets could be interior of triangles and so on. There is no restriction.

We shall prove one theorem that will allow us to reduce all the double integrals to integrals of one variable 'at a time'. this is analogue of the following theorem we proved: if f is continuous on a closed rectangle, than the repeated integrals are equal and in fact they equal the double integral.

Theorem: Let Q be a rectangle $[a,b] \times [c,d]$ and f be a bounded integrable function. Define for each x,

$$H(x) = \underline{\int_c^d} f(x,y)dy; \qquad G(x) = \overline{\int_c^d} f(x,y)dy.$$

Then G, H are integrable on $[a,b]$ and

$$\int_Q f = \int_a^b G(x)dx = \int_a^b H(x)dx.$$

The notation

$$\overline{\int_c^d} \varphi(y)dy.$$

is the upper integral of the function φ, it is the lower bound of all upper sums of the function. Since our hypothesis is only that the function f is integrable on Q and it does not imply that for each x the function $y \mapsto f(x,y)$ is integrable we need to take upper integral. similarly lower integral is the sup of all lower sums.

Also you can consider the other iterated integrals too, that is, lower and upper integrals w.r.t. x first. The corresponding statement is also be true.

Proof is simple. Let $\epsilon > 0$. Since f is integrable, take a product partition $\pi_1 \times \pi_2 = \Pi$ such that

$$U(f,\Pi) - L(f,\Pi) < \epsilon.$$

let us see what will be $U(H,\pi_1)$. Take a rectangle $T \times S \in \Pi$ Then

$$m_{T\times S}(f) \leq f(x,y); \quad (x,y) \in T \times S.$$

$$m_{T\times S}(f)|S| \leq \underline{\int_S} f(x,y)dy.$$

$$\sum_{S\in\pi_2} m_{T\times S}(f)|S| \leq \sum_{S\in\pi_2} \underline{\int_S} f(x,y)dy \leq \underline{\int_c^d} f(x,y)dy.$$

(Justify this last inequality) Hence

$$\sum_{S\in\pi_2} m_{T\times S}(f)|S| \leq H(x); \quad x \in T$$

$$\sum_{S\in\pi_2} m_{T\times S}(f)|S| \leq m_T(H)$$

$$\sum_{T\in\pi_1}\sum_{S\in\pi_2} m_{T\times S}(f)|S||T| \leq \sum_{T\in\pi_1} m_T(H)|T|$$

$$L(f,\Pi) \leq L(H,\pi_1)$$

Similarly

$$U(f,\Pi_1) \geq U(H,\pi_1).$$

Thus
$$L(f,\Pi) \leq L(H,\pi_1) \leq U(H,\pi_1) \leq U(f,\Pi).$$
First of all this shows, since $\epsilon > 0$ is arbitrary and the two extremities differ by at most ϵ, that H is integrable. The same inequalities show that
$$\int_Q f = \int_a^b H$$
Similarly argument holds for G, completing the proof.

In practice we have a continuous function $f(x,y)$ defined on a region of the following type
$$S = \{(x,y) : a \leq x \leq b; \varphi(x) \leq \psi(x).\}$$
where φ and ψ are continuous functions defined on the interval $[a,b]$ Thus the boundary of S consists of the graphs of φ, ψ and the two vertical lines at a and b. here it is assumed that $\varphi(x) \leq \psi(x)$ for all $x \in [a,b]$. Since the boundary of S is small and f is continuous we first conclude that f is integrable.

To integrate we can apply the previous theorem. Of course, you need not complicate life because for each x this function $y \mapsto f(x,y)$ is integrable. To be more precise, if you go by the rule book, you will put S in a rectangle, apply previous result, then if you look at the vertical line at x the function $y \mapsto f(x,y)$ is continuous except possibly at the two points $y = \varphi(x)$ and $y = \psi(x)$. this is integrable. You need not make fuss about upper and lower integrals.

thus we conclude
$$\int_S f = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right] dx.$$
The main point is that double integral is reduced to integrating one variable at a time, something we learnt last semester.

Let us work out one example: Find the volume of the ellipsoid,
$$\{(x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$$
Here $a,b,c > 0$. Consider the region in R^2
$$S = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$$

and define the function

$$f(x,y) = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

The ellipsoid is the region bounded by the function $-f(x,y)$ and $f(x,y)$ on the region S. Thus the volume required is found by calculating

$$2\int_S f$$

This is done by the above method and we get

$$\frac{4}{3}\pi abc.$$

In particular, if we have the sphere of radius r , that is $a = b = c = r$ we get

$$\frac{4}{3}\pi r^3.$$

Change of variables:

We shall now proceed to analogue of the change of variable formula.

Recall that in one dimension it states the following. Let V be a bounded interval. Let φ be a C^1 function on V, which is one-to-one and onto $\varphi(V)$ another bounded interval. Let f be a bounded continuous function on $\varphi(V)$. Then

$$\int_V f(\varphi(x))|\varphi'(x)|dx = \int_{\varphi(V)} f(y)dy.$$

of course we did not put modulus sign, stated it when φ is increasing and when it is decreasing separately. In practice, this translates to: put $y = \varphi(x)$ so that $dy = \varphi'(x)dx$.

The exact same formula remains true even in R^2. In one dimension, we had an easy proof of this formula taking recourse to the chain rule of differentiation and fundamental theorem of calculus. Here we do have chain rule, but at this moment we do not have fundamental theorem of calculus. We have to take recourse to a different method.

In one dimensions, a small interval around x is transformed to an interval around $y = \varphi(x)$ and the length of this interval is 'approximately' $\varphi'(x)dx$.

So first we need to understand how areas change under mappings. Of course the simplest mappings are linear mappings. Just bear in mind that R^2 is column vectors. For typographical convenience we are showing as rows. usually books put a transpose, but we are not taxing ourselves with this. This may be confusing, but as long as you know what you are talking about, it will not be confusing.

Consider the map: *interchange coordinates*

$$T(x, y) = (y, x)$$

This is given by the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Assuming you will not get confused, we are using the same symbol for the linear map as well as its matrix representation. If you take any rectangle Q (sides parallel to the axes, to start with) then

$$|T(Q)| = |Q|$$

In other words the area is multiplied by $|T|$, the absolute value of the determinant of the matrix representing T.

Consider the map: *multiply a coordinate*

$$T(x, y) = (31x, y).$$

This is given by the matrix

$$T = \begin{pmatrix} 31 & 0 \\ 0 & 1 \end{pmatrix}$$

if you take a rectangle Q (sides parallel to the axes, to start with)

$$|T(Q)| = 31|Q|.$$

again the area is multiplied by $|T|$, absolute value of the determinant of the matrix representing T. Try multiplication by -31 too.

Consider the map: *add one coordinate to the first one*

$$T(x,y) = (x+y,y).$$

This is given by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

We see for rectangle (sides parallel etc)

$$|T(Q)| = |Q|$$

again the area gets multiplied by $|T|$. of course this is not as obvious as the earlier ones. take first the unit square, that is the rectangle $[0,1] \times [0,1]$, verify that it is transformed to a parallelogram. Then take other rectangles.

It is believable that this should be true for any linear transformation. Let us consider only non-singular transformations because these are the only things we will be interested in. (In any case if you take a singular linear transformation, the statement we are going to make is true and trivial.)

Fact: If T is a non-singular linear transformation of R^2 to itself with matrix representation T, then for any bounded rectangle with sides parallel to the axes

$$|T(Q)| = |T||Q|.$$

This is actually done because any non-singular transformation is a composition of the above three types!

We have learnt the method of substitution for integrating functions of one variable. We are now in the process of getting an analogue of that method for functions of two variables. This goes by the name of change of variable formula or the Jacobian rule. We start with the simplest case.

1.
We shall prove the following:

Theorem (simple Jacobian rule):
Let T be a non-singular linear transformation of R^2 to itself. Let $|T|$ denote the modulus of the determinant of the matrix of the linear transformation. T. Then the following are true:

(i) For any rectangle $Q = [a,b] \times [c,d]$; TQ has small boundary and $|TQ| = |T||Q|$.

(ii) (when (i) holds for a linear transformation T) For any bounded open set V with small boundary; TV is an open set with small boundary and $|TV| = |T||V|$.

(iii) (whenever (i) and (ii) hold for a linear transformation T) For any bounded continuous function f on V,

$$\int_{TV} f = \int_V foT\ |T|$$

that is,

$$\int_{TV} f(u,v)dudv = \int_V f(T(x,y))|T|dxdy.$$

In other words, if you make the substitution. $T(x,y) = (u,v)$ on the right side then $|T|dxdy = dudv$. You get left side. This is like, putting $T(x) = u$ in one dimensions and saying $T'(x)dx = du$. Here observe that the derivative of the map $(x,y) \mapsto (u,v)$ is indeed T.

Having said in part (i) that something is true for every linear transform T; we started part (ii) with the phrase 'whenever (i) is true'. It appears puzzling. The reason is the following.

It is *not* that we prove (i) for every T and then prove (ii) for every T etc. We prove (i) for simple transforms; then (ii) is available for that transform, we use this (ii) to prove (i) for more general transforms — finally covering all linear transforms. That is why it is stated in that fashion.

to put it differently, we consider all linear transformations T for which (i), (ii) and (iii) hold. We show that for certain basic transformations this is true. Then we show that if it is true for two transformations, then it is true for their composition. Then we realize that every linear transformation is a composition of the basic ones. This is executed in a different order below.

2.

If T is any one of the following linear transformations we showed that $|TQ| = |T||Q|$ for any rectangle Q (with sides parallel to the axes). Recall that $|A|$ denotes area of A.

(i) interchange of coordinates:

$T(x, y) = (y, x)$ or T is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) multiply first coordinate :

$T(x, y) = (ax, y)$ or T is given by the matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ where $a \neq 0$.

(iii) Add the second coordinate to the first;

$T(x, y) = (x + y, y)$ or T is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Keep in mind that R^2 consists of column vectors, for typographical reasons we are showing it rows; we do not want to tax our reading by putting transpose symbol all the time.

3.

We now claim that every non-singular transformation is a composition of the above three transformations. Here is how. Instead of giving a general proof that works for every R^n, I give a hands-on proof that works just for R^2. Such proofs are considered bad because you can not generalize quickly. In what follows T stands for non-singular linear transformation on R^2.

Suppose that T is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, that is

$$T(x, y) = (ax + by, cx + dy)$$

case (i): All a, b, c, d are non-zero.

$$(x, y) \to (ax, y) \to (y, ax) \to (by, ax) \to (ax, by) \to (ax + by, by)$$

$$\to (cx + c\frac{b}{a}y, by) \to (by, cx + \frac{cb}{a}y) \to ((d - \frac{cb}{a})y, cx + \frac{cb}{a}y)$$

$$\to (cx + dy, cx + \frac{cb}{a}y) \to (cx + \frac{cb}{a}y, cx + dy) \to (ax + by, cx + dy).$$

case (ii) suppose that exactly one of a, b, c, d is zero. say $d = 0$ and others are non-zero. Thus $T(x, y) = (ax + by, cx)$.

$$(x, y) \to (ax, y) \to (y, ax) \to (by, ax)$$

$$\to (ax + by, ax) \to (ax, ax + by) \to (cx, ax + by) \to (ax + by, cx).$$

Similarly, the other cases when exactly one of them is zero is done.

case(iii) Exactly two of them zero. Since T is non singular the only possibilities are $a = d = 0$ or $b = c = 0$. Let us consider the case $a = d = 0$. Thus the transformation is $T(x, y) = (by, cx)$.

$$(x, y) \to (cx, y) \to (y, cx) \to (by, cx).$$

Since there can not be three zeros this completes all the cases. thus every non-singular transformation is a composition of the three basic ones.

if you look at the proof closely, you can organise carefully and prove for linear transformations on R^n with the help of induction.

4.

If U is a bounded open set, then so is its image $T(U)$ and moreover, $T(\partial U) = \partial(TU)$.

Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $k = \max\{|a|, |b|, |c|, |d|\}$. If $(x, y) \in U$ then by Cauchy-Schwarz,

$$|ax + by| \leq \sqrt{a^2 + b^2}\sqrt{x^2 + y^2} \leq \sqrt{2}\ \ k\ \ ||(x, y)||.$$

Exactly the same inequality holds for $cx + dy$ so that

$$||T(x, y)|| \leq 2k\ \ ||(x, y)||.$$

Thus if U is bounded then so is TU.

Since T^{-1} is also linear transformation, it is continuous and thus inverse image of open sets are open (definition of continuity). In other words TU is open.

To prove the last sentence, let $p \in \partial U$. If you take any open set V containing Tp, then $T^{-1}V$ is an open set containing p and hence has points from U as well as from U^c (definition of boundary point). By looking at their images and remembering that T is one-one, we conclude that V contains points from TU as well as from $T(U^c) = (TU)^c$. This proves that if $p \in \partial U$ then $Tp \in \partial(TU)$.

similarly one argues that if $p \notin \partial U$ then $Tp \notin \partial(TU)$. This proves $T(\partial U) = \partial(TU)$.

5.

Let us say that a transformation T is good if $|TQ| = |T||Q|$ for every rectangle Q with sides parallel to the axes. Let us also abbreviate **rspa** to say 'rectangle with sides parallel to the axes'.

Let U be a bounded open set with small boundary. Then so is its image $T(U)$ under any good transformation.

The only thing to be proved, in view of the above, is that TU has small boundary. Since U is bounded, its boundary ∂U is a compact set. since it is small we can cover it by finitely many rspa $\{Q_i\}$ with total area smaller than ϵ for any pre-assigned $\epsilon > 0$. We know that their images $\{TQ_i\}$ (need not be rspa) have total area at most $|T|\epsilon$ because T is good. Also these images cover ∂U. Thus ∂U can be covered by sets whose total area is smaller than any desired quantity. This shows it is small.

We covered $\partial(TU)$ with images of rspa's. If you are uncomfortable because we did not cover $\partial(TU)$ with rspa of small total area, you can proceed as follows. Cover each TQ_i with rspa having total area at most $2|TQ_i| = 2|T||Q_i|$. (Can you do that? Yes, recall definition of area as integral of the function 1 and recall integral as inf of upper sums etc. You need to remember the full story). Consider the collection so obtained and argue.

6.

For any bounded open set V with small boundary, $|TV| = |T||V|$ under any good T.

the above discussion says that TV has small boundary too and hence both V and TV have areas. So the conclusion makes sense. We prove its truth as follows.

Recall that area is integral of the function 1. Take $\epsilon > 0$. Put V in a rspa and get a partition $\Pi = \{Q_i\}$ with rspa so that

$$U - L < \epsilon; \quad L < |V| < U. \qquad (*)$$

Since we are integrating the function 1, L is nothing but the sum of areas of rectangles in Π which are contained in V, denote these by $\mathcal{L}$. On the other hand, U is sum of areas of rectangles in Π which intersect V, denote these rectangles by $\mathcal{U}$.

Clearly,

$$\bigcup\{TQ : Q \in \mathcal{L}\} \subset TV \subset \bigcup\{TQ : Q \in \mathcal{U}\}.$$

Use the fact that T is good, area is monotone, area is additive for disjoint sets (we proved this earlier using integral is additive). Since all sets that we are considering have areas, we conclude

$$|T|L < |TV| < |T|U, \qquad |T|U - |T|L < |T|\epsilon. \qquad (**)$$

These inequalities $(*)$, $(**)$ and the fact that $\epsilon > 0$ is arbitrary will prove the result.

7.

Every non-singular transformation is good.

We knew that the three basic linear transforms are good and every non-singular transformation is a composition of these.

We shall now complete the proof of the statement by showing that composition of good transforms is again good. so let T_1 and T_2 be good. Take any rspa Q. Interior of Q, say V, is a bounded open set with small boundary. From the results above you conclude that T_1V is an bounded open set with small boundary and so, in turn, T_2 being good we conclude that $T_2(T_1V)$ is a bounded open set with small boundary.

Also $|T_2T_1(V)| = |T_2||T_1||V|$. Since determinants multiply under composition of transforms we have $|T_2 o T_1| = |T_2||T_1|$. Denoting $T = T_2 o T_1$, we thus have $|TV| = |T||V|$. Clearly $\partial TV = T(\partial V)$ and is small. We conclude that

$$|TQ| = |T||Q|.$$

In other words, composition of good transforms is again good. This completes proof that every non-singular transform is good.

We know that T_2 being good, it transforms rspa nicely. Since T_1Q need not be a rspa (though a quadrilateral), we are forced to use that T_2 transforms such sets also nicely. This is precisely, the content of part (ii) of the there. As soon as you are told T_2 is good you can deduce it transforms T_1Q also appropriately. Of course, once we know that the composition T is good, it transforms not only rspa nicely, but also bounded open sets with small boundary nicely and the formula for areas holds.

8.

Thus so far we have proved parts (i) and (ii) (**7, 6** above) of the theorem for every non-singular transform. To complete the proof of the theorem, we need to prove part (iii); integral formula for continuous functions.

Let us first assume that *the function is defined on* $\overline{TV}$. This will allow us to use uniform continuity etc. Since both V and TV have small boundaries we know the functions are integrable. That is, f is integrable on TV and foT is integrable on V. We only need to show the equality of integrals. Let $\epsilon > 0$ Let f be bounded by M.

By an earlier theorem, there is an $\delta_1 > 0$ such that if we take any good partition of TV with norm smaller than δ_1, then upper and lower sums are close to $\int\limits_{TV} f$ upto $\epsilon/4$.

Denote $g = foT$ on $\overline{V}$. This is sensible because, $T(\overline{V}) = \overline{TV}$. Choose $\delta > 0$ so that the following two hold: (i) $|Tp - Tq| < \delta_1$ whenever $|p - q| < \delta$. (ii) For any good partition of V with norm smaller than δ_1, the upper and lower sums differ from $\int\limits_V g$ by at most $\epsilon/4$.

Now let us take a partition Π of V with $||\Pi|| < \delta$. For example you can take grid of rspa. Then from what had been done so far, we conclude the following. Firstly, $T\Pi$ is a good partition of TV. Secondly norm of this partition $T\Pi$ is at most δ_1. Observe that for any set $S \in \Pi$ the sup of g over S is same as sup of f over $TS \in T\Pi$. similarly for inf. Denote by L, U the upper and lower sums of g on V for the partition Π. Similarly L_1 and U_1 for the similar sums for f on TV for the partition $T\Pi$ on TV. Then we have

$$L \leq \int_V g \leq U; \quad U - L < \epsilon/4; \quad L_1 \leq \int_{TV} f \leq U_1; \quad U_1 - L_1 < \epsilon/4.$$

$$U_1 = |T|U; \quad L_1 = |T|L.$$

These inequalities are good enough to conclude

$$|T| \int_V foT = \int_{TV} f.$$

Since $|T|$ is a number, you can put it inside integral too.

This completes proof of the theorem.

Thus the theorem is proved when f is defined on $\overline{TV}$, in fact, when f is uniformly continuous on TV.

There are several ways of deducing the result for bounded continuous functions from the above. Here is a way. Fix any $\epsilon > 0$; get finitely many rspa $\{Q_i\}$ whose interiors cover boundary of TV and have total area smaller than ϵ/M (Recall, M is the bound for f). Set

$$W_1 = TV - \cup\{Q_i\}; \qquad V_1 = T^{-1}W_1$$

Then, V_1 is a bounded open set with small boundary; $TV_1 = W_1$; Restrict f to $\overline{W_1}$; apply the earlier result. Convince yourself that the difference between this and original integrals are small.

9

We shall prove the following which generalizes the above result. This goes by the name of Jacobian rule. This is very useful in evaluating integrals.

Theorem (Jacobian rule):

Let $\Omega \subset R^2$ be a bounded open set. Let $T : \Omega \to R^2$ be a one-to-one C^1 map with non-singular derivative at every point of Ω.

At every point $(x, y) \in \Omega$ let $|T'(x, y)|$ denote the modulus of the determinant of the derivative matrix.

Then the following are true:

(i) For any rectangle $Q = [a, b] \times [c, d] \subset \Omega$; TQ has small boundary and

$$|TQ| = \int_Q |T'|.$$

that is

$$|TQ| = \int \int_Q |T'(x, y)| dxdy.$$

(ii) (when (i) holds for a T) For any open set V with $\overline{V} \subset \Omega$ with small boundary; TV is an open set with small boundary and

$$|TV| = \int_V |T'|.$$

that is,

$$|TV| = \int\int_V |T'(x,y)|dxdy.$$

(iii) (whenever (i) and (ii) hold for a T) Let V be as above. For any continuous function f on $T\overline{V} = \overline{TV}$,

$$\int_{TV} f = \int_V foT \ \ |T'|$$

that is,

$$\int_{TV} f(u,v)dudv = \int_V f(T(x,y))|T'(x,y)|dxdy.$$

If you feel uncomfortable with function given on closure $\overline{TV}$ but integrals on open set TV, you can take integral too over $\overline{TV}$. It makes no difference as we saw earlier once.

Note that when T is a linear transform defined on all of R^2, this theorem reduces to the earlier one because the derivative is now the same matrix at every point, namely, T.

10

We first deal with two special cases. Then we argue that the statement remains true under compositions. Finally show any map is composition of maps of the special kind. This last step is true only 'locally'. Then a 'patching' will prove general case without much work.

Here is a special case, y-coordinate is not changed by T.

Suppose $Q = [a,b] \times [c,d]$ is an rspa. Let $g(x,y)$ be a real C^1 function on Q such that

$$\varphi(x,y) = (g(x,y), y)$$

is a one one function on Q with non-singular derivative.

$$\varphi'(x,y) = \begin{pmatrix} g_1 & g_2 \\ 0 & 1 \end{pmatrix}$$

Thus non-singular derivative, simply means that g_1 is not zero.

We show

$$|\varphi Q| = \int_c^d \int_a^b |\varphi'|.$$

There is just one subtle point. We always talked about derivatives at points in an open set. Here we started with a closed rectangle. Since any way the present theorem is not our main focus, we want to apply this for rectangles contained in Ω, you can as well pretend that g is defined on an open set containing this rectangle. You can make other interpretations too (that do not look outside the rectangle), but let us not get distracted by this point.

Note that $|\varphi'| = |g_1|$. Since g_1 is continuous and Q is convex, mean value theorem implies that g_1 keeps same sign through out Q. We shall assume that $g_1 > 0$. Thus $|g_1| = g_1$ and we need to show

$$|\varphi Q| = \int_c^d \int_a^b g_1.$$

We can actually identify $\varphi(Q)$. You should draw the picture.

The line $[a,b] \times \{c\}$ is mapped in an increasing manner (because $g_1 > 0$) to the line segment $\{(g(x,c),c) : a \leq x \leq b\}$. Note that continuity forces that the image is a line segment and since y-coordinate is unchanged, the image is part of the horizontal line $y = c$.

The line $[a,b] \times \{d\}$ is mapped to the line segment $\{(g(x,d),d) : a \leq x \leq b\}$.

The line $\{a\} \times [c,d]$ is mapped to the curve $\{(g(a,y),y) : c \leq y \leq d\}$ and the line $\{b\} \times [c,d]$ is mapped to the curve $\{(g(b,y),y) : c \leq y \leq d\}$

By the the theorem on iterated integration,

$$\int_{\varphi(Q)} 1 = \int_c^d \int_{g(a,y)}^{g(b,y)} dxdy = \int_c^d [g(b,y) - g(a,y)]dy.$$

$$\int_Q g_1 = \int_c^d \left(\int_a^b g_1 dx \right) dy = \int_c^d [g(b,y) - g(a,y)]dy.$$

and hence they are equal. for the second integral we used the fundamental theorem of Calculus. For each fixed y, g as a function of x is a primitive for g_1.

11.

Let $Q = [a,b] \times [c,d]$ be as above. Let $h(x,y)$ be a real C^1 function on Q such that

$$\varphi(x,y) = (x, h(x,y))$$

is a one one function on Q with non-singular derivative.

$$\varphi'(x,y) = \begin{pmatrix} 1 & 0 \\ h_1 & h_2 \end{pmatrix}$$

Thus non-singular derivative, simply means that h_2 is not zero.

We can show exactly as above

$$|\varphi Q| = \int_c^d \int_a^b |h_2|.$$

12.

Let now Ω be an bounded open set and

$$\varphi(x,y) = (\xi(x,y), \eta(x,y))$$

be a one-to-one C^1 function on Ω with derivative non-singular at every point.

$$\varphi'(x,y) = \begin{pmatrix} \xi_1(x,y) & \xi_2(x,y) \\ \eta_1(x,y) & \eta_2(x,y) \end{pmatrix}$$

$|\varphi'|$ is the modulus of the determinant of the above derivative matrix. The matrix above is referred to as Jacobian. We show that for any rectangle $Q = [a,b] \times [c,d] \subset \Omega$

$$|\varphi Q| = \int_Q |\varphi'|. \qquad (\spadesuit)$$

The main idea is that φ is 'locally' a composition of the above two kinds. Here is the precise statement.

Step 1: For every point x there is an open set V containing x such that ($\spadesuit$) holds for rectangles contained in V.

Step 2: Step 1 implies the full ($\spadesuit$).

First let us argue step 2. Let $Q = [a,b] \times [c,d] \subset V$ be any rectangle. for each $p \in Q$ there is a V_p as above. We can take the V_p to be a rspa with p at its centre. Let W_p be the rectangle with sides consisting of the middle halfs of sides of V_x. Take finitely many of these, say $\{W_i\}$ such that they cover Q; Remember Q is compact. Note that these $\{W_i\}$ and the Q determine a partition of Q into rspa. Since each of these sets of the partition is contained in one single V_i we can apply the result coming from step 1 and add up. Remember areas add up over disjoint regions and so do integrals.

We shall execute step 1.

Take a point $(x_0, y_0) \in \Omega$. Since $\varphi'(x_0, y_0)$ is non-singular, its first column can not be zero. We must have either $\xi_1(x_0, y_0) \neq 0$ or $\eta_1(x_0, y_0) \neq 0$. There is no loss to assume that $\xi_1(x_0, y_0) \neq 0$. Otherwise you need to apply interchange of coordinates in the range space. If you think about it you will see. But first just assume this and proceed.

Since ξ_1 is a continuous function, there is no loss to assume that in a ball around (x_0, y_0) it is strictly positive. If it is negative, similar proof applies. consider the map

$$\Psi(x, y) = (\xi(x, y), y)$$

Then

$$\Psi' = \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & 1 \end{pmatrix}$$

is clearly non-singular in this ball and hence there is an open set, say V, containing (x_0, y_0) such that Ψ maps this onto an open set W and the inverse map Ψ^* is differentiable etc. This is by the inverse function theorem. Note that Ψ changes only one coordinate, namely, the first coordinate. Also observe

$$|\Psi'| = |\xi_1| = \xi_1. \qquad (\bullet)$$

Now define on the open set W^* the following map;

$$\zeta(a, b) = (a, \eta(\Psi^*(a, b))); \qquad (a, b) \in W$$

Thus ζ changes only one coordinate, namely the second coordinate.

Let us see what is the composition $\zeta(\Psi(x, y))$ on V. Observe $\Psi(x, y) = (\xi(x, y), y)$. Since ζ does not change the first coordinate we see that first coordinate of the composition, namely $\zeta(\Psi(x, y))$ is $\xi(x, y)$.

Regarding second coordinate of the composition, to calculate ζ of this point (a, b) we need to apply Ψ^* which is inverse of Ψ and so you get back (x, y), that is, (x, y) with $\Psi(x, y) = (a, b)$ and then you should look at eta of this, you get $\eta(x, y)$. Thus second coordinate of the composition $\zeta o \Psi$ is η. Thus

$$\zeta(\Psi(x, y)) = (\xi(x, y), \eta(x, y)) = \varphi(x, y).$$

Let us also note that

$$\zeta'(a, b) = \begin{pmatrix} 1 & 0 \\ ? & ?? \end{pmatrix} \qquad (\dagger)$$

where the second row is $\nabla\eta o\Psi^*$ and so by chain rule

$$(?,\ ??) = \eta'(\Psi^*(a,b))(\Psi^*)'(a,b)$$

$$= (\eta_1(\Psi^*(a,b)), \eta_2(\Psi^*(a,b)))\ \ [\Psi'(\Psi^*(a,b))]^{-1}$$

But

$$\Psi'(P) = \begin{pmatrix} \xi_1(P) & \xi_2(P) \\ 0 & 1 \end{pmatrix}$$

$$[\Psi'(P)]^{-1} = \begin{pmatrix} 1/\xi_1(P) & -\xi_2(P)/\xi_1(P) \\ 0 & 1 \end{pmatrix}$$

So

$$(?,\ ??)\ \ =$$

$$(\eta_1(\Psi^*(a,b)), \eta_2(\Psi^*(a,b)))\begin{pmatrix} 1/\xi_1(\Psi^*(a,b)) & -\xi_2(\Psi^*(a,b))/\xi_1(\Psi^*(a,b)) \\ 0 & 1 \end{pmatrix} \tag{††}$$

Since both Ψ and ζ change only one coordinate, they are good, that is we can use theorem for them. Note that (i) of the theorem holds for each of them and hence the other parts too. Thus if you take any rspa $Q \subset V$ then $\Psi(Q) \subset W$. We can no longer say that this later set is a rectangle. However (ii) and (iii) of the theorem can be used. Thus

$$|\varphi(Q)| = |\zeta[\Psi Q]| = \int_{\Psi Q} |\zeta'(u,v)| dudv$$

by part (ii) of the theorem applied to the transform ζ and the set ΨQ. We now apply part (iii) to the transform Ψ and the integrand $|\zeta'|$ to see

$$|\varphi(Q)| = \int_Q |\zeta'(\Psi(x,y))||\Psi'(x,y)| dxdy.$$

To calculate the first term in the integrand, we use (†) and (††). Remember Ψ^* is inverse of Ψ and evaluate (††) at $(a,b) = \Psi(x,y)$ substitute in (†) to get

$$|\zeta'(\Psi(x,y))| = \left|\frac{\eta_1(x,y)\xi_2(x,y)}{\xi_1(x,y)} - \eta_2(x,y)\right|$$

From (•)

$$|\Psi'(x,y)| = \xi_1(x,y)$$

Substituting these two in the above integral we get

$$|\varphi Q| = \int_Q |\eta_1 \xi_2 - \eta_2 \xi_1| = \int_Q |\varphi'|.$$

This completes the proof of (♠). What the above shows is that (♠) holds for all rectangles contained in the open set V that we have exhibited. Thus around every point there is a open set such that this holds for all rspa contained in that open set. As argued in step 2, this is enough.

Remember the above proof used the fact that (a) the result is true for transformations which change only one coordinate and (b) when part (i) of the theorem is proved for a transformation then you can use other parts for that transformation. The result (a) has already been proved earlier.

13.

Thus we should now prove parts (ii) and (iii) of the theorem assuming (i) holds.

Let V be an open set with small boundary. The inverse function theorem already assures that the image TV is an open set. To show it has small boundary, we repeat the same argument that we had when T is a linear transformation (item (5) above). We cover ∂V with finitely many rspa of small total area and use the fact that $|T'|$ is bounded on compact sets to get an estimate for the area of the image.

14.

We shall now prove part (iii) of the theorem assuming part (ii) for T.

I have avoided Riemann sums so far in two dimensions. It is better to have them handy now, to avoid a nuisance.

Let f be a bounded function defined on $S = \overline{V}$. In the following; partition Π of S is either with rspa or a good partition. We defined upper and lower sums. We now define Riemann sums. A selection ξ for Π is a selection of one point p_A from each set A of the partition. Given a partition and selection we define the Riemann sum

$$R(\Pi, \xi) = \sum_{A \in \Pi} f(p_A)|A|.$$

Thus instead of taking inf or sup over a set we evaluate the function at a point in that set. These Riemann sums also converge to the integral. More precisely, Given $\epsilon > 0$ there is a $\delta > 0$ such that the following holds:

$$||\Pi|| < \delta \text{ good; } \xi \text{ selection for } \Pi \Rightarrow \left| R(\Pi, \xi) - \int f \right| < \epsilon. \quad (\dagger)$$

Integral is over V. There is nothing new in this. We already knew of a δ so that the upper sum, lower sum, integral are close as soon as $||\Pi|| < \delta$. It is obvious that the Riemann sum is in between the lower and upper sums and hence the above conclusion holds for it, no matter what the selection is.

We shall slightly generalise the notion of above Riemann sum when we have a product function. Suppose we have two continuous functions f and g on $\overline{V}$. We want to integrate the product fg. Let Π be a good partition of V and a selection ξ for it. We make an interesting Riemann sum. For a set A of the partition, instead of taking the term $fg(p_A)|A| = f(p_A)g(p_A)|A|$ we take $f(p_A) \int_A g$. After all, if the set A is small, value of g at any point of A is close to $g(p_A)$ and so the integral is indeed close to $g(p_A)|A|$. Thus we define

$$R_1(\Pi, \xi) = \sum_{A \in \Pi} f(p_A) \int_A g.$$

This makes sense because our partition is good and g is continuous bounded. We claim that, as $||\Pi|| \to 0$, these sums also converge to the integral $\int fg$. More precise statement is the following.

Given $\epsilon > 0$ there is a $\delta > 0$ such that

$$||\Pi|| < \delta \text{ good; } \xi \text{ selection for } \Pi \Rightarrow \left| R_1(\Pi, \xi) - \int fg \right| < \epsilon. \qquad (\dagger\dagger)$$

This is proved as follows. fix $\epsilon > 0$. Let M be a bound for $|f|$ on $\overline{V}$. Choose $\delta > 0$ such that

$$||\Pi|| < \delta \text{ good; } \xi \text{ selection;} \Rightarrow \left| R(fg, \Pi, \xi) - \int_V fg \right| < \epsilon/4.$$

and

$$p, q \in \overline{V}; \ \ ||p - q|| < \delta \Rightarrow |g(p) - g(q)| < \epsilon/(4M|V|).$$

Now take any good partition Π; with $||\Pi|| < \delta$ and any selection ξ. Let $A \in \Pi$.

$$p, q \in A \Rightarrow ||p - q|| < \delta \Rightarrow |g(p) - g(q)| < \epsilon/c; \quad c = 4M|V|.$$

So

$$\left| g(p_A)|A| - \int_A g \right| = \left| \int_A [g - g(p_A)] \right| \leq \frac{\epsilon|A|}{c}.$$

hence

$$\left|f(p_A)g(p_A)|A| - f(p_A)\int_A g\right| \le \frac{\epsilon M|A|}{c} = \frac{\epsilon M}{4M|V|}|A|$$

so that

$$\left|\sum_A f(p_A)g(p_A)|A| - \sum_A f(p_A)\int_A g\right| \le \sum_A \frac{\epsilon}{4|V|}|A| = \frac{\epsilon}{4}.$$

Thus

$$|R - R_1| < \epsilon/4.$$

But by choice of δ

$$|R - \int fg| < \epsilon/4.$$

combining these last two inequalities we have

$$\left|R_1 - \int fg\right| < \epsilon/2$$

as required. this completes the proof of the claim.

Let us now return to our problem. We want to show

$$\int_{TV} f = \int_V foT\ |T'|. \qquad (**)$$

Let $\epsilon > 0$ be fixed. We show that the difference between the two quantities above is at most ϵ.

Use (†) with the set $\overline{TV}$ and function f and $\epsilon/4$. Get $\delta > 0$ so that

$$||\Pi|| < \delta \ \text{ good for } \ \overline{TV}; \ \ \xi \ \text{ selection for } \ \Pi \Rightarrow \left|R(\Pi,\xi) - \int_{TV} f\right| < \epsilon/4 \ \ (\bullet)$$

Use (††) with the set $\overline{V}$; function f as foT and function g as $|T'|$ and $\epsilon/4$. Get $\delta_1 > 0$ so that

$$||\Pi|| < \delta_1 \ \text{ good for } \ \overline{V}; \ \ \xi \ \text{ selection for } \ \Pi \Rightarrow \left|R_1(\Pi,\xi) - \int_V foT\ |T'|\right| < \epsilon/4 \ \ (\bullet\bullet)$$

Take δ_1 smaller if necessary so that the following holds

$$p, q \in \overline{V}; \ \ ||p - q|| < \delta_1 \Rightarrow ||Tp - Tq|| < \delta/2. \ \ (\bullet\bullet\bullet)$$

Now let us take a good partition Π of $\overline{V}$ with $||\Pi|| < \delta_1$ and a selection ξ for it. Let $T\Pi$ be the partition of $\overline{TV}$ given by $\{TA : A \in \Pi\}$. Let $T\xi$ be the selection for it given by $p_{TA} = Tp_A$.

It is not difficult to show that $T\Pi$ is a good partition of $\overline{TV}$. By $(\bullet\bullet\bullet)$ we see $||T\Pi|| < \delta$ and so by $(\bullet)$ we have

$$\left| R(T\Pi,\ \ T\xi,\ \ f)\ \ -\ \ \int_{TV} f \right|\ \ <\ \ \epsilon/4.$$

Also

$$\left| R_1(\Pi,\ \ \xi,\ \ foT\ |T'|)\ \ -\ \ \int_V foT\ \ |T'| \right|\ \ <\ \ \epsilon/4$$

For any set $A \in \Pi$, we already know from part (ii) of the theorem that $|TA| = \int_A |T'|$ so that it is easy to see that

$$R(T\Pi,\ \ T\xi,\ \ f)\ \ =\ \ R_1(\Pi,\ \ \xi,\ \ foT\ |T'|)$$

Hence the two inequalities show that

$$\left| \int_V foT\ \ |T'|\ \ -\ \ \int_{TV} f \right| < \epsilon/2.$$

This completes proof of $(**)$ and thus proof of part (iii) of the theorem.

This completes proof of all the three parts of the theorem.

15.

We have completed the proof the Jacobian formula as stated.

it appears unsatisfactory. Would have been nice if part (iii) is stated for any bounded continuous function on Ω rather than for open sets V with $\overline{V} \subset \Omega$. Actually this is how I stated in the classes starting with Ω which is bounded open set with small boundary. But I proved only this much.

Yes, it is also true if you started with Ω bounded open set with small boundary. Proof is simple. You 'approximate' U with sets V as above. This can be made precise as follows. Cover $\partial\Omega$ with finitely many open rspa, take their closures and take V to be the part of Ω outside these finitely many closed rectangles. Since you can make total area of these rectangles as small as you please you ken get the result for Ω too.

However once you understand the main ideas, certain beautifications you can do yourself. At this stage you need not bother (unless you wish to).

16.

Improper integrals are dealt with as in the one dimensional case. This is a long but routine route and would appear boring if spelt out with all details. You may have a bounded region but the function is unbounded like $1/||x||$. Or the function may be bounded but the region is unbounded, like $\exp\{-||x||^2\}$ on R^2.

Just to give you a feel let us discuss the case of a function f given on R^2. We say that the integral

$$\int_{R^2} f$$

exists if whenever you take a sequence of rspa $Q_n \uparrow R^2$ the numbers

$$\int_{Q_n} f$$

converge to a limit and the limit is independent of the sequence of rectangles taken. Then this common value is called the value of the integral.

For example, if

$$\sup_n \int_{[-n,n]\times[-n,n]} |f(x,y)|dxdy < \infty$$

Then f is integrable. In other words there is a number c such that no matter what rectangles (or regions) you take which increase to R^2, your integrals converge to c.

We shall work out some examples rather than developing the theory of improper integrals.

Normal integral:

The following integral appears in several contexts.

$$I = \int_{-\infty}^{\infty} e^{-t^2/2} dt.$$

We can calculate this integral by using some tricks. But the simplest is to valculate

$$I^2 = \int_{R^2} e^{-(x^2+y^2)/2} dxdy.$$

Note that by iterated integral process, if you integrate w.r.t. y first you get I and then you integrate w.r.t. x to get I^2.

We shall use the jacobian rule. Put

$$x = r\cos\theta; \quad y = r\sin\theta \qquad (\spadesuit)$$

A little give and take is needed here because this transformation is not really C^1, in fact not even continuous. Let us first precisely define the transformation.

Given any point (x, y) different from zero, we set $r = +\sqrt{x^2+y^2}$. Thus the given point is $r(x/r, y/r)$ We know from last semester, there is a unique angle $\theta \in [0, 2\pi)$ such that

$$\frac{x}{r} = \cos\theta; \qquad \frac{y}{r} = \sin\theta.$$

Thus given any point different from $(0, 0)$ in R^2 there is a unique pair of numbers $r > 0$ and $0 \le \theta < 2\pi$ satisfying $(\spadesuit)$. These are called polar coordinates of the cartesian point (x, y).

Thus given a point P, the number r is the distance of the point P from the origin and θ is the angle determined by the positive x-axis and the line joining origin to P.
to the point.

For the point $(0, 0)$ we can and should take $r = 0$ but any θ would do. For other points we have a unique choice. It is easy to show uniqueness.

If a sequence (P_n) of points from the fourth quadrant approach a point P on the x-axis, then $\theta(P_n)$ approaches 2π where as $\theta(P) = 0$.

Let us consider the open set

$$\Omega = R^2 - \{(x, y) : y + 0, x \geq 0\}$$

We remove the non-negative x-axis from R^2. Now it is easy to see that

$$T(x, y) = (r, \theta); \quad \Omega \mapsto (0, \infty) \times (0, 2\pi)$$

is one-one and is in fact C^1 map. Also Jacobian at a polar point (r, θ) is given by

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$|T'| = r.$$

Remember that the set removed is a small set and hence if we integrate over Ω instead of R^2 we still gate I^2. Thus the integral reduces to

$$I^2 = \int_\Omega = \int_0^\infty \int_0^{2\pi} e^{-r^2/2} r dr d\theta$$

Integrate w.r.t. —theta and then w.r.t r to get

$$I^2 = 2\pi, \qquad I = \sqrt{2\pi}$$

Thus we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1.$$

This integrand is called standard normal density.

The way we evaluated is the integral is not right. We treated the whole space as if we have a bounded region. We should actually calculate over bounded regions and take limit. Thus take $R > 0$ and consider

$$\Omega(R) = \{(x, y) : x^2 + y^2 \leq R^2\}$$

We calculate the integral over this region. We have a nice bounded continuous function on this bounded region. From earlier theorems you can give up a little bit, this integral is same as integral over the region with non-negative x-axis (the part which is in the region) removed. You need to calculate integral, after transforming to polar coordinates, over the region

$$\{(r, \theta) : 0 < r < R; 0 < \theta < 2\pi.\}$$

This can be easily done and you can take the limit to get the same answer as above.

Even this is not really right. Imagine calculating integral of the function $f(x, y) = x$ over this region. We will get zero. Are we then going to say integral of f over R^2 is zero?

We should first show that the integral exists and only then proceed to calculate the integral Since our integrand is positive, we can afford not to do this. If over one sequence of regions increasing to R^2 the integrals of the positive continuous integrand are bounded, then the integral exists and you can choose your own convenient regions increasing to R^2 to calculate the integral

We have observed in the process

$$\int_{R^2} e^{-(x^2+y^2)/2} dxdy = 2\pi.$$

suppose you now take $\mu = (\mu_1, \mu_2) \in R^2$. Then we have

$$\int_{R^2} e^{-\{(x-\mu_1)^2+(y-\mu_2)^2\}/2} dxdy = 2\pi.$$

This follows from the Jacobian rule again. Simply change the variables. This can be rewritten as

$$\int_{R^2} e^{-(x-\mu)^t(x-\mu)/2} dx = 2\pi.$$

Remember vectors in R^2 are column vectors. Also we use the notation $x = (x_1, x_2)$ for pints of R^2 rather than (x, y). Also we use dx for $dx_1 dx_2$.

Now suppose that you have a symmetric 2×2 positive definite matrix Σ. Then

$$\int_{R^2} e^{-(x-\mu)^t \Sigma^{-1}(x-\mu)/2} dx = 2\pi\sqrt{|\Sigma|}.$$

This again follows from Jacobian rule. First, Get a symmetric positive definite matrix B with $B^t B = B^2 = \Sigma$.

First change the variables $u_1 = x_1 - \mu_1$ and $u_2 = x_2 - \mu_2$. that is $u = x - \mu$ then change $u = Bv$. Jacobian is B and $|B| = \sqrt{|\Sigma|}$

Beta and Gamma:

Recall that for numbers $a, b > 0$

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

and

$$\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx$$

We shall now show

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Start with

$$\Gamma(a)\Gamma(b) = \int_\Omega e^{-(x+y)}x^{a-1}y^{b-1}dxdy$$

where

$$\Omega = (0, \infty) \times (0, \infty).$$

This is true because if you perform the integration w.r.t. y first and then w.r.t. x you get it. Now we shall change the variable

$$x + y = u; y = y$$

The range set is

$$\Omega' = \{(u, v) : 0 < y < u < \infty\}$$

In other words, Ω' is an open set and on that there is the C^1 map $T(u, y) = (u - y, y)$ which will take you to Ω and you are integrating a function on Ω Here the Jacobian is one. Thus

$$\Gamma(a)\Gamma(b) = \int_{\Omega'} e^{-u}(u-y)^{a-1}y^{b-1}dudy$$

$$= \int_0^\infty e^{-u}\left[\int_0^u (u-y)^{a-1}y^{b-1}dy\right]du$$

Integrate w.r.t. y by substituting $y = vu$ so that dy v du$andrangeof$viszero to 1. You get

$$\Gamma(a)\Gamma(b) = \beta(a, b)\Gamma(a+b).$$

See how all the theorems are at work.

We have simply written the range of (u, y) without explanation. here it is. x, y range over Ω Thus first of all

$$0 < u < \infty; 0 < y < \infty$$

But not every such u, y come from a point of Ω if you want this pair to come from Ω, since the only pair from which it can come is $(u-y, y)$ e must have

$$0 < u - y < \infty; 0 < y < \infty$$

These four inequalities will tell us $0 < u < \infty$. And if you take such a u, then y ranges over

$$0 < y < \infty; 0 < u - y < \infty$$

If you want both these to be satisfied we must have

$$0 < y < u.$$

Of course here again the integrand could be unbounded and actually we should go through integrals over (ϵ, ∞). But we do not do.

$\int ||\mathbf{x}||^{-\alpha}$:

Let us consider $\alpha > 0$. We want to find out if the following integral exists.

$$\int_\Omega ||x||^{-\alpha} dx$$

where

$$\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 < 1.\}$$

If you take $0 < \epsilon < 1$ and

$$\Omega_\epsilon = \{(x_1, x_2) : \epsilon^2 < x_1^2 + x_2^2 < 1.\}$$

then change to polar coordinates (you need to remove the line segment $\{(x_1, 0) : \epsilon < x_1 < 1\}$) we get

$$\int_{\Omega_\epsilon} = \int_\epsilon^1 \int_0^{2\pi} r^{-\alpha} r dr d\theta.$$

You can explicittly calculate this integral and see that a limit as $\epsilon \to 0$ exists iff $\alpha < 2$.

Normal integral again:

Here is a tricky way of calculating normal integral. Let us put

$$a_n = \int_0^\infty e^{-t^2/2}\, t^n \;\; dt; \qquad n = 0, 1, 2, \cdots .$$

We do not know a_0 but can explain all others using it. Integration by parts gives

$$a_1 = 1; \quad a_n = (n-1)a_{n-2} \quad n = 2, 3, 4, \cdots .$$

This immediately gives

$$a_{2m} = (2m-1)(2m-3)\cdots 1 a_0.$$

$$a_{2m+1} = 2m(2m-2)\cdots 2 \cdot 1.$$

Now comes a high school idea. Note that for any λ we have

$$\int_0^\infty e^{-t^2/2}\, t^n\, (\lambda + t)^2 \;\; dt > 0$$

simply because the integrand is positive. That is, for every λ

$$a_{n+2} + 2\lambda a_{n+1} + \lambda^2 a_n > 0$$

Thus

$$a_{n+1} \le \sqrt{a_n a_{n+2}}.$$

In particular, for every m we have

$$a_{2m} < \sqrt{a_{2m-1} a_{2m+1}}.$$

giving us

$$a_0 \le \sqrt{2m}\frac{(2m-2)(2m-4)\cdots 2 \cdot 1}{(2m-1)(2m-3)\cdots 1}$$

This is true for every m and taking limits and appealing to Walli we get

$$a_0 \le \sqrt{\pi/2}.$$

We also have

$$a_{2m+1} < \sqrt{a_{2m} a_{2m+2}}.$$

Analogous to the above, on simplification, this inequality gives us

$$a_0 \geq \sqrt{\pi/2}.$$

These two inequalities give us

$$\int_0^\infty e^{-t^2/2} \ dt = \sqrt{\frac{\pi}{2}}.$$

From this we get

$$\int_{-\infty}^\infty e^{-t^2/2} \ dt = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}.$$

Again returning back to the integral on the positive side, we substitute $t^2/2 = u$ so that $tdt = du$ or $dt = du/\sqrt{2u}$ we see

$$\int_0^\infty e^{-u}u^{-1/2}du = \sqrt{\pi}.$$

That is

$$\Gamma(1/2) = \sqrt{\pi}.$$

We know that for integers $n > 1$, $\Gamma(n+1) = n!$. We can use the above result to calculate gamma values of half integers. Recall that

$$\Gamma(a+1) = a\Gamma(a).$$

Thus for example

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}.$$

Volume of unit ball:

Let us consider, for $R > 0$, the ball

$$B_R = \Big\{(x_1, x_2, \cdots, x_n) : \sum x_i^2 \leq R\Big\}$$

Let $V_n(R)$ denote the volume of this ball in R^n. That is

$$V_n(R) = \int_{B_R} 1 \ dx_1 \ dx_2 \ \cdots dx_n.$$

First let us note one thing.

$$V_n(R) = R^n V_n; \qquad V_n = V_n(1).$$

This is because the linear transform $Tx = Rx$ takes unit ball onto the R-ball.

To calculate $V_n(R)$ let us integrate the variables $(x_2, x_3, \cdots, x_n)$. Clearly they range over the set

$$\sum_{3}^{n} x_i^2 \leq R^2 - x_1^2 - x_2^2.$$

Thus if you integrate these $(n-2)$ variables over this set we get

$$\left(\sqrt{R^2 - x_1^2 - x_2^2}\right)^{n-2} V_{n-2}.$$

Note that V_{n-2} is a number and does not depend on x_1, x_2. Now let us integrate w.r.t. x_1, x_2. These variables range over

$$S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq R^2\}.$$

Thus

$$V_n(R) = V_{n-2} \int_S \left(R^2 - x_1^2 - x_2^2\right)^{(n-2)/2} dx_1 dx_2.$$

Changing to polar coordinates

$$x = r\cos\theta; y = r\sin\theta; \quad 0 \leq r \leq R;\ 0 < \theta < 2\pi.$$

Noting that the Jacobian is r we get

$$V_n(R) = \int_{r=0}^{R} \int_{\theta=0}^{2\pi} (R^2 - r^2)^{(n-2)/2} r \ d\theta \ dr.$$

The presence of $r\ dr$ allows us the substitution $r^2 = u$ and we have

$$V_n(R) = \frac{2\pi}{n} R^n V_n = \frac{2\pi R^2}{n} V_{n-2}(R).$$

Note that $V_1(R) = 2R$ and $V_2(R) = \pi R^2$. Thus the above relation gives us

$$V_{2k}(R) = \frac{\pi^k}{k!} R^{2k}$$

$$V_{2k+1} = \frac{2^{k+1}\pi^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)} R^{2k+1}.$$

As far as unit ball is concerned you can combine the even and odd formulae to get

$$V_n = \pi^{n/2} \Big/ \Gamma(\frac{n}{2} + 1)\ .$$

It is interesting to note that V_k converges to zero as $k \to \infty$. As the dimension grows, the volume of any fixed ball shrinks.

You can use the above result to calculate the volume of the ellipse. Fix strictly positive numbers $a_1, a_2, \cdots, a_n$. Consider the region

$$E = \{(x_1, x_2, \cdots, x_n) : \sum \frac{x_i^2}{a_i^2} \leq 1\}.$$

Clearly the unit ball is mapped onto this by the transformation

$$(x_1, x_2, \cdots, x_n) \mapsto (a_1 x_1, a_2 x_2, \cdots, a_n x_n).$$

Thus the Jacobian rule tells us that

$$|E| = a_1 a_2 \cdots a_n V_n.$$

Dirichlet integral:

We know that

$$\int_0^1 x^{a-1}(1-x)^{b-1} dx = \beta(a,b)$$

Here and in what follows the parameters a, b or a_i are strictly positive numbers.

Let

$$S_n = \{(x_1, x_2, \cdots, x_n) : \ \ x_i > 0 \ \ \forall \ \ i; \ \ \sum x_i < 1\}$$

This is called simplex. Fix numbers $a_1, a_2, \cdots, a_{n+1}$ all strictly positive.

We wish to calculate

$$I_n = \int_{S_n} x_1^{a_1 - 1} \, x_2^{a_2 - 1} \ \cdots x_n^{a_n - 1} \, (1 - \sum x_i)^{a_{n+1} - 1} \ \ dx.$$

(see how we changed x_1, x_2 to x, y and a_1, a_2, a_3 to a, b, c)

Let us consider $n = 3$.

$$I_3 = \int_{S_3} x^{a-1} y^{b-1} (1 - x - y)^{c-1} dx dy$$

$$= \int_0^1 x^{a-1} \left[\int_0^{1-x} y^{b-1} (1 - x - y)^{c-1} dy \right] dx$$

If you substitute $y = (1-x)u$ in the y-integral and simplify we get

$$I_3 = \beta(b,c) \beta(a, b+c)$$

This is not in recognizable (and symmetric) form. Let us now use a result proved earlier
$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$
We get
$$I_3 = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}.$$

You can show by induction
$$I_n = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{n+1})}{\Gamma(\sum a_i)}.$$
This is called Dirichlet integral.

Normal integral again:

Let us define on R^n,
$$\varphi(x) = (2\pi)^{-n/2} \;\; e^{x^t x}.$$
More precisely,
$$\varphi(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2} \exp\left\{\frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)\right\}.$$
This is called the standard normal density in n variables.
$$\int_{R^n} \varphi(x)dx = 1.$$
We used the abbreviation $dx = dx_1\, dx_2 \cdots dx_n$. This is because the integrand 'splits' and allows you to integrate variables one after the other (all integrals giving you the value one).

You should remember that vectors in R^n are column vectors. Thus $x^t x$ is a number. Suppose we take a vector $\mu \in R^n$. Then
$$\int_{R^n} \varphi(x-\mu)dx = 1.$$
That is
$$\int_{R^n} \frac{1}{(\sqrt{2\pi})^n} e^{-\{(x_1-\mu_1)^2 + (x_2-\mu_2)^2 + \cdots\}/2} dx = 1.$$
Here again the integral splits. Or you can make change of variable $y = x - \mu$.

Let us take a symmetric positive definite matrix Σ. Then

$$\int_{R^n} \frac{1}{(\sqrt{2\pi})^n \sqrt{|\det \Sigma|}} \;\; e^{x^t \Sigma^{-1} x} \;\; dx = 1.$$

To prove this you need some matrix theory. You know that there is a (necessarily non-singular) matrix A such that

$$A^{-1}\Sigma A = D; \qquad (i.e.) \quad \Sigma = ADA^{-1}.$$

where D is diagonal matrix. (You know $\underline{A}^{-1}\Sigma\underline{A} = D$, I am just renaming. Let us start with this.

First observe that the above equation says

$$\Sigma A = AD$$

Let v be the first column of A. Then the first column of left side equals Σv. If λ_1 is the first diagonal entry of the diagonal matrix D, then the first column of the right side equals $\lambda_1 v$.

In other words the jth column, say v^j of A, gives you eigen vector for the eigen value corresponding to j-th diagonal entry of D, say λ_j . Since A is invertible these eigen vectors form a basis.

In other words the above representation of Σ gives you the eigen values and basis of eigen vectors.

Also if $\lambda_1 \neq \lambda_2$ are two distinct eigen values (distinct diagonal entries of D) then the corresponding columns v_1, v_2 are orthogonal. Indeed using that Σ is symmetric,

$$\lambda_1\langle v_1, v_2\rangle = \langle \lambda_1 v_1, v_2\rangle = \langle \Sigma v_1, v_2\rangle = \langle v_1, \Sigma v_2\rangle = \langle v_1, \lambda_2 v_2\rangle = \lambda_2\langle v_1, v_2\rangle$$

Since $\lambda_1 \neq \lambda_2$ we conclude that

$$\langle v_1, v_2\rangle = 0.$$

If a λ repeats, say $\lambda_1, \lambda_3, \lambda_7$ are all equal to λ, then v^1, v^3, v^7 gives you a three dimensional subspace S (remember columns of A form a basis and hence they are independent vectors). Since for all these vectors $\Sigma v = \lambda v$ we see that this is true for all vectors in this three dimensional subspace S. If you replace these vectors by choosing three orthogonal vectors from this subspace S then the above equation still remains true. Multiply and see. In other words if you take orthogonal vectors w^1, w^3, w^7 from S and replace

v^1, v^3, v^7 columns of A by these w's then the matrix A is still non-singular and the equation $\Sigma A = AD$ still remains true.

Finally, if you multiply each column by norm of that vector then also the equation above remains true. All this amounts to saying that you can safely assume that the columns of A are orthogonal and each column is a unit vector. But then a direct multiplication shows that $A^tA = I$, the identity matrix. In other words we have an orthogonal matrix A with

$$\Sigma A = AD; \qquad A^tA = AA^t = I$$

Define $B = A\sqrt{D}A^{-1}$ where $\sqrt{D}$ is the diagonal matrix with entry-wise square roots. Remember Σ being positive definite, these λ are positive.

In other words Σ is the matrix which sends the vector v^j to $\lambda_j v^j$ and B sends v^j to $\sqrt{\lambda_j}\, v^j$.

Easy to see that B commutes with Σ. Hence B^t commutes with $\Sigma^t = \Sigma$. Also $B^tB = \Sigma$, multiply and see.

Finally, returning to our integral, if we substitute

$$x = By$$

then,

$$x^t\Sigma^{-1}x = y^tB^t\Sigma^{-1}By = y^ty$$

where we used that B and B^t commute with Σ and $B^tB = \Sigma$.

Also the Jacobian $dx = |B|dy = \sqrt{|\Sigma|}dy$. In other words this integral is reduced to the previous one without Σ.

We can combine both the processes, namely of introducing μ and Σ. Let as earlier $\mu \in R^n$ and Σ be a positive definete matrix. Then

$$\int_{R^n} \frac{1}{(\sqrt{2\pi})^n\sqrt{|\det \Sigma|}} \; e^{-(x-\mu)^t\Sigma^{-1}(x-\mu)/2} \; dx = 1.$$

You can do it in two steps. First substitute $z = x - \mu$ and then $z = By$.

volume of simplex:

To find the volume of the region, called simplex,

$$S = \{(x_1, x_2, \cdots, x_n) : x_i \geq 0 \;\; \forall \;\; i; \;\; \sum x_i \leq 1.\}$$

This is simple

$$v_n = \int_S 1 dx$$

Range is

$$0 \leq x_1 \leq 1; \quad 0 \leq x_2 \leq 1 - x_1; \quad 0 \leq x_3 \leq 1 - x_1 - x_2; \cdots$$

$$0 \leq x_n \leq 1 - x_1 - x_2 - \cdots - x_{n-1}.$$

You can successively integrate x_n and then x_{n-1} etc to get sucessively

$$(1 - \sum_1^{n-1} x_i); \quad (1 - \sum_1^{n-2} x_i)^2/2!; \quad (1 - \sum_1^{n-3} x_i)^3/3! \cdots \cdots 1/n!.$$

Thus the volume is $1/n!$ Of course, you use induction.

As suggested by one of you, you can do it neatly using Dirichlet integral. In a sense, this is special case of the Dirichlet integral where all the a_i are one.

You can use the above result to find volume of the following simplex. Fix strictly positive numbers $a_1, a_2, \cdots a_n$. Consider the region,

$$\{(x_1, x_2, \cdots, x_n) : \forall \ i \ \ x_i > 0; \ \sum \frac{x_i}{a_i} \leq 1\}.$$

Higher dimensions:

We started with general R^n norm, convergence of sequences and so on. Continuity and differentiability were also discussed in general and soon after those definitions we specialized to R^2 and sometimes to R^3. This is only to get a better feel and actually see things. all the results remain true in general. We shall mention only some.

You should get a full and clear picture. There is no need to be able to write complete proofs. The philosophy is simple. If you understand R^2 and R^3; both results as well as proofs; then you can carry out the details in R^n too.

Inverse function theorem:

suppose $\Omega \subset R^n$ is an open set and $f : \Omega \to R^n$ be a C^1 function. Let $a \in \Omega$ with $f'(a)$ non-singular. Recall that $f(x)$ being an n-tuple we can write

$$f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$$

$f'(x)$ is the matrix

$$\begin{pmatrix} \nabla f_1(x) \\ \nabla f_2(x) \\ \cdot \\ \cdot \\ \nabla f_n(x) \end{pmatrix}$$

or equivalently the matrix

$$(D_j f_i(x) : 1 \leq i, j \leq n)$$

Thus the matrix $f'(a)$ is non-singular. then there is an open set $V \subset \Omega$ such that $a \in V$ and an open set $W \subset R^n$ such that the following is true:

(i) f is one-one on V onto W.

(ii) The inverse map $g : W \to V$ is C^1 map.

(iii) For $y = f(x) \in W$ we have $g'(y) = [f'(x)]^{-1}$.

Proof goes along similar lines as in two dimensions.

<u>implicit function theorem:</u>

Suppose $\Omega \subset R^n \times R^m$ is an open set and $f : \Omega \to R^m$ is an C^1 function. Let $(a, b) \in \Omega$. The notation is $a \in R^n$ and $b \in R^m$. suppose that $f(a, b) = 0$. suppose that $f_2(a, b)$ is non-singular. here f_2 is the derivative w.r.t. the last set of m coordinates.

More precisely, let us write, still using the notation, $(x.y)$ for points of Ω with the understanding $x \in R^n$ and $y \in R^m$;

$$f(x, y) = (f_1(x, y), f_2(x, y), \cdots, f_m(x, y))$$

Then $f_2(a, b)$ is the $m \times m$ matrix

$$\begin{pmatrix} \nabla_y f_1(a, b) \\ \nabla_y f_2(a, b) \\ \cdot \\ \cdot \\ \nabla_y f_m(a, b) \end{pmatrix}$$

Then there is an open set $V \subset R^n$; open set $W \subset R^m$ such that $(a, b) \in V \times W \subset \Omega$ and an C^1 function $\varphi : V \to W$ such that $f(x, \varphi(x)) = 0$ for all $x \in V$.

In fact matters are so arranged that for each $x \in V$ there is just one $y \in W$ such that $f(x, y) = 0$ and this unique y is defined as $\varphi(x)$.

The proof we saw in the case $m = n = 1$ is a hands on proof and did not use the inverse function theorem. However the proof for R^n uses inverse function theorem.

integration:

We define (rspa) rectangles with sides parallel to the axes to be a set of the form

$$Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

These are also called boxes or cubes. Volume of this set is the number

$$|Q| = \prod_1^n (b_i - a_i).$$

Let f be a bounded function on such a Q. we partition each side $[a_i, b_i]$ and take the product partition of Q. Calculate inf and sup in each set of the partition, multiply by the volume of the set and add up. These are $L(\pi)$ and $U(\pi)$; lower and upper sums. The inf of upper sums and sup of lower sums are calculated. when they are equal we say f is integrable and this common value is the integral.

A set is small if you can cover it by countably many rspa with total volume as small as desired. All the theorems that we had earlier, remain valid: iterated integration, integrability of continuous functions; relation of integrability to smallness of the set of discontinuities.

We define volumes of more general sets as integral of the constant function one and so on. We define integrals over sets more general than Q. The proofs are nearly same. We say nearly because you need to use induction.

One can then proceed to integrals of unbounded functions or of bounded functions on an unbounded set or unbounded functions on unbounded sets etc.

Jacobian rule:

Let T be a non-singular linear transform of R^n to itself. If A is a bounded set, then TA is bounded. If A is open, then TA is open. Boundaries are preserved, that is, $T(\partial A) = \partial(TA)$. If A is small then so is TA.

Thus bounded open sets with small boundaries are transformed to bounded open sets with small boundaries. and the formula $|TV| = |T||V|$ remains valid.

The Jacobian rule remains valid. Here it is.

Let $\Omega \subset R^n$ be a bounded open set. Let $T : \Omega \to R^n$ be a one-to-one C^1 map with non-singular derivative at every point of Ω.

At every point $x \in \Omega$ let $|T'(x)|$ denote the modulus of the determinant of the derivative matrix. Recall if

$$T(x) = (T_1(x), T_2(x), \cdots, T_n(x))$$

$$T'(x) = ((D_j T_i(x)))_{1 \le i,j \le n}$$

Then the following are true:

(i) For any rectangle $Q = \prod[a_i, b_i] \subset \Omega$; TQ has small boundary and

$$|TQ| = \int_Q |T'|.$$

that is

$$|TQ| = \int\int_Q |T'(x)|dx.$$

Here $dx = dx_1 dx_2 \cdots dx_n$.

(ii) (when (i) holds for a T) For any open set $V \subset \Omega$ with small boundary; TV is an open set with small boundary and

$$|TV| = \int_V |T'|.$$

that is,

$$|TV| = \int\int_V |T'(x)|.$$

(iii) (whenever (i) and (ii) hold for a T) For any bounded continuous function f on TV,

$$\int_{TV} f = \int_V f|T|$$

that is,

$$\int_{TV} f(u)du = \int_V f(T(x))|T'(x)|dx.$$

polar coordinates:

Every point $(x, y) \in R^2$ other than $(0, 0)$ can be uniquely expressed as $x = r\cos\theta; y = r\sin\theta$ for some (r, θ) with $0 < r < \infty$ and $0 \leq \theta < 2\pi$. these (r, θ) are called polar coordinates of the cartesian point (x, y). The Jacobian of the transformation is r. Thus when you integrate $dxdy$ is transformed to $rdrd\theta$.

Every point $(x, y, z) \in R^3$ other than $(0, 0, 0)$ can be uniquely expressed as

$$x = r\cos\theta; \quad y = r\sin\theta\cos\psi; \quad z = r\sin\theta\sin\psi.$$

where

$$0 < r < \infty; \quad 0 \leq \theta \leq \pi; \quad 0 \leq \psi < 2\pi.$$

These (r, θ, ψ) are called the spherical or polar coordinates of (x, y, z). The Jacibian is $r^2 \sin\theta$. Thus when you integrate

$$f(x, y, z)\; dx\; dy\; dz$$

is transformed to

$$f(r, \theta, \psi)\; r^2\; \sin\theta\; dr\; d\theta\; d\psi.$$

In n dimensions lo there is a similar transformation to polar coordinates. Every point

$$(x_1, x_2, \cdots, x_n)$$

can be uniquely expressed as

$$x_1 = r\cos\theta_1;$$

$$x_2 = r\sin\theta_1\cos\theta_2;$$

$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3;$$

$$\vdots$$

$$x_{n-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1};$$

$$x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1};$$

where

$$0 < r < \infty;\; \theta_1, \theta_2, \cdots \theta_{n-2} \in [0, \pi];\; \theta_{n-1} \in [0, 2\pi).$$

The Jacobian of this transformation is

$$r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}$$

The proof is by induction on n. You can write the Jacobian and expand as sum of two determinants for each of which the induction hypothesis applies.

Thus

$$dx_1 dx_2 \cdots dx_n$$

is transformed to

$$r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\; dr d\theta_1 \cdots d\theta_{n-1}.$$

Here the sphere

$$\{x : 0 < ||x|| \leq R\}$$

is transformed to the rectangle

$$[0, R] \times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi).$$

The representation itself is also proved by induction. You need to put

$$r = \sqrt{\sum x_i^2}.$$

you need only represent the unit vector $(u_i) = (x_i/r)$. You get a θ_1 such that

$$u_1 = \cos\theta_1; \quad \sqrt{\sum_2^n u_i^2} = \sin\theta_1.$$

Since both these numbers are non-negative a number $\theta_1 \in [0, \pi]$ exists and it is unique. Now use induction hypothesis.

cylindrical coordinates:

Given a point $(x, y, z) \in R^3$ such that $(x, y) \neq (0, 0)$ there is a unique number $0 < r < \infty$ and $0 \leq \theta < 2\pi$ such that $x = r\cos\theta; y = r\sin\theta$. the triple (r, θ, z) are called the cylindrical coordinates of the point (x, y, z). jacobian of this transformation is r. Thus

$$dxdydz$$

is transformed to

$$r dr d\theta dz.$$

this is easy to establish. Here the cylinder (z-axis deleted)

$$\{(x, y, z) : 0 < x^2 + y^2 \leq R^2; 0 \leq z \leq h\}$$

is transformed to the rectangle

$$\{(r, \theta, z) : 0 < r \leq R; 0 \leq \theta <, 2\pi; 0 \leq z \leq h\}.$$

Lagrange multipliers:

We have discussed analogue of maxima and minima in two variables. Many a times you need to optimize (find local extrema of) a function subject to certain constraints. For example you want to find out in the positive quadrant

$$\max(x + y) \quad \text{subject to} \quad x + y = 1$$

(actually you need not say positive quadrant, it follows even if you did not say it).

Of course in the above problem, you can say $y = 1 - x$ and we need to maximize $x(1-x)$. Manytimes such a simplification is not possible. Rewrite the above problem as

$$\max f(x, y) \quad \text{subject to} \quad \varphi(x, y) = 0$$

where $f(x, y) = xy$ and $\varphi(x, y) = x + y - 1$. Lagrange found out that at such a point (x_0, y_0) where there is a max (or min) there is a number λ such that

$$\nabla f + \lambda \nabla \varphi = 0.$$

This method works even in complicated situations when you can not explicitly express one variable as a function of the other variable. More over the above equations maintain a symmetry without expressing one variable as a function of the other. Thus you can solve the three equations, namely, above system of two equations along with the one equation $\varphi = 0$ to obtain the three numbers (x_0, y_0, λ).

Of course you would ask what is the worth of this? You need to find out precisely the points of max and min and Lagrange only tells us what happens if you already know that (x_0, y_0) is such a point. You can utilise the above discovery to first solve the system of equations and get all solutions

(x, y, λ) and reduce your search to check only among these points. Usually, the system does not have too many solutions and so it is easy to check this.

Of course, you need to check whether a solution you obtained is max or min or neither. There is no simple criterion. remember Lagrange only tells you at a point of (constrained) local extremum something happens. he does not say that if such a thing happens then the point is an extremum. We see examples.

to proceed further, let us define what is meant by constrained extremum. Let f and φ be two real valued functions on an open set $\Omega \subset R^2$. A point $P \in \Omega$ is a local maximum of f subject to the constraint $\varphi = 0$ if there is an $\epsilon > 0$ such that the following happens:

$$\varphi(P) = 0;$$

and

$$Q \in \Omega; \ \varphi(Q) = 0; \ ||P - Q|| < \epsilon \ \Rightarrow f(Q) \leq f(P).$$

Thus locally at P there is no other point satisfying the constraint and gives a larger value for f.

Similarly we define local minimum subject to the constraint. These are called local extrema subject to the given constraint.

Theorem: f and φ be C^1 functions on an open set $\Omega \subset R^2$; $P = (x_0, y_0) \in \Omega$ is a constrained local extremum. Assume that $\varphi_y(P) \neq 0$.

Then there is a number λ such that

$$\nabla f(P) + \lambda \nabla \varphi(P) = 0.$$

Proof is simple. Assume we have a local max.

Since $\varphi(P) = 0$ and $\varphi_y(P) \neq 0$, the implicit function theorem applies. We have a rectangle $(a, b) \times (c, d)$ which includes P and has the following property. For every $x \in (a, b)$ there is a unique $y \in (c, d)$ such that $\varphi(x, y) = 0$. If you define $g(x)$ as this unique y, then g defines a C^1 function on (a, b). In other words the set of points $(x, g(x))$ captures *all* zeros of the φ in this rectangle.

If necessary we can take a smaller rectangle so that $\varphi(P)$ is max in this rectangle. Since g captures all zeros of φ in the sense described above we conclude that the function $f(x, g(x))$ assumes its max value on the interval (a, b) at the point x_0 and hence its derivative is zero at this point. Thus chain rule for the map

$$x \mapsto (x, g(x)) \mapsto f(x, g(x)).$$

We get

$$f_1(P) + g'(x_0) f_2(P) = 0. \qquad (\bullet)$$

How do we get rid of g that we introduced from the above. apply chain rule to the map

$$x \mapsto (x, g(x)) \mapsto \varphi(x, g(x)) \equiv 0.$$

$$\varphi_x(P) + g'(x_0)\varphi_y(P) = 0. \qquad (*)$$

(You can apply the formula for derivative of function defined implicitly). Combining $(\bullet)$ and $(*)$

$$\begin{pmatrix} f_1(P) & f_2(P) \\ \varphi_x(P) & \varphi_y(P) \end{pmatrix} \begin{pmatrix} 1 \\ g'(x_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we have a matrix equation $Av = 0$ where the vector v is non-zero. Thus the rows of the matrix must be linearly dependent. Since we know $\varphi_y(P) \neq 0$, the second row is non-zero row. So the first row must be a multiple of the second row. The constant denoted by $(-\lambda)$ satisfies the requirement.

This completes the proof.

Let us see some examples.

(1) maximize $f(x, y) = xy$ subject to $x^2 + y^2 = 1$.

Here $\varphi(x, y) = x^2 + y^2 - 1$. The equations reduce to

$$y + 2\lambda x = 0; \quad x + 2\lambda y = 0; \quad x^2 + y^2 = 1.$$

Multiply first eqn by x, second by y; add; use third to see $xy + \lambda = 0$. Use this in the first. You will see the only solutions are

$$(x, y, \lambda) = (1, 0, 0); (0, 1, 0); (1/\sqrt{2}, 1/\sqrt{2}, 1/2).$$

The first two are not extrema. third is.

(2) Maximize $f(x, y, z) = xyz$ subject to $x^2 + y^2 + z^2 = 1$.

Exactly the same procedure leads to

$$yz + 2\lambda x = 0; \quad xy + 2\lambda z = 0; \quad zx + 2\lambda y = 0; \quad x^2 + y^2 + z^2 = 1$$

so that Multiply first eqn by x etc and add; use the constraint to see

$$2\lambda = -3xyz.$$

substitute in the above three eqns to get the solutions

$$(\pm 1, 0, 0) \ \ (0, \pm 1, 0) \ \ (0, 0, \pm 1) \quad \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right).$$

It is easy to see that the first six solutions are neither max nor min. At these points f is zero. In the neighbourhood of these points f takes values positive and also negative. Of the next four points four are maxima and four minima.

In an extremal problem where $\nabla f(P) = 0$, if there are points Q in every neighbourhood of P such that $f(P) > f(Q)$ as well as there are points Q in every neighbourhood of P with $f(P) < f(Q)$; then we say p is a saddle point.

Similarly in a constrained extremal problem suppose we have a solution (P, λ) for the Lagrange method, that is, satisfying $\nabla f(P) + \lambda\varphi(P) = 0$. Suppose that in every neighbourhood of P there are points Q satisfying the constraint and $f(P) > f(Q)$ as well as points Q satisfying the constraint such that $f(P) < f(Q)$. Then we say that P is saddle point for the constrained problem.

In the above problem, the first six points are saddle points.

(3) We are given an $n \times n$ symmetric matrix A.

maximize $x^t Ax$ subject to $||x||^2 = 1$.

Note that the set $||x|| = 1$ is a compact set and and the expression has a maximum.

The expression $f(x) = x^t Ax = \sum_{i,j} a_{ij} x_i x_j$ is called a quadratic form. Here $\varphi(x) = \sum x_i^2 - 1$.

The equations are, denoting the lagrange constant by $(-\lambda)$,

$$2\sum_j a_{ij} x_j - 2\lambda x_i = 0; \quad i = 1, 2, \cdots, n.$$

That is

$$Ax - \lambda x = 0; \quad Ax = \lambda x.$$

In other words the Lagrange constant λ is an eigen value and x corresponding eigen vector. But which eigen value is it? At this point

$$x^t Ax = x^t \lambda x = \lambda.$$

Thus Lagrange method helps you to identify the largest eigen value as maximum of the quadratic form. Of course, there are several eigen values and corresponding eigen vectors. We know a priori that there is a maximum and hence this procedure gives it along with others.

(4) Find box with sides parallel to the coordinate planes which has the largest volume and is contained in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

A box (or rectangular parallelopiped) is a figure with rectangular sides. It may not be having sides parallel to the coordinate planes. For example start with a box with sides parallel to the coordinate planes and apply a rotation. What you get is still a box with rectangular faces. It should be possible to show that such a box contained in the above ellipsoid, having largest volume must actually have sides parallel to the coordinate planes. I do not have a proof right now.

Returning to our problem let us consider boxes as stated. If none of the corners is on the boundary of the ellipsoid, then you can enlarge the box increasing the volume. If one corner (x_0, y_0, z_0) is on the boundary then the box must have corners $(\pm x_0, \pm y_0, \pm z_0)$ and hence volume $8x_0y_0z_0$. Thus the geometric problem can be formulated as the following analytical problem.

Maximize $8xyz$ subject to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Proceeding exactly as above you get corners to be $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$. and hence the volume is $8abc/(3\sqrt{3})$.

Lagrange with two constraints:

Lagrange method works with any variables with reasonable number of constraints. I shall state the general theorem. But before that I shall state for three variables with two constraints.

Maximize $f(x, y, z)$ subject to $\varphi(x, y, z) = 0$; $\psi(x, y, z) = 0$.

We assume all are C^1 functions defined on an open set Ω. We assume that $P = (x_0, y_0, z_0) \in \Omega$ is a solution for the problem. We show there are numbers λ and μ such that

$$\nabla f(P) + \lambda \nabla \varphi(P) + \mu \nabla \psi(P) = 0.$$

Of course, this comes with a price. We assume that

$$\begin{pmatrix} \varphi_y(P) & \varphi_z(P) \\ \psi_y(P) & \psi_z(P) \end{pmatrix} \qquad (\spadesuit)$$

is non-singular, that is, the determinant is non zero.

Under the hypothesis, we can apply implicit function theorem. get an interval $I = (x_0 - \delta, x_0 + \delta)$ and an box (rspa) Q such that

(i) $(y_0, z_0) \in Q$; $\quad I \times Q \subset \Omega$.

(ii) for $x \in I$ there is unique $(y, z) \in Q$ such that $\varphi(x, y, z) = 0$ and $\psi(x, y, z) = 0$. Moreover the functions $g : I \to Q$ that maps $x \mapsto (y, z)$ is a C^1 map. Denote $g(x) = (g_1(x), g_2(x))$. Thus $g(x_0) = (y_0, z_0)$.

Thus region $I \times Q$ contains (x_0, y_0, z_0) and all common zeros of φ and ψ are captured by $\{(x, g(x)) : x \in I\}$ in this region. In other words x_0 is a extremal point for the function $f(x, g(x))$. so derivative of this function must be zero at x_0. applying chain rule

$$x \mapsto (x, g_1(x), g_2(x)) \mapsto f(x, g_1(x), g_2(x))$$

$$f_x(P) + f_y(P) g_1'(x_0) + f_z(P) g_2'(x_0) = 0. \quad (*)$$

Applying chain rule to

$$x \mapsto \langle x, g_1(x), g_2(x) \rangle$$
$$\mapsto \langle \varphi(x, g_1(x), g_2(x)), \psi(x, g_1(x), g_2(x)) \rangle \equiv \langle 0, 0 \rangle.$$

$$\begin{pmatrix} \varphi_x(P) & \varphi_y(P) & \varphi_z(P) \\ \psi_x(P) & \psi_y(P) & \psi_z(P) \end{pmatrix} \begin{pmatrix} 1 \\ g_1'(x_0) \\ g_2'(x_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (**)$$

combining $(*)$ and $(**)$ we see

$$\begin{pmatrix} f_x(P) & f_y(P) & f_z(P) \\ \varphi_x(P) & \varphi_y(P) & \varphi_z(P) \\ \psi_x(P) & \psi_y(P) & \psi_z(P) \end{pmatrix} \begin{pmatrix} 1 \\ g_1'(x_0) \\ g_2'(x_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We have a matrix equation $Av = 0$ with a non-zero vector v. so A has dependent rows. Last two rows are independent, because y hypothesis, the 2×2 bottom corner is non-singular. Thus the first row must be a linear combination of the last two. This gives λ and μ.

Instead of (♠) we could have assumed

$$\begin{pmatrix} \varphi_z(P) & \varphi_x(P) \\ \psi_z(P) & \psi_x(P) \end{pmatrix}$$

is non-singular. Then we get an interval around y_0 etc. the same proof works. Or we could assume, instead,

$$\begin{pmatrix} \varphi_x(P) & \varphi_y(P) \\ \psi_x(P) & \psi_y(P) \end{pmatrix}$$

is non-singular.

general Lagrange:

Let $f, \varphi_1, \cdots \varphi_m$ be all real valued C^1 functions defined on an open set $\Omega \subset R^n$. Assume $m < n$. Suppose $x^0 \in \Omega$ is an extremal for f subject to

$$\varphi_i(x) = 0; \quad i = 1, 2, \cdots, m.$$

Then there are numbers $(\lambda_i : 1 \leq i \leq m)$ such that

$$\nabla f(x^0) + \lambda_1 \nabla \varphi_1(x^0) + \lambda_2 \nabla \varphi_2(x^0) + \cdots + \lambda_m \nabla \varphi_m(x^0) = 0.$$

This comes with a price. We need to assume that the matrix

$$\left(\frac{\partial}{\partial x_j} \varphi_i(x^0) : 1 \leq i, j \leq m \right)$$

is non-singular. we have considered the $m \times m$ matrix by differentiating the constraining function w.r.t. the first m coordinates. You can take any m coordinates and use them for all the functions. Demand that this matrix be non-singular. Remember we are evaluating the matrix at the point x^0.

you should keep in mind two things. the method reduces the search for extremals among solutions of the following $n + m$ equations.

$$\nabla f(x) + \lambda_1 \nabla \varphi_1(x) + \lambda_2 \nabla \varphi_2(x) + \cdots + \lambda_m \nabla \varphi_m(x) = 0.$$

$$\varphi(x) = 0; \varphi_2(x) = 0; \cdots \varphi_m(x) = 0.$$

solve these $n + m$ equations to get solutions

$$(x, \lambda) : x \in R^n; \lambda \in R^m$$

and search those x for max or min. of course not all may be extremals. But extremals are definitely contained among these.

an estimation problem:

I have a die, biased. I do not know the chances of the faces appearing in a throw. I roll the die n times and observe that face i appeared n_i times for $1 \leq i \leq 6$. of course, $\sum n_i = n$. let us assume that each $n_i > 0$.

Question: Looking at the data how can I decide the chances for the faces. In other words, let $p_i > 0$ be the chances of face i for $1 \leq i \leq 6$. How do I estimate these numbers?

You say that intuitively n_i/n should be an estimate for p_i. Yes, right, that is what intuition suggests. But does any principle also suggest that. If so, I can use that principle when my intuition fails. Also probably I can develop a theory about such a principle.

The answer is yes, there is a principle which will give you the same answer that your intuition suggested. To begin, remember that in probability theory, you build models; assume some probabilities for outcomes and see the consequences, calculate other interesting probabilities and so on. When you build such models, you have the feeling that an outcome with higher probability is more likely to be seen than an outcome with smaller probability.

So when it comes to estimation, we turn things around and say that since I have seen something, it must have larger probability than others. Of course, you do not reduce it to ridiculous level and say that what I have seen has probability one. Then the numbers (p_i) disappear and there is only one outcome, namely, $(n_i; 1 \leq i \leq 6)$ which has probability one. Our premise is that each $p_i > 0$, that is, the chances for each face are strictly positive. And of course, this entails that the experiment is *not* deterministic. The data could be any thing consistent with multinomial.

In other words we propose the following principle: Our estimate of the parameters (p_i) are those numbers that maximize the probability of our observation. The multinomial probabilities tell you that in case (p_i) are the unknown probabilities, then chances of coming up with our observation are

$$f(p_1, \cdots, p_6) = \frac{n!}{n_1!n_2!n_3!n_4!n_5!n_6!}\, p_1^{n_1}\, p_2^{n_2}\, p_3^{n_3}\, p_4^{n_4}\, p_5^{n_5}\, p_6^{n_6}. \quad (*)$$

Thus our estimate are those numbers that maximise the above function of the (p_i). Of course, there is a constraint, namely we must have

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1.$$

Thus we are in Lagrange's set up. Since n_i are given numbers, maximizing $(*)$ is same as maximizing the second term there, forgetting the factorials. But again maximizing this is same as maximizing its logarithm. Note that we are assuming that each $p_i > 0$. So logarithm makes sense. Thus the problem reduces to maximizing

$$L(p_1, \cdots, p_n) = \sum n_i \log p_i$$

subject to the constraint $\sum p_i = 1$. Lagrange tells that at an extremum point we must have

$$\frac{n_i}{p_i} - \lambda = 0 \quad 1 \leq i \leq 6.$$

We have used $-\lambda$ for the Lagrange constant instead of $+\lambda$. In other words $p_i = n_i/\lambda$ and now $\sum p_i = 1$ tells us $\lambda = n$ and finally $p_i = n_i/n$.

Of course you need to verify if it is really a point of extremum at all, if so if it is maximum or minimum. In this case you can be sure it is maximum. It can not be minimum and there is a maximum. We shall not detail the argument. it is not difficult. There is no minimum because at the boundary, that is, when one of the $p_i = 0$ this function assumes the value zero. For the same reason combined with compactness, it has a maximum.

Hadamard inequality:

Consider an $n \times n$ matrix (real entries), say, A. Let us denote its rows by $r_1, r_2, \cdots, r_n$. Then Hadamard's inequality says

$$|\det(A)| \leq ||r_1||\, ||r_2|| \,\cdots\, ||r_n||.$$

here norm of a vector is the usual Euclidean norm: $\sqrt{\sum v_i^2}$.

If one of the norms is zero, there is nothing to do, that row is zero and done. Let us assume that all are strictly positive. thus the problem reduces to the following. We are given n strictly positive numbers $d_1, \cdots, d_n$. Let us, as above, denote rows of matrix by r_i.

$$\text{maximize} \quad |A|; \quad \text{subject to} \quad ||r_i|| = d_i; \quad 1 \le i \le n.$$

We are in Lagrange set up. We have a function of n^2 variables, instead of denoting the variables with usual linear index, name the variables (x_{ij}). Define

$$f(\{x_{ij}\}) = \det((x_{ij})).$$

name the n constraints

$$\varphi_i(\{x_{ij}\}) = d_i^2 \qquad 1 \le i \le n.$$

where

$$\varphi_i(\{x_{ij}\}) = \sum_j x_{ij}^2$$

See that the each constraint depends on variables only in one row.

Thus Lagrange tells us that at an extremum

$$X_{ij} - \lambda_i 2x_{ij} = 0 \qquad 1 \le i, j \le n.$$

We denoted by X_{ij} the cofactor of x_{ij}; which is $(-1)^{i+j}$ times the determinant obtained by deleting the row and column containing the entry x_{ij} (thus delete i-th row and j-th column). If you expand the determinant with the help of i-th row, then it is easy to calculate the derivative of f w.r.t. the variables in that row.

Let us recall two facts about the cofactors.

$$\sum_i x_{ij} X_{ij} = \det(A).$$

$$\sum_p x_{pj} X_{ij} = 0 \qquad p \ne i.$$

Combined with the above Lagrange equations we get

$$\det(A) = \lambda_i 2d_i^2; \qquad 1 \le i \le n$$

$$\lambda_i \sum_j x_{pj} x_{ij} = 0; \;\; p \ne j.$$

Note that at a maximum $\lambda_i \ne 0$ for all i. This is because, If some $\lambda_i = 0$ then the first equation above tells us that $\det(A) = 0$ but the diagonal matrix

$(d_1, d_2, \cdots, d_n)$ tells us that this can not be maximum.

Thus at a point of maximum, we must have the rows of A must be orthogonal. In other words at a maximum the matrix A is orthogonal matrix. But then for such a matrix A where the maximum is attained we must have

$$|\det(A)|^2 = \det(AA^t) = \mathrm{diag}\{d_1^2, d_2^2, \cdots, d_n^2\}.$$

In other words, at a maximum we have the Hadamard inequality. This proves the inequality.

See how we did not solve the equations and discuss which is max and which is min. We found the equations to be satisfied at an extremum. Then we discarded some and found the equations that should be satisfied at the maximum. This told us that rows must be orthogonal. This was enough to conclude the required inequality. But actually we have found all solutions where the equality is achieved in the Hadamard inequality.

recapitulation of the development:

(1). Points in R are x. Points in R^2 are (x, y); points in R^3 are (x, y, z).

If $f : R \to R$ then $f(x)$ is value of f at x. If $f : R^2 \to R$ then $f(x, y)$ is value of f at (x, y). Similarly we have for $f : R^3 \to R$

If $f : R \to R$ then its graph is a subset of R^2; consists of all points (x, y) such that $f(x) = y$. If $f : R^2 \to R$ then its graph is a subset of R^3; consists of all points (x, y, z) such that $f(x, y) = z$. Similarly for $f : R^3 \to R$, its graph is a subset of R^4.

(2). Derivative at a point x is denoted by $f'(x)$ in case of R.

In case of R^2 it is denoted at a point (x, y) by $f'(x, y)$ or $\nabla f(x, y)$ or $(f_1(x, y), f_2(x, y))$ or sometimes by $(f_x(x, y), f_y(x, y))$. But this last notation is confusing with x appearing as suffix and also argument. Also it has the disadvantage of leading to meaningless things $f_3(3, 4)$ (substitute blindly $x = 3, y = 4$). However if we have a fixed point (a, b) then we could, without fear of confusion, denote $f_x(a, b)$. Similar notation in dimension three holds.

(3) In one dimension we write

$$dy = f'(x)dx; \qquad df = f'(x)dx.$$

In two dimensions

$$dz = f_x dx + f_y dy; \quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Similar notation holds in three dimensions

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

all these notations have their origins in the mean value theorems. df denotes the change in the value of the function for a small change in the value of the arguments. For example in one dimension if x is changed to $x + \epsilon$ then $f(x)$ is changed to $f(x+\epsilon)$ and thus the change in its value is

$$df = f(x+\epsilon) - f(x) = f'(?)\epsilon.$$

where (?) is a point in between, $x+\epsilon$ and x assuming that f is C^1 and noting that ϵ is very small this number $f'(?)$ is approximately $f'(x)$. if you now read the equation above with ϵ replaced by dx you get the interpretation given above.

similarly for functions of two variables

$$f(x+\epsilon_1, y+\epsilon_2) - f(x,y)$$

$$= \{f(x+\epsilon_1, y+\epsilon_2) - f(x, y+\epsilon_2)\} + \{f(x, y+\epsilon_2) - f(x,y)\}$$

$$= f_1(?, y+\epsilon_2)\, \epsilon_1 + f_2(x, ??)\, \epsilon_2.$$

Again assuming C^1 we can argue as above.

(4). The main and very important change of attitude that put all these definitions on one single platform is to think of these derivatives as linear maps. Think of elements in R^2 as columns and these derivatives are rows. the row operates from left on a column to give a number. This also helped later to give a unified method of thinking of derivatives when we have functions from an R^m to an R^n. Derivative is essentially 'linearizing' a given function at a point.

(5). If you have a point (x_0, y_0) on the graph of a function $f : R \to R$ then tangent at that point to the curve is the affine map on the real line

$$Ax = y_0 + f'(x_0)(x - x_0)$$

Customarily this is written as

$$y - y_0 = f'(x_0)(x - x_0).$$

You think of, not only the analytical expression, but also the graph of this map, namely, straight line.

Of course this line is nothing but translate of the line (subspace) through the origin $y = f'(x_0)x$ by the point (x_0, y_0).

Similarly for function of two variables $f : R^2 \to R$ and a point (x_0, y_0, z_0) on its graph (it is now called surface instead of curve) we define tangent plane as the set of points satisfying

$$z - z_0 = f_1(x_0, y_0)(x - x_0) + f_2(x_0, y_0)(y - y_0)$$

or

$$z - z_0 = \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0).$$

again, this plane is nothing but the translate of the subspace

$$z = \nabla f(x_0, y_0) \cdot (x, y)$$

by the point (x_0, y_0, z_0).

(6). Chain rule is nothing but composition of linear maps.

In one dimensions if we have a map

$$t \mapsto x(t) \mapsto f(x(t))$$

then we have

$$\frac{d}{dt} f(x(t)) = f'(x(t))x'(t); \quad \text{or} \quad \frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

In two dimensions

$$t \mapsto (x(t), y(t)) \mapsto f(x(t), y(t))$$

then

$$\frac{d}{dt} f(x(t), y(t)) = f'(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle.$$

Remember f' is a row vector and the other is column vector. Sometimes this is written as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

You see the usage of d and ∂. When you use d you are indicating you have a function of one variable and you are differentiating w.r.t. that one variable.

When you use ∂ you are indicating you have a function of more than one variable and you are differentiating w.r.t. the variable indicated.
Similarly if we have $f(x(t), y(t), z(t))$,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Or if you denote

$$\varphi(t) = (x(t), y(t), z(t)) : R \to R^3$$

and

$$g(x, y, z) : R^3 \to R$$

and

$$f(t) = g(\varphi(t))$$

then

$$f' = g'(\varphi(t))\varphi'(t)$$

When you think of the derivative as linear map, the distinction between column and row disappears. You do not need to put dot (scalar product) etc. It is all composition of linear maps. But of course, you should know what linear map you are talking about and its domain/range.

(7) Level sets for a function are points of constancy. Thus for $f : R \to R$ its level set corresponding to the number a is the set $\{x : f(x) = a\}$. Of course this may be empty. Thus remember level set is not one set. It depends on the number a.

For $f : R^2 \to R$ and a number $a \in R$ the level set of f corresponding to the value a is the set $\{(x, y) : f(x, y) = a\}$. Thus if f is the temperature defined on a geographical region $\Omega \subset R^2$ then level sets are nothing but isothermal lines. This will give an idea of the function.

what next:

We have developed the basic concepts and ideas to understand functions of several variables. There are several directions one can take.

We could develop improper integrals completely. We have started once and I gave up after explaining some ideas. I felt that it was not going well.

You asked about the fundamental theorem of calculus. Yes, this is one topic that could be developed. yes, there is an analogue of fundamental theorem. In one dimensions it said the following. If $F' = f$ then

$$\int_{[a,b]} f = F(b) - F(a).$$

Do not think of this theorem as explaining full evaluation of the integral. A space of dimension less than one (whatever it may mean) is a set of two points $\{a < b\}$, the boundary of $[a, b]$ If we interpret the right side above as integral of F over this set, then the fundamental theorem is giving you a reduction of the complicated integral to a lower dimension. To see the beauty, let us denote derivative by not prime, but ∂. Thus instead of $F' = f$ we write $\partial F = f$. Thus the above equality can be written as follows. Let us abbreviate the interval to I

$$\int_I \partial F = \int_{\partial I} F.$$

Did you see how beautifully the ∂ travelled from top to bottom. Left side is integral of a probably simple function on a complicated set, interval. The right side is integral of a probably complicated function but on a simpler set.

If you stare at the above equation it makes perfect sense even in R^2. For example you could take a rectangle for I. You know its boundary. You could take a nice function f on the rectangle. Explain what could be the meaning of ∂F to make the equation above correct and discover that F. In other words, the idea is to make a good definition so that the beauty seen above is preserved. This can be done. This you will learn in the next course on vector calculus.

One can go to develop some concepts of geometry of curves and surfaces. We may not be able to do much, but we use integration and develop notions of lengths of curves. We shall do that.

We can also develop some vector calculus terminology that is useful in physics. We shall do that. One can develop some physics that explains these concepts, but we may not do that.

There is a profound symphony, complex analysis, which you will learn later. We can make a beginning and tie it up with what we learnt so far. We shall do that.

lengths of curves:

A curve is a continuous function f defined on a closed bounded interval $[a, b] \subset R$. Sometimes we have curves defined on an open interval too. But we do not need this generality. If the function takes values in R^2 then it is a curve in plane; if it takes values in R^3 then it is a curve in space. We do not discuss other curves.

The first question is whether the curve is the function or the set of points consisting the image. Thus, for example if you consider the curve

$$f(t) = (\cos t, \sin t); \quad 0 \leq t \leq 2\pi.$$

is the curve this function or is it the circle that you see in the plane described this set of points? Actually, it is both. Suppose you consider the function

$$g(t) = (\cos t, \sin t); \quad 0 \leq t \leq 4\pi.$$

Then this also gives the same circle, but the curve goes along the circle twice. So thinking of the set of points alone does not completely help. Or consider the function

$$h(t) = (\cos t/2, \sin t/2); \quad 0 \leq t \leq 4\pi.$$

This is also the same circle, also goes around exactly once but rather slowly. so you should keep both in mind; the function and the geometric picture. But in case of a dispute, it is the definition that wins, the curve is the function.

How shall we define length of a curve? To make a beginning, let us recall how we defined area under a graph. We all agreed upon, without any dispute, the area of a rectangle, namely, it is the product of lengths of its sides. So we approximated the area by rectangles. We shall do the same thing for length. after all, we have agreement on length of a straight line. The line joining two points (x_1, y_1) and (x_2, y_2) in the plane has length

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Given a curve $\varphi : [a, b] \to R^2$, we take points

$$t_0 = a < t_1 < t_2 < \cdots < t_k = b.$$

Consider the points on the curve

$$\varphi(t_0) = P_0; \varphi(t_1) = P_1; \varphi(t_2) = P_2; \cdots, \varphi(t_k) = P_k.$$

Length of the curve is at least as much as the sum

$$\sum_{i=0}^{k-1} ||P_{i+1} - P_i||.$$

We agree to this because if you have two points, the smallest distance between them is the length of the line joining them. Any other curve joining them must have a larger length.

Given the curve φ, and a partition π of the interval, we define

$$L(\varphi, \pi) = \sum ||\varphi(t_{i+1}) - \varphi(t_i)||$$

Just now we agreed that length of the curve is at least as much as the above. We define

$$L(\varphi) = \sup\{L(\varphi, \pi) : \pi \text{ partition of } [a, b]\}.$$

Thus length of the curve must be at least the above quantity. As the partition becomes finer, the lines joining the partition points on the curve move closer and closer to the curve; and finally trace the curve (of course, there is nothing like 'finally'). Thus this sup is length of the curve. However this sup may be ∞. We say that the curve has length if this is finite and then we define length of the curve to be this number.

Technical name for curves which have length is 'rectifiable'. Of course even if the above quantity is infinity, we could have said that the curve has length and its length is infinity. it makes no difference, but we shall not do it.

For example consider the function

$$f(t) = t \sin\left\{\frac{\pi}{t}\right\}; \quad 0 < t \leq 1; \quad f(0) = 0;$$

and the curve is its graph, that is,

$$\varphi(t) = (t, f(t)); \quad 0 \leq t \leq 1.$$

This curve is not rectifiable. That is, the sup will be infinity. Observe that this is a continuous function. This is easy to see. if you take the partition

$$\{0, 1/100, 1/99, \cdots, 1/2, 1\}.$$

Note that at successive points of the partition the points on the curve can be easily calculated and you get partial sum of the series $\sum(1/n)$.

We shall show that for a C^1 curve, length exists (and so the curve is rectifiable). We shall also give a formula for its length. Let us denote the curve by

$$\varphi(t) = (x(t), y(t)); \quad a \le t \le b.$$

The curve is C^1 means that these functions x and y are C^1 functions. If you took a partition π and calculate $L(\varphi, \pi)$ then mean value theorem tells you

$$L(\varphi, \pi) = \sum \sqrt{[x'(\xi_i)]^2 + [y'(\eta_i)]^2} \; (t_{i=1} - t_i).$$

Here (ξ_i) and (η_i) are selections for the partition. This should remind you of

$$\int_a^b ||\varphi'(t)|| \; dt.$$

We shall show that this is actually true. Where is the problem? We have two selections, if there is only one selection (ξ_i) for both x' and y' then the above sum is exactly the Riemann sum for the later integral.

Riemann sums revisited:

We need a fact. suppose we have continuous functions as follows:

$$x : [a, b] \to R; \quad y : [a, b] \to R$$

$$g : R^2 \to R$$

We define

$$f(t) = g(x(t), y(t)) : [a, b] \to R$$

From last semester we know the following. Suppose we have a sequence of partitions π_n of $[a, b]$ with $||\pi_n|| \to 0$ and for each n, a selection ξ_n for the partition π_n. Then

$$R(f, \pi_n, \xi_n) = \sum g(x(\xi_i), y(\xi_i))(t_{i+1} - t_i) \to \int_a^b f(t)dt.$$

Of course, here the sum is over the partition points of π_n and also what we denoted by ξ_i is actually $\xi_n(i)$, from the selection ξ_n. We did not burden the notation. But if you have confusion, you should write for yourself completely and clearly.

What we shall now claim is the following. Suppose that we have the above situation and two selections ξ_n and η_n for each n. Then

$$R(f, \pi_n, \xi_n, \eta_n) = \sum g(x(\xi_i), y(\eta_i))(t_{i+1} - t_i) \to \int_a^b f(t)dt.$$

Integral is robust, it will accommodate you if you deviate little bit. after all, since the norms of the partitions are getting smaller, if you take two selections, then they must be close and it should make no difference. Here is the proof.

We only need to show

$$\sum \{g(x(\xi_i), y(\eta_i)) - g(x(\xi_i), y(\xi_i))\} (t_{i+1} - t_i) \to 0. \quad (*)$$

Please note that the ξ and η appearing above are the n-th selections ξ_n and η_n.

Let $\epsilon > 0$ be given. Choose δ_1 such that

$$|a_1 - a_2| < \delta_1; |b_1 - b_2| < \delta_1 \Rightarrow |g(a_1, b_1) - g(a_2, b_2)| < \epsilon/(b - a).$$

choose $\delta > 0$ so that

$$|s - t| < \delta \Rightarrow |y(s) - y(t)| < \delta_1$$

Both are possible by uniform continuity of the functions involved. Now

$$||\pi_n|| < \delta \Rightarrow | \{g(x(\xi_i), y(\eta_i)) - g(x(\xi_i), y(\xi_i))\} | < \epsilon/(b - a)$$

so that the sum in $(*)$ is at most ϵ.

This completes the proof of the claim.

return to length:

the result proved just now shows

$$||\pi_n|| \to 0 \quad \Rightarrow \quad L(\varphi, \pi_n) \to \int_a^b ||\varphi'(t)||dt.$$

Thus every C^1 curve has length and is given by

$$L = \int_a^b ||\varphi'(t)||dt.$$

You might be wondering what happened to the sup we took of the $L(\varphi, \pi)$. There are several ways to see this. Here is a way.

Take a sequence of partitions π_n such that $L(\varphi, \pi_n)$ converges to the sup. Remember sup of any set is actually limit of a carefully chosen sequence of points from the set. Note that if π_1 has one extra point than π then triangle inequality immediately gives you $L(\varphi, \pi) \leq L(\varphi, \pi_1)$. Thus by considering the partition

$$\Pi_n = \pi_1 \vee \pi_2 \vee \cdots \vee \pi_n$$

you see that

$$||\Pi_n|| \quad \downarrow; \quad L(\varphi, \Pi_n) \to L.$$

If $||\Pi_n|| \not\to 0$, add extra points to make it converge to zero. Thus you have a sequence of partitions Π_n with norm converging to zero and the corresponding chord length sums converge to L. But these chord lengths are precisely the Riemann sums converging to the integral. This completes the proof.

You can calculate length of circle and see that it coincides with what you felt.

You saw two examples of 'parametrizing' the circle, both go through exactly once but with different speeds. It is possible to bring in uniform code for curves so that the parametrisation traces the curve at uniform speed. This is how we make it precise.

Let $\varphi : [a, b] \to R^2$ be a C^1 curve with length L. Let us assume that φ' does not completely vanish during any interval. In other words, in any given interval, $||\varphi'(t)|| \neq 0$ for at least one t. For each $a \leq t \leq b$, let $L(t)$ be the length of the curve upto t. That is, it is length of the curve

$$\gamma(s) = \varphi(s); \quad a \leq s \leq t.$$

you will see that its length is nothing but

$$L(t) = \int_0^t ||\varphi'(t)||dt.$$

Thus L is a strictly increasing C^1 function on $[a, b]$ onto $[0, L]$. Let its inverse be denoted by L^{-1}. Define the curve

$$\Psi(s) = \varphi(L^{-1}(s)); \quad 0 \leq s \leq L$$

Then it is not difficult to see that Ψ describes the same curve, has the same length L. The interesting point is the following. At any time s if you ask: what is the length of the curve Ψ traced so far; the answer is s. This is called parametrisation of the curve by arc length.

vector calculus:

Actually R^n is a vector space and what we have been doing is indeed calculus on (finite dimensional real) vector spaces. However, the word vector calculus is usually referred to certain notions that are found useful in physics. We have discussed some of these in a homework.

Physicists have a good nomenclature for functions to distinguish: real valued or vector valued. if f is real valued, they say it is a scalar function. If f takes values in R^2 or $R^3$and so on, they say it is a vector field. Of course, from a mathematical point real numbers are also vectors, one dimensional vectors. (And when you study vector spaces, you have a underlying field and R also plays the role of that field and you refer to elements of the field as scalars.) Usually small letters f, g etc.are used for scalar functions and capital letters F, G etc. are used for vector fields.

Let $f : R^3 \to R$ be a scalar function then ∇f is a vector field. for every point P it associates the vector $(f_x(P), f_y(P), f_z(P))$. If you think of ∇ as the symbolic operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Thus you can think of ∇ operating on f and giving ∇f.

suppose that $F : R^3 \to R^3$ is a vector field. Thus

$$F(P) = (F_1(P), F_2(P), F_3(P)).$$

Since now ∇ and F are vectors of size three it makes sense to talk about their scalar product (inner product) and their vector product. They are very important and are defined s follows.

$$\text{div}F(P) = \nabla \cdot F(P) = \frac{\partial F_1}{\partial x}(P) + \frac{\partial F_2}{\partial y}(P) + \frac{\partial F_3}{\partial z}(P)$$

This is called divergence of the vector field F. This is a scalar function.

$$\text{curl } F = \nabla \times F = (D_2F_3 - D_3F_2, D_3F_1 - D_1F_3, D_1F_2 - D_2F_1).$$

This is again a vector field. For every point P it associates a vector, namely the right side where the derivatives are evaluated at the point P.

There is another important operator

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

This is called Laplacian. For a scalar C^2 function f, its Laplacian is Δf another scalar function.

There are several important relations among them. Here are some. Most of them are routine to verify.

(1) Curl(grad f) =0; that is, $\nabla \times \nabla f = 0$

(2) div (Curl F) =0; that is $\nabla \cdot (\nabla \times F) = 0$.

(3) $\text{div}(fF) = F \cdot \nabla f + f \cdot \text{div}F$

(4) $\text{Curl}(fF) = \nabla f \times F + f\ \text{curl}(F)$.

(5) $\text{div}(F \times G) = G \cdot \text{curl}(F) - F \cdot \text{Curl}(G)$.

(6) $\Delta(F) = \nabla(\text{div}(F)) - \text{Curl}(\text{Curl}(F))$.
Here Laplacian for the field F is coordinate-wise.

complex derivative:

Recall that we have made R^2 into a field by defining addition as usual coordinate-wise. Multiplication is defined as follows:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

to be clear whether we are talking about R^2 as a vector space or as a field with the above multiplication, we denote R^2 by C when we think of it as a field. We also write $z = x + iy$ instead of (x, y). Thus as a set they are same but the way we think about them depends on what we are talking about. For example when you are thinking of it as a vector space, it makes no sense to divide by a vector. When we are thinking of it as a field, division definitely makes sense.

For a complex number $z = x + iy$ we refer to x as the real part of z and y as the imaginary part of z. We also use $Re(z)$ and $Im(z)$ for the real and imaginary parts of z. Thus, remember the real and imaginary parts are real numbers. Conjugate of a complex number $z = x + iy$ is the number $x - iy$ denoted by $\overline{z}$.

When the imaginary part is zero, we would not take the trouble of writing $x + i0$ but just write x. similarly, when the real part is zero we shall not write $0 + iy$, but simply as iy.

The real number system is a subset of C, identified as complex numbers with imaginary part zero. The multiplication and additions agree. By the way multiplication is defined we have $i \times i = -1$. Thus sometimes we also write $\sqrt{-1}$ or $+\sqrt{-1}$ for i.

Since you are familiar and also we discussed in an homework, we shall not spend time on routine matters.,

The definition of convergence is same as that of R^2. Thus a sequence of complex numbers z_n converge to a complex number z if the real parts converge to the real part of z and similarly the imaginary parts converge. Functions from C to C are thus essentially functions from R^2 to R^2. Continuity is just the same. Thus $f : C \to C$ is continuous if $f(z_n) \to f(z)$ whenever $z_n \to z$.

Let now $\Omega \subset C$ be an open set. We say that f is (complex) differentiable at a point $z_0 \in \Omega$ if there is a complex number α such that

$$z_n \to z_0; (\forall n) z_n \neq z_0 \quad \Rightarrow \quad \frac{f(z_n) - f(z_0)}{z_n - z_0} \to \alpha$$

or equivalently

$$h_n \in C; h_n \to 0, (\forall n) h_n \neq 0 \quad \Rightarrow \quad \frac{f(z + h_n) - f(z)}{h_n} \to \alpha.$$

The important question is the following. Set theoreticlly C is same as R^2. How does this derivative relate to what we have learnt? if you think of f as $f(x, y)$ (from R^2 to R^2) then does it have derivative? Is that all?

We shall now see that the real and imaginary parts of f do have partial derivatives and are related. Complex analysis has intrinsic beauty, if f is C^1 in a region then it is actually C^∞! You will have a course in complex analysis later when you will learn these. Right now our interest is only the relation to what we have learnt so far.

so let $f : C \to C$ be differentiable at a point $z_0 = x_0 + iy_0$. Let the derivative be $a + ib$. Let us denote

$$f(x + iy) = u(x + iy) + iv(x + iy).$$

Here u and v are the real and imaginary parts. Let us take a sequence of real numbers (h_n) all different from zero. Thus we have

$$\frac{u(x_0 + h_n, y_0) + iv(x_0 + h_n, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h_n} \to a + ib.$$

Since the real parts and imaginary parts converge, this implies u and v have partial derivatives w.r.t. x and

$$u_x(x_0, y_0) = a; \quad v_x(x_0, y_0) = b. \qquad (*)$$

since $ih_n \to 0$ whenever $h_n \to 0$ we also have

$$\frac{u(x_0, y_0 + h_n) + iv(x_0, y_0 + h_n) - u(x_0, y_0) - iv(x_0, y_0)}{ih_n} \to a + ib.$$

This means,

$$\frac{u(x_0, y_0 + h_n) + iv(x_0, y_0 + h_n) - u(x_0, y_0) - iv(x_0, y_0)}{h_n} \to ia - b.$$

This means that u and v have partial derivatives w.r.t. y and

$$v_y(x_0, y_0) = a; \qquad u_y(x_0.y_0) = -b. \quad (**)$$

comparing $(*)$ and $(**)$ we see

$$u_x = v_y; \quad u_y = -v_x \qquad (\spadesuit)$$

at the point (x_0, y_0). The equations above are called **Cauchy-Riemann equations**.

Thus any complex differentiable function satisfies the Cauchy Riemann equations; this means the real and imaginary parts of the complex function satisfy the above equations. Interestingly enough, if u and v are real valued C^1 functions on Ω and satisfy the above equations, then the function $f(z) = u(z) + iv(z)$ is a complex differentiable function.

holomorphic functions:

We have seen last time that if $f : C \to C$ is complex differentiable, then Cauchy-Riemann equations are satisfied by the real and imaginary parts of the function. We shall now show that conversely if two real functions u and v are C^1 functions satisfying the Cauchy Riemann equations then $f = u + iv$ is complex differentiable.

Functions which are complex differentiable are called holomorphic functions or analytic functions. As mentioned earlier, if $f : C \to C$ is differentiable at every point of a region $\Omega \subset C$ then it is differentiable any number of times. Not only that it has a power series expansion around every point. This means the following. if $z_0 \in \omega$ then there is an $r > 0$ such that the ball $S(z_0, r) \subset \Omega$ and there are numbers c_0 such that

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots; \qquad z \in S(z_0, r).$$

Functions having this property; namely given a point in the domain there is a neighbourhood around it where the function has power series representation; are called analytic functions. Thus differentiable functions are not only infinitely differentiable, but also have such power series representation. This is for $f : C \to C$.

For functions of real variables such results are not true. There are functions $f(x, y)$ of two variables which are differentiable just as many times as we want and no more. Even if it is infinitely differentiable, such power series expansions need not exist.

So let $\Omega \subset C$ be an open set and u, v be two real valued C^1 functions on Ω such that the Cauchy Riemann equations hold

$$u_1(x, y) = v_2(x, y); \quad u_2(x, y) = -v_1(x, y) \quad (x, y) \in \Omega.$$

We now show that

$$f(x + iy) = u(x, y) + iv(x, y)$$

is a holomorphic function.

Let us fix a $z \in C$. By the mean value theorem applied to u, there is some point ξ on the line joining the two points $z+h$ and z; that is the points $(x+h_1, y+h_2)$ and (x, y) such that

$$u(x+h_1, y+h_2) - u(x,y) = (u_1(\xi), u_2(\xi)) \cdot (h_1, h_2)$$

$$= u_1(z)h_1 + u_2(z)h_2 + \varphi(h)$$

where

$$\varphi(h) = (u_1(\xi) - u_1(z), u_2(\xi) - u_2(z)) \cdot (h_1, h_2)$$

so that

$$|\varphi(h)| \leq ||\nabla u(\xi) - \nabla u(z)||\ |h|$$

Since u is C^1 and ξ is on the line joining $z+h$ and z we see that

$$h \to 0 \quad \Rightarrow \quad \varphi(h)/|h| \ \to \ 0.$$

similarly

$$v(x+h_1, y+h_2) - v(x,y) = u_1(z)h_1 + u_2(z)h_2 + \psi(h)$$

where

$$\psi(h)/|h| \ \to \ 0.$$

Thus,

$$f(z+h) - f(z) = u(z+h) - u(z) + i[v(z+h) - v(z)]$$

$$= u_1(z)h_1 + u_2(z)h_2 + \varphi(h) + i[v_1(z)h_1 + v_2(z)h_2 + \varphi(h)]$$

$$= [u_1(z) + iv_1(z)]h_1 + [v_2(z) - iu_2(z)]ih_2 + \varphi(h) + \psi(h)$$

using Cauchy Riemann equations

$$= [u_1(z) + iv_1(z)](h_1 + ih_2) + \varphi(h) + \psi(h)$$

Hence

$$\frac{f(z+h) - f(z)}{h} = u_1(z) + iv_1(z) + \frac{\varphi(h) + \psi(h)}{h}$$

since the second term on the right converges to zero as $h \to 0$ we conclude that f is complex differentiable with

$$f'(z) = u_1(z) + iv_1(z).$$

We can regard f as a function from R^2 to R^2, namely,

$$f(x,y) = (u(x,y), v(x,y)).$$

If f is complex differentiable then as a function from R^2 to R^2, it is differentiable and the derivative is given by

$$f'(x,y) = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \quad = \quad \begin{pmatrix} u_1(x,y) & -v_1(x,y) \\ v_1(x,y) & u_1(x,y) \end{pmatrix}$$

where we have used he C-R equations. This is non-singular matrix unless both $f'(z) = 0$.

These simple results already have non-trivial consequences giving a glimpse into complex analysis. For instance if $f = u + iv$ and $g = u + iw$ are holomorphic on C (or on any connected open set) then v and w differ by a constant. This is because $f - g$ is holomorphic and its real part is zero and C-R equations tell imaginary part of $f - g$ also has zero partial derivatives and hence must be constant. Thus the real part (u) of a holomorphic function uniquely determines its imaginary part (v); of course, up to a constant.

similarly the imaginary part also determines its real part up to a constant.

In particular, if f is holomorphic and is purely real then it must be a constant. That is, if range f is contained in x-axis or y-axis, then f must be constant.

Can a holomorphic function (on C or on a connected open set) have range contained in the unit circle? That is $|f| \equiv 1$. No, unless it is a constant. If at some point $f'(z)$ is non-zero, then regarding f as a function from R^2 to R^2 combined with inverse function theorem tells that range f contains non-empty open sets. Of course, if the derivative is zero at all points then it is a constant.

Polynomials $P(z)$ are holomorphic. The function

$$f(z) \quad = \quad \overline{z},$$

conjugate of z is not holomorphic.

curves again:

Let us consider planar curves. let

$$\varphi : [a,b] \to R^2$$

be a curve. According to our definition, curves are continuous functions.

The curve φ is said to be simple if it is one-to-one except possibly on $\{a, b\}$. equivalently, if

$$s \neq t \in [a, b]; \varphi(s) = \varphi(t) \Rightarrow \{s, t\} = \{a, b\}.$$

Thus when you trace a simple curve, you do not pass through any point second time except when you reach the finish and finish with the starting point.

The curve φ is closed if $\varphi(a) = \varphi(b)$. Thus a closed curve ends at the starting point.

For example circle is a closed curve. If you trace it only once; $\varphi(t) = (\cos t, \sin t); 0 \leq t \leq 2\pi$; then it is simple closed curve. You can think of many examples of curves which are not simple or not closed.

If you think of simple closed curves, like, circle, triangle, polygon, and so on; you see that the plane is divided into three parts: points on the curve, outside the curve, inside the curve. Of course we used the word outside/inside by visual feeling. We can not exactly define these terms in general. Of course, if you take concrete curves like the ones mentioned above you can, in each specific case, precisely define.

Here is a theorem which is intuitively obvious but is non-trivial to prove. This is called Jordan curve theorem. Let φ be a simple closed plane curve. Then we can express

$$R^2 \ = \ I \ \cup \ \Gamma \ \cup \ E.$$

where (i) the union is disjoint; (ii) Γ is range of φ, that is, it is the curve; (iii) I and E are connected sets. Further such a decomposition is unique. Further exactly one of I and E is bounded and the other unbounded. If we think of I as the bounded part, then it is called the interior of the curve. The set E, unbounded component is called the exterior.

We have discussed C^1 curves φ and proved that they have length, given by the formula

$$L = \int_a^b ||\varphi'(t)||dt.$$

Many times we come across curves that are piece-wise smooth, but not smooth. A curve φ defined on $[a, b]$ is said to be piece-wise C^1 if the following holds. There are finitely many points

$$a = a_0 < a_1 < a_2 < \cdots < a_k = b$$

such that φ is C^1 on each piece $[a_i, a_{i+1}]$. Thus we demand left derivative at a_{i+1} and right derivative at a_i for each i. You can easily give examples (rectangle, triangle etc).

even for a piece-wise C^1 curve, length exists and is given by the same formula as above. The only thing you should take note is that at the points a_i the integrand possibly has two values (or has no value, depending on how you think). But it makes no difference. We have seen last semester that such bounded functions which are continuous except for finitely many discontinuities are integrable.

vector calculus again:

The concept of divergence $divF$ and curl $curlF$ are useful in discussing flow of fluids. Apparently, divergence measures tendency for the flow to dissipate/diverge in its plane of motion. On the other hand curl explains the tendency of the flow to move out of its plane of motion (like when it forms a whirlpool etc).

I shall only explain a nice interpretation of vector product for two vectors in R^3. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two vectors then their scalar product or inner product

$$|u \cdot v| = ||u|| \; ||v|| \; \cos\theta$$

where θ is the angle between the vectors. Similarly the vector product

$$||u \times v|| = ||u|| \; ||v|| \; |\sin\theta|.$$

This follows from

$$||u \times v||^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \quad - \quad (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$= ||u||^2 \; ||v||^2 \;\; [1 - (u \cdot v)^2].$$

normal integral again:

The theorems we learnt regarding differentiation under integral sign (Bounded interval) can be used to give a third method of evaluating the normal integral.

Consider the function

$$f(x) \;=\; \left(\int_0^x e^{-t^2/2}dt\right)^2 \;+\; 2\int_0^1 \frac{e^{-x^2(t^2+1)/2}}{t^2+1}dt.$$

Our theorems allow us to differentiate under the integral sign giving us, after differentiation $f' = 0$ so that f is a constant function. But you see $f(0) = \pi/2$. Thus f is the constant function $\pi/2$. But

$$\lim_{x\to\infty} f(x) = \left(\int_0^\infty e^{-t^2/2}dt\right)^2$$

giving us

$$\int_0^x e^{-t^2/2}dt = \sqrt{\pi/2}.$$

In the above calculation we used that the second integral converges to zero as $x \to \infty$. Of course the integrand converges to zero for every t, but this is not enough reason to conclude that the integral converges to zero. However the integrand decreases to zero point wise. In this case you can see directly that the integrand uniformly decreases to zero, it is smaller than $(1/2)\exp\{-x^2\}$.

This completes the justification for the second term converging to zero and hence evaluation of the normal integral.

It is interesting to note that if you have a sequence of continuous functions on $[0,1]$ that decrease to zero point wise then they do so uniformly. This is not difficult and is seen as follows.

Let $f_n \downarrow 0$ point wise. Let $\epsilon > 0$ be given. since the functions are decreasing, it is enough to show one N such that $f_N(x) < \epsilon$ for all x. Take any $x \in [0,1]$. since $f_n(x) \downarrow 0$, get $n(x)$ such that $f_{n(x)}(x) < \epsilon$. But this function $f_{n(x)}$ is a continuous function. So get an interval $I(x)$, such that $x \in I(x)$ and $f_{n(x)}(y) < \epsilon$ for all points y of $[0,1]$ which are in this interval $I(x)$. All these open intervals cover the compact $[0,1]$. Get finitely many of these intervals that cover, say $I(x_1), I(x_2), \cdots, I(x_k)$. it is easy to see that $N = \max\{n(x_1), n(x_2), \cdots, n(x_k)\}$.

This theorem is Dini's theorem.

uniform convergence of integrals:

We now make another attempt on understanding differentiation under the integral sign, when the integrals are over infinite interval. If we have a bounded interval,then uniform convergence was useful in establishing change of order of integration with integration/differentiation. Just now we saw how such things help us.

Let $f(x, y)$ be a continuous function on $[0, \infty) \times [c, d]$. Suppose that for each $y \in [c, d]$ the integral

$$\int_0^\infty f(x, y)dy \qquad (\spadesuit)$$

converges. Remember, this means

$$\lim_{A \to \infty} \int_0^A f(x, y)dx$$

converges to a number. This means that given $\epsilon > 0$, here is a number A_0 such that

$$A > A_0 \quad \Rightarrow \quad |\int_A^\infty f(x, y)dx| < \epsilon.$$

We say that the integerl $(\spadesuit)$ converges uniformly over $[c, d]$ if given $\epsilon > 0$, there is $A_0 > 0$ such that

$$A > A_0, \;\; y \in [c, d] \quad \Rightarrow \quad |\int_A^\infty f(x, y)dx| < \epsilon.$$

In other words the A_0 does not depend on the number y. Equivalently, the 'tail areas' (?) are uniformly small.

Since the condition that the tail area should be small involves again integral over infinite interval, usually the above condition is stated in the following equivalent form. the advantage is that it involves integral over finite interval. the complication is that you need to bring in A and B, two characters.

$$B > A > A_0, \;\; y \in [c, d] \quad \Rightarrow \quad |\int_A^B f(x, y)dx| < \epsilon.$$

Here are three useful theorems.

1. (**continuity**)
Suppose that $f : [0, \infty) \times [c, d] \to R$ is continuous. Suppose that $\int_0^\infty f(x, y)dx$ converges uniformly. Then the function

$$\varphi(y) = \int_0^\infty f(x, y)dx; \qquad y \in [c, d]$$

is c continuous function.

2. (**change of order of integration**)
Same conditions as above. Then

$$\int_c^d \varphi(y)dy \ = \ \int_0^\infty \left(\int_c^d f(x,y)dy\right) dx.$$

equivalently

$$\int_c^d \left(\int_0^\infty f(x,y)dx\right) dy \ = \ \int_0^\infty \left(\int_c^d f(x,y)dy\right) dx.$$

3. (**change of order of integration and differentiation**)
Same conditions as above. let us assume that for each x the function $y \mapsto f(x,y)$ is C^1 function on $[c,d]$ with derivative $g(x,y)$ and assume that the integral $\int_0^\infty g(x,y)dx$ is uniformly convergent. Then φ is differentiable and

$$\varphi'(y) \ = \ \int_0^\infty g(x,y)dx.$$

Equivalently,

$$\frac{d}{dy}\int_0^\infty f(x,y)dx \ = \ \int_0^\infty \frac{\partial f}{\partial y}(x,y)dx.$$

In other words, you can push the differentiation under the integral sign. Since the left side is a function of one variable y, we use d/dy. Since the integrand on right side is a function of two variables, we use $\partial/\partial y$.

First let us see some uses of these results.

Normal distribution again:

Evaluate

$$\varphi(t) \ = \ \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cos(tx)\, dx. \qquad (\clubsuit)$$

This integral arises in several areas of mathematics. Of course it arises in Probability and is called the characteristic function of the standard normal variable. strictly speaking the following is the characteristic function.

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, e^{itx}\, dx$$

$$= \ \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cos(tx)\, dx + i\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sin(tx)\, dx.$$

We have used the fact $\exp(i\theta) = \cos\theta + i\sin\theta$ for real numbers θ. The second integral on the right side exists (integrand being continuous and dominated by the normal integrand). But integrand being odd function the integral is zero. Thus only the first term remains.

This integral arises in Fourier analysis and is called the Fourier transform of the normal density function.

So how do we evaluate the integral. if only some one assures us that it van be differentiated under integral sign, then differentiating under the integral sign and evaluating the resulting integral by parts we see

$$\varphi'(t) = -t\varphi(t); \quad \varphi'(t) + t\varphi(t) = 0$$

multiplying by $\exp\{t^2/2\}$ we see

$$\left[e^{t^2/2}\varphi(t)\right]' = 0 : \quad \varphi(t) = Ce^{-t^2/2}.$$

Since $\varphi(0)$ can be explicitly evaluated and seen to be one we finally get

$$\varphi(t) = e^{-t^2/2}.$$

Heat equation:

Imagine an infinite rod, think of it as real line R. I supply a certain amount of heat, say $\varphi(y)$ at the point y of the rod. We assume that the function φ is a bounded continuous function on R. If you have any reservations about my supplying heat at *every* point of the *infinite* rod, you can assume that the function φ is zero outside a bounded interval.

How does it diffuse over time? How does it distributed over the rod? Thus, explain the amount of heat at time $t > 0$ at the point x of the rod. The answer is the following. Let $u(t, x)$ denote the amount of heat at time t at point x of the rod. Then

$$u(t, x) \text{ is a continuous function on } [0, \infty) \times R.$$

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t, x); \quad (t, x) \in (0, \infty) \times R.$$
$$u(0, x) = \varphi(x).$$

The equation

$$u_t = \frac{1}{2} u_{xx}$$

is called *heat equation* already derived by Newton. I have taken 1/2 on the right side, but generally it is taken as some constant c.

It is not difficult to verify that if

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} \quad (t,x) \in (0,\infty) \times R$$

satisfies the heat equation If you draw the normal curves for various values of t, you observe that as $t \downarrow 0$ the curves get more and more concentrated around zero. Thus it is plausible that this function gives the heat distribution if initial supply was unit amount at the point $y = 0$. Thus if you supply unit amount at the point y then you expect the heat distribution over time to be given by

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{(x-y)^2/2t} \quad (t,x) \in (0,\infty) \times R$$

If you supply amount $\varphi(y)$ at y you expect the distribution at time t to be

$$\int_{-\infty}^{\infty} p(t,x,y)\varphi(y)dy.$$

Thus the suggestion is

$$u(t,x) = \int_{-\infty}^{\infty} p(t,x,y)\varphi(y)dy; \quad u(0,x) = \varphi(x).$$

solves the problem of heat conduction. Yes it is true. Note that for each y the function $p(t,x,y)$ satisfies the heat equation as a function of $(t,x) \in (0,\infty) \times R$. That is

$$p_t = \frac{1}{2} p_{xx}.$$

If only someone allows you differentiate under the integral sign, then you see

$$u_t = \int p_t \, \varphi \qquad u_{xx} = \int p_{xx} \, \varphi$$

and hence conclude that

$$u_t = \frac{1}{2} \, u_{xx}.$$

Yes, the theorems stated above allow you to happily do this. Some more work needs to be done to see that $u(t,x)$ converges to $\varphi(x_0)$ as $(t,x) \to (0,x_0)$. We shall not do.

the main purpose of all this discussion is to explain to you why it is useful if some one tells you that you can differentiate before integrating (when you actually are suppose to differentiate after integrating.)

We shall complete our discussion on uniform convergence of integrals and close the chapter. I hope you have got a feeling for the applications I have outlined last time, namely, to evaluate the characteristic function of the normal random variable, solving the heat equation. Of course, apart from these you can think of continuity of the gamma function and its derivatives; which we explained earlier.

While discussing power series, especially its continuity and term by term differentiation we have understood how crucial it was to have uniform convergence and how it immediately provides the answers. Just to impress upon you that there are no new ideas to treat integrals, I shall repeat discussion of series also as we go along.

uniform convergence:

So let f_n be continuous functions defined on $[c, d]$. Recall that the series $\sum f_n$ converges to φ pointwise if for every number $y \in [c, d]$ the series of numbers $\sum f_n(y)$ converges to the number $\varphi(y)$. This means that given y, the partial sums $s_n(y) = \sum\limits_1^n f_i(y)$ converge to $\varphi(y)$. This, in turn, means that given a point y and $\epsilon > 0$, there is an integer N such that $|s_n(y) - \varphi(y)| < \epsilon$ for all $n \geq N$. Of course this integer N depends on the number ϵ and also on the point y.

We say that $\sum f_n$ converges uniformly to φ if given $\epsilon > 0$, there is an N such that $|s_n(y) - \varphi(y)| < \epsilon$ for all $n \geq N$ and also all $y \in [c, d]$. Thus the integer N does not depend on any y in the interval. There is one N that works for all y. This is the spirit of uniformity.

Similarly, let $f(x, y)$ be a continuous function defined on $[0\infty) \times [c, d]$. Recall that the integral $\int\limits_0^\infty f(x, y)dx$ converges to $\varphi(y)$ point wise if for every number $y \in [c, d]$ the integral $\int\limits_0^\infty f(x, y)dx$ converges to the number $\varphi(y)$. This means that given y, the 'partial' integrals $s_A(y) = \int\limits_0^A f(x, y)dx$ converge to $\varphi(y)$ as $A \to \infty$. This, in turn, means that given a point y and $\epsilon > 0$, there is an A_0 such that $|s_A(y) - \varphi(y)| < \epsilon$ for all $A \geq A_0$. Of course this A_0

depends on the number ϵ and also on the point y.

We say that $\int\limits_0^\infty f(x,y)dx$ converges uniformly to φ if given $\epsilon > 0$, there is an A_0 such that $|s_A(y) - \varphi(y)| < \epsilon$ for all $A \geq A_0$ and also all $y \in [c,d]$. Thus the number A_0 does not depend on any y in the interval. There is one A_0 that works for all y. This is the spirit of uniformity as said above.

Let us return to series and use the earlier notation. Assume $\sum f_n$ converge to φ uniformly. Let $\epsilon > 0$ be given and N be as above. then we have

$$n \geq N \Rightarrow |s_n(y) - \varphi(y)| < \epsilon.$$

This implies

$$n > m \geq N \Rightarrow |s_n(y) - s_m(y)| = |s_n(y) - \varphi(y)| + |\varphi(y) - s_m(y)| < 2\epsilon.$$

In other words

$$n > m \geq N \Rightarrow \left|\sum_{n+1}^{m} f_i(y)\right| < 2\epsilon.$$

Since $\sum\limits_1^\infty f_i(y)$ converges, so do all the sums $\sum\limits_m^\infty f_i(y)$. Thus the above inequality also implies that

$$n > N \Rightarrow \left|\sum_{n}^{\infty} f_i(y)\right| \leq 2\epsilon.$$

Returning to integrals let us continue with earlier notation. Assume $\int\limits_0^\infty f(x,y)dx$ converge to φ uniformly. Let $\epsilon > 0$ be given and A_0 be as above. then we have

$$A \geq A_0 \Rightarrow |s_A(y) - \varphi(y)| < \epsilon.$$

This implies

$$B > A \geq A_0 \Rightarrow |s_B(y) - s_A(y)| = |s_B(y) - \varphi(y)| + |\varphi(y) - s_A(y)| < 2\epsilon.$$

In other words

$$B > A \geq A_0 \Rightarrow \left|\int_A^B f(x,y)dx\right| < 2\epsilon.$$

Since $\int\limits_0^\infty f(x,y)dx$ converges, so do all the integrals $\int\limits_A^\infty f(x,y)dx$. Thus the above inequality also implies that

$$A > A_0 \Rightarrow \left|\int_A^\infty f(x,y)dx\right| \leq 2\epsilon.$$

continuity:

Suppose that the series of continuous functions $\sum f_i$ converges to φ uniformly on $[c, d]$. We show φ is continuous. We show uniform continuity of φ. let $\epsilon > 0$ be given. Choose N so that

$$n \geq N \Rightarrow |s_n(y) - \varphi(y)| < \epsilon/3; \qquad \forall y.$$

Let us not bother on $n \geq N$, but just consider this integer N. Since s_N is a finite sum of continuous functions it is continuous and hence uniformly continuous on $[c, d]$. So choose $\delta > 0$ so that

$$|y_1 - y_2| < \delta \Rightarrow |s_N(y_1) - s_N(y_2)| < \epsilon/3.$$

If we take this δ then

$$|y_1 - y_2| < \delta \Rightarrow |\varphi(y_1) - \varphi(y_2)|$$

$$\leq |\varphi(y_1) - s_N(y_1)| + |s_N(y_1) - s_N(y_2)| + |s_N(y_2) - \varphi(y_2)|$$

$$\leq \epsilon.$$

Let us now return to integrals. Suppose that we have continuous functions $f(x, y)$ as above with $\int\limits_0^\infty f(x, y)dx$ converging to φ uniformly on $[c, d]$. We show φ is continuous. We show uniform continuity of φ. let $\epsilon > 0$ be given. Choose A_0 so that

$$A \geq A_0 \Rightarrow |s_A(y) - \varphi(y)| < \epsilon/3; \qquad \forall y.$$

Let us not bother on $A \geq A_0$, but just fix just one number $A \geq A_0$. Since

$$s_A(y) = \int_0^A f(x, y)dx$$

is integral over a finite interval, we know from earlier theorems that s_A is a continuous function and hence uniformly continuous on $[c, d]$. So choose $\delta > 0$ so that

$$|y_1 - y_2| < \delta \Rightarrow |s_A(y_1) - s_A(y_2)| < \epsilon/3.$$

If we take this δ then

$$|y_1 - y_2| < \delta \Rightarrow |\varphi(y_1) - \varphi(y_2)|$$

$$\leq |\varphi(y_1) - s_A(y_1)| + |s_A(y_1) - s_A(y_2)| + |s_A(y_2) - \varphi(y_2)|$$

$$\leq \epsilon.$$

This shows that the uniformly convergent improper integral defines a continuous function (if f is continuous).

differentiation:

Let us first consider series. Let us now assume that f_i are C^1 functions on $[c,d]$ with derivative g_i. Assume that $\sum f_i$ converges to φ uniformly and $\sum g_i$ converges to ψ uniformly. then φ is differentiable and $\varphi' = \psi$. In other words derivative of sum equals sum of derivatives.

Of course we do not need $\sum f_i$ to converge uniformly, enough if it converges point wise so that we have a function φ to talk about. But uniform convergence of the series $\sum g_i$ is important. But this remark can be ignored because usually you have uniform convergence of both the series $\sum f_i$ and $\sum g_i$.

Let us fix a point y_0. Let us fix $\epsilon > 0$. Need to show $\delta > 0$ so that

$$0 < |h| < \delta \Rightarrow \left| \frac{\varphi(y_0 + h) - \varphi(y_0)}{h} - \psi(y_0) \right| < \epsilon.$$

(while reading you should assume that y_0 as well as $y_0 + h \in [c,d]$.

First fix N so that

$$n > m \geq N \Rightarrow \left| \sum_m^n g_i(y) \right| < \epsilon/3; \quad \forall y.$$

In particular note that

$$\left| \sum_{N+1}^{\infty} g_i(y) \right| < \epsilon/3.$$

Let us concentrate on this N. Since s_N is a finite sum of C^1 functions, it is differentiable. So fix $\delta > 0$ such that

$$0 < |h| < \delta \Rightarrow \left| \frac{s_N(y_0 + h) - s_N(y_0)}{h} - \sum_1^N g_i(y_0) \right| < \epsilon/3.$$

Now

$$0 < |h| < \delta \Rightarrow \left| \frac{\varphi(y_0 + h) - \varphi(y_0)}{h} - \psi(y_0) \right|$$

$$\leq \left| \frac{s_N(y_0 + h) - s_N(y_0)}{h} - \sum_1^N g_i(y_0) \right|$$

$$+\left|\frac{\sum\limits_{N+1}^{\infty} f_i(y_0+h) - \sum\limits_{N+1}^{\infty} f_i(y_0)}{h}\right| + \left|\sum_{N+1}^{\infty} g_i(y_0)\right|$$

First and last terms are at most $\epsilon/3$ by choice of δ and N respectively. Regarding the middle term observe that the mean value theorem applied to the C^1 function $\sum\limits_{N+1}^{m} f_i$ we get

$$\left|\frac{\sum\limits_{N+1}^{m} f_i(y_0+h) - \sum\limits_{N+1}^{m} f_i(y_0)}{h}\right| = \left|\sum_{N+1}^{m} g_i(\theta)\right| < \epsilon/3.$$

This being true for every $m > N$ we conclude that the middle term is also at most $\epsilon/3$ completing the proof.

You will now see that exactly the same proof works for integrals.

Let us now assume that there is a continuous function g on $[0, \infty) \times [c, d]$ such that for each x as a function of y it is derivative of $y \mapsto f(x, y)$. Assume that $\int\limits_0^{\infty} f(x, y)dx$ converges to φ uniformly and $\int\limits_0^{\infty} g(x, y)dx$ converges to ψ uniformly. then φ is differentiable and $\varphi' = \psi$. In other words derivative of integral equals integral of derivative.

As in the case of series, we do not need $\int_0^{\infty} f(x, y)dx$ to converge uniformly, enough if it converges point wise so that we have a function φ to talk about. But uniform convergence of $\int\limits_0^{\infty} g(x, y)dx$ is important. But this remark can be ignored because usually you have uniform convergence of both integrals.

Let us fix a point y_0. Let us fix $\epsilon > 0$. Need to show $\delta > 0$ so that

$$0 < |h| < \delta \Rightarrow \left|\frac{\varphi(y_0+h) - \varphi(y_0)}{h} - \psi(y_0)\right| < \epsilon.$$

(we assume that $y_0 + h \in [c, d]$).

First fix N so that

$$B > A \geq N \Rightarrow \left|\int_A^B g(x, y)dx\right| < \epsilon/3; \quad \forall y.$$

In particular note that
$$\left|\int_N^\infty g(x,y)dx\right| < \epsilon/3.$$
Let us concentrate on this N. Since s_N is integral over finite interval, from the results we proved earlier, it is differentiable. So fix $\delta > 0$ such that
$$0 < |h| < \delta \Rightarrow \left|\frac{s_N(y_0+h) - s_N(y_0)}{h} - \int_0^N g(x,y_0)dx\right| < \epsilon/3.$$
Now
$$0 < |h| < \delta \Rightarrow \left|\frac{\varphi(y_0+h) - \varphi(y_0)}{h} - \psi(y_0)\right|$$
$$\leq \left|\frac{s_N(y_0+h) - s_N(y_0)}{h} - \int_0^N g(x,y_0)dx\right|$$
$$+\left|\frac{\int\limits_N^\infty f(x,y_0+h)dx - \int\limits_N^\infty f(x,y_0)dx}{h}\right| + \left|\int_N^\infty g(x,y_0)dx\right|$$
First and last terms are at most $\epsilon/3$ by choice of δ and N respectively. Regarding the middle term observe that the mean value theorem applied to the C^1 function $\int\limits_N^B f(x,y)dx$ (note that range of integration is finite) we get
$$\left|\frac{\int\limits_N^B f(x,y_0+h)dx - \int\limits_N^B f(x,y_0)dx}{h}\right| = \left|\int_N^B g(x,\xi)dx\right| < \epsilon/3.$$
This being true for every $B > N$ we conclude that the middle term in the earlier string of terms is also at most $\epsilon/3$ completing the proof.

integration:
With the same hypothesis as in continuity assume that the series $\sum f_i$ uniformly converges to φ. Then
$$\sum_i \int_c^d f_i(y)dy = \int_c^d \varphi(y)dy = \int_c^d \sum_i f_i(y)dy.$$
In other words infinite sum and integral can be interchanged. Similarly with the same hypothesis as in continuity,
$$\int_0^\infty \left(\int_c^d f(x,y)dy\right)dx = \int_c^d \varphi(y)dy = \int_c^d \left(\int_0^\infty f(x,y)dx\right)dy.$$

The integrals can be interchanged. Proof is similar to the above, you only need to make the tail sum (or tail integral) smaller than $\epsilon/(d-c)$ and argue carefully.

criterion for uniform convergence:

Consider series $\sum f_n$. If there are numbers M_n such that $|f_n| \leq M_n$ for each n and $\sum M_n$ converges then the series $\sum f_n$ converges uniformly. Given $\epsilon > 0$ you only need to choose N so that $\sum\limits_N^\infty M_n < \epsilon$. This is known as Weierstrass M-test.

Similarly, suppose we have continuous function $f(x, y)$ on $[0, \infty) \times [c, d]$. Suppose that there is a function $M(x)$ such that $|f(x, y)| \leq M(x)$ for every (x, y) and $\int\limits_0^\infty M(x)dx$ converges. then the integral $\int\limits_0^\infty f(x, y)dx$ converges uniformly.

GOOD LUCK

1. Suppose f and g are continuous functions on an interval $[a, b]$. Suppose that $g(x) \geq 0$ for every x. Show that there is a number $\theta \in (a, b)$ such that
$$\int_a^b f(x)g(x)dx = f(\theta)\int_a^b g(x)dx.$$
[In case $\int g = 0$ argue that $g \equiv 0$ and any theta would do. If not, consider the ratio of the two integrals and intermediate value theorem for f. Remember to show $a < \theta < b$.] This is called *mean value theorem for integrals.*

Let f be a function on an interval $[a, b]$ which is n times continuously differentiable. Show that for $a \leq x \leq b$,
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt. \qquad (\spadesuit)$$
[integration by parts]

The above equality is stated as follows
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n;$$
where
$$R_n = \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt.$$
R_n is called the remainder term. Thus ($\spadesuit$) is called *Taylor expansion with integral form of remainder.*

Use appropriate g in the mean value theorem to show
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + f^{(n)}(\theta)\frac{(x-a)^n}{n!}.$$
Here θ is a number between a and x. This is called *Taylor expansion with Lagrange form of remainder.*

Use appropriate g in mean value theorem (♠) to show

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + f^{(n)}(\theta)\frac{(x-\theta)^{n-1}}{(n-1)!}(x-a).$$

Here θ is a number between a and x. This is called *Taylor expansion with Cauchy form of remainder.*

2. Consider R^3.

   Can you imagine the set of points

$$\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1\}.$$

   For each fixed z, its (x,y)-section is either an ellipse or a single point or empty set depending on whether $|z| < 1$ or $|z| = 1$ or $|z| > 1$. (hold on, what is *section*?) This is called an ellipsoid. More generally, for fixed numbers a, b, c (none zero) the set of points satisfying $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ is called an ellipsoid.

   Can you imagine the set of points

$$\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1\}.$$

   For each fixed z, its (x,y)-section is an ellipse. It is smallest when $z = 0$, keeps on becoming bigger as $|z|$ increases. This is called a hyperboloid of one sheet. More generally, for fixed numbers a, b, c (none zero) the set of points satisfying $(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$ is called a hyperboloid of one sheet.

   Can you imagine the set of points

$$\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} - z^2 = -1\}.$$

   For each fixed z its (x,y)-section is either an ellipse or a single point or empty set depending on whether $|z| > 1$ or $|z| = 1$ or $|z| < 1$. In particular the set is in two parts, one in the region $(z \geq 1)$ and another in the region $(z \leq -1)$. This is called hyperboloid of two sheets. More generally, for fixed numbers a, b, c (none zero) the set of points satisfying $(x/a)^2 + (y/b)^2 - (z/c)^2 = -1$ is called a hyperboloid of two sheets.

3. Define the function f on R^2 as follows. If x and y have same sign then $f(x, y) = xy$ and otherwise $f(x, y)$ is zero. Understand the function. What is its value at $(0, 5)$? at $(-3, 0)$?

   is this a continuous function?

   Where is the function $f(x, y) = \sqrt{x} - \sqrt{y}$ defined? is it continuous at those points?

   Let $f(x, y)$ equal 1 on the two axes and zero for points not on the axes. Where is this function continuous?

   $f(x, y) = (x^2 + y^2)/(x^2 - y^2)$. Where is this defined? Is it continuous there?

   $f(x_1, x_2, x_3, x_4) = \sin x_1 \cos x_2 - x_4 e^{x_3}$ is a continuous function on R^4.

4. Let A be an $n \times n$ matrix. Define f on R^n by $f(x) = Ax$. Is it continuous?

   Let us think of points in R^n as column vectors and let x^t denote transpose of x; so x^t is a row vector. Define $f(x) = x^t Ax$. is it continuous function?

5. Let $S \subset R^n$. Let $C(S)$ denote the set of all real valued continuous functions on S. Sow that this is a linear space. show that product of two continuous functions is again continuous.

   Suppose that I have n^2 continuous functions on S denoted as f_{ij} for $1 \leq i, j \leq n$. Define $f(x)$ to be the determinant of the matrix $((f_{ij}(x)))$ for each $x \in S$. Show that f is a continuous function on S.

   Let f be a continuous function on R^n. Let π be a permutation of the set $\{1, 2, \cdots, n\}$. Define a function g on R^n by

   $$g(x_1, x_2, \cdots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)}).$$

   Show g is continuous.

   Let f be a continuous function on R^n. Let $f_1, \cdots, f_n$ be continuous functions on R to R. Define g on R^n by

   $$g(x_1, x_2, \cdots, x_n) = f(f_1(x_1), f_2(x_2), \cdots f_n(x_n)).$$

   then g is a continuous function on R^n. More generally, let $f_1, f_2, \cdots, f_m$ be continuous functions on R^n and f be a continuous function on R^m. Define g on R^n by

   $$g(x) = f(f_1(x), f_2(x), \cdots, f_m(x)); \quad x \in R^n.$$

   Show g is continuous on R^n.

6. Suppose f is a continuous function on R^5. Let us fix 2 real numbers a_1, a_2. Define a function on R^3 by

$$g(x_1, x_2, x_3) = f(x_1, a_1, x_2, x_3, a_2).$$

Show that g is a continuous function on R^3. do you think the numbers 3 and 5 are special or they an be replaced by any integers $1 \leq m < n$. How does such a generalization read?

7. Suppose f is a function on R^3 and g is a function on R^4. We define a function h on R^7 as follows (think of it as *external product*).

$$h(x_1, x_2, \cdots x_7) = f(x_1, x_2, x_3) g(x_4, \cdots, x_7).$$

Show that h is a continuous function on R^7if f, g are continuous. do you think 3 and 4 are special or you can replace them by any integers $m \geq 1$ and $n \geq 1$. How does such a generalization read?

8. This problem concerns the way of thinking of our spaces themselves.

Let $(x, y) \in R^2$ and $\sqrt{x^2 + y^2} = 1$. Show that there is a number $\theta \in [0, 2\pi)$ such that $\cos\theta = x; \sin\theta = y$.

Show that every non-zero vector $(x, y) \in R^2$ can be uniquely represented as $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ via $x = r\cos\theta, y = r\sin\theta$. These numbers (r, θ) are called the polar coordinates of the point whose Cartesian coordinates are (x, y).

Show that every non-zero vector $(x, y, z) \in R^3$ can be uniquely represented as $(r, \theta, z) \in (0, \infty) \times [0, 2\pi) \times R$ via $x = r\cos\theta, y = r\sin\theta, z = z$. These numbers (r, θ, z) are called cylindrical coordinates of the point (x, y, z).

Show that every non-zero vector $(x, , y, z) \in R^3$ can be uniquely represented as $(r, \theta, \phi) \in (0, \infty) \times [0, \pi] \times [0, 2\pi)$ via

$$x = r\cos\phi\sin\theta;\ y = r\sin\phi\sin\theta;\ z = r\cos\theta.$$

These numbers (r, θ, ϕ) are called spherical coordinates of (x, y, z).

9. Last time we said that behind the norm is the 'linear' inner product. Let us verify that inner product is actually linear (actually bilinear or linear in each argument).

   Show $\langle cx, y\rangle = c\langle x, y\rangle = \langle x, cy\rangle$.

   Show that $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$

   Show the polarisation identity:

$$\frac{||x + y||^2 - ||x - y||^2}{4} = \langle x, y\rangle.$$

10. Argue rigorously that the following set is an open set.

$$\{x \in R^d : ||x - a|| < r\}.$$

   Here $a \in R^d$ and $r > 0$.

11. Let $A \subset R^d$ (you think of R^2 and proceed till the finish and then look back).

   Given a point $x \in R^d$ there are exactly three possibilities:

   either there is an $r > 0$ such that $B(x, r) \subset A$,

   or there is an $r > 0$ such that $B(x, r) \subset A^c$,

   or $B(x, r)$ intersects both A and A^c for ever $r > 0$.

   In the first case say that x is an interior point of A, in the second case say that x is an exterior point of A, in the third case say that x is a boundary point of A.

   Show that the set A^o of interior points of A is an open set. Show that it is the largest open set contained in A.

   Show that the set of exterior points of A is an open set. It is the largest open set disjoint with A.

   Show that the set ∂A of boundary points of A is a closed set.

   Show that $\overline{A} = A \cup \partial A$ is a closed set. Show that it is the smallest closed set containing A. This is called closure of A.

12. If $A \subset R$ is the set of rational numbers, calculate A^o and ∂A.

    Do the same if $A = \{(x, y) : \ x^2 + y^2 < 1\} \subset R^2$.

    Do the same if $A = \{(x, y) : \ x > 0, y > 0\} \subset R^2$.

13. I have a closed subset of the real line. I know that every rational number is in my set. What do you think my set could be?

14. Let $f : R^2 \to R$. Suppose that $f(x, y_1) = f(x, y_2)$ for any three real numbers x, y_1, y_2.

    Show that there is a function $g : R \to R$ such that $f(x, y) = g(x)$ for every (x, y).

15. Let $f(x, y)$ be one or zero according as y is rational or not. Does f_1 exist? what is it? Is f continuous at any point?

16. Find derivatives of the following functions.

    $f(x, y) = x^y$ defined on $\{(x, y) : x > 0, y > 0\}$.

    $f(x, y, z) = x^{(y^z)}$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y, z) = (x^y)^z$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y, z) = x^{(y+z)}$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y, z) = (x + y)^z$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y) = \sin(x \cos y)$.

    $f(x, y, z) = \sin(x \cos[y \sin z])$.

    $f(x, y, z) = \sin(xyz)$

    $f(x, y, z) = \sin x + \sin y + \sin z$.

17. Let $n \geq 3$. Put

    $$f(x_1, x_2, \cdots, x_n) = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^{(n-2)/2}}.$$

    Show

    $$f_{11} + f_{22} + \cdots + f_{nn} \equiv 0.$$

18. Let

    $$f(x, y) = e^x \cos y; \qquad g(x, y) = e^x \sin y.$$

    Show $f_{11} + f_{22} \equiv 0$ and similar for g.

19. Find if limit as $(x, y) \to (0, 0)$ exists for the following and find when the limit exists.

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}; \qquad g(x, y) = \frac{\sin(x^4 + y^4)}{x^2 + y^2}$$

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^4 + y^4}; \qquad g(x, y) = \frac{e^{-1/(x^2+y^2)}}{x^4 + y^4}$$

20. Suppose that ξ and η are C^1 functions on R to R. Find the derivatives of the following.

$$f(x, y) = \xi(x + y); \qquad g(x, y) = \eta(xy).$$

$$f(x, y) = \xi(x)\eta(y); \qquad g(x, y) = \eta(x).$$

21. $f : R^2 \to R$. Assume that the two partial derivatives exist and $f_1 \equiv 0$ and $f_2 \equiv 0$. Show that f is a constant.

[sometimes I feel that some of you are not serious and are violating the trust put on you by the Institute. If it persists, and if you use your freedom not to attend classes, I would have to use my freedom not to allow you to sit for my exam. But let us hope sense prevails and I do not have to do this.]

We have finally changed our attitude towards derivative, it is not a number, not a vector, but a linear transformation — the best that you can think of for your function near your point of interest. It takes time to get used to this idea. Do not worry. But what you should realize is that we came to the conclusion in order to bring some order in a chaotic situation. I am not giving many problems in this set so that you have time to settle down to the new ideas.

22. The following functions have the interesting property that they are continuous 'along any line' at $(0,0)$, but they are not continuous at $(0,0)$.

$$f(x,y) = \frac{x^4y^4}{(x^2+y^4)^3}, \quad (x,y) \neq (0,0); \quad f(0,0) = 0.$$

$$g(x,y) = \frac{x^2}{x^2+y^2-x}, \quad (x,y) \neq (0,0); \quad f(0,0) = 0.$$

(I have not defined the phrase 'along any line'; what could it mean?)

23. Sometimes a function can be expressed both in cartesian coordinates and also in polar coordinates. for example the function $u(x,y) = x^2+y^2$ is 'same as' $u(r,\theta) = r^2$. It would be confusing at first sight, using the same notation u. But let us indulge in such a confusion.

Express cartesian coordinate derivatives in terms of polar coordinate derivatives:

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}.$$

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

$$u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2}u_\theta^2.$$

$$u_{xx} = u_{rr}\cos^2\theta + u_{\theta\theta}\frac{\sin^2\theta}{r} - 2u_{r\theta}\frac{\cos\theta\sin\theta}{r} +$$

$$u_r\frac{\sin^2\theta}{r} + 2u_\theta\frac{\cos\theta\sin\theta}{r^2}.$$

$$u_{yy} = u_{rr}\sin^2\theta + u_{\theta\theta}\frac{\cos^2\theta}{r} + 2u_{r\theta}\frac{\cos\theta\sin\theta}{r} +$$

$$u_r\frac{\cos^2\theta}{r} - 2u_\theta\frac{\cos\theta\sin\theta}{r^2}.$$

$$u_{xy} = u_{rr}\cos\theta\sin\theta - u_{\theta\theta}\frac{\cos\theta\sin\theta}{r^2} + u_{r\theta}\frac{\cos^2\theta - \sin^2\theta}{r} -$$

$$u_r\frac{\sin\theta\cos\theta}{r} + u_\theta\frac{\sin^2\theta - \cos^2\theta}{r^2}.$$

$$u_{xx} + u_{yy} = u_{rr} + u_{\theta\theta}\frac{1}{r^2} = u_r\frac{1}{r}.$$

$$= \frac{1}{r^2}\left\{r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right\}.$$

Here is how you express polar coordinate derivatives in tees of cartesian coordinate derivatives.

$$u_r = u_x\cos\theta + u_y\sin\theta.$$

$$u_\theta = -u_x r\sin\theta + u_y r\cos\theta.$$

24. f and g are two C^2 functions on R. Put $h(x,y) = f(x-y) + g(x+y)$. Show $h_{xx} - h_{yy} = 0$.

25. When coordinates x, y are changed to

$$\xi = a + \alpha x + \beta y; \quad \eta = b - \beta x + \alpha y;$$

where $\alpha^2 + \beta^2 = 1$; the function $u(x,y)$ is changed to $U(\xi,\eta)$. Show

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 = u_x u_{yy} - u_{xy}^2.$$

As I keep on mentioning, it is the ideas and the ability to think and grasp what is going on that plays crucial role. The mathematics is not difficult, probably imitation (with proper notation) of one variable results. This is so at the initial stages of the development but gradually it changes we will have things that are not imitations of one variable results.

Though most of the problems below are stated for R^n. It is alright if you do not solve it for general n. But you should work out the two cases $n = 2$ and $n = 3$ and also get a feel. However, you must understand what it says for n.

26. There are two C^1 functions $f : R \to R^n$ and $g : R \to R^n$. Define

$$h(t) = \langle f(t), g(t) \rangle; \quad t \in R.$$

Calculate $h'(t)$.

27. Let f be a C^1 function from R to R^n. Suppose $||f(t)|| = 1$ for each t. Show that $f'(t) \cdot f(t) \equiv 0$

In other words the function is orthogonal to its gradient at every point.

For example, $f(t) = (\cos t, \sin t)$ is such a function.

28. Suppose f is a real valued C^1 function of two variables; g, h are real valued C^1 functions of one variable, then I define real valued function of three variables:

$$F(x, y, z) = f(g(x + y), h(y + z)).$$

Calculate F'.

29. I have three real valued C^1-functions: f is a function of one variable, g is a function of two variables, h is a function of three variables. I have a C^1 function φ of four variables. I define a function of three variables;

$$F(x, y, z) = \varphi(x, f(x), g(x, y), h(x, y, z))$$

find F'.

30. Suppose I have n^2 many C^1 functions $f_{ij}(t)$ defined on R to R. Here $1 \leq i \leq n$ and $1 \leq j \leq n$.

I consider the $n \times n$ matrix $A(t) = (\ f_{ij}(t)\ )$. Consider the real valued function $F(t)$ defined on the real line by the formula

$$F(t) = \mathrm{Det} A(t).$$

Show that F is C^1 function and $F'(t)$ is sum of determinants of n matrices $\{A_i :\leq i \leq n\}$ where A_i is obtained by replacing the i-th row of $A(t)$ by derivative of that row; more precisely, replace the i-th row of $A(t)$ by

$$(f'_{i1}(t), f'_{i2}(t), \cdots, f'_{in}(t)).$$

31. This exercise is long. Take fifteen/twenty minutes to understand. To solve, you need five minutes.

It is difficult to keep on writing R^{n^2} all the time. Let us use an abbreviation just for this exercise: $\mathbf{d} = \mathbf{n^2}$.

The space of $n \times n$ matrices can be thought of as R^d in the following way: First row followed by second row etc. More precisely if A is $n \times n$ matrix (a_{ij}) then it is the point in R^d defined as

$$(a_{11}, a_{12}, \cdots, a_{1n},\ \ a_{21}, a_{22}, \cdots, a_{2n},\ \ \cdots,\ \ a_{n1}, a_{n2}, \cdots, a_{nn}).$$

We regard R^d as column vectors, but for this exercise and for typographical considerations imagine rows. It is possible to stick to column vectors but that will confuse you.

When you read a matrix, you read first row and then second row etc. This is precisely how we identified it as a point of R^d.

Let me say it differently to familiarize you. If you have matrix with rows $r_1, r_2, \cdots, r_n$ then the point of R^d to which it corresponds is

$$(r_1, r_2, \cdots, r_n).$$

Be careful. Here r_i is not a number it is an n-tuple of numbers, the i-th row of A, thus all these n many n-tuples of numbers make up an n^2-tuple.

Similarly, if you have a point x of R^d it corresponds to the matrix whose first row is the first n entries of x, second row is the next n entries of x and so on. Define

$$F(x) = \mathrm{Det}(x); \qquad x \in R^d.$$

Thus when we wrote Det, we are regarding the x as a $n \times n$ matrix. Then show that F is C^1 (defined on R^d to R).

If you take a point $a \in R^d$, then the derivative of F at a is a linear transformation $F'(a) : R^d \to R$. What is it? If you take $x \in R^d$ then value of the linear transformation $F'(a)$ at the point x is the sum of n determinants of matrices $A_1, A_2, \cdots A_n$ where A_i has all rows except the i row same as those of a, but the i-th row is the i-th row of x.

To say differently, if

$$a = (a_1, a_2, \cdots, a_n); \quad x = (x_1, x_2, \cdots, x_n) \in R^d$$

(pause, do you understand what are these a's and x's?) then

$$F'(a)(x) = \sum_1^n \mathrm{Det}(a_1, \cdots, a_{i-1}, x_i, a_{i+1} \cdots, a_n).$$

32. z is a function of two variables (x, y) and I know $xy + yz - zx \equiv 2$.

    Using implicit function rules calculate z_1 and z_2, the partial derivatives. Also explicitly solve for the function z and calculate.

33. Find a number θ such that $0 < \theta < 1$ and

$$f\left(1, \frac{1}{2}, \frac{1}{3}\right) = f_x\left(\theta, \frac{\theta}{2}, \frac{\theta}{3}\right) + \frac{1}{2} f_y\left(\theta, \frac{\theta}{2}, \frac{\theta}{3}\right) + \frac{1}{3} f_z\left(\theta, \frac{\theta}{2}, \frac{\theta}{3}\right)$$

when

$$f(x, y, z) = xyz.$$

$$f(x, y, z) = xy + yz + zx.$$

34. Let $\Omega \subset R^2$ be a region which has the following property:

$$(x, y) \in \Omega; \lambda > 0 \Rightarrow (\lambda x, \lambda y) \in \Omega.$$

For example any quadrant or all of R^2 satisfies this condition. If you do not like this, think of R^2. Let n be an integer.

A function $f : R^2 \to R$ is *homogeneous* of degree n if the following holds:

$$(x, y) \in \Omega; \lambda > 0 \Rightarrow f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

Show that the following functions are homogeneous and find the order.

$$f(x, y) = ||(x, y)||; \qquad \Omega = R^2.$$

$$f(x, y) = \log(y/x); \qquad \text{First quadrant, axes not included}$$

$$f(x, y) = ax^2 + bxy + cy^2.$$

Let f be real valued C^1 function on R^2.

If f homogeneous of degree n, show that $xf_1 + yf_2 = nf$.

(Hint: differentiate w.r.t. λ, the equation that defines homogeneity.)

If $xf_1 + yf_2 = nf$ then show that f is homogeneous of degree n.

(Hint: show $\varphi(\lambda) = f(\lambda a, \lambda b)$ satisfies $\lambda\varphi'(\lambda) = n\varphi(\lambda)$ and $\varphi(\lambda)\lambda^{-n}$ is a constant.)

35. Find derivative of

$$f(x) = \int_{x^2}^{x^3} \frac{1}{x+t} dt.$$

Evaluate integral and differentiate and also do by using the formula derived by us.

$$f(x) = \int_{-x}^{x} \frac{1}{x^2+t+1} dt. \qquad f(x) = \int_{x^2}^{-sinx} e^{xt} dt.$$

$$f(x) = \log\left(\int_0^{x^2} \frac{\sin xt}{t} dt\right)$$

36. Calculate $f_1(x,y)$ and $f_2(x,y)$

$$f(x,y) = \int_{xy}^{x^2+y} \frac{\sin(t+y)}{t^2+y^2} dt.$$

37. Calculate

$$\lim_{x\to 0} \frac{1}{x^3} \int_0^x (x-t)^2 f(t) dt.$$

38. This is something we should have done last semester. Recall, we showed that the integral

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt$$

is finite for $x > 0; y > 0$. Value of the integral was denoted by $\beta(x,y)$. show

$$\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\ dt.$$

(Put $t/(1+t) = u$)

Show

$$\beta(x,y) = 2\int_0^{\pi/2} (\sin t)^{2x-1}(\cos t)^{2y-1}\ dt.$$

Show that for any two integers $x \geq 1$ and $y \geq 1$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Actually this is true for any $x, y > 0$ but its proof should wait till we develop integration of functions of two variables.

39.

$$\frac{\partial^n e^{xy}}{\partial x^m \partial y^{n-m}}(0,0) = \begin{cases} 0 & \text{if} \quad 2m \neq n \\ m! & \text{if} \quad 2m = n \end{cases}$$

40. Consider the complement of the closed fourth quadrant in R^2, that is U = complement of the set $\{(x,y) : x \geq 0, y \leq 0\}$.
Let f be be a real valued continuous function on U such that both partial derivatives are zero. Show f is a constant.

Suppose that U is the set of all points (x,y) such that both x, y are strictly positive or bot are strictly negative. Show that U is an open set. Suppose that both partial derivatives are zero. Can you conclude that f is a constant function?

41. suppose that U is an open set in R^2 and $f : U \to R^2$ is a C^1 function. Assume that f is C^1, f is one-to-one, $f')(x$ is non-singular for all x.

Show that range of f is again an open set. In fact show that whenever $V \subset U$ is an open set then $f(V)$ is an open set.

42. This exercise is simple but you should work out fully.

We consider R^2. In my mind I think of (x, y) as $x+iy$ where $i = \sqrt{-1}$. So what?

Define addition on R^2 as usual

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

We define multiplication on R^2 as follows.

$$(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

We made it clear in the class that you can not divide one vector by another vector when you are considering R^n with $n \geq 2$. There is going to be confusion and chaos if we keep on using the same symbol R^2. Let use the symbol C to denote the same set R^2 when we consider the above multiplication.

Thus as a set C is same as R^2, but when we use C it means that we are allowed to use the operation of multiplication. We then refer to elements of C as complex numbers. If $z = (x, y)$ is a complex number we also refer to x as the real part of z and y as the imaginary part of z (instead of calling them as first and second coordinates of z).

Show the following in C. Let $z_1 = (x_1, y_1)$ $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3)$.

$$z_1 \times z_2 = z_2 \times z_1.$$

Thus multiplication is commutative.

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3.$$

Thus multiplication is distributive w.r.t. addition. Let e stand for the complex number $(1, 0)$. Of course 0 stands for $(0, 0)$.

$$z \times e = z \quad \forall z.$$

Thus e is the identity element for multiplication.

$$(\forall z,\ z \neq 0)\ (\exists!\ w)\ zw = e.$$

Thus every non-zero complex number has an (multiplicative) inverse.

We define modulus $|z|$ of a complex number $z = (x, y)$ as norm of (x, y), more precisely, $|z| = \sqrt{x^2 + y^2}$. Show that the usual rules hold:

$$|z_1 + z_2| \leq |z_1| + |z_2|; \quad |z_1 \times z_2| = |z_1||z_2|.$$

We want to identify a complex number whose imaginary part is zero with real number, namely, the real part of that number. In other words the x-axis is identified with real numbers. Thus when we write 'the complex number 5' we mean $(5, 0)$.

Show that the complex multiplication we defined above, when restricted to x-axis coincides with usual multiplication of real numbers.

43. In the following repeated integrals, draw a picture of the region in R^2 where the integration is being carried out. Change the order of integration. There is nothing for you to evaluate.

$$\int_0^1 \left[\int_{x^2}^1 f(x,y)dy\right] dx. \qquad \int_{-2}^1 \left[\int_x^{x^2} f(x,y)dy\right] dx.$$

$$\int_{1/3}^{2/3} \left[\int_{y^2}^{\sqrt{y}} f(x,y)dx\right] dy.$$

44. Show

$$\int_a^b \left[\int_a^x f(x,y)dy\right] dx = \int_a^b \left[\int_y^b f(x,y)dx\right] dy.$$

This is called Dirichlet's formula.

Show

$$2! \int_a^b f(x) \left[\int_x^b f(y)dy\right] dx = \left[\int_a^b f(x)dx\right]^2.$$

Expressing this repeated integral as double integral will help.

45. Consider the solid bounded by the planes $z = x + a$; $z = -x - a$; and the cylinder $x^2 + y^2 = a^2$. Express its volume as a double integral.

Consider the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ where a, b, c are positive numbers. Express its volume as a double integral and evaluate.

46. You are given n points $\{(x_i, y_i) : 1 \le i \le n\}$ in R^2. You should fit the best straight line. That is straight line $y = ax + b$ so that

$$f(a,b) = \sum_1^n (ax_i + b - y_i)^2$$

is minimum. This is called method of least squares.

47. f and g are C^1 functions on R^n to R. Assume that g does not take the value zero. Show

$$\nabla(f/g) = \frac{g\nabla f \ - \ f\nabla g}{g^2}.$$

48. f is a C^1 function on R^3 to R. At every point (x, y, z) the vectors $\nabla f(x, y, z)$ and (x, y, z) are parallel. Show that

$$f(0, 0, z) \equiv f(0, 0, -z).$$

49. Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be points of R^3. Define

$$f(x) = \langle x \times a, x \times b \rangle; \qquad x \in R^3.$$

Here $\times$ denotes vector product. show

$$\nabla f(x) = a \times (x \times b) + b \times (x \times a).$$

50. Let F be a C^2 function and G be a C^1 function on R. Let c be a real number. Define a function $f(x, t)$ on R^2 by

$$f(x, t) = \frac{F(x + ct) - F(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(s) ds. \qquad (*)$$

Show that the function satisfies

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}. \qquad (**)$$

and

$$f(x, 0) = F(x); \quad f_2(x, 0) = G(x). \qquad (***)$$

The (partial) differential equation (**) is called one-dimensional wave equation. The solution (*) is called D'Alembert's solution. The conditions (***) are called initial conditions.

51. Find stationary points (zero gradient) for the following functions and classify if they are maxima or minima or neither.
$f(x, y) = x^2 + (y - 1)^2$.
$f(x, y) = x^2 - (y - 1)^2$.
$f(x, y) = \sin x \sin y \sin(x + y)$.
$f(x, y) = \sin x \cosh y$.

52. If $f : R^3 \to R^3$ be C^1 function, $f = (f_1, f_2, f_3)$, divergence of f is defined by

$$div(f) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \nabla \cdot f.$$

Show $div(f + g) = div(f) + div(g)$. if $\varphi : R^3 \to R$ be C^1, then show $div(\varphi f) = \varphi(div(f)) + \nabla\varphi \cdot f$.

53. calculate the following integrals:

(a) $\int\int_{\Omega} e^{x+y} dxdy$ over $\Omega = \{|x| + |y| \leq 1\}$.

(b) $\int\int_{\Omega} x^2y^2 dxdy$ over $x \geq 0, y \geq 0, xy = 1, xy = 2, y = x, y = 4x$.

(c) $\int\int_{\Omega} (x^2 + y^2) dxdy$ over $\Omega = \{|x| \leq 1, |y| \leq 1\}$.

(d) $\int\int_{\Omega} (3x + y) dxdy$ over $\Omega = \{x \geq 0, y \geq 0, 4x^2 + y^2 \leq 36\}$.

54. In each of the following, a region Ω is given, a transformation $(x, y) \mapsto (u, v)$ is given. Show that it is one-one on Ω; describe the transformed region in two ways

(i) possible values of u and for every u, the possible values of v;

(ii) possible values of v and for every v, the possible values of u.

(a) $\Omega = [0, 1] \times [0, 1]$; $u = x + y$, $v = x - y$.

(b) $\Omega = [-1, +1] \times [-1, +1]$; u, v as above.

(c) $\Omega = [0, 1] \times [0, 1]$; $u = x + y$, $v = x$.

(d) $\Omega = (0, \infty) \times (0, \infty)$; $u = xy$, $v = y$.

(e) $\Omega = (0, \infty) \times (0, \infty)$; $u = xy$, $v = x/y$.

(f) $\Omega = (0, \infty) \times (0, \infty)$; $u = x + y$, $v = y$.

(g) $\Omega = (0, \infty) \times (0, \infty)$; $u = x + y$, $v = y/(x + y)$.

(h) $\Omega = (0, \infty) \times (0, \infty)$; $u = x^2 + y^2$, $v = y$.

(i) $\Omega = (0, \infty) \times (0, \infty)$; $u = x^2/y^2$, $v = y^2$.

in the following verify the map is one-to-one and describe the range of u and for each u, the possible values of v and for every u, v the possible values of w.

(j) $\Omega = (0, \infty) \times (0, \infty) \times (0, \infty)$;
$u = x + y + z$, $v = x + y$, $w = x$.

(k) $\Omega = (0, \infty) \times (0, \infty) \times (0, \infty)$;
$u = x + y + z$, $v = x/(x + y + z)$, $w = z/(x + y + z)$.

(l) $\Omega = (0,\infty) \times (0,\infty) \times (0,\infty)$;
$u = x^2 + y^2 + z^2$, $v = x/\sqrt{x^2+y^2+z^2}$, $w = y/\sqrt{x^2+y^2+z^2}$

Discuss the following map (what does this mean?).

(m) $\Omega = [0,\infty) \times [0,\pi) \times [0,2\pi)$
points here are denoted by (r, θ, ϕ).
$x = r\cos\theta$, $y = r\sin\theta\cos\phi$, $z = r\sin\theta\sin\phi$.

(n) $\Omega = [0,\infty) \times [0,\pi) \times [0,2\pi) \times [0,2\pi)$
points here are denoted by (r, θ, ϕ, ψ).
$x = r\cos\theta$, $y = r\sin\theta\cos\phi$, $z = r\sin\theta\sin\phi\cos\psi.$, $w = r\sin\theta\sin\phi\sin\psi$.

55. If you have a bounded region $\Omega \subset R^2$ with small boundary, then centroid of Ω is the point $(\overline{x}, \overline{y})$ defined by

$$\overline{x} = \int\int_{\Omega} x dx dy; \quad \overline{y} = \int\int_{\Omega} y dx dy.$$

calculate centroids of the regions shown (?) below.

(a) unit disc
(b) the triangle with vertices $(-1,0)$, $(0,-1)$, $(1,1)$.
(c) the unit square $[0,1] \times [0,1]$
(d) region bounded by $y = x^2; x + y = 2$
(e) region bounded by $y = \sin^2 x$, $y = 0$, $0 \leq x \leq \pi$.
(f) region bounded by $y = \sin x$, $y = \cos x$, $0 \leq x \leq \pi/4$.
(g) region bounded by $x > 0$, $y > 0$, $\sqrt{x} + \sqrt{y} = 1$.

56. Let A be a symmetric 2×2 matrix with strictly positive eigen values. This is same as saying that A is symmetric positive definite matrix. Show that there is a symmetric positive definite matrix B such that

$$B^2 = B^T B = A$$

Generalize to higher dimensions (why should we care?)

57. Calculate the Taylor expansion around the origin upto third order.
   (a) $f(x, y) = \exp\{\sin y\}$.
   (b) $f(x, y) = \cos(xy)$.
   (c) $f(x, y, z) = \sin\{e^x + y^2 = z^3\}$.

58. In the following calculate the integrals over the regions indicated.
   (a) $Q = [-1, 1] \times [0, 2]$. $\int\int_Q \sqrt{|y - x^2|}$.
   (b) $Q = [0, \pi/2] \times [0, \pi/2]$. $\int\int_Q \sin(x + y)$.
   (c) $Q = [0, \pi] \times [0, \pi]$. $\int\int_Q \cos(x + y)$.
   (d) $Q = [0, 3] \times [0, 2]$. $\int\int_Q [x + y]$;

   where $[a]$ is the greatest integer not exceeding a.
   (e) $Q = [0, 1] \times [0, 1]$. $\int\int_Q f$;

   where f is defined on Q as follows. $f(x, y) = x+y$ if $x^2 \leq y \leq 2x^2$; and zero otherwise.
   (f) $\Omega = \{4x^2 + 9y^2 \leq 36; x > 0; y > 0\}$, $\int\int_S (3x + y)$.
   (g) $\Omega = \{x^2 + y^2 \leq 16\}$, $\int\int_S (20 + 2x + y)$.
   (h) Ω is the region bounded by $xy = 1$; $xy = 2$; $y = x$ and $y = 4x$, $\int\int_S (x^2y^2)$.
   (i) Let $D = \{x \in R^2 : 0 < ||x|| < 1\}$. Show that the integral $\int\int_D \log ||x||$ exists and calculate it.

   Same problem in other dimensions.
   (j) Let D be as above. When is the integral $\int\int_D ||x||^{-p}$ convergent $(p > 0)$.

   Same problem in other dimensions.
   (k) Let $\Omega = \{||x|| > 1\}$. When is the integral $\int\int_\Omega ||x||^p$ convergent. Here $p > 0$. Calculate the integral.

59. For the function $f(x,y) = xy(1-x^2-y^2)$ on $[0,1]\times[0,1]$ calculate local maxima, local minima, global maxima, global minima, saddle points.

60. Do the same problem for the function $f(x,y,z) = x^4+y^4+z^4-4xyz$.

61. Suppose that $f(x,y,z)$ is a C^2 function with a stationary point P. Suppose that $f''(P)$, that is, the second derivative matrix, has two diagonal entries with opposite sign.

    Show that the point P is a saddle point.

62. For the function $f(x,y) = ax^2+2bxy+cy^2+2dx+2ey+h$ $(a,b,\cdots$ are constants) with $a>0$ and $b^2<ac$ show that there is a global minimum. Calculate it.

63. Which straight line is close to $f(x)=x^2$ on $[0,1]$. What about on $[1,2]$. (What does the question mean?)

64. Suppose $a^1, a^2, \cdots, a^{100}$ are points in R^{121}. Show the centroid minimizes $f(x) = \sum\limits_i ||x-a^i||^2$.

65. Let A and B be $n\times n$ symmetric matrices and B is positive definite. Solve: $\max x'Ax$ subject to $x'Bx=1$.

    First show that the later set is compact and so the problem has a solution.

    Show that if x and λ are obtained by Lagrange method, then $Ax=\lambda Bx$ and λ is the max.

66. Given k points $\{(x_i,y_i): 1\le i\le k\}$ in R^2 (x_i are distinct), fit the best straight-line by least squares method. Give formulae for the parameters of the line.

    Find the best quadratic function by least square method. Find formulae for the parameters.

67. I have a die with 9 faces, the i-th face having probability $p_i>0$ in a throw. I rolled the die 100 times and got the observation (?)

$$(8,12,4,11,20,6,14,9,16)$$

    What do you think is the most likely values of the (p_i)? (I believe that what I observed must be having maximum probability).

68. Are the following functions continuous? if not describe the set of points where they are discontinuous?

(a)
$$f(x,y) = \sqrt{1 - x^2 + y^2}; \quad \text{if} \quad x^2 + y^2 \le 1$$
$$f(x,y) = (1 - x^2 + y^2)^5 \quad \text{if} \quad x^2 + y^2 > 1$$

(b)
$$f(x,y) = \sqrt[86]{xy} \quad \text{if} \quad (x \ge 0, y \ge 0)\text{or}(x \le 0, y \le 0)$$
$$f(x,y) = \sqrt[59]{xy} \quad \text{otherwise.}$$

(c)
$$f(x,y) = \frac{\sin(xy)}{x} \quad x \ne 0$$

and $f(x,y) = 1$ when $x = 0$.

(d)
$$f(x,y) = \begin{cases} \frac{1}{2y} & \text{if} \quad |x| < |y|; y \ne 0 \\ 0 & \text{if} \quad |x| > |y|; y \ne 0 \\ +1 & \text{if} \quad x = 0, y = 0 \\ 0 & \text{if} \quad x \ne 0; y = 0. \end{cases}$$

69. Describe the set where the functions are defined (called domain of definition) and compute ∇f at the interior points of the set.

(a)
$$f(x,y) = \exp\left\{\frac{x}{y} + \frac{y}{x}\right\}$$

(b)
$$f(x,y) = \sin^{-1}(x+y)$$

(c)
$$f(x,y) = \tan^{-1}\sqrt{\frac{x^2 - y^2}{x^2 + y^2}}.$$

(d)
$$f(x,y) = e^x \log y + \sin y \log x.$$

(e)

$$f(x,y) = \log \frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2=y^2}+x}$$

(f)

$$f(x,y) = \frac{x}{y}.$$

70. Find the directional derivatives.

(a) $f(x,y) = 2x^2 - y^2$ at the point $(1,2)$ in the direction of the line joining $(1,2)$ to $(4,0)$.

(b) $f(x,y) = x^3+3xy+4y^2$ at $(0,0)$ in the direction of the line making 60^o with the x-axis.

(c) $f(x,y) = x^2+y^2-3xy$ at the point $(1,2)$ in the direction of the tangent to the curve $y = x^2$ at $(0,0)$.

(d) $f(x,y) = 5x^2-3x-y-1$ at $(2,1)$ in the direction of the line from this line to $(5,5)$.

(e) $f(x,y) = x^3+3x^2+4xy+y^2$ at the point $(2/3, \quad -4/3)$ in all directions.

71. Define a function $u(x,t)$ for $0 < t < \infty$ and $-\infty < x < \infty$ by

$$u(x,t) = \frac{1}{t^{171}} \int_{-t}^{t} e^{-(x+y)}(t^2-y^2)^{85} dy.$$

Show that

$$u_{xx} = \frac{172}{t} u_t + u_{tt}.$$

72. For the curves find their length. parametric so that arc length is the parameter.

(a) Consider the circular helix

$$x_1 = \cos t; \quad x_2 = \sin t; \quad x_3 = t$$

for $0 \le t \le 100$.

(b)

$$x_1(t) = \frac{\sin t}{\sqrt{2}}; \quad x_2 = \frac{\sin t}{\sqrt{2}}; \quad x_3 = \cos t.$$

(c)

$$x_1 = 6t; \quad x_2 = 3t^2 \quad x_3 = t^3$$

for $0 \le t \le 1$.

(d)

$$x_1 = e^t; \quad x_2 = e^{-t} \quad x_3 = \sqrt{2}t.$$

(e)

$$x_1 = \frac{\sqrt{t^2+4}+t}{2} \quad x_2 = \frac{\sqrt{t^2+4}-t}{2}$$

$$x_3 = \sqrt{2}\log\frac{\sqrt{t^2+4}+t}{2}.$$

73. Show that the plane passing through the points $\{(x_i, y_i, z_,) : i = 1, 2, 3\}$ has equation

$$\begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0.$$

74. suppose f, f^2, g, g^2 are continuous or just bounded integrable functions on $[a, b]$. Show

$$\frac{1}{2}\int_a^b \left[\int_a^b \left| \begin{array}{cc} f(x) & g(x) \\ f(y) & g(y) \end{array} \right|^2 dy\right] dx \quad = \quad \int_a^b f^2 \int_a^b g^2 - \left(\int_a^b fg\right)^2 .$$

Deduce Cauchy-Schwarz inequality for integrals (what is it?)

75. Show

$$\begin{aligned} &\frac{1}{2}\int_a^b \left[\int_a^b [f(y) - f(x)][g(y) - g(x)]dy\right] dx \\ &= \quad (b-a)\int_a^b fg - \int_a^b f \int_a^b g. \end{aligned}$$

76. Let f be a non-negative continuous function on $[a, b]$ with $\sup f = M$. Show

$$\lim_n \left\{\int_a^b [f(x)]^n dx\right\}^{1/n} = M.$$

The following is known as Laplace principle. This will locate minimum of a function.

Let $h : [0, 1] \to R$ be a continuous function. Then

$$\lim_n \frac{1}{n} \log \int_0^1 e^{-nh(x)} dx = -\min h.$$

77. Circular Helix is the curve in R^3 defined by

$$x - \rho\cos t; \quad y = \rho\sin t; \quad z = kt; \quad 0 \le t, \infty.$$

Here $k > 0$. Calculate its length from 0 to $t = 28$.

78. Twisted cubic is the curve given by

$$x = at; \quad y = bt^2; \quad z = ct^3; \quad 0 \le t < \infty.$$

Here the product $abc \neq 0$. Calculate the tangent vector at each point of the curve.

79. Find length of the curve

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),\ z = 4a\sin(\theta/2), \quad 0 \le \theta \le 2\pi.$$

80. Design a cylindrical can with lid to contain one litre (=1000 cm^3) of water using minimum amount of metal. Assume that there is a minimum.

    (diameter = length = $20/\sqrt[3]{2\pi}$)

81. A parcel delivery service requires that size of rectangular box be such that length + 2 width =2 height be no more than 108 inches. What is the volume of the largest box the company delivers?

    (11,664 cubic inches)

82. The well known Cobb-Douglas model of economy, simplest one, takes the production function as

$$Q(K, L) = A\ K^{\alpha}\ L^{(1-\alpha)}$$

    where $A > 0$ and $0 < \alpha < 1$ are constants. K denotes units of capital and L denotes units of labour. Assume that the price of one unit of capital is q rupees and price of one unit of labour is p rupees. If the total cash available is B rupees, maximize the production.

    $(\alpha B/q; (1 - \alpha)B/p)$

83. The following is apparently known as Lagrange's identity. Prove it.

$$(r \times s) \cdot (t \times u) = (r \cdot t)(s \cdot u) - (r \cdot u)(s \cdot t).$$

84. Show that the function

$$f(x + iy) = \sqrt{|x||y|}$$

    satisfies Cauchy Riemann equations at $(0, 0)$. Is it (complex) differentiable at this point?

85. Describe the following sets of points in the complex plane. (draw the picture).

    (a) $Re(z) = 3. \quad Im(z) = -1.$

    (b) $|z - c| = |z - d|$ where c and d are complex numbers.

    (c) $Re(z) + Im(z) = 0$

    (d) $|z| = Re(z) + Im(z)$.

86. Show that the function $f(z) = 1/z$ is differentiable on $(|z| \neq 0)$.

87. Considering the function

$$f(x, y) = \sin x \cos y$$

show that there is a number θ between zero and one such that

$$\frac{3}{4} = \frac{\pi}{3} \cos \frac{\pi\theta}{3} \cos \frac{\pi\theta}{6} - \frac{\pi}{6} \sin \frac{\pi\theta}{3} \sin \frac{\pi\theta}{6}.$$

88. Calculate $\nabla \times F$ for the following.

(a)

$$F(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right)$$

(b)

$$F(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, 0 \right)$$

(c)

$$F(x, y, z) = (\sin x, \ \sin y, \ \sin z)$$

89. Using

$$\int_0^\infty \frac{\sin x}{x} dx = \pi/2$$

evaluate

$$f(x, y) = \int_0^\infty \frac{\sin xt \quad \cos yt}{t} dt.$$

for each (x, y).

90. Suppose that f is a bounded continuous function on $(-\infty, +\infty)$. Define a function

$$u(y, t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-(y-x)^2/4t} f(x) dx$$

for

$$0 < t < \infty; \quad 0 - \infty < y, \infty.$$

show that

$$u_{yy} = u_t.$$

Do not need to justify differentiation under the integral sign.

91. Show that the function

$$\varphi(s) = \int_0^\infty e^{-st} \frac{1}{1+t^2} dt; \quad s \geq 0$$

is a continuous function.

92. Let

$$\varphi(y) = \int_0^\infty e^{-yt} \frac{\sin t}{t} dt; \quad y > 0$$

is a continuous function. Show that for $y > 0$

$$\varphi'(y) = -\frac{1}{1+y^2}$$

Show that

$$\varphi(y) = \frac{\pi}{2} - \tan^{-1} y \quad y > 0.$$

(What happens if you let $y \to 0$).

Chennai Mathematical Institute

Midsemestral Examination BSc (Second Semester)

27-02-2014 Calculus II duration: Two hours

Justify your statements. Classwork can be used, by quoting what you are using.

1. Let A be an open set in R^2 and $a \in A$. Prove, by using definition of open set, $A - \{a\}$, (that is, the set A with the point a removed) is an open set. [5]

2. If $F(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}; \quad (x, y, z) \neq (0, 0, 0)$.
show $F_{11} + F_{22} + F_{33} \equiv 0$. [5]

3. Let $a = (a, b, c) \in R^3$ be given. define $F : R^3 \to R^3$ by
$$F(x, y, z) = (yc - zb, za - xc, xb - ya).$$
Calculate F'. [5]

4. Let $F : R^2 \to R^2$ be a C^1 function with $F'(x)$ non-singular at every point $x \in R^2$. Show that the set $F(U) = \{F(x) : x \in U\}$ is an open set for every open $U \subset R^2$. [5]

5. If
$$f(x, y) = \frac{x^3}{3} - y^2 x.$$
show [5]
$$||\nabla(||\nabla f||)||^2 = 4||\nabla f||.$$

6. Consider the function
$$f(x, y) = (3 - x)(3 - y)(x + y - 3); \quad (x, y) \in R^2.$$
Draw the axes (x-axis and y-axis) and plot the set of points where $f(x, y) \geq 0$. [5]

Describe the set of points (x, y) where $f'(x, y) = (0, 0)$. [5]

Describe which of the above points are local maximum and which are local minimum. [5]

GOOD LUCK

Chennai Mathematical Institute

Semestral Examination BSc (Second Semester)

01-05-2014 Calculus II duration: $2\frac{1}{2}$ hours

Be precise and clear in your statements and justify them.
You can use theorems proved in class, but quote what you use.

1. Complete the following sentences. [9]

   (a) Let $f : R^2 \to R$.
   f is not continuous if ....

   (b) Let $f : R^3 \to R$.
   f is not uniformly continuous if ....

   (c) Let $S \subset [0,1] \times [0,1]$.
   S does not have area if ....

2. Let $f(x,y) = e^{xy}$. Calculate $\dfrac{\partial^{30} f}{\partial^{15} x \partial^{15} y}$ at the point $(1,1)$. [4]

3. Let $f(x,y) = |xy|$. For *each* point $\mathbf{a} \in \mathbf{R^2}$ and for *each* unit vector $\mathbf{v} \in R^2$ state whether the directional derivative at the point $\mathbf{a}$ in the direction $\mathbf{v}$ exists and if so calculate. [10]

4. Define f for $(x,y) \in Q = [1,3] \times [1,4]$ by
$$f(x,y) = (x+y) \quad \text{if} \quad x \le y \le 2x$$
   and $f(x,y) = 0$ otherwise. Calculate $\int_Q f$. You need not prove that the integral exists. [12]

5. Find the length of the shortest line from the point $(0,0,25/9)$ to the surface $z = xy$. [10]

6. Evaluate [10]

$$\left(\int_0^4 e^{-t^2/2}dt\right)^2 \ + \ 2\int_0^1 \frac{e^{-16(t^2+1)/2}}{t^2+1}dt.$$

7. Introduce arc length as parameter for the curve below. [5]

$$x = e^t; \quad y = e^{-t}; \quad z = \sqrt{2}t; \quad 0 \le t \le 9.$$

G O O D L U C K