1. Solve the following LP using the simplex tableau method. **(5 marks)**

$$\begin{array}{llcll} \text{maximize} & 5x_1 & + & 4x_2 & \\ \\ \text{subject to} & x_1 & + & 2x_2 & \leq 6 \\ & -2x_1 & + & x_2 & \leq 4 \\ & 5x_1 & + & 3x_2 & \leq 15 \\ & x_1 & , & x_2 & \geq 0 \end{array}$$

2. Prove or disprove the following statement: if the primal has a unique optimum, then the dual has a unique optimum (an LP is said to have a unique optimum if there is exactly one feasible solution giving the optimum value). **(6 marks)**

3. Suppose x^* is the unique optimum of an LP. Show that the second best extreme point must be adjacent to x^*. **(6 marks)**

4. Show that if the LP maximize $c^T x$ subject to $Ax = b$ (where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$) is unbounded, there is a rational vector $\alpha \in \mathbb{R}^n$ such that (a) $c^T \alpha > 0$ and (b) for every feasible point x and $k > 0$, $x + k\alpha$ is feasible. **(6 marks)**

5. We are given a complete undirected graph with positive costs on edges. For every pair of vertices (i, j) we are given a positive integer t_{ij} which indicates the minimum number of edge-disjoint paths required between i and j.

   The problem is to select a minimum cost subset of edges that satisfies the connectivity criterion.

   (a) Write an LP relaxation for this problem. **(4 marks)**

   (b) Give a polynomial-time separation oracle. **(5 marks)**

6. Consider the Steiner forest problem: given a complete undirected graph with non-negative edge weights and disjoint subsets $S_1, \ldots, S_k$ of its vertices, find a minimum cost subset of edges so that vertices in each S_i are in a connected component induced by the picked edges.

   This problem is NP-hard.

   (a) Write an ILP for this problem. **(4 marks)**

   (b) Give the relaxed LP and its dual. **(3 marks)**

   (c) Describe a primal-dual algorithm which gives a 2-approximation. Provide the analysis of your method. **(6 marks)**

7. Where does the ellipsoid method (discussed in class) fail in the case of ILPs? **(5 marks)**