COMPLEX ANALYSIS

CMI UNDERGRADUATE COURSE NOTES

T.R. RAMADAS

1. Holomorphic Functions

A *domain* is a connected open subset of $\mathbb{C}$. (In this case connected=path-connected.) We will consider functions $f : D \to \mathbb{C}$, where D is a domain. We will often write $f(x+iy) = u(x,y) + iv(x,y)$ with u and v real-valued functions. We can define continuity of f directly, as a map between metric spaces. Check that this is equivalent to the continuity of u and v.

Let $D \subset \mathbb{C}$ be a domain. A function $f : D \to \mathbb{C}$ is *(complex) differentiable* at $\xi \in D$, with derivative $f'(\xi) \in \mathbb{C}$, if

$$\lim_{h\to 0} \frac{f(\xi+h) - f(\xi)}{h} = f'(\xi)$$

Check that if f is complex differentiable at ξ, it is "continuous at ξ". That is to say, if $f'(\xi)$ exists then

$$\lim_{h\to 0} f(\xi+h) = f(\xi)$$

The function f is *holomorphic* on D if f is complex differentiable at each $\xi \in D$. In this case, we have a new function f' defined on D: $\xi \mapsto f'(\xi)$, $\xi \in D$ which is *a priori* not continuous, though in fact this (and more) is true. I prefer not not use the phrase "holomorphic at ξ" even though I did use it in class.

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, we say that f is *entire*.

The theorems of Cauchy and Goursat (to come later). We are now in a position to prove the most important theorem in the subject – Cauchy's Theorem, and an important special case, Goursat's Theorem. However, we postpone this to §11 for pedagogical reasons. We *will* state and use two consequences of this theory, making sure to avoid circular reasoning. In other words, the proofs in section §11 will not use any of the results proved between this point and then.

Consequence B of Cauchy's Theorem: If $f : D \to \mathbb{C}$ is holomorphic, then f' is holomorphic on D. Consequence B is part of Consequence A ("holomorphic $\implies$ analytic") which we will come to in a while.

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Clearly then, if f is holomorphic, we can define complex derivatives of all orders. We will denote by $f^{(n)}$ the n^{th} derivative of f, setting $f^{(0)} = f$. Thus (for $z \in D$)

$$f^{(0)}(z) = f(z), \text{ and}$$
$$f^{(n)}(z) = [f^{(n-1)}]'(z), \ n > 0$$

Corollary 1.1. *The function $f = u + iv$ is holomorphic with derivative $f'(z) = u_x + iv_x = -iu_y + v_y$ on D iff u and v are infinitely differentiable and satisfy the Cauchy-Riemann equations:*

$$u_x = v_y$$
$$u_y = -v_x$$

where u_x, etc. denote partial derivatives.

Proof. Suppose first that f is complex differentiable at $z_0 = x_0 + iy_0 \in D$. Then, given $\epsilon > 0$ small enough that $D(z_0, \epsilon) \subset D$, we have

$$\begin{aligned}\lim_{t\to 0} \frac{f(z_0 + t\epsilon) - f(z_0)}{t\epsilon} &= \lim_{t\to 0} \frac{u(x_0 + t\epsilon, y_0) - u(x_0, y_0) + iv(x_0 + t\epsilon, y_0) - iv(x_0, y_0)}{t\epsilon} \\ &= \lim_{t\to 0} \frac{u(x_0 + t\epsilon, y_0) - u(x_0, y_0)}{t\epsilon} \\ &\quad + i \lim_{t\to 0} \frac{v(x_0 + t\epsilon, y_0) - v(x_0, y_0)}{t\epsilon}\end{aligned}$$

So the limit on the left exists iff $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ exist; and furthermore $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. Similarly,

$$\begin{aligned}\lim_{t\to 0} \frac{f(z_0 + it\epsilon) - f(z_0)}{it\epsilon} &= \lim_{t\to 0} \frac{u(x_0, y_0 + t\epsilon) - u(x_0, y_0) + iv(x_0, y_0 + t\epsilon) - iv(x_0, y_0)}{it\epsilon} \\ &= -i \lim_{t\to 0} \frac{u(x_0, y_0 + t\epsilon) - u(x_0, y_0)}{t\epsilon} \\ &\quad + \lim_{t\to 0} \frac{v(x_0, y_0 + t\epsilon) - v(x_0, y_0)}{t\epsilon}\end{aligned}$$

So the limit on the left exists iff $u_y(x_0, y_0)$ and $v_y(x_0, y_0)$ exist; and furthermore $f'(z_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0)$. So we conclude that if f is holomorphic on D, the partial derivatives u_x, u_y, v_x, v_y exist on D and obey the CR equations.

Note the formulae

$$f'(z) = u_x + iv_x, \ if'(z) = u_y + iv_y$$

Exercise: Prove by induction easy induction that (since f is infinitely complex differentiable) u and v are C^∞.

Conversely suppose u and v are C^1 on D and obey the CR equations. We will prove that f is holomorphic on D, with derivative $f'(z) = u_x + iv_x = -iu_y + v_y$. (This is a stronger than what is claimed in the corollary since we are only assuming that u, v are C^1; note that as a consequence we see that if u and v are C^1 and satisfy the CR equations, they are in fact C^∞.)

Let $z_0 = x_0 + iy_0 \in D$, and suppose let $r > 0$ be such that $D(z_0, r) \subset D$. For $h = h_x + ih_y$ such that $|h| < r$, we have

$$\begin{aligned}
f(z_0+h) - f(z_0) &= u(x_0+h_x, y_0+h_y) - u(x_0,y_0) + i(v(x_0+h_x,y_0+h_y) - v(x_0,y_0)) \\
&= u(x_0+h_x,y_0+h_y) - u(x_0+h_x,y_0) + u(x_0+h_x,y_0) - u(x_0,y_0) \\
&+ i(v(x_0+h_x,y_0+h_y) - v(x_0,y_0+h_y) + v(x_0,y_0+h_y) - v(x_0,y_0)) \\
&= u_y(x_0+h_x, y_0+t_1h_y)h_y + u_x(x_0+t_2h_x,y_0)h_x \\
&+ iv_x(x_0+t_3h_x,y_0+h_y)_x h_x + iv_y(x_0,y_0+t_4h_y)h_y
\end{aligned}$$

for some $t_i \in [0,1]$, using the (real, one-variable) mean-value Theorem. Note that

$$\begin{aligned}
f'(z_0)h &= (u_x(x_0,y_0) + iv_x(x_0,y_0))(h_x+ih_y) \\
&= u_x(x_0,y_0)h_x - v_x(x_0,y_0))h_y + i(u_x(x_0,y_0)h_y + v_x(x_0,y_0))h_x)
\end{aligned}$$

This gives (using the CR equations)

$$\begin{aligned}
f(z_0+h) - f(z_0) - f'(z_0)h &= [u_y(x_0+h_x,y_0+t_1h_y) - u_y(x_0,y_0)]h_y \\
&+ [u_x(x_0+t_2h_x,y_0) - u_x(x_0,y_0)]h_x \\
&+ i[v_x(x_0+t_3h_x,y_0+h_y) - v_x(x_0,y_0)]h_x \\
&+ i[v_y(x_0,y_0+t_4h_y) - v_y(x_0,y_0)]h_y
\end{aligned}$$

Finally, dividing by h and using the inequalities $|h_x| \le |h|$ and $|h_y| \le |h|$

$$\begin{aligned}
\left|\frac{f(z_0+h)-f(z_0)}{h} - f'(z_0)\right| \le & \sup_{t_1\in[0,1]} |u_y(x_0+h_x,y_0+t_1h_y) - u_y(x_0,y_0)| \\
&+ \sup_{t_2\in[0,1]} |u_x(x_0+t_2h_x,y_0) - u_x(x_0,y_0)| \\
&+ \sup_{t_3\in[0,1]} |v_x(x_0+t_3h_x,y_0+h_y) - v_x(x_0,y_0)| \\
&+ \sup_{t_4\in[0,1]} |v_y(x_0,y_0+t_4h_y) - v_y(x_0,y_0)|
\end{aligned}$$

Since u and v are assumed to be C^1, the RHS clearly tends to zero as $h \to 0$. $\square$

1.1. Alternate proof à la Stein-Shakarchi.

We begin with some generalities. For any $z_0 = x_0 + iy_0 \in D$ and $r > 0$ and small enough that $D(z_0, r) \subset D$, define a function ψ_u by

$$\psi_u(h_x,h_y) = \frac{u(x_o+h_x,y_0+h_y) - u(x_0,y_0) - u_x(x_0,y_0)h_x - u_y(x_0,y_0)h_y}{|h|}$$

This is defined for $h = h_x + ih_y$ such that $0 < |h| < r$. Since u is assumed to be C^1, $\psi_u \to 0$ as $|h| \to 0$. Define ψ_v similarly. Then using the CR equations and collecting terms, we see that

$$f(z_o + h) - f(z_0) = \lambda h + (\psi_u(h) + \psi_v(h))|h|$$

where $\lambda = u_x(x_0, y_0) - iu_y(x_0, y_0)$. This shows that f is complex differentiable at z_0 and

$$f'(z_0) = u_x(x_0, y_0) - iu_y(x_0, y_0) \ .$$

One introduces the real and complex parts of f and the CR equations only to make contact with calculus in several (real) variables. For most of this course, we will stick to complex differentiability.

2. Geometry of the CR equations

This will requite some familiarity with multivariable calculus. Let $\tilde{D}$ be a connected open set in $\mathbb{R}^2$ and consider a map $\tilde{f} : \tilde{D} \subset \mathbb{R}^2 \to \mathbb{R}^2$:

$$\tilde{f}(\tilde{\xi}) = \begin{pmatrix} u(\tilde{\xi}) \\ v(\tilde{\xi}) \end{pmatrix}$$

Recall that such a map is differentiable at a point $\tilde{\xi}$, with derivative $\tilde{f}'(\tilde{\xi})$ a 2×2 real matrix, provided, given any vector $\tilde{h} = \begin{pmatrix} h' \\ h'' \end{pmatrix}$, we have

$$\lim_{t \to 0} \frac{\tilde{f}(\tilde{\xi} + t\tilde{h}) - \tilde{f}(\tilde{\xi})}{t} = \tilde{f}'(\xi)\tilde{h}$$

The coordinate functions on $\mathbb{R}^2$ will be denoted x and y. It is a basic result in multivariable calculus that the partial derivatives u_x, u_y, v_x, v_y exist and are continuous iff the derivative $\tilde{f}'(\tilde{\xi})$ exists at each point $\tilde{\xi}$ and the map

$$\tilde{\xi} \mapsto \tilde{f}'(\tilde{\xi})$$

from $\tilde{D}$ to the vector space of 2×2 matrices is continuous. In this case we say that $\tilde{f}$ is C^1 ('continuously differentiable'). The matrix $\tilde{f}'(\tilde{\xi})$ is given by

$$\tilde{f}'(\tilde{\xi}) = \begin{pmatrix} u_x(\tilde{\xi}) & u_y(\tilde{\xi}) \\ v_x(\tilde{\xi}) & v_y(\tilde{\xi}) \end{pmatrix}$$

Note that under the isomorphism of real vector spaces, $\mathbb{C} \to \mathbb{R}^2$, given by

$$x + iy \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

the operation of multiplication by i goes over to multiplication by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. That is, if $z = x + iy$, and $z' = iz = x' + iy'$, then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Exercise: $\tilde{f}'(\xi)$ commutes with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ iff u_x, u_y, v_x, v_y satisfy the Cauchy-Riemann equations at ξ.

In other words, a function the function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic iff the derivative of the function $\tilde{f}(x,y) = (u(x,y), v(x,y))$ is a $\mathbb{C}$-linear map $\mathbb{R}^2 \to \mathbb{R}^2$, under the natural identification $\mathbb{C} \sim \mathbb{R}^2$.

3. Examples of holomorphic functions

Begin with the most trivial case: choose a complex number c (for "constant") and define f by $f(z) = c\ \forall z \in \mathbb{C}$. (So the domain in question is $\mathbb{C}$ itself.) This is complex differentiable at every ξ, since for any $h \neq 0$

$$\frac{f(\xi+h) - f(\xi)}{h} = 0$$

So in particular

$$\lim_{h\to 0} \frac{f(\xi+h) - f(\xi)}{h} = 0$$

and hence f is complex differentiable at every point ξ, with $f'(\xi) = 0$. So f' is the zero function.

Consider next $f(z) = z$. (This is short-hand for: define f by $f(z) = z\ \forall z \in \mathbb{C}$.) For any $h \neq 0$

$$\frac{f(\xi+h) - f(\xi)}{h} = h$$

So in particular

$$\lim_{h\to 0} \frac{f(\xi+h) - f(\xi)}{h} = 1$$

and hence f is complex differentiable at every point ξ, with $f'(\xi) = 1$. So f' is the constant function, $f'(z) = 1$.

For a change, consider $f(z) = \overline{z}$. In this case, For any $h \neq 0$

$$\frac{f(\xi+h) - f(\xi)}{h} = \frac{\overline{h}}{h}$$

If h is real the RHS is 1, and if h is purely imaginary, the RHS is -1

$$\lim_{h\to 0} \frac{f(\xi+h) - f(\xi)}{h}$$

cannot exist. So f is *not* complex differentiable at any point ξ. This can be also checked using the CR equations. For, $f = u + iv$, with $u(x,y) = x$ and $v(x,y) = -1$; this yields $u_x = 1 \neq v_y = -1$.

As with the case of real-valued functions of a real variable, we can check that sums and products of holomorphic functions are holomorphic. Further, if f is holomorphic on D and nowhere zero, so is $\frac{1}{f}$, and

$$(\frac{1}{f})' = -\frac{f'}{f^2}$$

The function $f(z) = z$ is an entire function, and by the above remarks so is any polynomial p:

$$p(z) = \sum_{n=0}^{N} a_n z^n$$

with a_i complex numbers. We recall from algebra that a polynomial of degree N has at most N zeroes. If p and q are polynomials with no common zeroes, the function $z \mapsto p(z)/q(z)$ defined in the complement of the zero set of q and is holomorphic there. Such a function is called *rational*.

If $f : D \to \mathbb{C}$ and $\tilde{f} : \tilde{D} \to \mathbb{C}$ are holomorphic and $f(D) \subset \tilde{D}$, the composite function $\tilde{f} \circ f : D \to \mathbb{C}$ can be defined: $(\tilde{f} \circ f)(z) \equiv \tilde{f}(f(z))$. The function $\tilde{f} \circ f$ is holomorphic and its derivative is given by the Chain Rule:

$$(\tilde{f} \circ f)'(z) \equiv \tilde{f}'(f(z))f'(z)$$

The proofs of the above claims are left as an exercise.

Exercise 3.1. The function $z \mapsto \bar{z}$ is *not* holomorphic, nor are the functions $z = x + iy \mapsto x$, $z = x + iy \mapsto y$, $z \mapsto |z|$.

Exercise 3.2. Use the CR equations to prove that if f is a holomorphic function and $f(z) = u(x, y)$ (respectively, $f(z) = iv(x, y)$) with u (resp., v) real and C^∞, then f is constant.

Exercise 3.3. Use the CR equations to prove that if f is a holomorphic function and the image of f is contained in the diagonal $\{(u+iv|u = v\}$, then f is constant.

4. Sequences and series

A *sequence* $\{a_n\}_{n\geq 0}$ of complex numbers is a map $n \mapsto a_n$ from the set of non-negative integers to $\mathbb{C}$; by abuse of notation, we refer to the sequence $\{a_n\}$. (This notation is ambiguous because traditionally it signifies the set of values that the sequence takes.) The sequence has a limit $a \in \mathbb{C}$ iff given any $\epsilon > 0$, there exists n_0 (possibly depending on ϵ) such that $|a_n - a| < \epsilon$ for $n > n_0$. A limit, if it exists is unique, and we write:

$$\lim_n a_n = \lim_{n\to\infty} a_n = a$$

A limit exists iff the sequence is Cauchy (since $\mathbb{C}$ is a complete metric space).

In the case of a sequence $\{a_n\}$ of *nonnegative real* numbers we can also define the notion of lim sup as follows:

(1) If the sequence *is not* bounded above, we set $\limsup_{n>0} a_n = +\infty$,

(2) If the sequence *is* bounded above, we set $\limsup_{n>0} a_n = L$, where L is characterised by:
   - given any $\epsilon > 0$, $a_n \in (L - \epsilon, L + \epsilon)$ infinitely often, and
   - $a_n < L+\epsilon$ for all but finitely many n, or equivalently, there exists n_0 possibly depending on ϵ such that $a_n < L + \epsilon$ for $n > n_0$.

Note that $\limsup_{n>0} a_n = 0$ iff $\lim_n a_n = 0$. The notion of lim sup and lim inf can be defined more generally, but the above will suffice for us.

A *series* is a formal expression $\sum_n a_n \equiv \sum_{n=0}^{\infty} a_n$, with $\{a_n\}$ a sequence of complex numbers. This stands for the limit, when it exists, of the sequence of partial sums:

$$S_m \equiv \sum_{n=0}^{m} a_n$$

In other words, if $\lim_m S_m$ exists (in which case we say that the series converges), we write $\lim_m S_m = \sum_{n=0}^{\infty} a_n$. Note that

$$\lim_m S_m \text{ exists } \implies \lim_n a_n = 0$$

A *(formal) power series* is a formal expression $\sum_n a_n z^n \equiv \sum_{n=0}^{\infty} a_n z^n$, with $\{a_n\}$ a sequence of complex numbers and z a symbol, which can eventually be a variable or a complex number. The partial sum

$$S_m(z) \equiv \sum_{n=0}^{m} a_n z^n$$

will then be a polynomial in the variable z or the value of such a polynomial at a complex number.

In the rest of this section, let $\{a_n\}$ be a sequence of *non-negative real numbers*, and consider the series $\sum_n a_n r^n$; let $S_m^*(r) = \sum_{n=0}^{m} a_n r^n$. When r is also real and non-negative partial sums $S_m^*(r)$ are real and non-negative, and

$$m \geq l \implies S_m^*(r) \geq S_l^*(r)$$
$$r' \geq r \implies S_m^*(r') \geq S_m^*(r)$$

This implies that (a) for fixed (real and nonnegative) r, either the sequence $S_m^*(r)$ is unbounded or it has a limit, and (b) the set

$$\{r \text{ real and nonnegative} | \sum_n a_n r^n \text{ converges}\}$$

is an interval of the form $[0, \rho)$ or $[0, \rho]$[1]. Define the *radius of convergence* ρ of the series $\sum_n a_n r^n$ by

$$\rho = sup\{r \geq 0 | \sum_{n=0}^{\infty} a_n r^n \text{ converges}\}$$

[1]Both possibilities occur – consider $\sum_n r^n$ and $\sum_n \frac{r^n}{n^2}$. (Thanks to Mohan Swaminathan for the second example.)

(So ρ is a non-negative real number, or $+\infty$, depending on the sequence $\{a_n\}$.) For real r such that $0 \le r < \rho$, define the function S^* by:

$$S^*(r) = \lim_m S^*_m(r) \equiv \sum_{n=0}^{\infty} a_n r^n$$

Note that if only finitely many a_n are nonzero, the function S^* is differentiable and

$$S^{*\prime}(r) = \sum_{n=0}^{\infty} n a_n r^{n-1}$$

Theorem 4.1. *(Hadamard) Given a series as above,*

1. *if* $\lim_{n>0} a_n^{1/n} = 0$, *then* $\rho = \infty$,
2. *if the sequence* $a_n^{1/n}$ *is bounded, but* $a_n^{1/n}$ *does not tend to zero, then*
$$\frac{1}{\rho} = \limsup_{n>0}\ a_n^{1/n}$$
3. *if the sequence* $a_n^{1/n}$ *is* not *bounded, then* $\rho = 0$.

Proof. Suppose first that $\limsup_{n>0}\ a_n^{1/n} = L < \infty$. (Note that $L = 0$ iff $\lim_{n>0}\ a_n^{1/n} = 0$.) Let $\epsilon > 0$. We have $a_n^{1/n} < L + \epsilon$ for $n > n_0$, with n_0 depending on ϵ. Given any $r \ge 0$, set $u \equiv r(L+\epsilon)$. We have

$$a_n^{1/n} r < r(L+\epsilon) = u$$

for $n > n_0$. We have

$$\sum_n a_n r^n \le \sum_{n \le n_0} a_n r^n + \sum_n u^n$$

which is finite provided $u < 1$. This shows that $\rho = \infty$ if $L = 0$ and $\rho \ge L^{-1}$ if $L > 0$. By the definition of L, we have a subsequence $\{a_{n_i}\}$ s.t. $a_{n_i}^{1/n_i} > L - \epsilon$. So if $L > 0$ and $u' \equiv r(L-\epsilon)$, we have

$$\sum_i a_{n_i} r^{n_i} > \sum_i u'^{n_i}$$

which diverges if $u' \ge 1$. So we conclude that $\rho = L^{-1}$.

If $\limsup_{n>0}\ |a_n|^{1/n} = \infty$, let $\{a_{n_i}\}$ be a subsequence such that $a_{n_i}^{1/n_i} \ge i,\ i = 0, 1, \ldots,$. Then, given any $r > 0$,

$$a_{n_i} r^{n_i} \ge i^{n_i} r^{n_i} = (ir)^{n_i}$$

Since $ir > 1$ for $i > \frac{1}{r}$, the sum $\sum_{n=0}^{\infty} a_n r^n$ necessarily diverges. This shows that if the sequence $a_n^{1/n}$ is unbounded, then $\rho = 0$. $\square$

Corollary 4.2. *The radius of convergence of the series* $\sum_n n a_n r^{n-1}$ *is the same as that of the parent series* $\sum_n a_n r^n$.

Proof. The radius of convergence of the "derivative series" $\sum_n na_n r^{n-1}$ is clearly the same as that of $\sum_n na_n r^n$. So the claim will be proved once we prove that $\limsup_{n>0}(na_n)^{1/n} = \limsup_{n>0}(a_n)^{1/n}$. To see this, let $L = \limsup_{n>0}(a_n)^{1/n}$ and $M = \limsup_{n>0}(na_n)^{1/n}$. Let $\{a_{n_i}\}_i$ be a subsequence such that $\limsup_{n>0}(a_{n_i})^{1/n_i} = L$. Then

$$\lim_{i>0}(n_i a_{n_i})^{1/n_i} =_a \lim_{i>0}(n_i)^{1/n_i} \lim_{i>0}(a_{n_i})^{1/n_i} =_b L$$

where the equality a holds because of the continuity of the multiplication map $(x,y) \mapsto xy$, and b holds because $\lim_{n>0} n^{1/n} = 1$. This shows that $M \geq L$. Similarly one proves that $M \leq L$. □

5. Power series in z

Let $z_0 \in \mathbb{C}$. Given a real number $r > 0$, we let $D(z_0, r) \equiv \{z \in \mathbb{C} | |z - z_0| < r\}$ denote the open disc of radius r centred at z_0; we let $\overline{D}(z_0, r) \equiv \{z \in \mathbb{C} | |z - z_0| \leq r\}$ denote the closed disc of radius r centred at z_0.

A *power series centred at* z_0 is a formal expression

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n,\ n = 0, 1 \ldots$ is a sequence of complex numbers. To such a series, one attaches its partial sums

$$S_m(z) = \sum_{n=0}^{m} a_n (z - z_0)^n$$

which are polynomials, and in particular, entire functions of the variable z. We wish to associate to the power series a holomorphic function which is a limit, in a suitable sense, of these partial sums.

Definition 5.1. The *radius of convergence* ρ of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is by definition the radius of convergence of the associated series $\sum_{n=0}^{\infty} |a_n| r^n$. In other words,

$$\rho = sup\{r \geq 0 | \sum_{n=0}^{\infty} |a_n| r^n < \infty\}$$

Note that Hadamard's Theorem gives the radius of convergence in terms of the behaviour of the sequence $\{|a_n|\}$, and that for $0 \leq r < \rho$, we have $S_m^*(r) \equiv \sum_{n=0}^{m} |a_n| r^n \to S^*(r) \equiv \sum_{n=0}^{\infty} |a_n| r^n$. We start by stating and proving a basic

Proposition 5.2. *Suppose* $\rho > 0$. *The sequence of partial sums* $S_m(\xi)$ *converges for all* $\xi \in D(z_0, \rho)$, *and diverges for all* $\xi \notin \overline{D}(z_0, \rho)$.

Proof. If $|z - z_0| \equiv r < \rho$,

$$|S_{m+l}(z) - S_m(z)| \le \sum_{n=m+1}^{m+l} |a_n| r^n = S^*_{m+l}(r) - S^*_m(r)$$

By assumption the sequence $S^*_m(r) = \sum_{n=0}^{m} |a_n| r^n$ is Cauchy. The above equality then shows that so is the sequence $S_m(z)$, and convergence follows.

Conversely, if the sequence $S_m(\xi)$ converges for some $\xi \neq z_0$, clearly $a_n(\xi - z_0)^n \to 0$. Let $0 < r < |\xi - z_0|$, and set $t = \frac{r}{|\xi - z_0|}$. Then $|a_n| r^n = |a_n||\xi - z_0|^n (\frac{r}{|\xi - z_0|})^n <_* t^n$, where the inequality $<_*$ holds for all but finitely many n. Since $t < 1$, $\sum_n |a_n| r^n < \infty$ and we conclude that $\rho \ge |\xi - z_0|$. □

Theorem 5.3. *Suppose $\rho > 0$. Then there is a unique continuous function S defined on the domain $D(z_0, \rho)$ such that $S_m \to S$ uniformly on $\overline{D}(z_0, r)$ for every $0 < r < \rho$. The function S is holomorphic, and in fact $S'_m \to S'$ uniformly on the closed discs $\overline{D}(z_0, r)$.*

Proof. Define the function $S : D(z_0, \rho) \to \mathbb{C}$ by

$$S(z) = \lim_m S_m(z)$$

We need to first show that S is continuous and $S_m \to S$ uniformly on $\overline{D}(z_0, r)$ for every $0 < r < \rho$. Note that continuity of S follows once we prove the uniform convergence. To show uniform convergence, note that

$$S(z) - S_m(z) = \sum_{m+1}^{\infty} a_n (z - z_0)^n$$

But if $z \in \overline{D}(z_0, r)$,

$$|\sum_{m+1}^{m+l} a_n (z - z_0)^n| \le \sum_{m+1}^{m+l} |a_n||z - z_0|^n \le \sum_{m+1}^{m+l} |a_n| r^n = S^*_{m+l}(r) - S^*_m(r)$$

This shows that

$$|S(z) - S_m(z)| \le S^*(r) - S^*_m(r) = |S^*(r) - S^*_m(r)|$$

provided $z \in \overline{D}(z_0, r)$, proving the uniform convergence.

Consider next the series $\sum_n n a_n z^{n-1}$, with partial sums $S'_m(z)$. We already know that this has the same radius of convergence. Let g denote the limit, which exists (by what we have already proved in the previous paragraph) on $D(z_0, \rho)$. We claim that S is complex differentiable at each point $\xi \in D(z_0, \rho)$, and $S'(\xi) = g(\xi)$. This follows from general arguments (see below) but we give a direct proof, following [GIR]. Choose $0 < r < \rho$ such that $\xi \in D(z_0, r)$, and $h \neq 0$ be such that $\xi + h \in D(z_0, r)$ as well. Set $\xi - z_0 = \tilde{\xi}$, and set $\Delta(h) = \frac{S(\xi+h)-S(\xi)}{h} - g(\xi)$. We have

$$
\begin{aligned}
\Delta(h) &= \sum_n a_n \{ \frac{(\tilde{\xi}+h)^n - \tilde{\xi}^n}{h} - n\tilde{\xi}^{n-1} \} \\
&= \sum_n a_n \{ \sum_{l=1}^{n-1} (\tilde{\xi}+h)^l \tilde{\xi}^{n-1-l} - n\tilde{\xi}^{n-1} \}
\end{aligned} \tag{1}
$$

which yields

$$|\Delta(h)| \le \sum_n 2n|a_n| r^{n-1} < \infty$$

Let $\epsilon > 0$ be given. Choose N such that $\sum_{n>N} 2n|a_n|r^{n-1} < \epsilon/2$. Again using the computation (1), we get

$$|\Delta(h)| \le \sum_{n\le N} |a_n| | \sum_{l=1}^{n-1} |(\tilde{\xi}+h)^l \tilde{\xi}^{n-1-l} - \tilde{\xi}^{n-1}| + \sum_{n>N} 2n|a_n|r^{n-1}$$

The first (finite) sum is a continuous function of h that vanishes when $h = 0$; choose h small enough that this is less than $\epsilon/2$; this achieves

$$|\Delta(h)| < \epsilon$$

□

The above Theorem can be rephrased and extended as follows:

Theorem 5.4. *Suppose given a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with radius of convergence $\rho > 0$. Define the function S on $D(z_0, \rho)$ by*

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

This is a holomorphic function, and

$$S'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$$

where the series on the right has radius of convergence ρ. By induction, S is infinitely (complex) differentiable on $D(z_0,\rho)$ and

$$S^{(n)}(z_0) = n! a_n$$

In particular the coefficients a_n are determined by the function S.

6. Postponed business

6.1. **Uniform limits.** Let f_n be a sequence of continuous complex-valued functions on a metric space X. We say that f_n tends uniformly to a function f if given $\tilde{\delta} > 0$, there is $n_0(\tilde{\delta})$ such that for all $x \in X$, we have $|f(x) - f_n(x)| < \tilde{\delta}$ provided $n > n_0(\tilde{\delta})$. It is a fact that such a "uniform limit of continuous functions is continuous". Proof: Let $x_0 \in X$, and let $\epsilon > 0$ be given. For $n > n_0(\epsilon/3)$, we have $|f(x) - f_n(x)| < \epsilon/3 \ \forall x \in X$. Fix such an n; then by continuity of f_n we have $\delta > 0$ such that $|f_n(x) - f(x_0)| < \epsilon/3$ for $d(x, x_0) < \delta$, where d is the distance function on $X \times X$. Then, provided $d(x, x_0) < \delta$, we also have

$$\begin{aligned}|f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon\end{aligned}$$

6.2. **Uniform C^1 limits.** Suppose now that $X = U$, an open set in $\mathbb{R}^n$, and f_n a sequence of C^1 ($\mathbb{R}^m$-valued) functions converging uniformly on compact subsets of U to a function f, and suppose further that the derivatives f'_n converge uniformly on compact subsets of U to a continuous function g from U to the space of $m \times n$ matrices. Then f is C^1 as well, and $f' = g$.

Proof. The proof uses the following version of the mean-value theorem: suppose f is a differentiable on U (notation as above), $x, y \in U$ and the line segment $\{ty + (1-t)x | t \in [0,1]\}$ is contained in U. Then

$$||f(y) - f(x)|| \le ||y - x|| \sup_{t\in[0,1]} ||f'(ty + (1-t)x)||$$

Here we have fixed norms on $\mathbb{R}^m$ and $\mathbb{R}^n$ and the norm on $m \times n$-matrices is the appropriate operator norm.

Fix x, and let c be any $m \times n$ matrix. Applying the above inequality to the function $y \mapsto \tilde{f}(y) = f(y) - c(y - x)$ yields

$$||f(y) - f(x) - c(y - x)|| \le ||y - x|| \sup_{t\in[0,1]} ||f'(ty + (1-t)x) - c||$$

Suppose now that $f_n \to f$ and $f'_n \to g$. Fix x as above, and consider the functions $y \mapsto \tilde{f}_n(y) = f(y) - g(x)(y - x)$. We have

$$||f_n(y) - f_n(x) - g(x)(y - x)|| \le ||y - x|| \sup_{t\in[0,1]} ||f'_n(ty + (1-t)x) - g(x)||$$

If $y \ne x$, we get

$$\begin{aligned}\frac{||f_n(y) - f_n(x) - g(x)(y - x)||}{||y - x||} &\le \sup_{t\in[0,1]} ||f'_n(ty + (1-t)x) - g(x)| \\ &\le \sup_{t\in[0,1]} ||f'_n(ty + (1-t)x) - g(ty + (1-t)x)| \\ &+ \sup_{t\in[0,1]} ||g(ty + (1-t)x) - g(x)|\end{aligned}$$

As $n \to \infty$ $f'_n \to g$ uniformly on compacts so we get

$$\frac{||f(y) - f(x) - g(x)(y - x)||}{||y - x||} \le \sup_{t\in[0,1]} ||g(ty + (1-t)x) - g(x)|$$

which shows, since g is continuous, that the LHS goes to zero as y tends to x. $\square$

7. Analytic Functions

Consider, for the moment, a function $f : I \to \mathbb{C}$ where $I \subset \mathbb{R}$ is a (non-empty) open interval. We say that f is *real-analytic* on I, if given $x_0 \in I$, it is the uniform limit, in some sub-interval $[x_0 - \epsilon, x_0 + \epsilon] \subset I$ (with $\epsilon > 0$) of a power series centred at x_0:

$$\sum_n a_n (x - x_0)^n$$

We can imitate the reasoning above to see that f is C^∞ and

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Given any C^∞ function f on an interval I, and $x_0 \in I$, the "Taylor series" of f at x_0 is the formal series:

$$\sum_n \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

An alternate definition of a real analytic function is as a C^∞ function f whose Taylor series at any point has a nonzero radius of convergence, within which it converges to f.

Consider next a connected open set $U \subset \mathbb{R}^l$, and a complex-valued function f defined on it. As above, we say that f is *real-analytic* on U, if given $x_0 \in U$, it is the uniform limit, in some closed ball $\{x \in U \| \|x - x_0)\| < \epsilon\} \subset U$ (with $\epsilon > 0$) of a power series centred at x_0:

$$\sum_\alpha a_\alpha (x - x_0)^\alpha$$

We can imitate the reasoning above to see that f is C^∞ and

$$a_\alpha = \frac{f^{(\alpha)}(z_0)}{n!}$$

We have used the following notation above. $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a multi-index (the α_i are non-negative integers), and

$$x^\alpha = x_1^{\alpha_1} \times \cdots \times x_l^{\alpha_l}$$
$$f^{(\alpha)} = D_1^{\alpha_1} \ldots D_l^{\alpha_l} f$$
$$\alpha! = \alpha_1! \times \cdots \times \alpha_l!$$

and $D_1 = \frac{\partial}{\partial x_1}$, etc.

Definition 7.1. Given a domain D, a function $f : D \to \mathbb{C}$ is said to be *(complex-)analytic* if given $z_0 \in D$, it is the uniform limit, in some closed disc $\overline{D}(z_0, \epsilon) \subset D$ (with $\epsilon > 0$), of a power series centred at x_0:

$$\sum_n a_n (z - z_0)^n$$

We have seen that an analytic function is holomorphic, and in fact infinitely (complex) differentiable.

Consequence A of Cauchy's Theorem: If $f : D \to \mathbb{C}$ is holomorphic (i.e., complex-differentiable at every point in D), then f is (complex-)analytic on D. In fact, given any point $z_0 \in D$ the "Taylor series" of f at z_0:

$$\sum_n \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

has a nonzero radius of convergence ρ, and the sum represents f in a neighbourhood of z_0. **Furthermore, if** $D(z_0, r) \subset D$**, then** $\rho > r$. In particular, "holomorphic" = "(complex-)analytic". As easy corollaries:

(1) f is infinitely complex differentiable.

(2) u and v are real-analytic. (Exercise)

From now on we use "holomorphic" and "analytic" interchangeably. This is in contrast to [GIR] who are careful to distinguish the two notions until Cauchy's Theorem has been proved.

A number of facts follow as a consequence of the fact that holomorphic functions have convergent power series representations. In what follows, let f be a holomorphic function on a (connected!) domain D.

Corollary 7.2. *If all derivatives of f vanish at $z_0 \in D$, then $f \equiv 0$.*

Proof. Consider the $D' \subset D$ where all derivatives of f vanish. This is closed, being the intersection of closed sets. On the other hand, given $z_0 \in D'$, the Taylor series centred at z_0 vanishes, so the function itself vanishes in a neighbourhood. So D' is also open. Since D is assumed connected, $D = D'$ □

Corollary 7.3. *If f vanishes on a nonempty open subset $U \subset D$, then $f \equiv 0$. Two holomorphic functions which coincide on a nonempty open subset U coincide everywhere.*

Proposition 7.4. *Suppose $f^{(n)}(z_0) = 0$ for $n = 0, \ldots, n_0$ for some $n_0 \geq 0$ and $f^{(n_0+1)}(z_0) \neq 0$ ("f vanishes up to order n_0''). There exists a unique holomorphic function g on D, nonvanishing at z_0, such that*

$$f(z) = (z - z_0)^{n_0+1} g(z)$$

Proof. The Taylor series of f is of the form

$$\sum_{n=n_0+1} a_n (z - z_0)^n$$

Consider the series $\sum_{n>n_0} a_n (z - z_0)^{n-n_0-1}$. This has the same radius of convergence; let $\tilde{g}$ denote the limit, which is defined in an open disc $D(z_0, \rho)$.

This need not be contained in D, but $D \cap D(z_0, \rho)$ is a nonempty open set, which therefore contains an open disc D_1 centred at z_0. We claim that

$$f(z) = (z - z_0)^{n_0+1}\tilde{g}(z)$$

on D_1. (Both sides have the same derivatives, to all orders, at z_0.) Define the function g on D by

$$g(z) = \begin{cases} f(z)/(z - z_0)^{n_0+1} & \text{if } z \neq z_0 \\ \tilde{g}(z_0) & \text{if } z = z_0 \end{cases}$$

This is holomorphic on D, and does the job. □

Note that $g(z_0) \neq 0$, so that by continuity g is non-vanishing in a neighbourhood of z. This yields

Corollary 7.5. *If f is not identically zero and $f(z_0) = 0$ for some $z_0 \in \mathbb{C}$, then there is $\epsilon > 0$ such that $D(z_0, \epsilon) \subset D$ and $f(z) \neq 0$ for $0 < |z - z_0| < \epsilon$. ("The zeroes of a nonzero holomorphic function are isolated.") In other words, if $\{z_n\}_n$ is a sequence of distinct points in D tending to $z \in D$ and $f(z_n) = 0$ for all n, then $f \equiv 0$.*

Here is a very important fact about nonconstant holomorphic maps which uses the above discussion crucially.

Theorem 7.6. (The Open Mapping Theorem.) *Let D be a domain (in particular, connected). If $f : D \to \mathbb{C}$ is holomorphic and not constant, then f is an open map. That is, if $U \subset D$ is open, so is $f(U)$. In particular $f(D)$ is a domain.*

Proof. Let $z_0 \in D$, and set $w_0 = f(z_0)$. By considering the function $z \mapsto f(z+z_0) - w_0$, we can assume $z_0 = 0$ and $w_0 = 0$. We need to show that there exists $\epsilon > 0$ such that if $|w| < \epsilon$ we have $w = f(z)$ for some $z \in D$. If $f'(0) \neq 0$, this follows from the Inverse Function Theorem from multi-variable calculus. If $f'(z_0) = 0$ we have by Proposition 7.4, that $f(z) = z^n g(z)$ for $n > 0$ and some holomorphic g, with $g(0) \neq 0$. By the next lemma there exists a holomorphic function h defined in a neighbourhood of 0, such that $g(z) = h(z)^n$. The function f can be written, in this neighbourhood, as a composition

$$z \mapsto zh(z) \mapsto z^n h(z)^n$$

The first map is open by the inverse function theorem, and the second one is clearly open. (See the remark following the proof of the next Lemma.)

□

Lemma 7.7. *Let D be an open disc centred at $z_0 \in \mathbb{C}$, and $g : D \to \mathbb{C}$ a nowhere vanishing continuous (respectively, holomorphic) function and n a positive integer. Then there exists an open subdisc D' centred at z_0 and* *a continuous (respectively, holomorphic) function h such that $g = h^n$, i.e., $g(z) = h(z)^n$, $z \in D'$*[2]*.*

Proof. As a preliminary, consider the map $P_n : \mathbb{C} \to \mathbb{C}$:

$$P_n(u) = u^n$$

This takes the origin to itself. Given any nonzero $w_0 \in \mathbb{C}$, it has n pre-images. Consider the open set $U = \{r(\cos\theta + i\sin\theta) | 1 - \epsilon_1 < r < 1 + \epsilon_1, |\theta| < \epsilon_2\}$. Check by hand that for ϵ small enough (depending on n) (a) the inverse image of $w_0 U$ under P_n consists of n disjoint connected open sets $u_0 U'$ (for U' of the same form as U), each containing a unique n^{th} root u_0 of w_0, and (b) if we fix any one of these open sets, say $u_0 U'$, the map $P_n : u_0 U' \to w_0 U$ is biholomorphic. That is to say, $P_n|_{u_0 U'}$ is bijective with holomorphic inverse; let Q denote the inverse.

Set $g(z_0) = w_0$. Since g is continuous, $g^{-1}(w_0 U)$ is an open set, and it contains z_0. Let D be an open disc centred at z_0 and contained in $g^{-1}(w_0 U)$. On D, set $h = Q \circ g$. Clearly h is continuous (resp., holomorphic) if g is. Further, if $z \in D$,

$$h(z)^n = Q(g(z))^n = P_n(Q(g(z))) = g(z)$$

$\square$

Remark 7.8. The map P_n is open. Proof: Let $\tilde{U} \in \mathbb{C}$ be open, and $u_0 U' \subset \tilde{U}$ be an open set containing u_0 and contained in $\tilde{U}$, with U' as above. Then $P_n(\tilde{U})$ contains $u_0^n P_n(U')$. But $P_n(U')$ is clearly an open subset of the form U above.

8. Bonus: inverse function theorem

The proof that follows is borrowed from Hörmander, and will work in the multivariable context, with minimal changes. *All discs below are centred on the origin.*

Theorem 8.1. *Let $f : D \to \mathbb{C}$ be a holomorphic function on an open disc D, and assume that $f(0) = 0, f'(0) = 1$. Then there exists an open disc $\tilde{D} \subset D$ such that $f|_{\tilde{D}}$ is injective and $U \equiv f(\tilde{D})$ is open. If $g : U \to \tilde{D}$ is the inverse function to $f|_{\tilde{D}}$, g is holomorphic*[3]*.*

Proof. By the analyticity of f, f' is continuous. Let $\delta > 0$ such that $|f'(z) - 1| < 1/2$ for $|z| \leq \delta$. (For future use, note that this implies $|f'(z)| > 1/2$.)

[2] In fact this holds without the need to shrink D, but we will not need this.

[3] Of course $g(0) = 0$ and (by the Chain Rule) $g'(0) = 1$.

Given z, z' with $|z| \leq \delta$ and $|z'| \leq \delta$, we have (by the MVT applied to $z' \mapsto f(z') - z'$)

$$\begin{aligned} |f(z') - f(z) - (z' - z)|| &\leq |z' - z| \sup_{t \in [0,1]} |f'(tz' + (1-t)z) - 1| \\ &\leq 1/2|z' - z| \end{aligned}$$

so that f is injective on $\{z||z| \leq \delta\}$.

Let w be such that $|w| < \delta/2$. Set $z_0 = w$, and $z_1, \dots,$ inductively by

$$z_{k+1} = z_k + w - f(z_k)$$

as long as $|z_k| < \delta$. Note that $z_1 = w$ and for that $k \geq 1$

$$|z_{k+1} - z_k| = |z_k - z_{k-1} - f(z_k) - f(z_{k-1})| \leq \frac{1}{2}|z_k - z_{k-1}|$$

This ensures that

$$|z_{k+1}| \leq |z_1| + |z_k - z_{k-1}| + |z_{k+1} - z_k| \leq (1 + \dots \frac{1}{2^k})|w| < 2|w|$$

so that indeed the sequence is well-defined for all k. It is a Cauchy sequence (check); let $z \in \tilde{D}_1$ denote the limit. Note that $|z| \leq 2|w| < \delta$. Clearly $z = z + w - f(z)$. In other words we have found a unique z, with $|z| < \delta$ such that

$$w = f(z)$$

In particular, $f(D)$ contains the open disc of radius $\delta/2$.

Define $g : \{w||w| < \delta/2\} \to \mathbb{C}$ by $g(w) = z$, with $f(z) = w$ and $|z| \leq \delta$. This function is *a priori* not even continuous, but $f(g(w)) = w$, so by the Chain Rule, if g were complex differentiable at w, we would have $g'(w) = \frac{1}{f'(z)}$. Set $f(z) = w$ and $f(z+h) = w+k$. By the MVT we have $|k| \leq \frac{3}{2}|h|$. By the MVT applied to $z' \mapsto f(z') - z'$, we have $|k-h| \leq \frac{1}{2}|h|$. Together, these inequalities yield:

$$\frac{1}{2}|h| \leq |k| \leq \frac{3}{2}|h|$$

Now

$$|\frac{h}{k} - \frac{1}{f'(z)}| = |\frac{h}{kf'(z)}||f'(z) - k/h| \leq 4|f'(z) - k/h|$$

Since $h \to 0$ as $k \to 0$ and $|f'(z) - k/h| \to 0$ as $h \to 0$, the above inequality shows that $|\frac{h}{k} - \frac{1}{f'(z)}| \to 0$ as $k \to 0$.

To summarise what we have achieved so far: there exists $\delta > 0$ such that $f|_{\{z||z| \leq \delta\}}$ is injective, the image contains the open disc $\{w||w| < \delta/2\}$, and the inverse map $g : \{w||w| < \delta/2\} \to \{z||z| < \delta\}$ is C^1. Take $\tilde{D} = f^{-1}(\{w||w| < \delta/2\}) \cap \{z||z| < \delta\}$. $\square$

9. The exponential function, trigonometric functions, π

9.1. **The exponential function.** We look for an entire holomorphic function f, satisfying $f(0) = 1$ and the differential equation:

$$f' = f$$

Assuming "holomorphic=analytic", f given by a convergent power series in a neighbourhood of the origin. It follows by induction that

$$f^{(n)}(0) = 1$$

so that f is given by the power series

$$f(z) = \sum_n \frac{1}{n!} z^n$$

with infinite radius of convergence since $\lim_{n>0} \frac{1}{(n!)^{1/n}} = 0$. We define this entire function to be the exponential function:

$$z \mapsto \exp z$$

We will see in a minute that $\exp z$ does not vanish at any point. If g is any entire function satisfying the same differential equation and initial condition

$$\left(\frac{g(z)}{\exp z}\right)' = \frac{g'(z)\exp z - g(z)\exp z}{\exp^2 z} = 0$$

so that $g = \exp$. We set $e = \exp 1$.

We will now develop a number of important properties of this function:

(1) $\exp z \exp -z = 1$. Proof: By differentiating, we see that $(\exp z \exp -z)' = 0$. So the function is constant. Evaluating at 0, we see that this constant is 1.

(2) In particular exp is nowhere vanishing. Since exp is clearly real-valued on the x axis and $\exp(0) = 1$, we see that $\exp x > 0$ for x real.

(3) $\exp z + \tilde{z} = \exp z \exp \tilde{z}$, $z.\tilde{z} \in \mathbb{C}$. Proof: Treat $\tilde{z}$ as constant. Both sides satisfy the same differential equation in z, and have the same initial value at 1, and both sides are nowhere-vanishing. Now use the uniqueness argument above.

(4) $\overline{\exp z} = \exp \overline{z}$. Proof: This follows from the power series representation. Here's another argument. Check that for any holomorphic function $f : D \to \mathbb{C}$, the function $\tilde{f} : \overline{D} \to \mathbb{C}$, defined by

$$\tilde{f}(z) = \overline{f(\overline{z})}$$

is holomorphic, with $\overline{D}$ suitably defined. Check that $\widetilde{\exp}$ satisfies the same differential equation and initial condition as exp.

(5) Restricted to the real axis, exp is a smooth (even real-analytic) map $(-\infty, +\infty) \to (0, +\infty)$. Since $\exp' x = \exp x > 0$, the function $\exp x$ is a (real-valued and) strictly increasing function of x, for x real. Clearly $\exp x \to \infty$ as $x \to \infty$, so $\exp x \to 0$ as $x \to -\infty$. Thus the map is surjective, and since the derivative is everywhere positive, the function is strictly increasing and therefore a bijection. By the same token, the inverse map log is C^∞, and in fact real-analytic. (This will follow from what we will see below.)

9.2. **Trigonometric functions of z.** We define entire functions cos and sin by

$$\cos z = \frac{\exp iz + \exp -iz}{2}$$
$$\sin z = \frac{\exp iz - \exp -iz}{2i}$$

From the definition it follows that

(1) the functions cos and sin have power series representations (with infinite radius of convergence)

$$\cos z = \sum_{n\geq 0} (-1)^n \frac{1}{(2n)!} z^{2n}$$
$$\sin z = \sum_{n\geq 0} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}$$

(2) cos and sin are solutions of the differential equation:

$$f'' + f = 0$$

(In fact they span the two-dimensional space of solutions.)

(3) cos and sin obey the addition formulae

$$\cos(z + \tilde{z}) = \cos z \cos \tilde{z} - \sin z \sin \tilde{z}$$
$$\sin(z + \tilde{z}) = \sin z \cos \tilde{z} + \cos z \sin \tilde{z}$$

(4) $\cos^2 z + \sin^2 z = 1$.

(5) $\exp iz = \cos z + i \sin z$. Note that cos and sin are (entire) holomorphic functions on their own right; this is not the "$u + iv$" decomposition of exp.

9.3. **Trigonometric functions of x, definition of π, periodicity of** exp**, (mostly following Rudin's *Real and Complex Analysis*).** When we specialise to the real-axis, we recover the "old" trigonometric functions, see (9) below.

(6) cos and sin take real values on the x-axis; these are real-analytic functions.

(7) if $z = x + iy$, with x, y real, then

(2) $$\exp z = \exp x \cos y + i \exp x \sin y$$

and this *is* the "$u + iv$" decomposition of exp.

(8) In particular, with θ real, $\exp i\theta = \cos\theta + i \sin\theta$.(The switch from y to θ is because we are respecting centuries of tradition.)

(9) That cos and sin agree with trigonometric functions defined in trigonometry follows from the power series representations. Alternatively, we argue as follows:

Define $\mathcal{P} \subset \mathbb{R}$ by

$$\mathcal{P} = \{\alpha | \exp i(\theta + \alpha) = \exp i\theta \ \forall \theta \in \mathbb{R}\}$$

This is the set of *periods* of the function $\theta \mapsto \exp i\theta$. From the definition it follows that $\mathcal{P}$ is an additive subgroup of $\mathbb{R}$. We have $\alpha \in \mathcal{P} \iff \exp i\alpha = 1 \iff \cos\alpha = 1$ *and* $\sin\alpha = 1 \iff \cos\alpha = 1$ so

$$\mathcal{P} = \{\alpha | \cos\alpha = 1\}$$

We will show that the set $\{x \in \mathbb{R} | \cos x = 0, x > 0\}$ is nonempty. Granted this, let x_0 be the smallest element of this set, and set $\pi = 2x_0$. We have $\sin x_0 = \pm 1$; on the other hand, $\sin 0 = 0$ and $\sin' x = \cos x > 0$ on $[0, x_0)$ so $\sin x_0 = 1$. (For future use remark that since $\sin x$ increases from 0 to 1in the range $[0, 1]$, $\sin x > 0$ on $(0, 1)$.) This yields

$$\exp \pi i/2 = i$$

which yields (squaring both sides) the wonderful Euler's identity:

$$\exp \pi i + 1 = 0$$

Proceeding, we also get

$$\exp 2\pi i = 1$$

and therefore $\mathbb{Z}2\pi \subset \mathcal{P}$. To show that this is in fact an equality, we need to show that if if $\exp ix = 1$ then x is an integral multiple of 2π. We will in fact prove that

$$\exp ix \neq 1 \text{ for } 0 < x < 2\pi$$

To see this, set $\exp ix/4 = \cos x/4 + i \sin x/4$. We know that $\cos x/4 > 0$ and $\sin x/4 > 0$ on the interval $(0, 2\pi)$. For simplicity of notation, set $u = \cos x/4$ and $v = \sin x/4$. Then

$$\exp ix = (u + iv)^4 = u^4 - 6u^2v^2 + v^4 + 4i(-v^2 + u^2)uv$$

The RHS is real iff $u^2 = v^2$. Since $u^2 + v^2 = 1$, this forces $u^4 - 6u^2v^2 + v^4 = -4(1/4)^2 = -1 \neq 1$.

It remains to show that $\cos x$ vanishes at a positive value of x. Consider the power series for cos:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} +$$

When $x = 2$ the first three terms add up to $-1/3$. The succeeding terms alternate in sign and are strictly decreasing in magnitude. This shows that $\cos 2 < -1/3$. By continuity, there is a point $0 < x < 2$ where $\cos x$ vanishes. Recall that this is the point we have denoted by $\pi/2$.

Consider the functions $\theta \mapsto \sin\theta$ and $\theta \mapsto \cos\theta$, with θ real. To summarise what we have established.

- cos and sin are real valued functions of the real variable θ, periodic with period 2π, with $\pi/2$ being the smallest positive real number where cos vanishes.
- $\sin 0 = 0$ and sin increases from 0 to 1 as θ increases from 0 to $\pi/2$. $\cos 0 = 1$ and cos decreases from 1 to 0 in the same interval.

To complete the identification with standard trigonometric functions, consider the map[4] $z \mapsto \mathrm{iExp}\,(z) \equiv \exp iz$ from $\mathbb{C}$ to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This is clearly a group homomorphism:

$$\mathrm{iExp}\,(z + \tilde{z}) = \mathrm{iExp}\,(z)\mathrm{iExp}\,(\tilde{z}) \tag{3}$$

Restricted to the real axis, this maps to the unit circle $\mathbb{U} = \{w \in \mathbb{C} | |w| = 1\}$. Using (3), it is easy to show that the map iExp is injective on any interval $[\theta, \hat{\theta})$ provided $\hat{\theta} \le \theta + 2\pi$. Given $\theta \le 2\pi$, the length of the curve iExp from 1 to $\mathrm{iExp}\,\theta$ is

$$\int_0^\theta |\frac{d\mathrm{iExp}\,(t)}{dt}| dt = \int_\theta^{\hat{\theta}} |i\mathrm{iExp}\,(t)| dt = \theta$$

which completes the agreement with the geometric definitions.

We list some further properties of the function iExp.

(1) $\mathrm{iExp}\, z = \cos z + i \sin z$.

(2) iExp is periodic, with period 2π. That is $\mathrm{iExp}\,(z + c) = \mathrm{iExp}\, z \ \forall z$ iff $c \in 2\pi\mathbb{Z}$.

(3) Exercise: Figure out what the next sentence means, and prove the claim. "The map $\mathrm{iExp} : \mathbb{R} \to \mathbb{U}$ induces a isomorphism of groups $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{U}$, which becomes a homeomorphism of topological spaces when the first set is endowed with the quotient topology. "

(4) The map $\mathrm{iExp} : \mathbb{C} \to \mathbb{C}^*,\ z \mapsto \exp iz$ is surjective. To see this, note that

$$\exp i(x + iy) = \exp ix \exp -y$$

and then use the preceding discussion.

[4] ©Apple

(5) The map iExp induces an homeomorphism $\mathbb{C}/2\pi\mathbb{Z} \to \mathbb{C}^*$

Remark 9.1. "Polar coordinates": From the above discussion it follows that the map

$$\begin{aligned} \mathbb{R}^* \times \mathbb{R} \to \mathbb{R}^* \times \mathbb{U} \quad \sim \quad \mathbb{C}^* \\ (r, \theta) \mapsto (r, \exp i\theta) \mapsto r \exp i\theta \end{aligned}$$

is surjective (and a homomorphism of abelian groups). The variable r is actually a function on $\mathbb{C}^*$ (the radial distance from the origin in $\mathbb{C}$), but θ is a "multi-valued function".

9.4. **Exp and Log.** Recall that the real logarithm is the inverse function to the bijective map

$$\exp : \mathbb{R} \to \mathbb{R}_{>0}$$

We will be concerned with extending log to a function $\widetilde{\log} : D \to \mathbb{C}$, where D is some domain containing $\mathbb{R}_{>0}$, such that

(1) $\widetilde{\log}|_{\mathbb{R}} = \log$

(2) $\widetilde{\log}$ is holomorphic, and

(3) $\exp \circ \widetilde{\log} = Id|_D$.

Any such extension is called a "branch" of the (complex) logarithm. Note that since $\log x \to -\infty$ as $x \to 0+$, there is no hope of finding a D that contains the origin.

One such extension, called the "principal branch" of the logarithm, is entirely determined by the choice:

$$D = \mathbb{C} \setminus (-\infty, 0]$$

For, once D is such defined, it is clear that $\exp : \{z = x + iy | -\pi < y < \pi\} \to D$ is a bijective holomorphic map, so all that remains is to show that the inverse map, which we define to be $\widetilde{\log}$, is holomorphic. This is left as an exercise. There are other domains that will do as well, among them, the "slit domains"

$$D_u = \mathbb{C} \setminus \{ru | 0 \le r < \infty\}$$

with $u \in \mathbb{U}, u \neq 1$.

From now on, we set log to be $\widetilde{\log}$ as defined above.

9.5. **Power series for *log* centred at 1.** Differentiating the identity $\exp \circ \log(z) = z$, we get

$$\exp(\log(z)) \log' z = 1$$

which yields

$$\log' z = \frac{1}{z}$$

Using this, one can inductively compute the derivatives of log, to get

$$\log^{(n+1)}(z) = \frac{(-1)^n n!}{z^n},\ n \geq 0$$

Evaluating at $z = 1$, we get

$$\log^{(n+1)}(1) = (-1)^n n!,\ n \geq 0$$

So the Taylor series of log at 1 is (remembering that $\log 1 = 0$):

$$\begin{aligned} \log z &= \sum_{n>0} \frac{\log^{(n)}(1)}{n!}(z-1)^n \\ &= \sum_{n\geq 0} \frac{(-1)^n}{n+1}(z-1)^{n+1} \\ &= (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \dots \end{aligned}$$

It is usual to set $z = 1 - \tilde{z}$, and write

$$\log(1-\tilde{z}) = \tilde{z} + \frac{1}{2}\tilde{z}^2 + \frac{1}{3}\tilde{z}^3 + \dots$$

The power series

$$(z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \dots$$

converges in the open disc $\{z||z-1| < 1\}$ since $\lim_n(\frac{1}{2})^{\frac{1}{n}} = 1$. This is as it should be, since the domain corresponding to the principal branch contains this open disc.

9.6. **Fractional (and worse) powers of z.** Once a branch of log is chosen, with domain D, we can define the functions

$$z \mapsto z^a$$

with a an arbitrary complex number, by

$$z^a \equiv \exp(a \log z)$$

This is defined on the same domain D, and

(1) $z^a z^b = z^{a+b}$,

(2) $\log z^a = a \log z$ provided $z \in D$ and $z^a \in D$,

(3) $e^a = \exp a$, and

(4) $(z^a)' = a z^{a-1}$.

10. Conformal mappings

Let $D \in \mathbb{R}^2$ be a domain and $\phi : D \to \mathbb{R}^2$ a C^1 map. Recall that the derivative of ϕ at a point $w \in \mathbb{R}^2$ is a linear map $\phi'(w) : \mathbb{R}^2 \to \mathbb{R}^2$. Assume that $\phi'(w)$ is invertible at every v. Note that this implies (by the inverse function theorem) that ϕ is a local C^1 diffeomorphism. We say that

(1) ϕ is *orientation preserving* if *det* $\phi'(w) > 0$ at every w.

(2) ϕ is an *isometry* if $\phi'(w)$ is an orthogonal transformation at every w. (That is, $|\phi'(w)[v]|^2 = |v|^2$ for every vector v.)

(3) ϕ is a *conformal map* if $\phi'(w)$ is a conformal transformation at every w. That is (and check that this is equivalent to demanding that $\phi'(w)$ preserves angles), $|\phi'(w)[v]|^2 = \lambda(w)|v|^2$ for every vector v, with $w \mapsto \lambda(w)$ a function with values in the positive reals.

Proposition 10.1. *A map ϕ as above is holomorphic iff it is conformal and orientation preserving.*

The proof is an exercise. Also as an exercise, check by hand that exp is a conformal map. Give an example of a conformal map that is not holomorphic.

11. Line integrals, preparation for the Theorem of Cauchy

Until further notice, "holomorphic in a domain" will mean complex differentiable in that domain. We will not assume "holomorphic $\implies$ analytic".

In this and subsequent sections we will follow the book of Stein and Shakarchi quite closely, even to the extent of copying a diagram.

As above, let D be a domain. By a *parametrised piecewise-C^1 curve* in D, we will mean a continuous map $\gamma : [a,b] \to D$, with the following additional property. There exist finitely many real numbers $a = t_0 < t_1 < t_2 \cdots < t_n < t_{n+1} = b$ such that each $\gamma|_{[t_i,t_{i+1}]}$ is a C^1-map. (This means that the derivatives of the coordinate functions, pulled back to the interval $[t_i, t_{i+1}]$, are differentiable there and the derivatives are continuous; at the end-points we mean by derivative the suitable left/right derivative.)

Two parametrised curves $\gamma : [a,b] \to D$ and $\tilde{\gamma} : [c,d] \to D$ are *equivalent* if there is a C^1-bijection $[c,d] \to [a,b]$, $\; s \mapsto t(s), \; c \mapsto a, \; d \mapsto a$ such that $t'(s) > 0$ and such that

$$\gamma(t(s)) = \tilde{\gamma}(s)$$

An *oriented curve* Γ is a parametrised curve γ as above, up to the equivalence ("reparametrisation") also defined above. If γ is injective, one can identify

Γ with its image, together with an orientation. We do not want to make this precise, and keep the image for heuristics.

The points $\gamma(a), \gamma(b) \in D$ are independent of the parametrisation and are called the end-points; we say that the curve Γ goes from $\gamma(a)$ to $\gamma(b)$. The curve Γ is *closed* if $\gamma(a) = \gamma(b)$; it is *simple* if γ is injective, except possibly for the end-points. By a *contour* we mean (usually) a simple, closed curve upto reparametirsation.

Given Γ, we define another oriented curve $\tilde{\Gamma}$ by reversing the orientation. More precisely, given a parametrised oriented curve γ (as above) representing Γ, we define the "time-reversed" curve $\tilde{\gamma}$ by

$$\tilde{\gamma} : [a,b] \to D, t \mapsto \gamma(b+a-t)$$

and then $\tilde{\gamma}$ represents $\tilde{\Gamma}$. (Check that this is well-defined, independent of the choice of γ. Note that $\tilde{\Gamma}$ goes from the end-point of Γ to its starting point.)

Let f be a continuous complex-valued function on D. We define the *line-integral* $\int_\Gamma f(z)dz$ by

$$\int_\Gamma f(z)dz = \sum_{i=0}^{i=n} \int_{t_i}^{t_{i+1}} f(\gamma(t)) \frac{d\gamma(t)}{dt} dt$$

This is a good definition, because the RHS is invariant under reparametrisation. When γ and γ' are both C^1 (and not just piece-wise C^1) this amounts to:

$$\int_a^b f(\gamma(s)) \frac{d\gamma(s)}{ds} ds = \int_c^d f(\tilde{\gamma}(t)) \frac{d\gamma(t(s))}{dt} dt$$

which is true by the Chain Rule:

$$\frac{d\gamma(t(s))}{ds} = \frac{d\tilde{\gamma}(t)}{dt} \frac{dt(s)}{ds}$$

and because $\frac{d\gamma(t(s))}{dt} > 0$. Check that

$$\int_{\tilde{\Gamma}} f(z)dz = -\int_\Gamma f(z)dz$$

and

$$\int_\Gamma (\alpha f + \beta g)(z)dz = \alpha \int_\Gamma f(z)dz + \beta \int_\Gamma g(z)dz$$

if f, g are continuous functions and α, β complex constants. Finally

$$|\int_{\tilde{\Gamma}} f(z)dz| \le \sup_{t \in [a,b]} |f(\gamma(t))| \; length(\Gamma)$$

where $length(\Gamma)$ is defined by

$$length(\Gamma) = \int_a^b |\frac{d\gamma(t)}{dt}| dt$$

Check that this is independent of parametrisation, and $length(\Gamma) = length(\tilde{\Gamma})$.

Let us compute some line integrals of functions on $\mathbb{C}$. We first set up some curves

(1) $\gamma_1 : [0,1] \to \mathbb{C},\ \gamma_1(s) = s,$

(2) $\gamma_2 : [0,1] \to \mathbb{C},\ \gamma_1(s) = 1 - s + is,$

(3) $\gamma_3 : [0,1] \to \mathbb{C},\ \gamma_3(s) = i(1-s)$, and

(4) $\gamma_4 : [0,1] \to \mathbb{C},\ \gamma_3(s) = \cos 2\pi s + i \sin 2\pi s,$

First consider the entire function $f(z) = z$. Working from the definition,

(1) $\int_{\Gamma_1} f(z)dz = \int_0^1 sds = 1/2,$

(2) $\int_{\Gamma_2} f(z)dz = \int_0^1 (1-s+is)(-1+i)ds = (1/2+i/2)(-1+i) = -1,$

(3) $\int_{\Gamma_3} f(z)dz = \int_0^1 i(1-s)(-i)ds = 1/2$, and finally

Note (for future reference) that these three integrals add up to 0. As for the fourth integral,

$$\begin{aligned}\int_{\Gamma_4} f(z)dz &= 2\pi \int_0^1 (\cos 2\pi s + i \sin 2\pi s)(-\sin 2\pi s + i\cos 2\pi s)ds \\ &= 2\pi \int_0^1 (-2\cos 2\pi s \sin 2\pi s + i\cos^2 2\pi s - i \sin^2 2\pi s)ds \\ &= 0\end{aligned}$$

Consider the function $f(z) = \overline{z}$. Again working from the definition,

(1) $\int_{\Gamma_1} f(z)dz = \int_0^1 sds = 1/2,$

(2) $\int_{\Gamma_2} f(z)dz = \int_0^1 (1-s-is)(-1+i)ds = (1/2-i/2)(-1+i) = i,$

(3) $\int_{\Gamma_3} f(z)dz = \int_0^1 -i(1-s)(-i)ds = -1/2,$

In this case the three integrals *do not* add up to zero. The fourth integral gives

$$\begin{aligned}\int_{\Gamma_4} f(z)dz &= 2\pi \int_0^1 (\cos 2\pi s - i \sin 2\pi s)(-\sin 2\pi s + i\cos 2\pi s)ds \\ &= 2\pi \int_0^1 (i\cos^2 2\pi s + i \sin^2 2\pi s)ds \\ &= 2\pi i\end{aligned}$$

12. Line integrals and "Primitives" (="anti-derivatives")

Let f be a holomorphic function on a domain D. A function F is called a "primitive" for f if F is holomorphic and $F' = f$. A primitive may not exist on all of D, as we will show below using the example

$$z \mapsto \frac{1}{z}, \ z \in \mathbb{C}^*$$

Given z_0 in D and a disc $D(z_0, r)$ in which f is given by a power series (with radius of convergence $\rho \geq r$)

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

the power series

$$\sum_{n \geq 0} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

has the same radius of convergence and is a primitive for f on $D(z_0, r)$. So primitives exist *locally* for analytic functions, and assuming "holomorphic" = "analytic", for holomorphic functions. In fact, the existence of a local primitive is the key to Cauchy's theorem, and the rest is topology. The main tool for constructing a primitive is the line integral. We begin with

Proposition 12.1. *If f has a primitive F and Γ is a curve in D from A to B,*

$$\int_\Gamma f(z) dz = F(B) - F(A)$$

In particular,

$$\int_\Gamma f(z) dz = 0$$

for a closed curve Γ.

Proof. Let $\gamma : [0,1] \to D$ be a representative curve, with $\gamma(1) = B$ and $\gamma(0) = A$. Then

$$\begin{aligned}
\int_\Gamma f(z) dz &= \int_0^1 f(\gamma(t)) \frac{d\gamma(t)}{dt} dt \\
&= \int_0^1 F'(\gamma(t)) \frac{d\gamma(t)}{dt} dt \\
&= \int_0^1 \frac{dF(\gamma(t))}{dt} dt \\
&= F(\gamma(1)) - F(\gamma(0)) \\
&= F(B) - F(A)
\end{aligned}$$

□

Corollary 12.2. *Given a polynomial function f, we have $\int_\Gamma f(z) dz = 0$ for any closed path.*

13. Goursat's theorem

Let D be an open set in $\mathbb{C}$. (We do not assume D is connected since that is irrelevant in this context.) By a triangle in D, we mean an oriented triangle $T \subset D$, such that the interior of T is nonempty and also contained in D. We leave it to the reader to give a formal definition of T as a contour. Let $\mathcal{T}$ denoting the closure of the interior of T.

Theorem 13.1. *Let f be a function on a domain D. Suppose f is complex differentiable*[5] *at each point of D, with derivative $z \mapsto f'(z)$. Then for any triangle such that $\mathcal{T} \subset D$, we have*

$$\int_T f(z)dz = 0$$

Proof. The proof is by contradiction. Suppose $\int_T f(z) \neq 0$. Scale f so that $\int_T f(z) = 1$. Divide T into four triangles as in the figure (copied from S&S), with T renamed to $T^{(0)}$:

[5] We do not assume that f' is a continuous function, though this and more will follow once we prove (using the material to be developed) that f is analytic on D.

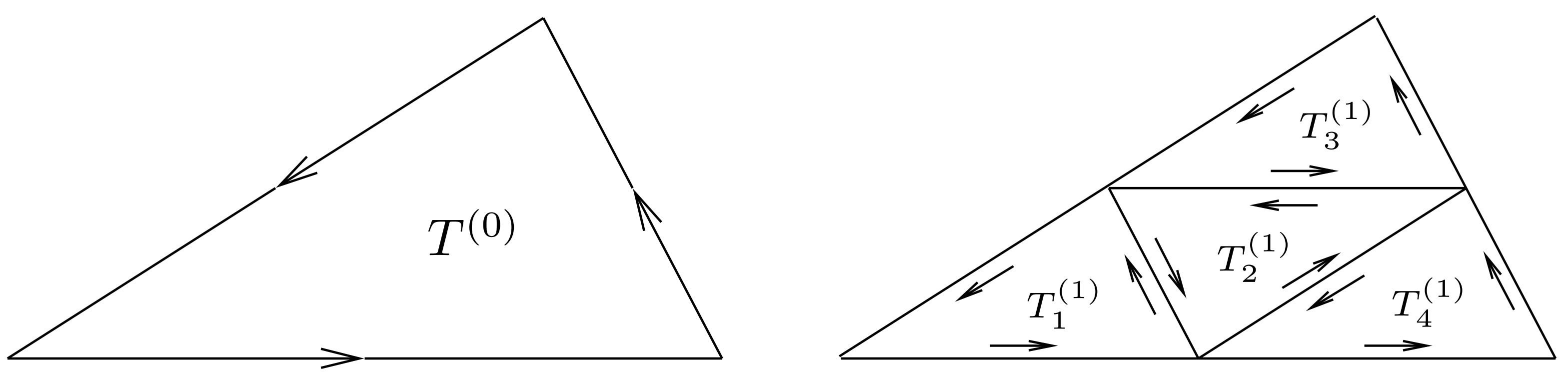


Figure 1. Bisection of $T^{(0)}$

We have

$$\int_{T}^{(0)} f(z)dz = \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_4^{(1)}} f(z)dz$$

so that

$$|\int_{T_{j_1}^{(1)}} f(z)dz| \geq \frac{1}{4}\text{for at least one } j_1$$

Continuing, we get a sequence of triangles $T^0, T_{j_1}^{(1)}, T_{j_2}^{(2)}, \ldots, T_{j_k}^{(k)} \ldots$ such that $\mathcal{T}^0 \supset \mathcal{T}_{j_1}^{(1)} \supset \mathcal{T}_{j_2}^{(2)} \supset \cdots \supset \mathcal{T}_{j_k}^{(k)} \ldots$, and such that

$$|\int_{T_{j_k}^{(k)}} f(z)dz| \geq \frac{1}{4^k}$$

Let $w_0, w_1, \ldots w_k, \ldots$ be a sequence of points with w_k in the interior of $\mathcal{T}_{j_k}^{(k)}$. This is a Cauchy sequence and tends to a limit z_0 which belongs to every $\mathcal{T}_{j_k}^{(k)}$.

Define the function $\psi : D \to \mathbb{C}$ by

$$\psi(z) = \begin{cases} \frac{f(z)-f(z_0)-f'(z_0)(z-z_0)}{z-z_0} \ if \ z \neq z_0 \\ 0 \ if \ z = z_0 \end{cases}$$

By the differentiability of f the function ψ is continuous everywhere on D, and

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

This yields (using Corollary 12.2)

$$|\int_{T_{j_k}^{(k)}} \psi(z)(z - z_0)dz| \geq \frac{1}{4^k}$$

On the other hand,

$$\begin{aligned} |\int_{T_{j_k}^{(k)}} \psi(z)(z - z_0)dz| &\leq \sup_{z \in T_{j_k}^{(k)}} |\psi(z)(z - z_0)| \times length(T_{j_k}^{(k)}) \\ &\leq \sup_{z \in T_{j_k}^{(k)}} |\psi(z)| \sup_{z \in T_{j_k}^{(k)}} |(z - z_0)| \times \ length(T_{j_k}^{(k)}) \end{aligned}$$

Now $\psi(z) \to 0$ as $z \to z_0$, so $\lim_{k\to\infty} \sup_{z \in T_{j_k}^{(k)}} |\psi(z)| = 0$. On the other hand, we have the estimates (whose proofs I leave as an exercise)

$$\sup_{z \in T_{j_k}^{(k)}} |(z - z_0)| \leq \frac{C_1}{2^k}, \quad length(T_{j_k}^{(k)}) \leq \frac{C_2}{2^k}$$

for some $C_1, C_2 > 0$. This gives the required contradiction. □

Corollary 13.2. *Let $R \subset D$ be a (oriented) rectangle, with D also containing the interior. Then*

$$\int_R f(z)dz = 0$$

Proof. We can write the integral along the oriented rectangle as a sum of integrals along two oriented triangles, and then appeal to Goursat. □

14. Primitives exist on a disc; Cauchy's Theorem on a disc

We begin by proving that every holomorphic function has, locally at least, a primitive.

Theorem 14.1. *Let f be complex differentiable on an open disc D. Then f has a primitive F.*

Proof. WLOG, assume that the disc is centred on the origin. Define the function F by

$$F(z) = \int_{\Gamma_z} f(z)dz$$

where, if $z = x + iy$, then Γ_z is the curve that goes horizontally from 0 to x and the vertically from x to $x + iy$. If z_0 and $z_0 + h$ are both in D, then (using Goursat+Corollary for rectangles)

$$F(z_0 + h) - F(z_0) = \int_{\Gamma_{z_0 \to z_0 + h}} f(z)dz$$

with $\Gamma_{z_0 \to z_0+h}$ denoting the curve that goes straight from z_0 to $z_0 + h$. Since f is continuous, we can write

$$f(z) = f(z_0) + \psi(z)$$

with ψ continuous and $\psi(z_0) = 0$. This yields

$$F(z_0 + h) - F(z_0) = f(z_0)h + \int_{\Gamma_{z_0 \to z_0 + h}} \psi(z)dz$$

which yields

$$\left|\frac{F(z_0 + h) - F(z_0)}{h} - f(z_0)\right| \leq \sup_{|z - z_0| \leq h} |\psi(z)|$$

Since the RHS goes to 0 as $h \to 0$, we see that $F'(z_0) = f(z_0)$. □

Corollary 14.2. (Cauchy's Theorem for a disc). *Let f, D be as above. Given any closed curve Γ in D,*

$$\int_\Gamma f(z)dz = 0$$

15. Cauchy's integral formula

We begin with a

Lemma 15.1. *Let D be a "slit" open disc*[6]*:*

$$D = D(z_0, R) \smallsetminus \{a \exp i\theta_0 | a_0 \leq a < R\}$$

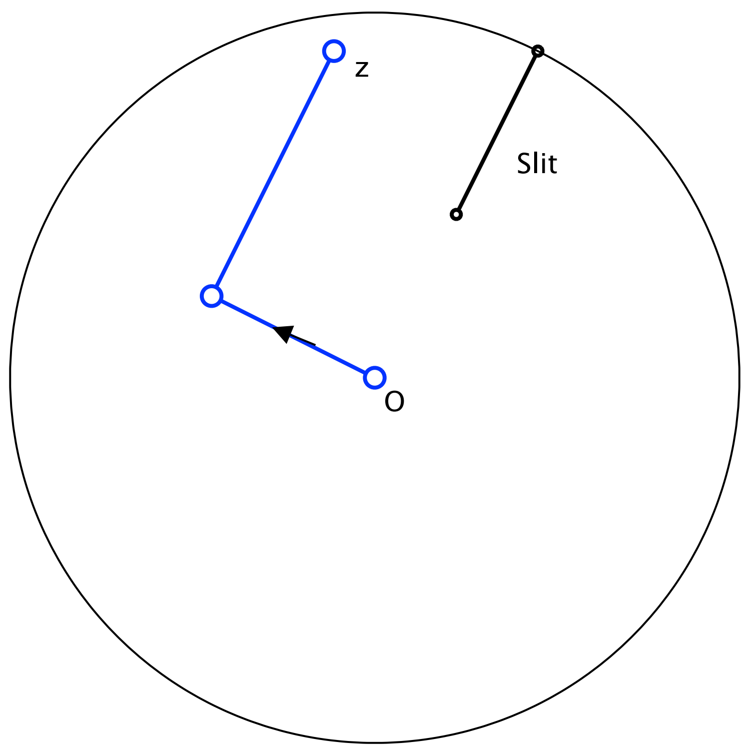
where a_0, θ_0 are real numbers and $0 \leq a_0 < R$, and f a holomorphic function on D. Then f has a primitive on D.

Proof. We construct a primitive just as we did in the case of a disc, by first choosing base-point O and then for each $z \in D$ a standard path Γ_z from O to z. The primitive F is then defined by

$$F(z) = \int_{\Gamma_z} f(z) dz$$

The choice of Γ_z is illustrated in this diagram:

[6]The open disc $D(z_0, R)$ has been "slit" along the line segment $\{a \exp i\theta_0 | a_0 \leq a < R\}$


z
Slit
O

□

Theorem 15.2. *Let $\overline{D}(z_0, R)$ be a closed disc with interior $D(z_0, R)$ and boundary C_R oriented counterclockwise. Suppose f is holomorphic on a domain containing $\overline{D}(z_0, R)$. Then*

$$f(\xi) = \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-\xi}dz$$

for any point $\xi \in D$.

Proof. Let C_r be circle of radius r centred on ξ, oriented conterclockwise and such that $\{z||z-\xi| \le r\} \subset D$. We checked in class that

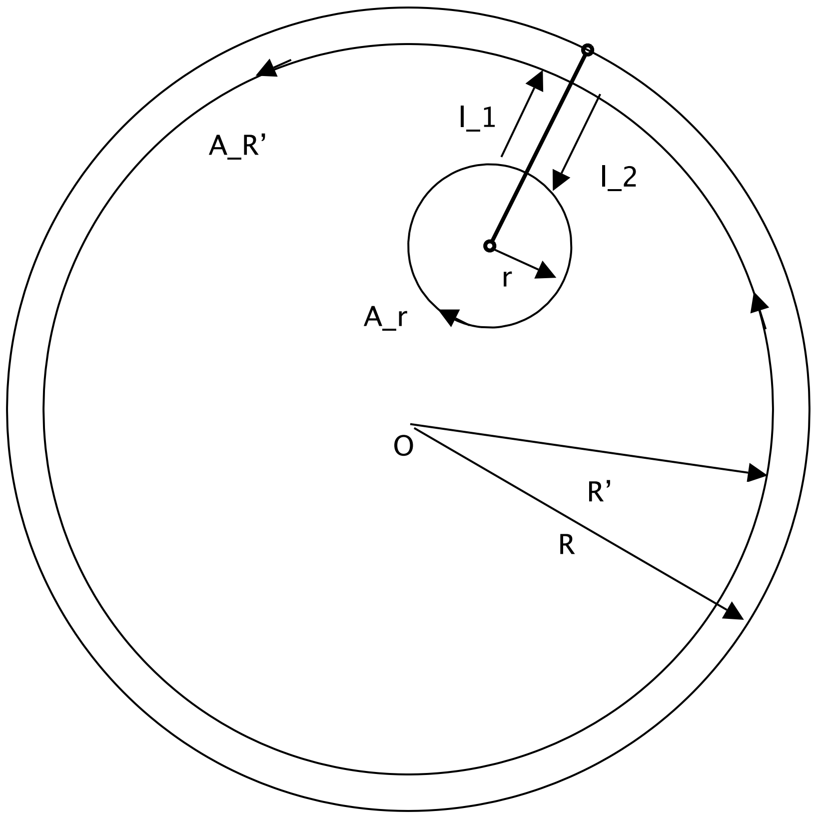
$$\frac{1}{2\pi i}\int_{C_r}\frac{1}{z-\xi}dz = 1$$

So

$$\begin{aligned}
|\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-\xi} - f(\xi)| &= |\frac{1}{2\pi i}\int_{C_r}\frac{f(z)-f(\xi)}{z-\xi}dz| \\
&\le \frac{1}{2\pi}\sup_{z\in C_r}|\frac{f(z)-f(\xi)}{z-\xi}|length(C_r) \\
&= \frac{1}{2\pi}\sup_{z\in C_r}|f(z)-f(\xi)|\frac{1}{r}2\pi r \\
&= \sup_{z\in C_r}|f(z)-f(\xi)|
\end{aligned}$$

By continuity of f, the last supremum goes to zero as $r \to 0$. We claim that the line integral does not change if we replace C_R by C_r, and this proves the Theorem.

Proof of the claim: Refer to the following diagram, where ξ is the centre of the disc of radius r:


A_R'
l_1
l_2
r
A_r
O
R'
R

Consider the line integral of the analytic function

$$z \mapsto \frac{1}{2\pi i}\frac{f(z)}{z-\xi}$$

along the oriented counter Γ consisting of the four pieces: the 'major arc' $A_{R'}$, the straight line segment l_2, the 'minor arc' A_r, and finally the line segment l_1. This is a closed curve in the slit disc of radius R, so the integral

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-\xi}dz = 0$$

As the two line segments are brought towards the slit (check these statements!),

$$\frac{1}{2\pi i}\int_{A_R}\frac{f(z)}{z-\xi}dz \to \frac{1}{2\pi i}\int_{C'_R}\frac{f(z)}{z-\xi}dz$$

$$\frac{1}{2\pi i}\int_{A_r}\frac{f(z)}{z-\xi}dz \to -\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-\xi}dz, \textit{ and}$$

$$\frac{1}{2\pi i}\int_{l_1}\frac{f(z)}{z-\xi}dz + \frac{1}{2\pi i}\int_{l_2}\frac{f(z)}{z-\xi}dz \to 0$$

so that

$$\frac{1}{2\pi i}\int_{C_{R'}}\frac{f(z)}{z-\xi}dz = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-\xi}dz$$

Finally, as $R' \to R$ we have

$$\frac{1}{2\pi i}\int_{C_{R'}}\frac{f(z)}{z-\xi}dz \to \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-\xi}dz$$

and we have already seen that as $r \to 0$

$$\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-\xi}dz \to f(\xi)$$

This finishes the proof. □

A major consequence of Cauchy's integral formula is

Theorem 15.3. *Let $\overline{D}(z_0, R)$ be a closed disc with interior $D(z_0, R)$ and boundary C_R oriented counterclockwise. Suppose f is holomorphic (=complex differentiable) on a domain containing $\overline{D}(z_0, R)$. Then f is infinitely complex differentiable and*

$$f^{(n)}(\xi) = \frac{n!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-\xi)^{n+1}}dz$$

for any point $\xi \in D$.

Proof. The proof is by induction on n, starting with $n = 0$. Let $m \geq 1$ and suppose the result is true for $n < m$. For h small enough that $\xi + h \in D(z_0, R)$, consider the difference $\Delta_h = f^{m-1}(\xi + h) - f^{m-1}(\xi)$ Using the inductive hypothesis, we have

$$
\begin{aligned}
\Delta_h = &= \frac{(m-1)!}{2\pi i}\int_{C_R}\left(\frac{f(z)}{(z-\xi-h)^m} - \frac{f(z)}{m(z-\xi)^m}\right)dz \\
&= \frac{(m-1)!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-\xi-h)^m}\left(1-\frac{(z-\xi-h)^m}{(z-\xi)^m}\right)dz \\
&= \frac{(m-1)!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-\xi-h)^m}\times\frac{h}{(z-\xi)}\left(1+\frac{(z-\xi-h)}{(z-\xi)}\cdots+\frac{(z-\xi-h)^{m-1}}{(z-\xi)^{m-1}}\right)dz
\end{aligned}
$$

Dividing by h, and letting $h \to 0$, the last integrand tends uniformly on C_R to $\frac{mf(z)}{(z-\xi)^{m+1}}$ so

$$\frac{\Delta_h}{h} \to \frac{m!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-\xi)^{m+1}}dz$$

□

Cauchy's estimate for Taylor coefficients follows:

Corollary 15.4. *Let $\overline{D}(z_0, R)$ be a closed disc with interior $D(z_0, R)$ and boundary C_R oriented counterclockwise. Suppose f is holomorphic (=complex differentiable) on a domain containing $\overline{D}(z_0, R)$. Then*

$$|f^{(n)}(z_0)| \le \frac{n!}{R^n}\sup_{z\in C_R}|f(z)|$$

Next, we prove Liouville's Theorem:

Theorem 15.5. *An entire bounded function is constant.*

Proof. Use the Cauchy integral formula on a large circle to estimate the derivative. □

Finally we can prove that "holomorphic implies analytic". We start with

Theorem 15.6. *Let $\overline{D}(z_0, R)$ be a closed disc with interior $D(z_0, R)$ and boundary C_R oriented counterclockwise. Suppose f is holomorphic (=complex differentiable) on a domain containing $\overline{D}(z_0, R)$. Then the Taylor series of f at z_0:*

$$\sum_{n\ge 0}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges to f at all points of $D(z_0, R)$.

Proof. Let ρ be the radius of convergence. We have

$$\rho^{-1} =_H \limsup_n\left(\frac{|f^{(n)}(z_0)|}{n!}\right)^{1/n} \le_C \limsup_n\left(\frac{1}{R^n}\right)^{1/n} = \frac{1}{R}$$

which shows that $\rho \geq R$. (Here the equality H uses Hadamard and the inequality C uses Cauchy's estimate.)

If S denotes the analytic function defined by the sum, both f and S have the same derivatives at z_0. However, as Mohan Swaminathan pointed out, we cannot conclude that they coincide on $D(z_0, R) \subset D(z_0, \rho)$ (as claimed in an earlier version of these notes). This is because we do not know (yet) that f is analytic. Going back to the drawing board (and essentially discarding the above approach), Cauchy's Formula is again the key.

For $\xi \in D(z_0, R)$, and $z \in C_R$, write

$$\begin{aligned}\frac{1}{z-\xi} &= \frac{1}{(z-z_0)}\frac{1}{(1-\frac{\xi-z_0}{z-z_0})} \\ &= \frac{1}{z-z_0}\{1+\frac{\xi-z_0}{z-z_0}+(\frac{\xi-z_0}{z-z_0})^2+\dots+(\frac{\xi-z_0}{z-z_0})^n+\dots\}\end{aligned}$$

where (for fixed ξ) the series converges uniformly for $z \in C_R$. This justifies the "interchange" step in the following computation which begins with Cauchy's formula for f and in the last step uses Cauchy's formula for derivatives:

$$\begin{aligned}f(\xi) &= \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-\xi}dz \\ &= \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}\{1+\frac{\xi-z_0}{z-z_0}+(\frac{\xi-z_0}{z-z_0})^2+\dots+(\frac{\xi-z_0}{z-z_0})^n+\dots\}dz \\ &=_{interchange} \quad \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}dz \\ &+ \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}\frac{\xi-z_0}{z-z_0}dz \\ &+ \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}(\frac{\xi-z_0}{z-z_0})^2dz+\dots \\ &+ \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}(\frac{\xi-z_0}{z-z_0})^n dz+\dots \\ &= f(z_0)+f'(z_0)(\xi-z_0)+\dots+\frac{f^{(n)}(z_0)}{n!}(\xi-z_0)^n+\dots\end{aligned}$$

Consider the sequence of functions on the circle C_R:

$$g_M(z) = \frac{f(z)}{(z-z_0)}+\dots+\frac{f(z)(\xi-z_0)^M}{(z-z_0)^{M+1}}$$

It is clear that

$$\lim_{M\to\infty} g_M(z) = \frac{f(z)}{z-\xi}\ .$$

We claim that the limit is uniform. To see this, note that (for $z \in C_R$)

$$\begin{aligned}
|g_{M+N}(z) - g_M(z)| &= |\frac{f(z)(\xi - z_0)^{M+1}}{(z-z_0)^{M+2}} + \cdots + \frac{f(z)(\xi - z_0)^{M+N}}{(z-z_0)^{M+N+1}}| \\
&\le \sup_{z \in C_R} |f(z)| \times (t^{M+1} + \cdots + t^{M+N})/R \\
&\le \sup_{z \in C_R} |f(z)| \frac{t^{M+1}}{R(1-t)}
\end{aligned}$$

where $t \equiv |\xi - z_0|/R < 1$. Letting N tend to infinity, we see that

$$|\frac{f(z)}{z-\xi} - g_M(z)| \le \frac{t^{M+1}}{R(1-t)}$$

which proves the claim, and justifies the "interchange" step above.

□

Yet another consequence is the Fundamental Theorem of Algebra:

Theorem 15.7. *A polynomial of degree ≥ 1 has a root in $\mathbb{C}$.*

Proof. Let $n > 0$ be an integer and $P(z) = a_0 + a_1 z + \ldots a_n z^n$ a polynomial function, with $a_i \in \mathbb{C}$ and $a_n \ne 0$. If P is nowhere-vanishing then $1/P$ is an entire function. On the other hand, if $z \ne 0$ we can write

$$P(z) = z^n R(z)$$

where $R(z) = a_0 z^{-n} + a_1 z^{-(n-1)} + \cdots + a_n$. We have

$$|a_n| \le |R(z)| + \sum_{i=0}^{n-1} |a_i| R^{-(n-i)}$$

provided $|z| \ge R$, so that also

$$|P(z)|^{-1} \le R^{-n} \{|a_n| - \sum_{i=0}^{n-1} |a_i| R^{-(n-i)}\}^{-1}$$

This shows that $1/P$ is bounded. By Liouville's Theorem it must be constant, and hence so must P be. This is a contradiction since its degree is at least one. □

16. Interlude: Real integrals *via* complex analysis

17. Morera's Theorem

This is a converse to Cauchy's Theorem.

Theorem 17.1. *Let f be a continuous complex valued function on an open disc, and suppose $\int_\Gamma f(z)dz = 0$ for every simple closed contour Γ in the disc. Then f is holomorphic.*

Proof. If you examine the proof that a holomorphic function has a primitive on a disc, you will see that it only uses the fact (Cauchy's Theorem for the disc) that the line integral of such a function vanishes on every closed simple contour in the disc. So the function f has a primitive F which is by definition complex differentiable. We have seen that the derivative of a holomorphic function is holomorphic (either by Cauchy's formula or via analyticity). ☐

Corollary 17.2. *Uniform limits of holomorphic functions are holomorphic. More precisely, if $f_n, n = 0, 1, 2, \ldots,$ is a sequence of holomorphic functions on a domain D, and $f_n(z) \to f(z)$ at every point $z \in D$, and for every compact $K \subset D$ we have*

$$\lim_n \sup_{z \in K} |f_n(z) - f(z)| = 0$$

then f is holomorphic, and[7] $\lim_n f'_n(z) = f'(z)$.

Proof. Let Γ be any simple closed contour in D. We have (applying Cauchy's Theorem to the integral of f_n)

$$|\int_\Gamma f(z)dz| = |\int_\Gamma (f(z) - f_n(z))dz| \le \sup_{z \in \gamma} |f(z) - f_n(z)| length(\Gamma)$$

Since Γ is a compact subset of D, the RHS goes to zero. Now apply Morera's Theorem. The pointwise convergence $f^{(n)}(z) \to f^{(n)}(z)$ follows from the uniform convergence (on compacts) $f_n \to f$ and Cauchy?s formula for derivatives. ☐

Corollary 17.3. *Let $D \subset \mathbb{C}$ be a domain, and $I = [a, b] \subset \mathbb{R}$ a finite closed interval, with $a < b$. Let $F : D \times I \to \mathbb{C}$ be a (jointly) continuous function such that for each fixed $t \in I$:*

$$z \mapsto F(z, t)$$

is holomorphic. Then the function $f : D \to \mathbb{C}$, defined by

$$f(z) = \int_I F(z, t)dt$$

is holomorphic.

Proof. Let $D' \subset D$ be an open disc. Let Γ be any simple closed contour in D', represented by $\gamma : [0, 1] \to D'$. Then

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_\Gamma \int_I F(z, t)dtdz \\ &=_{interchange} \int_I \int_\Gamma F(z, t)dzdt \\ &=_C 0\end{aligned}$$

[7] In fact this limit is also uniform on compact sets.

where the equality C follows from Cauchy's Theorem applied to f. Now use Morera.

The equality "interchange" has to be justified. One could use the Lebesgue integral and Fubini's Theorem, but since F is (jointly) continuous, integrals can be defined as limits of Riemann sums, and a simpler justification exists *à la* Stein-Shakarchi.

For $n \geq 1$, consider the Riemann sums:

$$f_n(z) = \frac{1}{n}\sum_{j=1}^{n} F(z, a + \frac{j}{n}(b-a))$$

Since F is jointly continuous in z and t, it is in particular continuous in t for each fixed z, so we have pointwise convergence:

$$f_n(z) \to \int_I F(z,t)dt = f(z)$$

We claim that in fact $f_n \to f(z)$ uniformly on compact subsets of D. By the earlier corollary of Morera's Theorem this proves that f is holomorphic.

Proof of Claim: We have for any $z \in D$,

$$f(z) - f_n(z) = \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} (F(z,t) - F(z, j/n))dt$$

Let $K \subset D$ be compact. If $z \in K$, we have

$$|f(z) - f_n(z)| \leq \frac{1}{n}\sum_{j=1}^{n} \sup_{z\in K,\ t\in[\frac{j-1}{n},\frac{j}{n}]} |F(z,t) - F(z,j/n)|$$

By joint continuity of F on $K \times I$, there exists, for each $\epsilon > 0$, a $n > 0$ such that for $z, z' \in K$ and $t, t' \in I$ we have $|F(z',t') - F(z,t)| < \epsilon$ provided $|z'-z| + |t'-t| < \frac{1}{n}$. (Uniform Continuity.) Then

$$|f(z) - f_n(z)| \leq \frac{1}{n} n\epsilon = \epsilon,\ \ z \in K\ ,$$

proving the Claim. □

18. Isolated singularities: removable/pole/essential

Suppose given a nonzero holomorphic function f on a "punctured disc" $D_r^* = \{z | 0 < |z| < r\} \subset D_r = \{z || z| < r\}$, where r could be $+\infty$. In this section we will show that there are three possibilities:

(1) *$|f|$ is bounded as $z \to 0$, in which case f extends as a holomorphic function to the "un-punctured" open disc D_r.* (We say that f has a *removable singularity* at the origin.) Note that in this case f actually has a limit as $z \to 0$.

(2) $|f| \to +\infty$ *as* $z \to 0$*, in which case* $f(z) = z^{-n}g(z)$ *for some* $n > 0$ *and holomorphic function* g *on* D_r *such that* $g(0) \neq 0$. (We say that f has a *pole of order* n at the origin.) In this case $1/f \to 0$ as $z \to 0$.

(3) $|f|$ does not have a limit as $z \to 0$. (We say that f has an *essential singularity* at the origin.) In this case neither f nor $1/f$ have a limit as $z \to 0$.

Let us deal with these cases in *reverse* order.

18.1. **An essential singularity.** Here is an example of an essential singularity.:

$$z \mapsto \exp\frac{1}{z}, \ z \in \mathbb{C}^*$$

Write $z = x + iy$, with x and y real. Then

(1) $\exp\frac{1}{x} \to +\infty$ as $x \to 0_+$,

(2) $\exp\frac{1}{x} \to 0$ as $x \to 0_-$, and

(3) $|\exp\frac{1}{iy}| = 1,\ y \neq 0$.

<u>Exercise:</u> Consider the above function restricted to D_r^*, for any $r > 0$. Prove that given any nonzero complex number w, it has infinitely many pre-images. That is, there exist infinitely many z, $0 < |z| < r$ such that

$$\exp\frac{1}{z} = w$$

The "Little Theorem" of Picard asserts that an analytic function assumes any complex value, with the possible exception of one value, in an arbitrarily small neighbourhood of an essential singularity.

18.2. **Pole.** If $|f(z)| \to +\infty$ as $z \to 0$, then f is nonvanishing on some punctured sub-disc $D_{r'}^*$ with $r' \leq r$. the function g defined by

$$h(z) = \frac{1}{f(z)}$$

is holomorphic on $D_{r'}^*$, and $|h(z)| \to 0$ as $|z| \to 0$. By the removable singularities theorem, which we will prove below, h extends to an holomorphic function on $D_{r'}$, which must of course vanish at $z = 0$. In this case we know that $h(z) = z^n\tilde{h}(z)$ for some $n > 0$ and $\tilde{h}$ such that $\tilde{h}(0) \neq 0$. Thus

$$z \mapsto g(z) \equiv z^n f(z)$$

is regular and nonvanishing on D_r.

18.3. **Removable singularity.** Finally, suppose $|f(z)|$ is bounded as $|z| \to 0$. We begin with an application of a corollary of Morera's Theorem that we proved earlier.

Lemma 18.1. *Let $\tilde{f}$ be a continuous complex-valued function defined on a circle C_R of radius $R > 0$, centred on the origin. Assume C_R oriented counterclockwise. The function $f : D_R \to \mathbb{C}$, defined by*

$$f(\xi) = \frac{1}{2\pi i} \int_{C_R} \frac{\tilde{f}(z)}{z - \xi} dz$$

is holomorphic.

Proof. Writing out the RHS, we have

$$f(\xi) = \int_0^1 \frac{\tilde{f}(Re^{2\pi it})}{Re^{2\pi it} - \xi} Re^{2\pi it} dt = \int_0^1 F(\xi, t) dt$$

where

$$F(\xi, t) = \frac{\tilde{f}(Re^{2\pi it})}{1 - \xi e^{-2\pi it}/R} .$$

Now use the result referred to above. □

Suppose now that f is a holomorphic function on D_r^*, and $|f(z)|$ bounded as $|z| \to 0$. Let $R < r$. We claim that

$$f(\xi) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z - \xi} dz$$

provided $0 < |\xi| < R$. On the other hand, by the previous Lemma, the RHS defines a holomorphic function on D_R, so we get the sought-after extension.

Consider the contour $\Gamma = A_R + l_2 + A'_s + l_4 + A_u + l_3 + A_s + l_1$ in the diagram on the next page. This is a closed contour contained in the slit disc of radius r, with the slit running from the origin to the boundary through the point ξ. The function

$$z \mapsto \frac{f(z)}{z - \xi}$$

is holomorphic on the slit disc, so

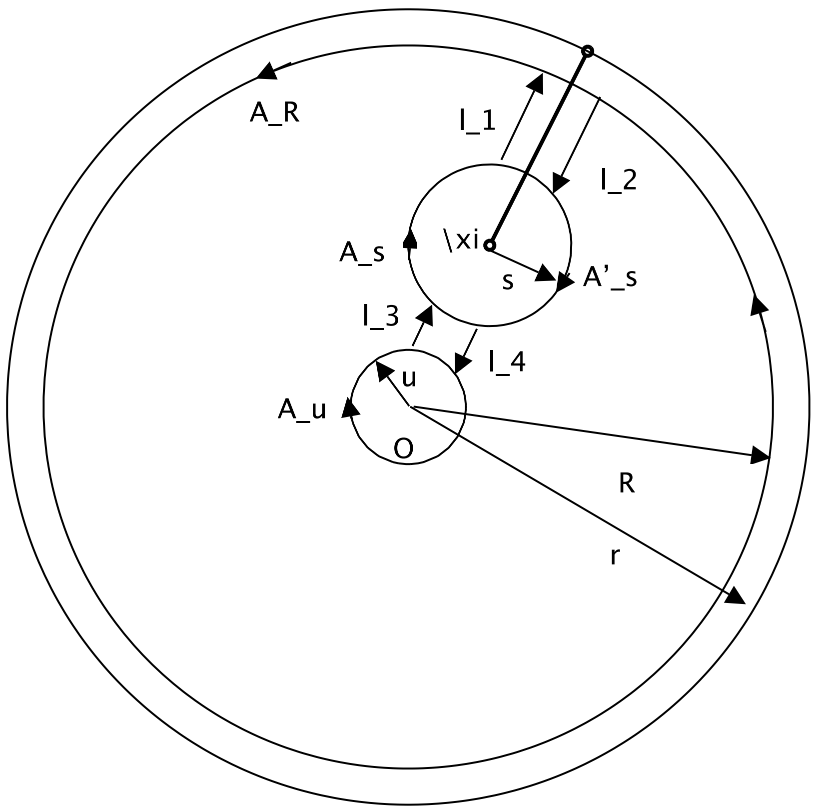
$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - \xi} dz = 0$$

Letting l_1 and l_2 approach each other, and l_3 and l_4 as well, we get

$$\frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z - \xi} dz = \frac{1}{2\pi i} \int_{C_s} \frac{f(z)}{z - \xi} dz + \frac{1}{2\pi i} \int_{C_u} \frac{f(z)}{z - \xi} dz$$

Here C_R, C_u are circles of radius R, u respectively centred on the origin, and C_s is the circle of radius s centred on ξ; all circles are oriented counterclockwise. The parameters u, s are chosen so that C_u and C_s are contained in D_r, and the corresponding discs do not intersect.

The first term on the right gives $f(\xi)$; on the other hand, the second term can be seen to go to zero as $u \to 0$ using the fact that $|f|$ remains bounded. This proves the claim.


A_R
l_1
l_2
A_s
\xi
s
A'_s
l_3
l_4
u
A_u
O
R
r

19. Meromorphic Functions

Let D be a domain. A *meromorphic function* on D is a holomorphic function on $D \setminus S$ where S is a subset of D. such that for each point $z_0 \in S$, there is a $r > 0$ such that the punctured disc $D^*(z_0, r) = \{z | 0 < |z - z_0| < r\}$ is contained in D and contains no point of S, and $f|_{D^*(z_0,r)}$ is either bounded near z_0 or has a pole at z_0. One of producing meromorphic functions is by taking a ratio f/g of holomorphic functions with $g \neq 0$.

Suppose given a meromorphic function f on $\mathbb{C}$. We say that f is *meromorphic at infinity* if there is a a real $R > 0$ such that the function

$$w \mapsto f(1/w)$$

is holomorphic on the punctured disc $D^*_{1/R}$, with possibly a pole (in particular, *not* an essential singularity) at the origin. The exponential function is an example of an entire function which is not meromorphic at infinity.

<u>Exercise</u>: Let P and $Q \neq 0$ be polynomials. Let S denote the set of zeroes of Q. Consider the holomorphic function on $\mathbb{C} \setminus S$. Prove that the function $R : \mathbb{C} \setminus S$:

$$z \mapsto R(z) \equiv P(z)/Q(z)$$

is meromorphic on $\mathbb{C}$ with at worst poles at S. (It could happen that Q has a zero at a point of S "cancelling" a zero of P, so that R has a removable singularity there. If we demand that P and Q are coprime in $\mathbb{C}[z]$, we can indeed ensure that R has a pole at every point of S.) Prove that R is meromorphic at infinity, and

(1) R extends to a holomorphic function at infinity (i.e., at $w = 0$) with a zero of order n if *degree* Q − *degree* $P = n > 0$,

(2) R extends to a holomorphic function at infinity (i.e., at $w = 0$), and is nonvanishing there if *degree* Q = *degree* P, and

(3) R has a pole of order n at infinity (i.e., at $w = 0$) if *degree* P − *degree* $Q = n > 0$.

Prove that any meromorphic function on $\mathbb{C}$ that is is meromorphic at infinity arises this way.

20. Automorphisms of $\mathbb{C}$ and the unit disc (=upper half plane); the Schwarz Lemma and the Maximum Modulus Principle

In this section we determine the group of (holomorphic) automorphisms of the complex plane and the unit disc (which is in fact biholomorphic to the upper half-plane, see below). The proofs depend on two tools, which are very important in themselves, namely the Schwarz Lemma and the Maximum Modulus Principle.

20.1. **Automorphisms of the complex plane.** In this section we ask first: what are the biholomorphic [8] maps $\phi : \mathbb{C} \to \mathbb{C}$. After translating and multiplying by a nonzero constant if needed, we can assume that $\phi(0) = 0$ and $\phi'(0) = 1$. Since ϕ is one-one, it is nowhere vanishing on $\mathbb{C}^*$. Also, for any $r > 0$, the image of the closed disc $\overline{D}_r$ by ϕ is a compact subset of $\mathbb{C}$, and (why?) contains an open disc around the origin. Therefore the function η defined by

$$w \mapsto \eta(w) = \frac{1}{\phi(1/w)} \ (w \neq 0)$$

is bounded in norm as $w \to 0$, and must therefore have a removable singularity at $w = 0$. By earlier results,

$$\eta(w) = w^n g(w)$$

in some neighbourhood of $w = 0$, for some $n > 0$ and g such that $g(w) \neq 0$. Clearly η will not be one-one if $n > 1$, so we conclude that $|\phi(z)/z|$ is bounded as $|z| \to \infty$. An estimate with Cauchy's formula then shows that ϕ' is bounded and hence constant. Given our normalisation, $\phi(z) = z$. We conclude: Any automorphism of $\mathbb{C}$ is of the from

$$z \mapsto az + b$$

where $a, b \in \mathbb{C}$, and a is nonzero.

Note that the set $Aut(\mathbb{C})$ of biholomorphic maps $\mathbb{C} \to \mathbb{C}$ is a group under composition. Write down the composition law in terms of the parameters (a, b), and check directly that this defines a group law on the set

$$\{(a, b) | 0 \neq a \in \mathbb{C}, b \in \mathbb{C}\}$$

20.2. **Automorphisms of the unit disc.** We ask next: what are the automorphisms of the unit disc $D_1 = \{z \in \mathbb{C} | |z| < 1\}$?

Given any point $\alpha \in D_1$, we exhibit an automorphism that takes α to 0. Consider the function $\psi_\alpha : D_1 \to D_1$ defined by

$$z \mapsto \psi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$

Note that the RHS well-defined if $|z| < 1$ because the denominator cannot vanish. (This is because $|\overline{\alpha} z| = |\alpha||z| < 1$). To check that the RHS indeed defines a point of D_1 we need to verify that

$$|1 - \overline{\alpha} z|^2 = |\alpha|^2 |z|^2 + 1 - \overline{\alpha} z - \alpha \overline{z} > |z|^2 + |\alpha|^2 - \overline{\alpha} z - \alpha \overline{z} = |z - \alpha|^2$$

This follows from the inequality

$$|\alpha|^2 |z|^2 + 1 - |z|^2 - |\alpha|^2 = (1 - |\alpha|^2)(1 - |z|^2) > 0$$

[8]A *biholomorphic* map is a one-one onto holomorphic map such that – in fact this last condition is automatically satisfied – the inverse is also holomorphic.

The computation

$$\psi_{-\alpha}\circ\psi_\alpha(z) = \frac{\frac{z-\alpha}{1-\overline{\alpha}z}+\alpha}{1+\overline{\alpha}\frac{z-\alpha}{1-\overline{\alpha}z}} = \frac{z-\alpha+\alpha-|\alpha|^2 z}{1-\overline{\alpha}z+\overline{\alpha}z-|\alpha|^2} = z$$

shows that ψ_α is biholomorphic.

<u>Exercise</u> Using what we have proved so far, show that $Aut(D_1)$ acts transitively on D_1.

There is another class of automorphisms that we have to consider, namely, rotations around the origin:

$$z \mapsto R_\theta(z) = \exp i\theta \times z$$

with $\theta \in [0, 2\pi)$. Note that $R_{\theta_1}\circ R_{\theta_2} = R_{\theta_1+\theta_2}$. On the other hand,

$$\begin{aligned}\psi_{\alpha_1}\circ\psi_{\alpha_2}(z) &= \frac{z-\alpha_1-\alpha_2+\alpha_2\overline{\alpha}_1 z}{1-\overline{\alpha}_1 z-\overline{\alpha}_2 z+\overline{\alpha}_2\alpha_1}\\ &= \frac{1+\alpha_2\overline{\alpha}_1}{1+\overline{\alpha}_2\alpha_1}\frac{z-\frac{\alpha_1+\alpha_2}{1+\alpha_2\overline{\alpha}_1}}{1-\frac{\overline{\alpha}_1+\overline{\alpha}_2}{1+\overline{\alpha}_2\alpha_1}z}\\ &= R_{\theta(\alpha_1,\alpha_2)}\psi_{\{\psi_{-\alpha_1}(\alpha_2)\}}(z)\end{aligned}$$

where $\exp i\theta(\alpha_1,\alpha_2) = \frac{1+\alpha_2\overline{\alpha}_1}{1+\overline{\alpha}_2\alpha_1}$. Finally,

$$\psi_\alpha\circ R_\theta(z) = \frac{\{\exp i\theta\}z-\alpha}{1-\overline{\alpha}\{\exp i\theta\}z} = \{\exp i\theta\}\frac{z-\{\exp -i\theta\}\alpha}{1-\overline{\{\exp -i\theta\}\alpha}z} = R_\theta\circ\psi_{\{\exp -i\theta\}\alpha}(z)$$

Composing the two kinds of automorphisms, we get an automorphism $\phi_{\theta,\alpha}: D_1 \to D_1$:

$$z \mapsto \phi_{\theta,\alpha}(z) = \exp i\theta \times \frac{z-\alpha}{1-\overline{\alpha}z}$$

and the above computations show that such automorphisms are closed under compositions.

Proposition 20.1. *Any automorphism of the unit disc is of the above form.*

Proof. Let ϕ be any automorphism, and suppose $\phi(0) = \alpha$. The automorphism $\psi_\alpha\circ\phi$ leaves the origin fixed. We claim that any automorphism that fixes the origin is of the form R_θ. Granting this claim, $\psi_\alpha\circ\phi = R_\theta$, which yields $\phi = \psi_{-\alpha}\circ R_\theta = R_\theta\circ\psi_{-\{\exp -i\theta\}\alpha}$.

Proof of claim: Let $\eta : D_1 \to D_1$ be an automorphism such that $\eta(0) = 0$. By the Schwarz Lemma (see below), we have $|\eta(z)| \le |z|$. Applying the same Lemma to η^{-1}, see that $|z| = |\eta^{-1}(\eta(z)) \le |\eta(z)|$. This yields $\eta(z) = |z|,\ z \in D$. Appealing to the lemma again, the claim is proved. □

20.3. **The Schwarz Lemma and the Maximum Modulus Principle.** The following lemma is the tip of a geometric iceberg. Or to mix metaphors:

"This could be the discovery of the century. Depending, of course, on how far down it goes."

Lemma 20.2. *Let $f : D_1 \to D_1$ be a holomorphic map, with $f(0) = 0$. Then*

1. $|f(z)| \le |z|, \ z \in D_1$,
2. *if* $|f(z_0)| = |z_0|$ *for some* $z_0 \neq 0$, *then* f *is a rotation, and*
3. *if* $|f'(0)| = 1$, *then* f *is a rotation.*

Proof. We know that $f(z) = zg(z)$ for some holomorphic function g. Let D_r be an open disc of radius $0 < r < 1$ with boundary circle C_r. For $z \in D_r$ we have

$$|g(z)| \le_M \sup_{w \in C_r} |g(w)| = \frac{1}{r} \sup_{w \in C_r} |f(w)| \le \frac{1}{r}$$

since f maps D_1 to D_1. Letting $r \to 1$, we get $|g(z)| \le 1, z \in D_r$, which yields $|f(z)| \le |z|$, proving (1). The inequality M follows from the Maximum principle (proved below)

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, $|g(z_0)| = 1$, which (again using the Maximum Principle) forces g to be constant (since we have proved $|g(z)| \le 1, \ z \in D$), of modulus 1, so that $f(z) = \exp i\theta z$ for some real θ. Note that $f'(0) = g(0)$, so if $|f'(0)| = 1$, again the same conclusion follows. □

Finally, we prove the Maximum Modulus Principle, which is a corollary of the Open Mapping Theorem. First a definition: let F be a continuous real-valued function, taking only non-negative values, on a domain D. We say that F *attains a local maximum at a point* $z_0 \in D$ if for some $r > 0$ we have $D(z_0, r) \subset D$ and

$$F(z) \le F(z_0), \ \forall z \in D(z_0, r)$$

Theorem 20.3. *Let f be a nonconstant holomorphic function on a domain D. Then $|f|$ cannot attain a local maximum at any point of D.*

Proof. Let $f(z_0) = w_0$. If f is not constant, it is an open map, so given any open disc $D(z_0, r) \subset D$, the image $f(D(z_0, r))$ is open and therefore contains an open disc around w_0. In particular there exists $w = f(z)$, with $z \in D(z_0, r)$, such that $|w| > |w_0|$. □

20.4. **The upper half plane.** Let $H = \{z = x + iy \in \mathbb{C} | y > 0\}$ denote the ("Poincaré") upper half-plane. This is biholomorphic to the unit disc, the maps in either directions being given by:

$$z \mapsto \frac{i - z}{i + z} \ \ (H \to D_1), \quad i\frac{1 - w}{1 + w} \leftarrow\!\shortmid w \ \ (H \leftarrow D_1)$$

So in principle, we know the automorphisms of H.

21. Homotopies and simply connected domains

We wish to characterise, in topological terms, domains in which every holomorphic function has a primitive.

21.1. **Simply connected domains.** Let D be a domain, P, Q two points in D.

Definition 21.1. Two continuous paths $\gamma_0 : [0,1] \to D$, $\gamma_1 : [0,1] \to D$ from P to Q[9] are said to be *homotopic* if there is, for each $s \in [0,1]$ a path γ_s from P to Q interpolating between γ_0 and γ_1 and such that the map $[0,1] \times [0,1] \to D$, $(s,t) \mapsto \gamma_s(t)$ is continuous.

Definition 21.2. A (path)-connected domain D is *simply connected* if given any two points P and Q and two paths γ_0 and γ_1 from P to Q, they are homotopic.

If D is simply connected and γ is a closed path starting and ending at a point P, γ is homotopic to the constant path $t \mapsto P$. This in turn implies: given a continuous map $\gamma : S^1 \to D$, it extends to a continuous map $\overline{D}_1 \to D$, where (given $r > 0$) D_r is the open disc of radius r centred at the origin, and $\overline{D}_r$ its closure, and S^1 is its boundary circle. We have used the fact that a closed path – a continuous map $\gamma : [0,1] \to D$ with $\gamma(0) = \gamma(1)$ – can be regarded as a map from S^1. (Exercise: show that this is in fact equivalent to simply-connectedness.)

Note that homotopy is defined for continuous paths, while line integrals – which we defined earlier, and we will define in greater generality below – require our paths to be at least C^1.

21.2. **One-forms; closed and exact 1-forms.** The natural framework for the following will involve p-forms on n-manifolds. For our purposes, it suffices to deal with domains in $\mathbb{R}^n$ $(n = 1,2)$, and with (C^∞) 0-forms (functions), and (C^∞) 1-forms, which we will define now. For the moment we deal with real-valued forms.

Definition 21.3. A 1-form on an open interval $I \subset \mathbb{R}$ (with coordinate function t) is a C^∞ function ω_t, packaged into the object $\tilde{\omega} = \tilde{\omega}_t(t)dt$. If $D \subset \mathbb{R}^2$ is a domain (with x, y the coordinate functions) a *1-form* on D is a pair of C^∞ functions ω_x, ω_y, packaged into the object $\omega = \omega_x dx + \omega_y dy$. The form is *exact* if there exists a C^∞ function f such that

$$\omega_x = f\frac{\partial f}{\partial x}, \ \omega_y = \frac{\partial f}{\partial y}$$

[9] Recall that this means $\gamma_0(0) = \gamma_1(0) = P$ and $\gamma_0(1) = \gamma_1(1) = Q$

(We write $\omega = df \equiv \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.) The form is *closed* if

$$\frac{\partial \omega_y}{\partial x} = \frac{\partial \omega_x}{\partial y}$$

By the equality of mixed partial derivatives of a function, any exact form is closed. We will see that the converse holds locally.

First, we define the *pull-back* of a form. Let $\tilde{D} \subset \mathbb{R}^n$ be a domain, and $\phi : \tilde{D} \to D$ smooth map. (We will need the cases $n = 1, 2$.) Given a 0-form (=function) f on D, its pull-back to $\tilde{D}$ is the function

$$\phi^* f = f \circ \phi \ .$$

That is to say, $\phi^* f(p) = f(\phi(p))$. Given a 1-form ω on D, its pull-back to $\tilde{D}$ is $\tilde{\omega} \equiv \phi^*\omega$ defined as follows:

(1) if $n = 1$ and $\tilde{D} = I$, an interval, then $\tilde{\omega} = \tilde{\omega}_t(t)dt$, where

$$\tilde{\omega}_t(t) = \omega_x(\phi(t))\frac{dx(t)}{dt} + \omega_y(\phi(t))\frac{dy(t)}{dt}, and$$

where we set $x(t) = x(\phi(t))$, and $y(t) = y(\phi(t))$.

(2) if $n = 2$ and $\tilde{D} \subset \mathbb{R}^2$, with coordinates X, Y, $\tilde{\omega} = \tilde{\omega}_X(X,Y)dX + \tilde{\omega}(X,Y)_Y dY$, where

$$\tilde{\omega}_X(X,Y) = \omega_x(\phi(X,Y))\frac{\partial x(X,Y)}{\partial X} + \omega_y(\phi(X,Y))\frac{\partial y(X,Y)}{\partial X}$$

and

$$\tilde{\omega}_Y(X,Y) = \omega_x(\phi(X,Y))\frac{\partial x(X,Y)}{\partial Y} + \omega_y(\phi(X,Y))\frac{\partial y(X,Y)}{\partial Y}$$

Proposition 21.4. *Let $n = 2$. The pullback of an exact form is exact. More precisely, $d(\phi^* f) = \phi^* df$. The pull-back of a closed form is closed.*

Proof. We leave the proof of the first claim as an exercise. As for the second, suppose ω is closed. Then

$$\begin{aligned}
\frac{\partial\tilde{\omega}_Y}{\partial X} &= \frac{\partial}{\partial X}\{\omega_x(\phi(X,Y))\frac{\partial x(X,Y)}{\partial Y} + \omega_y(\phi(X,Y))\frac{\partial y(X,Y)}{\partial Y}\} \\
&= \frac{\partial\omega_x}{\partial x}(\phi(X,Y))\frac{\partial x(X,Y)}{\partial X}\frac{\partial x(X,Y)}{\partial Y} \\
&+ \frac{\partial\omega_x}{\partial y}(\phi(X,Y))\frac{\partial y(X,Y)}{\partial X}\frac{\partial x(X,Y)}{\partial Y} \\
&+ \omega_x(\phi(X,Y))\frac{\partial^2 x(X,Y)}{\partial X\partial Y} \\
&+ \frac{\partial\omega_y}{\partial x}(\phi(X,Y))\frac{\partial x(X,Y)}{\partial X}\frac{\partial y(X,Y)}{\partial Y} \\
&+ \frac{\partial\omega_y}{\partial y}(\phi(X,Y))\frac{\partial y(X,Y)}{\partial X}\frac{\partial y(X,Y)}{\partial Y} \\
&+ \omega_y(\phi(X,Y))\frac{\partial^2 y(X,Y)}{\partial X\partial Y}
\end{aligned}$$

This yields

$$\begin{aligned}
\frac{\partial\tilde{\omega}_Y}{\partial X} - \frac{\partial\tilde{\omega}_X}{\partial Y} &= \frac{\partial\omega_x}{\partial y}(\phi(X,Y))\frac{\partial y(X,Y)}{\partial X}\frac{\partial x(X,Y)}{\partial Y} \\
&- \frac{\partial\omega_x}{\partial y}(\phi(X,Y))\frac{\partial y(X,Y)}{\partial Y}\frac{\partial x(X,Y)}{\partial X} \\
&+ \frac{\partial\omega_y}{\partial x}(\phi(X,Y))\frac{\partial x(X,Y)}{\partial X}\frac{\partial y(X,Y)}{\partial Y} \\
&- \frac{\partial\omega_y}{\partial x}(\phi(X,Y))\frac{\partial x(X,Y)}{\partial Y}\frac{\partial y(X,Y)}{\partial X} \\
&= \{\frac{\partial\omega_x}{\partial y}(\phi(X,Y)) - \frac{\partial\omega_y}{\partial x}(\phi(X,Y))\}\frac{\partial y(X,Y)}{\partial X}\frac{\partial x(X,Y)}{\partial Y} \\
&+ \{\frac{\partial\omega_y}{\partial x}(\phi(X,Y)) - \frac{\partial\omega_x}{\partial x}(\phi(X,Y))\}\frac{\partial x(X,Y)}{\partial X}\frac{\partial y(X,Y)}{\partial Y} \\
&= 0
\end{aligned}$$

□

21.3. **Line integrals.** Given a 1-form ω on an open interval I, and a compact subinterval $[a,b] \subset I$, we define the integral of ω on $[a,b]$ by

$$\int_{[a,b]} \omega = \int_a^b \omega_t(t)dt$$

Let $D \subset \mathbb{R}^2$ be a domain, and suppose given a 1-form ω on D. Given a C^∞ map $\gamma : [a,b] \to D$ (by which we mean a map which extends to a C^∞ map from an open interval I containing $[a,b]$), we define its integral along γ by

$$\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega = \int_a^b \{\omega_x(\gamma(t))\frac{dx(t)}{dt} + \omega_y(\gamma(t))\frac{dy(t)}{dt}\}dt$$

where, as usual, we write $x(t) = x(\gamma(t))$, etc. We extend the definition, as in the case of line integrals of holomorphic functions, to the case of piecewise smooth paths. *From now on, by path we mean piecewise smooth path.* As in the case of line integrals of holomorphic functions, the above integrals are invariant under reparametrisations of the path, but we will not belabour this.

Note that if ω is exact, $\omega = df$, we have

$$\int_\gamma \omega = \int_a^b \{\frac{\partial f}{\partial x}(\gamma(t))\frac{dx(t)}{dt} + \frac{\partial f}{\partial x}(\gamma(t))\frac{dy(t)}{dt}\}dt = \int_a^b \frac{d}{dt}(\gamma^* f)(t)dt = f(\gamma(b)) - f(\gamma(a))$$

so that $\int_\gamma \omega = 0$ for any closed path. Conversely (imitating what we did with holomorphic functions), if $\int_\gamma \omega = 0$ for any closed path γ, we can construct a function f such that $\omega = df$ by integrating from a chosen base-point. In other words,

Proposition 21.5. *A form ω on a domain $D \subset \mathbb{R}^2$ is exact iff $\int_\gamma \omega = 0$ for any closed path γ.*

21.4. **Locally, closed=exact.** The following is an analogue of Goursat's Theorem. Because we assume forms are C^∞ the proof is very simple.

Theorem 21.6. *Let D be a disc and ω a closed 1-form. Then the line integral of ω around any rectangle in D vanishes.*

Proof. For simplicity of notation, we assume the rectangle to have corners at $(0,0), (1,0), (1,1), (0,1)$, and we will assume that the integral is taken counterclockwise along the boundary, Γ. From the definition,

$$\int_\Gamma \omega = \int_0^1 \omega_x(s,0)ds + \int_0^1 \omega_y(1,t)dt - \int_0^1 \omega_x(s,1)ds - \int_0^1 \omega_y(0,t)dt$$

Now,

$$\begin{aligned}\int_0^1 \omega_y(1,t)dt - \int_0^1 \omega_y(0,t)dt &= \int_0^1 \{\omega_y(1,t) - \omega_y(0,t)\}dt \\ &= \int_0^1 \{\int_0^1 \frac{\partial \omega_y}{\partial x}(s,t)ds\}dt\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \omega_x(s,0)ds - \int_0^1 \omega_x(s,1)ds &= -\int_0^1 \{\omega_x(s,1) - \omega_x(s,0)\}ds \\ &= \int_0^1 \{\int_0^1 \frac{\partial \omega_x}{\partial y}(s,t)dt\}ds\end{aligned}$$

Changing the order of integration is certainly permissible since we are dealing with C^∞ functions, so we have the desired result. □

21.5. **On a simply connected domain, closed=exact.**

Theorem 21.7. *Let D be a simply-connected and ω a closed 1-form. Then the line integral of ω around any closed path in D vanishes.*

Proof. Let $\gamma : [0,1] \to D$ be a closed path. First, by "rounding the corners", we can assume that the path is a smooth, not just piecewise smooth, map from S^1, which we continue to denote by γ. Since D is assumed simply-connected, the map γ extends to a continuous map from the closed unit disc $\overline{D}_1$. But in fact [10] it extends to a smooth map, which we denote $\tilde{\gamma}$. Pulling back ω by $\tilde{\gamma}$ we get a closed 1-form on the open disc D_1, which we denote by $\tilde{\omega}$. The line integral of $\tilde{\gamma}$ around any closed path in D_1 vanishes, so in particular $\int_{S^1_{1-\epsilon}} \tilde{\omega} = 0$, where $S^1_{1-\epsilon}$ is the boundary of the disc of radius $1-\epsilon$. As $\epsilon \to 0$, this tends to $\int_{S^1} \gamma^*\omega$, which therefore has to vanish. □

Let us also note

Theorem 21.8. *Let D be a domain (not necessarily simply connected), and ω a closed 1-form. Let γ_0 and γ_1 be homotopic paths from P to Q. Then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.*

Proof. The path γ_0 followed by γ_1 can be thought of as a piecewise smooth map from S^1, and we need to show that $\int_{S_1} \omega = 0$. By assumption, the map from S^1 extends to a map from the closed unit disc. Now imitate the last proof. □

21.6. **Cauchy's theorem for simply connected domains.** We are now in a position to prove

Theorem 21.9. *Let f be a holomorphic function on a simply connected domain. Then f has a primitive. Equivalently, the line integral of f around any closed path vanishes.*

Proof. Write $f = u + iv$ as usual. Note that by analyticity of f, the real-valued functions u, v are C^∞. Consider the complex-valued form $f dz = (u + iv)(dx + idy) = (udx - vdy) + i(udy + vdx)$. The real valued forms $udx - vdy$ and $udy + vdx$ are both closed. (This is the content of the Cauchy-Riemann equations.) Therefore their line integrals around any closed path vanish. On the other hand, one checks from the definitions that

$$\int_\Gamma f dz = \int_\Gamma (udx - vdy) + i\int_\Gamma (udy + vdx)$$

[10] see `http://mathoverflow.net/questions/35198/smooth-homotopy-theory` and the references there, particularly, Hirsch's *Differential Topology*. By a smooth map from the closed disc we mean the restriction of a smooth map from a slightly larger open disc.

for any Γ, closed or otherwise. $\square$

21.7. **Simply connected domains.** To make use of the above results, we need to be able to recognise simply connected domains D. Here are some two sufficient criteria:

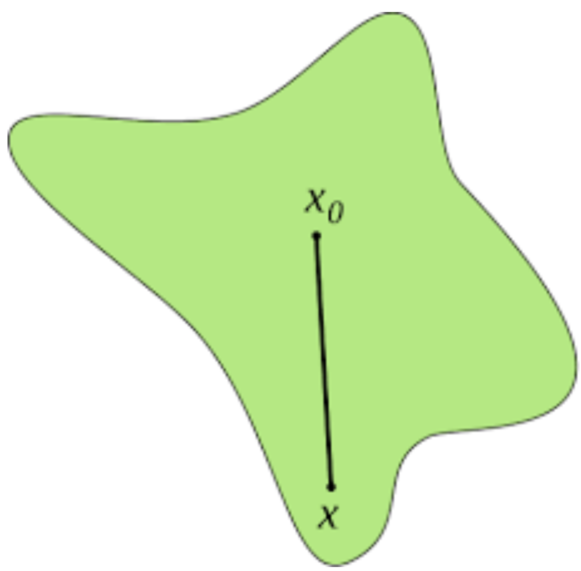
(1) if D is *starlike*, i.e., if there is a point $x_0 \in D$ such that contractions[11] centred on x_0 leave D invariant;

(2) if D is homeomorphic to a star-like domain.

The second criterion follows from the first. As for the first criterion: if D is starlike, and $\phi : S^1 \to D$ any map, $\phi_t = H_t \circ \phi$ gives a one-parameter family of maps interpolating between ϕ (when $t = 1$) and (when $t = 0$) the constant map which maps S^1 to x_0.

A disc is starlike, as is the following:

[11] that is to say, maps (parametrised by $t \in [0,1]$) $x \mapsto H_t(x) = x_0 + t(x - x_0)$


x_0
x

22. Residues, Residue Theorem

Suppose given a domain D, and sequence of distinct points $z_1, z_2, \dots, z_n, \dots$. For the moment we do not assume that the number of points is finite, but we do require that if the sequence has limit points, they do not belong to D. Suppose given a holomorphic function f on $D \setminus \{z_1, z_2, \dots, z_n, \dots\}$, with either poles or removable singularities at the points z_j.

We define the residue of f at z_j as follows. Let $\overline{D}_r$ be a closed disc of radius $r > 0$ centred at z_j, with r chosen small enough that all the points $\{z_k | k \neq j\}$ are in the complement of $\overline{D}_r$. Let S^1_r be the boundary of $\overline{D}_r$, oriented counterclockwise as usual. We define the *residue* of f at z_j by

$$Res_{z_j} f = \frac{1}{2\pi i} \int_{S^1_r} f(z) dz$$

We leave the following claims as exercises:

(1) The residue as defined above is independent of the choice of $r > 0$ as long as all the points $\{z_k | k \neq j\}$ are in the complement of $\overline{D}_r$,

(2) Suppose f has a pole of order n at z_j, with $n > 0$. By definition, this means that $(z - z_0)^n f(z) = g(z)$ is regular at z_j and $g(z_j) \neq 0$. Expanding g in a Taylor series, we see that f is of the form

$$f(z) = \frac{a_{-n}}{(z - z_0)^{-n}} + \dots + \frac{a_{-1}}{(z - z_0)} + h(z)$$

with h holomorphic in D_r. Then

$$Res_{z_j}(f) = a_{-1}$$

(3) f has a primitive on $D^*_r = D_r \setminus \{z_j\}$ iff $Res_{z_j}(f) = 0$.

We can now state and prove the Residue Theorem for discs. Let $D_R \subset D$ be a disc of radius $R > 0$ such that $\overline{D}_R \subset D$ and none of the points z_j lies on the boundary circle S^1_R (which, as always, we orient counterclockwise).

Theorem 22.1. *Let f be as above. Let Let $z_1, \dots, z_N$ be the poles of f contained in D_R. Then*

$$\frac{1}{2\pi i} \int_{S^1_R} f(z) dz = \sum_{j=1}^{N} Res_{z_j} f$$

Proof. To start with suppose that the points $z_1, \dots z_N$ have the following property: none of them is at the centre of D_R and for each z_j among them the slit ℓ_j joining z_j to the boundary circle S^1_R misses the other points $\{z_k | k \neq j, 1 \leq k \leq N\}$. The region consisting of D_R with the slits ℓ_j removed is starlike, and therefore simply connected. Hence the function f admits a primitive on this domain. By the “keyhole” argument that we have used before, the integral over the circle can be expressed as a sum of integrals over small circles centred at the z_j's, which yields the residue formula. If the points are not arranged in the above auspicious pattern, we use a deformation argument. $\square$

We can use residues to count zeroes and poles:

Theorem 22.2. *Let f and D_R be as above. Assume that none of the zeroes of f line on the circle S^1_R, and let $y_1, \dots, y_M$ be the zeroes of f in D_R. Then*

$$\frac{1}{2\pi i}\int_{S^1_R} f'(z)/f(z)dz = \sum_{h=1}^{M} \text{order of zero at } y_h - \sum_{j=1}^{N} \text{order of pole at } z_j$$