GAME THEORY

J. Von Neumann
'Theory of Games
and
Economic
Behavior'

Game of Mora

P_1 (1,1), (1,2), (2,1), (2,2).

P_2 (1,1), (1,2), (2,1) (2,2)

$$\begin{array}{cc} & P_2 \\ & \begin{array}{cccc}(1,1), & (1,2), & (2,1) & (2,2)\end{array} \\ P1 \begin{array}{c}(1,1)\\(1,2)\\(2,1)\\(2,2)\end{array} & \left[\begin{array}{cccc} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{array}\right]_{4\times 4} = A = -A^t \end{array}$$

$\downarrow$

Payoff Matrix wrt P_1

Aim:- P_1 wants to maximize his/her avg. and P_2 wants to minimize the same.

David Blackwell

Question:- Does $\exists$ a real number $\overset{\nearrow \text{value of the matrix game}}{v}$ for the given payoff matrix A of order $m \times n$, such that P1 has a 'strategy' which will fetch him/her on the average atleast v, no matter what P_2 does and can you say P_2 has a 'strategy' s.t which will restrict P1's avg. to at most v no matter what P1 does.

David Gale

Suppose $A = -A^T$, then $v = 0$.

Suppose $\exists$ a prob vector $x \in \mathbb{R}^n$.

such that $Ax \leq$ [illegible]. $\Rightarrow$ ~~$x^tA \leq 0$~~. $x^tA \geq 0$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq 0$$

$$\vdots$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n \leq 0$$

J. Von Neumann
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$= A = -A^t$

$\times 4$

the
matrix game
the given
, such that
fixed him/her
what to do
'strategy' s.t.
to at most

$A \geq 0$.

$\begin{pmatrix} \cdot \\ \cdot \\ \vdots \\ 0 \end{pmatrix}$.

This is called a mixed strategy.

$$\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} & \vdots & \\ & \vdots & \end{bmatrix} \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 1-x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2x_1+3(1-x_1) \\ 3x_1-4(1-x_1) \\ 0 \end{pmatrix}$$

$-5x_1+3 \leq 0 \Rightarrow x_1 \geq 3/5$

$7x_1-4 \leq 0$

$\Rightarrow x_1 \leq 4/7$

$3/5 \leq x_1 \leq 4/7$

– not possible

$$\begin{pmatrix} 0 \\ x_1 \\ 1-x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2x_1-3(1-x_1) \\ 0 \\ 0 \\ -3x_1+4(1-x_1) \end{pmatrix}$$

$5x_1 \leq 3$

$\Rightarrow x_1 \leq 3/5$

$4 \leq 7x_1$

$\Rightarrow x_1 \geq 4/7$

$\Rightarrow \frac{4}{7} \leq x_1 \leq 3/5$

$x_1 \in \left[\frac{4}{7}, \frac{3}{5}\right]$ is optimal.

G. Dantzig

Simplex method P_2

$$P_1 \quad \begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & 2 \\ P & 1 & 0 & -3 \\ S & -2 & 3 & 0 \end{array}_{3\times 3} = A = -A^t$$

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 & 3/6 \\ x_2 & 2/6 \\ 1-x_1-x_2 & 1/6 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(3/6, 2/6, 1/6)$ is the unique optimal strategy.

I. Kaplansky.

<u>Defn</u>:- Call $x = (x_1, x_2, \dots, x_m)$ is a mixed strategy if $x_i \geq 0 \ \forall \ i = 1, 2, \dots, m$. and $\sum_{i=1}^{m} x_i = 1$.

Call x a completely mixed strategy if $x_i > 0 \ \forall \ i = 1, 2, \dots, n$. and $\sum_{i=1}^{n} x_i = 1$

Call a matrix game A completely mixed if every optimal for either player is completely mixed.

GMTH (Lec 2)

P_1 : $\{1, 2, \dots, m\} = I$, $i \in I, j \in J$.
P_2 : $\{1, 2, \dots, n\} = J$.

$x = (x_1, x_2, \dots, x_n)$, $x_i \geq 0$, $\sum_{i=1}^{n} x_i = 1$.

(mixed strategy).

$y = (y_1, y_2, \dots, y_n)$, $y_j \geq 0$, $\sum_{j=1}^{n} y_j = 1$.

$A(x,y)$ = Expected payoff to P_1

$\overset{def}{=} \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i y_j$.

$X = \{x : x$ is a mixed strategy to P_1

$Y = \{y : y$ is a mixed strategy to P_2.

Obs:-

(1) X and Y are nonempty convex sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively

(2) X & Y are closed and bounded

(3) $A(x,y)$ is jointly cts over $X \times Y$.

upper value of the game.

$$U = \min_{y \in Y} \max_{x \in X} A(x,y).$$

$$L = \max_{x \in X} \min_{y \in Y} A(x,y)$$

Lemma:- $U \geq L$.

Fundamental result (two person matrix games):

Minimax theorem (Jon Neumann)

$U = L$.

Remark:- This common value $\vartheta = U = L$ is called the value of the game and it is unique.

J.

$i=1$.

rategy).

$=1$.

strategy to P_1

strategy to P_2.

$\mathbb{R}^m$ and $\mathbb{R}^n$

ed the

$$v = \min_y \max_x A(x,y) = \max_x \min_y A(x,y)$$

$$\therefore \quad = \max_x A(x,y^\circ) = \min_y A(x^\circ,y) = A(x^\circ,y^\circ)$$

Call y° optimal for P_2 if $A(x,y^\circ) \leq v \; \forall x$.

Call x° optimal for P_1 if $A(x^\circ,y) \geq v \; \forall y$.

__Dantzig's__ way __of looking at minimax "theorem"__

'Result of von Neumann': $\exists$ a unique real number v and a pair (x°, y°) such that

$$\sum_{i=1}^{m} a_{ij} x_i^\circ \geq v \quad \forall j = 1, 2, \ldots, n$$

$$\text{and } \sum_{j=1}^{n} a_{ij} y_j^\circ \leq v \quad \forall i = 1, 2, \ldots, m.$$

x, x'

$$\sum_{i=1}^{m} a_{ij} x_i \geq \lambda_1 \; \forall j. \quad \text{Suppose } \lambda_1 > \lambda_2.$$

$$\sum_{j=1}^{n} a_{ij} x_i' \geq \lambda_2 \; \forall j.$$

Wlog assume $a_{ij} > 0 \; \forall i,j$.

$\max \lambda$

subject to

$\sum a_{ij} x_i \geq \lambda \; \forall j = 1, 2, \ldots, n.$

$x_i \geq 0.$

$\sum_{i=1}^{n} x_i = 1$

$\lambda \geq 0.$

$\max \lambda.$

$$\sum_{i=1}^{m} a_{ij}\left(\frac{x_i}{\lambda}\right) \geq 1 \; \forall j.$$

$\xi_i = \frac{x_i}{\lambda}$

$\sum a_{ij} \xi_i \geq 0., \quad \xi_i \geq 0.$

$\sum \xi_i = \frac{1}{\lambda}.$

Reformulation:-

$$\min \Sigma \xi_i$$
subject to
$$\Sigma a_{ij} \xi_i \geq 1$$
$$\xi_i \geq 0.$$

Kaplansky Reference
(A contribution to Von Neumann)

Call $x = (x_1, x_2, \ldots, x_m)$ a completely mixed strategy if $x_i > 0 \ \forall \ i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_i = 1$.

Call "the matrix game A" is completely mixed if every optimal (for either player) is completely mixed.

Lemma 1: Let x^0 be an optimal strategy which is completely mixed.

Let y^0 be any optimal for P_2. then.

$$\sum_{j=1}^{n} a_{ij} y_j^0 \equiv v \ \forall \ i = 1, 2, \ldots, m.$$

Proof: Since y^0 is optimal for P_2, $\sum_{j=1}^{n} a_{ij} y_j \leq v \ \forall \ i$.

Suppose $\sum_{j=1}^{n} a_{ij} y_j^0 < v$.

$$\sum_{j=1}^{n} a_{ij} y_j^0 \leq v \text{ for } i = 1, 2, \ldots, i_0-1, i_0+1, \ldots, n$$

$A(x^0, y^0) = \Sigma\Sigma a_{ij} y_j^0 x_i^0 < v$. — Contradiction since $A(x^0, y^0) = v$.

Lemma 2: Assume every optimal of P_1 to be completely mixed.

Suppose $v = 0$.

then $m-1 \leq r(A) \leq (n-1)$.

"$r(A) \leq (n-1)$" follows from the previous lemma.

Corollary:-

If $m > n$, then P_1 has an optimal which is not completely mixed.

Lemma 3:- Suppose $m = n$. If P_1 has an optimal which is not completely mixed then P_2 also has an optimal which is not completely mixed.

$$\left(\frac{1}{2}, \frac{1}{2}\right) \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}_{2\times 3}$$

$(1/2, 1/2)$ is optimal and completely mixed.

$y' = (0,0,1)$ is optimal for P_2.

Main Theorem (Kaplansky)

Let $A \in \mathbb{R}^{m\times n}$. Assume $v = 0$.

Then the game is completely mixed if

(i) $m = n$.

(ii) $r(A) = n-1$

(iii) all the co-factors ~~a_{ij}~~ A_{ij} (of a_{ij}) are diff from zero and are of the same sign.

Ex: Let $A = -A^t$
Assume A is of even order
Show that A is not completely mixed