Brief outline

[Warning: This is unedited and uncorrected notes. B V Rao]

[179-197]

Stirling for $n!$; improper integrals; Trigonometric functions; improper integrals again [198-208]

improper integrals; gamma and beta integrals; $\int \frac{\sin x}{x}$ [209-220]

Home Assignment:

1. (221-226) 2. (227-230) 3. (231-235)

4. (236-239) 5. (240-244) 6. (245-248)

7. (249-253) 8 (254-258) 9. (259-263)

10. (264-267) 11. (268-271) 12. (272-276)

13 (277-280)

Midsem (281) semestral (??)

While “pleasure” and “enjoyment” are often used to characterize one’s efforts in science, failures, frustrations, and disappointments are equally, if not the more. common ingradeints of scientific experience. Overcoming difficulties, undoubtedly, contributes to one’s final enjoyment of success.

S. Chandrasekhar.

“If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.”

John von Neumann

Before beginning:

We are going to learn about real numbers and functions. After all, now a days you learn some calculus in school. Let us start at the beginning.

What are real numbers?

??

Am I a real number? Is this table a real number?

??

We appear unsure, Ok, let us see. What is the colour of your shirt?

Green.

Good. What is ‘colour”? What is ‘green colour’?

??

Again we appear unsure. Let us be clear, we know very few things. Indeed, colour is a difficult concept. We have learnt to identify green colour without really knowing what exactly it is. There is nothing wrong with it, except that sometimes we might go wrong.

If some one tried to explain what green colour is, he would probably start saying: you are able to see the shirt because light reflected from the shirt is reaching your eyes; you know light comes in several sizes, call it wave lengths or frequencies; the shirt material you have is absorbing all except a particular size. So just that size light is reflected reaching your eye, and this corresponds to green colour.

We would probably say to him: forget it, I know better, I do not need your explanation. I do not understand sizes of light and reflection and all that nonsense.

The same thing happens with real numbers too. We are familiar with them, we know how to use them and so on. Let us ask ourselves some simple questions.

Is there a number whose square is 3?

Yes. $\sqrt{3}$.

You are absolutely right, there is such a number, but I do not understand what you said.

It is $\sqrt{3}$.

Yes, I heard, but what is that number?

It is a number whose square is 3.

But this is precisely our question, whether there is such a number at all. If you first show me that there is such a number, then we all can name it $\sqrt{3}$.

Actually, we can write it: $1 \cdot 732 \cdots$.

If you multiply with itself, will you get 3?

Yes.

Can you write that number on the board and multiply with itself and show me you will get 3.

We can not write.

But do you know the number?

Yes.

Write it on the board.

It has infinitely many decimal places.

But do you know all the decimal places?

No.

You mean to say you do not know the number fully.

It is there, but we can not write it down.

How can I believe that it is there. You did not show that there is such a number: you can do this by writing the number OR by convincing me, by an argument, that there is such a number. You did neither.

??

OK, let us leave this discussion, accept that there is a number whose square is 3.

Is there a number whose square is -3?

No.

How come both questions appear similar, but for the first the answer is 'yes', while for the the second question it is 'no'.

Because square of any number is non-negative.

Good, but why is it so.

Because product of two positive numbers is positive and product of two negative numbers is also positive.

Very Good, we are heading some where. But why is this so?

We can actually multiply and see.

How do you multiply? We do not have any specific numbers before us.

We can take some numbers, multiply and see.

If you take some numbers and see the truth, why should others believe that it is *always* true?

??

Obviously we use certain properties of the collection of real numbers to arrive at these answers. What are those properties? Do we keep on bringing in new properties every time you want to answer a question or are there some properties listed once and for all (not depending on the question asked) which are used forever in the analysis? If so what are those rules? Does everything you know follow from those rules?

You have learnt in school, the meanings of 2^4, multiply 2 with itself four times (equals 16); or $(\sqrt{3})^5$, multiply $\sqrt{3}$ with itself five times (equals $9\sqrt{3}$ But what is the meaning of $(\sqrt{3})^{(\sqrt{5})}$? You can not say multiply $\sqrt{3}$ with itself $\sqrt{5}$ times!

We define it through limits.

Yes, So some of you know clearly, but many seem to be unsure.

Let us ask ourselves another question. you seem to know square roots exist for non-negative numbers. But do fifth roots exist? Is there a number x such that $x^{55} = 4$?

Yes, we use least upper bound to define.

Good, again some of you know exactly how we get those numbers we are looking for, but many are not sure.

You know how to add two numbers or a billion numbers. You also know

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = 2.$$

What is the meaning of adding infinitely many numbers?

We can keep adding, we get closer to 2, so the sum is nearly 2.

Wait a minute, is the sum exactly 2 or nearly 2?

Very close to 2.

How close?

Very close.

I do not understand what is meant by very close.

We can define using limits.

Yes, you are right. Our basic rules only tell us how to add finitely many numbers, adding infinitely many numbers is *not* part of basic rules. We have developed. the above series is simple. You can keep on adding and get

$$1,\ 3/2,\ 7/4,\ 15/16,\ 31/32\cdots$$

or equivalently

$$2-1,\ 2-\frac{1}{2},\ 2-\frac{1}{4},\ 2-\frac{1}{8},\ 2-\frac{1}{16},\ 2-\frac{1}{32}\cdots$$

the terms differ less and less from 2 as you keep adding. Here we are lucky, we could add numbers. Some times we can not add like this. Do you know if the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\cdots=?$$

converges?

Yes, it converges.

Now it is diffficult to keep on adding terms and see what is happening. Do you know the value of the sum.

Yes, $\pi^2/6$.

Very Good, you are right. Some of you have done extra reading in analysis, very nice.

You read about sine and cosine functions. Why do we need functions, why are they useful?

If you have a rod and heat it, you would like to know its temperature at time t, as a function of t.

Yes, good. You can think of many other things. Simplest is the simple pendulum. a bob hanging (nearly frictionless manner). You displace from vertical a little bit and see what happens. Its displacement at time t is an interesting quantity. Or when you discuss planetary motion, you describe using functions.

One of the first calculations made by people, was to find areas of plane figures. Archimedes (287 BC - 212 BC) — Greek mathematician, physicist, inventor — discussed the following problem, even before the concepts of real number, function, limits were invented.

Consider the plane, with x-axis and y-axis drawn as you did in high school. Now consider the region bounded by the positive x-axis, the line $x = 1$ and the parabola $y = x^2$ (between $x = 0$ and $x = 1$). What is its area?

This is how he argued.

[You should draw the x-axis, y-axis and the curve $y = x^2$ between $x = 0$ and $x = 1$.] We need the area, A, below this curve. Since we know areas of rectangles, let us see if we can get rectangles to approximate the area. Draw rectangles with bases

$$[0, 1/10],\ [1/10, 2/10],\ [2/10, 3/10], \cdots\cdots, [9/10, 10/10]$$

and heights

$$(1/10)^2,\ (2/10)^2,\ (3/10)^2,\ \cdots\cdots, (10/10)^2$$

respectively. You see that the area under the curve is covered by these rectangles. So

$$A \leq \frac{1}{10} \times (\frac{1}{10})^2 + \frac{1}{10} \times (\frac{2}{10})^2 + \frac{1}{10} \times (\frac{3}{10})^2 + \cdots + \frac{1}{10} \times (\frac{10}{10})^2.$$

Instead of ten rectangles, if you selected any integer n and covered by n rectangles with bases each of length $1/n$, we would obtain

$$A \leq \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}.$$

Archimedes knew

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Thus

$$A \leq \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

Archimedes knew that, whatever non-zero number is given, $1/n$ is smaller than that number for all large values of n. Using this he deduced that the right side above can be made arbitrarily close to 1/3. Thus $A \leq 1/3$. With a similar and smart argument, using the rectangles with the same bases but heights

$$(0/10)^2,\ (1/10)^2,\ (2/10)^2, \cdots\cdots, (9/10)^2.$$

he could deduce $A \geq 1/3$. He concluded that the area must equal 1/3.

Beginning:

Let us now start with the properties of real numbers that we shall use. Anything that we say must follow from these basic rules we agree now.

We have a set R with two operations + (addition: associates with every pair of elements of R an element of R) and . (multiplication; associates with every pair of elements of R an element of R) and a comparision relation $<$ (less than: this is not an operation as the other two, this only compares two elemets of R). The system is denoted by $(R, +, \cdot, <)$.

Axiom set I: Four rules for for addition (+).
(i) $x + y = y + x$;
(ii) $(x + y) + z = x + (y + z)$;
(iii) there is an element $a \in R$ such that $x + a = a + x = x$;
(iv) for each $x \in R$ there is an element y, depending on x, such that $x + y = y + x = a$.

The conditions above hold for all x, y, z in R. The a asserted in (iii) is unique. Indeed if there are two such, say a and b, then $a = a + b = b$. Here the first equality is by property of b and the second is by property of a. Since there is only one such, we denote it by 0. Also, for every x there is a unique y such that $x + y = 0$. For a given x, this y is denoted by $-x$.

Axiom set II: Four rules for multiplication $(\cdot)$.
(i) $x \cdot y = y \cdot x$;

(ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(iii) there is an element $c \in R, c \neq 0$ such that $x \cdot c = c \cdot x = x$;
(iv) for each $x \in R, x \neq 0$ there is an element y, depending on x, such that $x \cdot y = y \cdot x = c$.

As above we can show that the c asserted above is unique. This will be denoted by 1. For a given $x \neq 0$, the y asserted above is unique and is denoted by $1/x$ or $\frac{1}{x}$.

Axiom set III: one rule, says $(+, \cdot)$ are friendly.
$x \cdot (y + z) = x \cdot y + x \cdot z$.

Axiom set IV: two rules for $(<)$.
(i) For any x, y exactly one of $x < y$, $x = y$, $y < x$ holds.
(ii) If $x < y$ and $y < z$ then $x < z$.

Second rule says that the comparision is consistent, in the sense, if x is smaller than y and y smaller than z, then x is smaller than z. In practice, if three persons A, B, C are playing matches, it is quite possible that A wins over B and B wins over C, but this does not automatically imply that A wins over C. First rule says that any two elements can be compared. Taking y same as x, we see that we never have $x < x$. We use $x \leq y$ as abbreviation for '$x < y$ or $x = y$'. We also sometimes write $x > y$ to mean $y < x$. Similarly $x \geq y$ means $y \leq x$.

Axiom set V: Two rules, say $(<)$ is friendly with $(+)$ and $(\cdot)$.
(i) $y < z$ implies $x + y < x + z$;
(ii) $0 < x,\ 0 < y \Rightarrow 0 < x \cdot y$.

Finally Axiom set VI: one rule, says least upper bounds exist.
Let $S \subset R$ be non-empty.
Suppose S has an upper bound — which means $(\exists y)(\forall x \in S)(x \leq y)$.
Then S has a least upper bound — which means
$(\exists z)\ \{[\forall x \in S,\ x \leq z]\ \&\ [(\forall x \in S,\ x \leq y) \Rightarrow z \leq y]\ \}$.

This axiom is called least upper bound axiom or completeness axiom or continuity axiom. This axiom tells us that our geometric picture of real numbers as a line without breaks/gaps is justified.

Fix such a system $(R, +, \cdot, <)$ once and for all. Elements of R are called real numbers. sometimes we just xy for $x \cdot y$.

There is only one such system, in the sense, if there are two such systems you can establish a bijection between the elements of the two systems so that the operations as well as the order are preserved. Will such a system allow us to do everything we are used to do? Yes, instead of trying to convince you of *everything*, we shall do a few as a sample. There are two aspects you are used to. Algebraic (or arithmetic) manipulation of numbers and geometric visualization of numbers as a line. First we see some algebraic manipulations.

Arithmetic of numbers.

Fact 1. (for plus): (i) $x + y = x + z$ implies $y = z$. (ii) $x + y = x$ implies $y = 0$. (iii) If $x + y = 0$ then $y = (-x)$. (iv) $(-(-x)) = x$.

Here are the proofs: (i)

$$y = y + 0 = y + (x + (-x)) = (y + x) + (-x) = (x + y) + (-x)$$

$$= (x + z) + (-x) = (z + x) + (-x) = z + (x + (-x)) = z + 0 = z.$$

Decipher the rules used at each equality.

To see (ii) use (i) with $z = 0$ and to see (iii) use (i) with $z = -x$. To show (iv) observe $(-x) + x = 0$ and use (iii).

Fact 2. (for multiplication): (i) $x \cdot y = x \cdot z$ and $x \neq 0$ imply $y = z$. (ii) $x \cdot y = x$ and $x \neq 0$ imply $y = 1$. (iii) If $x \cdot y = 1$ and $x \neq 0$ then $y = (1/x)$. (iv) If $x \neq 0$, $(1/(1/x)) = x$.

Proof of this is similar to the above, carefully replace $(+)$ by $(\cdot)$. *Please* remember that just because I said that the proof is similar, you can not say *unless* you have gone through the proof and convinced yourself that it is indeed similar.

Fact 3. (1) $0 \cdot x = 0$. (ii) If $x \neq 0$ and $y \neq 0$ then $x \cdot y \neq 0$. (iii) $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$ (iv) $(-x) \cdot (-y) = xy$.

Proof: (i). We shall omit putting the symbol $\cdot$. Observe $0x = (0+0)x = 0x + 0x$ and use appropriate part of fact 1. To prove (ii), suppose $xy = 0$. Using $x \neq 0$

$$y = y1 = y(x\frac{1}{x}) = (yx)\frac{1}{x} = 0\frac{1}{x} = 0.$$

But $y \neq 0$. Thus $xy \neq 0$. To prove (iii), we need to show that $(-x)y$ is the additive inverse of xy. This follows from

$$xy + (-x)y = (x + (-x))y = 0y = 0$$

and appropriate part of fact 1. Other part of (iii) is similar. To prove (iv) use (iii) repeatedly and finally Fact 1.

$$(-x)(-y) = -(x(-y)) = -(-(xy)) = xy.$$

Fact 4. (for order): (i) $x > 0$ if and only if $-x < 0$. (ii) If $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$. If $x < 0$ and $y < z$ then $x \cdot y > x \cdot z$. (iv) $x \neq 0$ implies $x^2 > 0$. In particular $1 > 0$. (v) if $0 < x < y$ then $0 < (1/y) < (1/x)$.

Proof: Using an axiom, add (-x) to both sides of $x > 0$ to see $0 > -x$, to see part of (i). Other part is similar. Add $-y$ to both sides of $y < z$ to see $0 < (z - y)$. Now use $0 < x$ and axiom to see $0 < x(z - y) = xz - xy$ (why). Now add xy to both sides to get (ii). The other part of (ii) is similar. To see (iii), note that if $0 < x$ then axiom does it. If $x < 0$, then (i) tells us $(-x) > 0$. Hence $(-x)(-x) > 0$. But this is $x \cdot x$ by the last part of Fact 3. Since $0 < x$, $0 = x$ and $x < 0$ are the only possibilities this shows $x^2 > 0$. Since $1 = 1 \cdot 1$, and $1 \neq 0$, we get $1 > 0$.

Finally let $0 < x < y$. If $1/x < 0$, then $(-1/x) > 0$ so that $x(-1/x) = -(x \cdot 1/x) = -1 > 0$ which implies $1 < 0$, but both $1 > 0$ and $1 < 0$ can not simultaneously hold. Thus both $1/x > 0$ and $1/y > 0$ hold. Multiply both sides of $x < y$ with $1/x \cdot 1/y$ to see $1/y < 1/x$.

You should by now be convinced that much of the algebra you are using is actually a consequence of the few rules we have listed about the set R of real numbers.

Geometry of the number line.

Now let us see the geometric picture. We usually draw the number line as below.

———————————————————————————

We plot the number zero arbitrarily.

—————————————0 —————————————

If $0 < x$ we say x is positive, If $x < 0$ we say x is negative. Fact 4 tells us that x is positive if and only if $-x$ is negative. We make up our mind that numbers to the right of zero are positive and increase as you go away from zero. If you do not like, you can plot positive numbers to the left of zero.

So we plot 1 to the right of zero. Since $1 > 0$ we see $1+1 > 1+0 = 1$. We denote $1+1$ by 2, plot it to the right of 1. Since $2-1 = 1-0$, we plot 2 at the same distance from 1 as 1 is from zero. Now argue in the same way that $1+1+1 > 1+1$. Denoting $1+1+1$ by 3, wehave $3 > 2$. I leave it to your imagination now for negative numbers. There is one main problem: are all positive numbers somewhere on the right side of zero between the $1, 2, 3 \cdots$ we have plotted or are there numbers beyond all things we plotted? Luckily there is nothing beyond all these. To formulate this precisely, let us give the name

$$N = \{1, 2, 3, \cdots\cdots\}.$$

We denote elements of N by n, m etc for now. These are called natural numbers.

There is only one problem with this definition of the set N. I have used $\cdots\cdots$. But what do these dots denote, do we know what exactly is our set N? We should define N as the smallest subset of R which has the following two properties: (i) $1 \in N$ and (ii) $m \in N$ implies $m+1 \in N$. I think it is better to leave it at this stage. You can keep your mental picture that N consists of 1, $1+1$, $1+1+1$ etc etc (without understanding what is the meaning of etc). We shall return to this point at some stage later.

Fact 5. (Archimedean property of R) Let $x > 0$. Then there is an $n \in N$ such that $x < n$.

Suppose this is false. Then for every $n \in N$ we have $n \leq x$. Thus the set N is bounded above. Let s be its lub. But then $s - 1$ can not be upper bound of the set N. So $s - 1 < m$ for some $m \in N$ which means $s < m + 1$. But $m \in N$ tells us that $m + 1 \in N$. But then s can not be upper bound of N. This contradiction proves the result.

So there is no number beyond all the elements of N. We have the following picture.

_________-2_________-1_________0_________1_________2_________

We denote

$$Z = \{n : n = 0 \quad \text{or} \quad m \in N \quad \text{or} \quad -m \in N\}.$$

Elements of Z are called integers. We define

$$Q = \{\frac{m}{n} : m \in Z; n \in N\} = \{\frac{m}{n} : m, n \in Z, n \neq 0\}.$$

Elements of Q are called rational numbers. Just to remind you, m/n means $m \cdot (1/n)$ and of course $1/n$ is the multiplicative inverse of n. Now we start plotting these numbers on the line. Since $(1/2) + (1/2) = 1$ we plot $1/2$ midway between zero and 1. Now I leave the rest to your imagination.

_____-$\frac{3}{2}$_____-1_____-$\frac{1}{2}$_____0 _____$\frac{1}{2}$_____1_____$\frac{3}{2}$_____

The main point is that the mental picture about the arithmetic and geometry (whatever it may mean) of the number line we have, agrees with the picture given by the few rules we made above.

There are two very important questions now.
Does every real number correspond to a point on the line? Yes.
Does every point of the line correspond to a number? Yes.

The first answer comes out of the fact that every real number is lub of a set of rational numbers and the line is continuous, it has no gaps. The second answer comes out of the lub axiom. We shall not spend any more time on these matters.

From now on, we shall sometimes be brief with proofs. You should not just copy, you should understand the truth of each of the steps of the argument.

Fact 6: Given any two numbers $a < b$, there is a rational number x such that $a < x < b$.

Proof: Since $b - a > 0$ get, by archemidean property (apply with $x = 1/(b-a)$), an integer $n \in N$ such that $(b-a) > 1/n$. There is at least one integer k such that $k/n > a$ (apply archimedean peoperty with $x = na$, if $a < 0$, then $k = 0$ would do). Let us take least such integer m. Thus $m/n > a$ but $(m-1)/n \leq a$ Can $m/n > b$? If it were so then,

$$b - a \leq \frac{m}{n} - \frac{m-1}{n} = \frac{1}{n}.$$

But $(b-a) > 1/n$ leading to a contradiction. Thus we have $a < (m/n) < b$.

Are there numbers which are not rational? Let us assume for the time being, there is a positive number x such that $x^2 = 2$. Denote such a number by $\sqrt{2}$.

Fact 7. $\sqrt{2}$ is not a rational number.

You know this, please try to write a proof of this fact.

Fact 8. Given any two numbers $a < b$, there is an irrational number x such that $a < x < b$.

Apply fact 6 to the numbers $a/\sqrt{2}$ and $b/\sqrt{2}$ and argue.

So between any two numbers $a < b$, we can find both rational numbers as well as irrational numbers. Can we find out how many numbers of each kind are there? Both are infinite. What does this mean?

Let us say that emptyset has zero number of elements, $|\emptyset| = 0$. If A is a non-empty set, we say that A has n elements, in symbols $|A| = n$ if we can establish a map $f : A \to \{1, 2, \cdots, n\}$ which is one-to-one and onto. Here $n \in N$. We say that a set A is finite if $|A| = n$ for some $n = 0, 1, 2 \cdots$. Otherwise, we say that the set A is infinite.

We say that a set A is countably infinite if we can establish a map $f : A \to N$ which is one-to-one and onto. We say A is countable if either it is finite or countably infinite. Otherwise, we say that A is uncountable.

Here is a fact which you know and we shall prove soon:

Fact 9. The set of rational numbers is countably infinite, the set of irrational numbers is uncountable.

Given any number x, we define x^n for every $n \in N$ by induction as follows: $x^1 = x$. If we have defined x^n, we define $x^{n+1} = x^n \cdot x = x \cdot x^n$.

Fact 10. If $0 < x < y$ then $x^n < y^n$.

Prove it by induction. As suggested by one of you, you can also use the fact that product/sum of positive numbers is positive along with the formula

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}).$$

But since we need induction later on, it is better to know. Here it is:

Fact 11. (Mathematical induction)
Suppose that for every $n \in N$ we have a mathematical statement, say, P_n.
Suppose that P_1 is true.
Suppose that for every integer k, truth of $P_1, P_2, \cdots, P_k$ implies truth of P_{k+1}.
THEN: For every $n \in N$, the statement P_n is true.

There are two issues. Why should we believe this? How does it help us? Suppose that P_n is the statement $x^n < y^n$. Here x and y are fixed numbers

$0 < x < y$. This is a statement depending on n. Clearly P_1 is true. If we assume P_k is true, then

$$x^{k+1} = x^k x < y^k x < y^k y = y^{k+1}.$$

Mathematical induction is true for the following reason. Let

$$A = \{n : P_n \text{ is not true.}\}$$

If $A = \emptyset$, then P_n is true for every n. If $A \neq \emptyset$, then it has a first element (why?), say m. Can $m = 1$? No, because P_1 is true. Thus $m > 1$. Also m being the first element of A, we see $P_1, P_2, \cdots, P_{m-1}$ are true. But then P_m must also be true. Thus $m \notin A$ leading to a contradiction.

Fact 12. Fix $n \in N$. Let $x > 0$. Then there is a unique $y > 0$ such that $y^n = x$.

Proof: We show two things
(i) If $z > 0$ and $z^n < x$, then we can increase z a little bit so that its n-th power is still smaller than x. That is, there is $h > 0$ so that $(z+h)^n < x$.
(ii) If $z > 0$ and $z^n > x$, then we can decrease z a little bit so that its n-th power is still larger than x. That is, there is $h > 0$ so that $0 < z - h < z$ and $(z-h)^n > x$.

Let us see what happens if we have done this. Let

$$A = \{z > 0 : z^n < x\}$$

Is A non-empty? Yes, because if we take $z = x/(1+x)$, then $0 < z < 1$ so that $z^n < z < x$ and hence $z \in A$.

Is this set bounded above? Yes, because if $u = 1 + x$, then $u > 1$ so that $x < u < u^n$ and hence $u \notin A$. But then of course $v \notin A$ for every $v > u$; by fact 10. In other words u is an upper bound of the set A.

Thus A has lub, name it s.

Can $s^n < x$? No, because then by (i) above we can get $(s+h)^n < x$, which means $s + h \in A$. But then $s < s + h \in A$, contradicting that s is an upper bound of A.

Can $s^n > x$? No, because then by (ii), there is $h > 0$ with $0 < s - h < s$ so that $(s-h)^n > a$. This implies that $z < s - h$ for every $z \in A$. Thus $s - h$ is an upper bound of A contradicting that s is least upper bound.

There are only three possibilities: $s^n < x$, $s^n = x$, $s^n > x$. Since two of them are ruled out above, we must have $s^n = x$. completing the proof of Fact 12. Of course, the proof is complete provided we show the truth of the two statemens we made at the beginning of the proof.

Let us prove (i). Let $z > 0$ and $z^n < x$. We need $h > 0$ so that $(z+h)^n < x$. Let us make up our mind that we shall choose $0 < h < 1$.
Let us see what we want: $(z+h)^n - z^n < x - z^n$
That is $h[(z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1}] < x - z^n$.
This will be true if $hn(z+h)^{n-1} < x - z^n$.
This in turn will be true if we can choose $0 < h < 1$ so that $hn(z+1)^{n-1} < x - z^n$.
We are done with rough calculations. We say now, choose

$$0 < h < 1; \quad \text{so that} \quad h < \frac{x - z^n}{n(z+1)^{n-1}}.$$

Such a choice is possible, for example

$$h = \frac{1}{2}\min\left\{1, \frac{(x - z^n)}{n(z+1)^{n-1}}\right\}.$$

You can convert the above rough calculation into a proof of the fact that, with this choice of h, we do have $(z+h)^n < x$.

Let us finally prove (ii). Let $z > 0$ and $z^n > x$. We need h, so that $0 < z - h < z$ and $(z-h)^n > x$. Again let us do rough calculation.
We need $(z-h)^n - z^n > x - z^n$.
That is, $z^n - (z-h)^n < z^n - x$.
That is, $h[z^{n-1} + z^{n-2}(z-h) + \cdots + (z-h)^{n-1}] < z^n - x$.
This will be true if $0 < z - h < z$ and $hnz^{n-1} < z^n - x$.

Now rough calculation is over, we say choose

$$0 < h < z; \quad \text{and also} \quad h < \frac{z^n - x}{nz^{n-1}}.$$

As earlier, such a h can be chosen, for example, half of minimum of the two bounds we have above. The rough calculations above can be converted to showing that with this choice of h, (ii) is verified.

discussion of HA1.

Q2. For each $n \geq 1$, let A_n denote the set $\{1, 2, \cdots, n\}$.

We have a bijection $f : S \to A_{11}$ and a bijection $g : S \to A_{13}$. We want to show that this is not possible. What should we do?

The first step is to eliminate the unknown set S from the problem.

Step 1: We define a function $h : A_{11} \to A_{13}$ as follows. Take $x \in A_{11}$. There is one element $s \in S$ such that $g(s) = x$, because g is onto A_{11}. Also such a point is unique, because g is one-to-one. This s is denoted by $g^{-1}(x)$. We take $f(s)$ and this is our $h(x)$. Thus

$$h(x) = f(g^{-1}(x))$$

Let $y \in A_{13}$. There is one $s \in S$ with $f(s) = y$, because f is onto. Let $g(s) \in A_{11}$ be denoted by x. Then it is easy to see $h(x) = y$. In other words h is onto. Thus h is a function on A_{11} onto A_{13}. Next step shows there can not be such a function. [Actually this h is one-one too, though we do not need: If $x, y \in A_{11}$ and $x \neq y$, then $g^{-1}(x) \neq g^{-1}(y)$ and now use f is one-to-one to conclude $h(x) \neq h(y)$.]

Now we have a function from A_{11} ONTO A_{13}. Shall show this is not possible.

If we start associating with 1,2 etc of A_{11} an element of A_{13} we will run out of elements from A_{11} whereas elements of A_{13} are still left out.

Why? how do you know this happens?

Because A_{11} has 11 elements and A_{13} has 13 elements.

Firstly, I do not know ‘number of elements’. Even if I agree with you on a suitable definition, why should there be no such function? Actually, this was our main problem. We did, definitely, believe that A_{11} has 11 elements. So if we can establish a bijection between A_{11} with S then we felt eligible to say that S also has 11 elements, we decided to write $|S| = 11$. You are absolutely right. But to see that this is a sensible definition, we must make sure that there can not be bijection from S to both A_{11} and A_{13}. If this happens we do not know whether S has 11 elements or 13 elements.

But is it not obvious that A_{11} has 11 elements and A_{13} has 13 elements.

Hold on, I think we are going in circles. You are again using 'number of elements' and also the word obvious. Remember, to use an adjective like obvious, easy, trivial; you must first verfiy the truth of the statement; then depending on the nature of the argument needed to verify its truth (may be length, may be depth of facts used), you use these adjectives. We have not even verified the truth of the statement.

To make you appreciate the issue involved, let me rephrase the problem. Someone walks into this class room and tells us that he has a function from A_{11} onto A_{13}. If he shows you his function, you are sure, you can find an error. You might even think it is absolutely trivial for you to find his error. I believe you. But we have a road-block. He refuses to show us his function. Should we leave matters like this or are we capable of telling him that he is wrong, we can prove it to him without seeing his function? Yes, we accept the challenge of showing that he is wrong. Here is how we achieve it.

Step 2: For $m = 1, 2, \cdots$ let P_m be the following statement:
Whatever be $n > m$, we can not find a function on A_m *onto* A_n.
This is proved by induction.

<u>m=1</u>. Take $n > 1$. Let f be any map of A_1 to A_n. If $f(1) \neq 1$, then 1 is not in the range of f. If $f(1) = 1$, then 2 is not in the range of f; just remember $n \geq 2$ and $A_n = \{1, 2, \cdots, n\}$. Thus f can not be onto.

done for $m = 1, 2, \cdots, k-1$. Shall do for $\underline{m = k.}$ Let $n > k$ be fixed and if possible a function f on A_k onto A_n. We shall produce a function h on A_{k-1} onto A_{n-1}. But $n - 1 > k - 1$ and the statement P_{k-1} is true, leading to a contradiction.
Let $f(k) = a \in A_n$. We define h as follows. Take x with $1 \leq x \leq k-1$. If $f(x) < a$, then declare $h(x) = f(x)$; if $f(x) \geq a$ declare $h(x) = f(x) - 1$. It is clear that $h(x) \leq n-1$, so that h takes values in A_{n-1}. Is it onto A_{n-1}? Yes, to see this take y with $1 \leq y \leq n-1$. In case $y < a$, using the fact that f is onto A_n, we get x with $f(x) = y$. But then, $h(x) = y$ and of course $x \neq k$, so that $x \in A_{k-1}$. If $y \geq a$, then using the fact that $1 \leq y \leq n-1$ first conclude that $y + 1 \leq n$ and hence there is a $x \in A_k$ with $f(x) = y + 1$. Observe $x \neq k$ because $f(k) = a < y + 1$. Thus $x \in A_{k-1}$ and $h(x) = y$. Thus whatever be $y \in A_{n-1}$ there is $x \in A_{k-1}$ with $h(x) = y$.

Q3 . The reason we are going through this is the following. Many-a-times we 'think' we know certain things while we really do not know them. From school we know,

$$0.1 = \frac{1}{10}; \quad 0.12 = \frac{12}{100}; \quad 0.121 = \frac{121}{1000} \cdots\cdots.$$

Such an understanding is correct and enough in school. But then what is the meaning of $0.12121212 \cdots\cdots\cdots\cdots.$?

$$\frac{12121212 \cdots\cdots\cdots\cdots}{100000000 \cdots\cdots\cdots\cdots}$$

does not make sense. A perfectly equivalent, but probably looking complicated, way of putting the earlier understanding is the following.

$$0.1 = \frac{1}{10}; \quad 0.12 = \frac{1}{10} + \frac{2}{100}; \quad 0.121 = \frac{1}{10} + \frac{2}{100} + \frac{1}{1000}, \cdots\cdots.$$

With such an equivalent way of putting things, we can easily understand the infinite decimal expansion.

$$0.12121212 \cdots\cdots = \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \cdots\cdots.$$

Let us return to the problem. When $x = 0$, we see the required expansion by taking each $\epsilon_i = 0$. So let us now consider $0 < x \leq 1$.

How shall we get the decimal expansion?

??

You can give a try.

We take gif of 10x.

What is gif, I do not know.

greatest integer function.

Oh, let us denote it as $[x]$: the largest integer which is less than or equal to x. With this notation you are saying that $[10x]$ is the first decimal digit. What is the second digit?

We take $[10\{x - [10x]\}]$. This is second decimal digit.

Yes, you are right. But how do we show that the decimal expansion with these digits gives us the number x?

??

I shall put the exact same thing in a more picturesque way, you can see convergence too.

Given a number x with $0 < x \leq 1$, here is an algorithm for obtaining its decimal expansion. Denote by I_k the interval

$$I_k \quad = \quad \left(\frac{k}{10}, \ \frac{k+1}{10}\right]; \qquad k = 0, 1, \cdots, 9.$$

These intervals are disjoint, each having length 1/10. They make up all of (0,1]. Thus x must be in exactly one of these intervals. Let that interval be I_k and put $\epsilon_1 = k$. Since

$$\frac{\epsilon_1}{10} \quad < \quad x \quad \leq \quad \frac{\epsilon_1 + 1}{10},$$

we see that

$$0 \quad < \quad x - \frac{\epsilon_1}{10} \quad \leq \quad \frac{1}{10}.$$

Now divide this interval I_k (just remember now $k = \epsilon_1$) into ten parts by conisdering

$$I_{kl} \quad = \quad \left(\frac{k}{10} + \frac{l}{10^2}, \ \frac{k}{10} + \frac{l+1}{10^2}\right]; \qquad l = 0, 1, \cdots, 9.$$

Since $x \in I_k$ and the intervals $\{I_{kl} : 0 \leq l \leq 9\}$ are disjoint making up all of I_k, there is exactly one l such that $x \in I_{kl}$. Put $\epsilon_2 = l$ and immediately observe, as earlier, (recalling that $k = \epsilon_1$) that

$$0 \quad < \quad x \ - \ \frac{\epsilon_1}{10} \ - \ \frac{\epsilon_2}{10^2} \quad \leq \quad \frac{1}{10^2}.$$

By induction now one can obtain the required digits and show the stated properties too by using the inequalities deduced at each stage.

Suppose a number x has two different expansions

$$\cdot\epsilon_1\epsilon_2\epsilon_3 \cdots = x = \cdot\eta_1\eta_2\eta_3 \cdots.$$

Remember this means,

$$\frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2} + \frac{\epsilon_3}{10^3} + \cdots = x = \frac{\eta_1}{10} + \frac{\eta_2}{10^2} + \frac{\eta_3}{10^3} + \cdots.$$

Let $\epsilon_i = \eta_i$ for $i = 1, 2, \cdots k-1$ and $\epsilon_k < \eta_k$. (of course, η_k may be smaller. Either you can repeat the same argument, or, to start with itself, you can say the one having smaller value at the first digit where they differ, is denoted with epsilons). Thus, we see

$$\frac{\epsilon_k}{10^k} + \frac{\epsilon_{k+1}}{10^{k+1}} + \frac{\epsilon_{k+2}}{10^{k+2}} + \cdots = \frac{\eta_k}{10^k} + \frac{\eta_{k+1}}{10^{k+1}} + \frac{\eta_{k+2}}{10^{k+2}} + \cdots . (\spadesuit)$$

Observe that the left side of ($\spadesuit$) is at most, (use sum of geometric series)

$$\text{LHS of } (\spadesuit) \leq \frac{\epsilon_k + 1}{10^k}$$

and strict inequality holds if $\epsilon_m < 9$ for at least one $m > k$. Also

$$\text{RHS of } (\spadesuit) \geq \frac{\eta_k}{10^k}$$

and strict inequality holds if $\eta_m > 0$ for at least one $m > k$.

But LHS and RHS of ($\spadesuit$) are equal and $\epsilon_k < \eta_k$; we must have
$\epsilon_k + 1 = \eta_k$; and
$\epsilon_m = 9$ for each $m > k$; and
$\eta_m = 0$ for each $m > k$.

Q6. For the time being do not try the part about algebraic numbers. It is simple, but needs a new idea. Interestingly, if a and b are algebraic, we can not *simply* show that $a + b$ is algebraic; we need to show (at the same time) that several things are algebraic and $a + b$ is one of them.

exponentiation. (continued).

$\underline{x^n \text{ for } x \neq 0 \text{ and } n \in N.}$

This is defined by induction: $x^1 = x$ and if we have defined x^n for $n = 1, 2, \cdots k$ then we put $x^{k+1} = x^k \cdot x$. Do you know how to prove the law of indices: $x^{n+m} = x^n \cdot x^m$ and $(xy)^n = x^n y^n$. If you never saw a proof, now is the time to write a proof of this fact.

$\underline{x^n \text{ for } x \neq 0 \text{ and } n \in Z.}$

For $n \in N$ it is defined above. For $n = 0$, we put $x^0 = 1$. For $n < 0$ it is defined as $x^n = (1/x)^{-n}$. Prove the law of indices.

$x^{1/n}$ for $x > 0$ and $n \in N$.

In the last class we proved the existence of exactly one number $y > 0$ such that $y^n = x$. We define this y as $x^{1/n}$, also denoted as $\sqrt[n]{x}$.

If $0 < x < y$ then $x^{1/n} < y^{1/n}$. Also for any $x, y > 0$, $(xy)^{1/n} = x^{1/n}y^{1/n}$.

x^r for $x > 0$ and $r \in Q$

Let $r = m/n$ where m, n are integers and $n \geq 1$. we put $x^r = (x^m)^{1/n}$. This makes sense because $x^m > 0$ whatever be $m \in Z$. Is this well defined? Someone expreses the same rational m/n as $(km)/(kn)$ (here $k \in N$), and calculates. Will he get the same answer? In other words

$$(x^m)^{1/n} = (x^{km})^{1/(kn)}?$$

To see that this is indeed true, first observe that the right side is positive. If we show that its n-th power equals x^m, then the right side equals the left side (by uniqueness of n-th roots).

$$(x^{km})^{1/(kn)}(x^{km})^{1/(kn)} \cdots (n - \text{times}) = (x^{km}x^{km} \cdots (n - \text{times})^{1/kn}$$
$$= (x^{kmn})^{1/(kn)} = [(x^m)^{kn}]^{1/(kn)} = x^m.$$

Here the last equality is by uniqueness of (kn)-th root.

More generally, suppose a rational number is expressed as m/n and also p/q where all m, n, p, q are integers and $n, q \geq 1$. — for example $(4/6)$ and $(6/9)$. The question is whether

$$(x^m)^{1/n} = (x^p)^{1/q}?$$

This follows from

$$(x^m)^{1/n} = (x^{mq})^{1/(nq)}; \qquad (x^p)^{1/q} = (x^{np})^{1/(nq)}; \qquad mq = np.$$

Thus x^r is well defined for every $x > 0$ and rational number r. Verify the laws of indices:

$$x^{r+s} = x^r x^s; \qquad (xy)^r = x^r y^r; \qquad x^r = (1/x)^{-r}.$$

$$x > 1, r < s \Rightarrow x^r < x^s; \qquad x < 1, r < s \Rightarrow x^r > x^s.$$

x^a for $x > 1$ and $a \in R$

For $x > 1$, we define $x^a = lub\{x^r : r \in Q, r \leq x\}$. If we take any rational t, with $a-1 < t < a$, then x^t is in the above set; if we take any rational s with $a < s < a+1$ then x^s is an upper bound for that set. Thus lub is sensible. Also if a happens to be rational then this definition gives the same answer as the definition in the previous clause. Show this by using the last property (monotonicity) stated above.

x^a for $x > 0$ and $a \in R$

If $x > 1$ the above clause defines x^a. If $x = 1$, we put $x^a = 1$ whatever be a. If $0 < x < 1$, we put $x^a = (1/x)^{-a}$. This makes sense because, $1/x > 1$ and above clause applies.

The laws of indices still hold. We postpone this study. We need to develop some other stories of importance.

modulus.

For a real number x, we define $|x|$, modulus of x, as follows: if $0 \leq x$ then $|x| = x$ while we define $|x| = -x$ in case $x < 0$.

Fact: (i) $|x| \geq 0$; $|x| = 0$ iff $x = 0$. (ii) $|x| = |-x|$. (iii) $|x+y| \leq |x|+|y|$.

(i) If $x \neq 0$ then $-x \neq 0$ as well and hence if $x \neq 0$ then $|x| \neq 0$.
If $x > 0$, then $|x| = x > 0$. If $x < 0$, then we know $|x| = -x > 0$.
(ii) If $x > 0$, then $-x < 0$ so that $|-x| = -(-x) = x = |x|$.
If $x < 0$, then $-x > 0$ so that $|-x| = -x = |x|$.
If $x = 0$, then $-x = 0$ so that $|-x| = 0 = |x|$.
(iii) case 1: $x \geq 0$, $y \geq 0$. Then $x + y \geq 0$ so that

$$|x + y| = x + y = |x| + |y|.$$

case (ii): $x < 0$, $y < 0$. Then $x + y < 0$ so that

$$|x + y| = -(x + y) = (-x) + (-y) = |x| + |y|.$$

case (iii): $x > 0$ and $y < 0$. Then $x + y \geq 0$ and $x + y < 0$ are both possible. In case $x + y \geq 0$, use $y < -y$ to see

$$|x + y| = x + y \leq x + (-y) = |x| + |y|.$$

In case $x + y < 0$, use $-x < x$ to see

$$|x + y| = -(x + y) = (-x) + (-y) < x + (-y) = |x| + |y|.$$

case (iv): $x < 0$ and $y > 0$. argue as above.

cardinality.

We shall now return to cardinality of sets: Recall the definitions of finite, countable, uncountable sets. Generally, $|A|$ is called the cardinality of the set A. Of course, as of now this is defined for finite sets. For countably infinite set $\aleph_0$ denotes its cardinality. For R, the symbol is $\mathbf{c}$. Even among uncountable sets, there are several kinds; we shall not enter that topic. Here are some simple rules.

Fact:

(i) Let $n \geq 1$ be an integer. If $|A| = n$ and $f : A \to B$ a bijection, then $|B| = n$.

(ii) If A is countably infinite, $f : A \to B$ is a bijection, then B is also countably infinite.

(iii) The set $N = \{1, 2, 3, \cdots\}$ is countably infinite.

(iv) Any infinite subset of N is countably infinite. Any subset of a countable set is countable.

(v) The set of integers, Z is countable.

(vi) The set of pairs $S = \{(m, n) : m, n \in N\}$ is countable.

(vii) The set of positive rational numbers is countably infinite.

(viii) The set of rational numbers is countable.

(ix) If for each $n = 1, 2, 3, \cdots$ we have countable sets, then their union $\cup A_n$ is also countable set.

Proof: (i) Since $|A| = n$, fix a bijection $g : \{1, 2, \cdots, n\} \to A$. Then $h(x) = f(g(x))$ gives a bijection of $\{1, 2, \cdots, n\}$ to B.

(ii) Similar proof as above holds.

(iii) Identity map $f(x) = x$ shows this.

(iv) Let A be an infinite subset of N. Define, by induction, for each $n \in N$, an element $a_n \in A$ as follows:

$$a_1 = \min A; \qquad a_{n+1} = \min\left(A \cap \{a_1, \cdots, a_n\}^c\right); \quad n > 1.$$

Recall that if $A \subset N$ is not empty, then it has a first element; that is, $a \in A$ such that $x > a$ for all $x \in A; x \neq a$. This is denoted as min above. Also the set being infinite, this process can be continued for ever. Define

$$f : \{1, 2, \cdots\} \to A; \qquad f(n) = a_n.$$

This is one to one. Indeed, if $m < n$, then by definition of a_n we see that $a_n \neq a_m$. To see that this is onto, we proceed as follows. Observe that $a_1 \geq 1$. This, in turn, implies

$$x \in A,\ x \neq a_1 \Rightarrow x > 1.$$

Using induction, we can prove that $a_n \geq n$ for every n. This, in turn, gives

$$x \in A,\ x \neq a_1, a_2, \cdots, a_n \Rightarrow x > n.$$

Thus,

$$x \in A \Rightarrow x = a_n \quad \text{for some } n.$$

This shows that f is onto A.
The last part is easy now.

(v) Define $f : Z \to N$ by $f(n) = 2^n$ if $n \geq 1$; $f(n) = 3^{-n}$ if $n \leq -1$ and $f(0) = 1$. Then f is a one-to-one function. But of course it is *not onto* N. However it is onto its range, namely, the set $A = \{f(x) : x \in Z\}$. Since A is countably infinite, so is Z.

(vi) The function $f(m, n) = 2^m 3^n$ establishes a on-one function on S onto its range contained in N.

(vii) The set of strictly rational numbers is identified with the set of pairs (m, n) wher $m \geq 1$, $n \geq 1$ are integers and have no common factors. But then this is a subset of the earlier set.

(viii) Proof similar to (v). Fix a bijection f from strictly positive rationals to N. For $x \in Q$, put

$$g(x) = 2^{f(x)} \text{ if } x > 0; \quad g(x) = 3^{-f(-x)} \text{ if } x < 0; \quad g(0) = 1.$$

(ix) We can safely assume that $A_n \neq \emptyset$ for each n (why?). For each $n = 1, 2, \cdots$, fix a function $f_n : A_n \to N$ a one-one function. We are not saying 'onto' because in case A_n is finite it would be impossible. Define a function $f : \cup A_n \to N$ as follows. Take $x \in \cup A_n$. Let i be the first integer such that $x \in A_i$. Put

$$f(x) = 2^i 3^{f_i(x)}.$$

This establishes a one-one function onto its range contained in N.

Fact: The interval $(0, 1)$ is uncountable. The set of real numbers is uncountable. The set of irrational numbers is uncountable. The set of transcendental numbers is uncountable.

Proof: <u>the set $(0, 1)$ is not countable.</u>

If possible, let $f : N \to (0, 1)$ be any function. We show that there is an element of $(0, 1)$ which is not in the range of f; showing, in particular, that there can not be a bijection between the two sets. The method of proof is called Cantor's diagonal argument, which is simple, yet powerful.

For each n, fix a decimal expansion of the number $f(n)$. Let us define numbers ϵ_n for $n = 1, 2, \cdots$ as follows. $\epsilon_n = 7$ if the n-th digit in the decimal expansion of $f(n)$ is different from 7, while $\epsilon_n = 8$ if the n-th digit in the decimal expansion of $f(n)$ equals 7.

Let

$$a = \frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2} + \frac{\epsilon_3}{10^3} + \cdots.$$

The series converges and defines a number.
Is it between zero and one? Yes, actually it is between 7/10 and 9/10.
Does it differ from every number in the range of f? Yes, from the number $f(n)$ it differs at the n-th decimal digit.
Since some numbers have more than one expansion, how can we say that this number a is different from each $f(n)$ just because there is a difference in one decimal place? Since the expansion of a does not end either with zeros or with

nines, a has exactly one expansion and it does not agree with any of the $f(n)$.

R is not countable.
If R were countable its subset $(0,1)$ would be countable too.

Set of transcendental numbers is not countable.

The set of algebraic numbers is known to be countable; if the set of transcendental numbers is also countable then the union of these two sets would be countable set too.

discussion of HA2.

Q7: Expansions to base other than 10 (to which we are used to) are also important. The basic idea is same as for the decimal expansion. For example, here is how you get binary expansion. If $x = 0$ take all digits zero. Let us consider x, $0 < x \leq 1$.

Divide the interval $(0, 1]$ into two parts: $(0, 1/2]$ and $(1/2, 1]$. If x is in the first part (left) declare $\epsilon_1 = 0$ and if it is in the second part (right) declare it to be 1. Observe $|x - \epsilon/2| \leq 1/2$. Divide each of these two first level intervals into two halves. Declare $\epsilon_2 = 0$ or 1 according as the point x is in the left or right second level intervals within the first level interval. Observe $|x - \epsilon_1/2 - \epsilon_2/2^2| \leq 1/2^2$. continue and complete proof.

Again the division points are esentially the ones having more than one expansion.

Q9: To show that a non-empty set bounded below has greatest lower bound.

Take all lower bounds of S, denote this set by T.

Is T non-empty? Yes, by hypothesis there are lower bounds for S.

Is T bounded above? Yes, any point $x \in S$ is an upper bound, because if y is any lower bound of S then $y \leq x$ for every point x of S.

Let z be least upper bound of T.

Is z a lower bound of S? Yes, if there is a $x \in S$ such that $x < z$, then every poiny of S being an upper bound of T, we see z can not be least upper bound of T, a contradiction.

Is z greatest lower bound of S? Yes, If there is a lower bound y of S with $z < y$ then $y \in T$ and z is not an upper bound of T, a contradiction.

Q14: Given s is lub of A and t is lub of B, to show $s + t$ is the lub of $C = \{x + y : x \in A, y \in B\}$.

Is $s + t$ an upper bound of C? Yes, if $x \in A$ and $y \in B$, then $x \leq s$ and $y \leq t$ so that $x + y \leq s + t$.

Let $\epsilon > 0$. Are there points of C above $s + t - \epsilon$? (we are using the criterion already derived for lub). Yes, because there is $x \in A$ with $x > s - \epsilon/2$

and $y \in B$ with $y > t - \epsilon/2$. This $x + y \in C$ will do.

Q13: We now have $C = \{xy : x \in A; y \in B\}$. Of course we were told A and B have only positive numbers.

If A or B consists of only one point, namely, $\{0\}$ then easy to see that C also consists of one point $\{0\}$ and so the result is true. We assume that both A and B have strictly positive numbers. Hence $s > 0$ and $t > 0$.

Is st an upper bound of C? Yes everything being positive $x \leq s$ and $y \leq t$ implies that $xy \leq st$.

Let $\epsilon > 0$, Are there points of C above $st - \epsilon$. (we are using the criterion for lub derived already). We can safely assume that $st - \epsilon > 0$, otherwise any non-zero point of C would do. Since we are in the 'multiplicative set up' (an intuitive expression, just to motivate what we are doing), we shall write $st - \epsilon$ as αst with $0 < \alpha < 1$. We are just saying take $\alpha = (st - \epsilon)/st$. Express $\alpha = \alpha_1 \alpha_2$ with $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$. Can we do this? Yes. If you wish take $\sqrt{\alpha}$ as both α_1 and α_2. (Did you realize we are imitating the $\epsilon/2$ argument of the previous 'additive set up'?). Of course, you can take any α_2 with $0 < \alpha < \alpha_2 < 1$ and put $\alpha_1 = \alpha/\alpha_2$. Since $\alpha_1 s < s$ and $\alpha_2 t < t$ get $x \in A$ and $y \in B$ with $x > \alpha_1 s$ and $y > \alpha_2 t$ so that $xy > \alpha st$.

Is this result true without assuming that we have sets of positive numbers? Not necessarily, take $A = B = [-1, 0]$.

set operations.

In the last class we talked about cardinality of sets, countable union of sets etc. Since some of you are not sure about set theoretical jargon, let us briefly digress a little bit and recall certain definitions and notations.

As I said in the first class, we shall not define what is a set. We still proceed with our understanding that a set is a well-defined collection of objects. Given something, we should be able to say whether it is in the set or not. If we can not tell, then we do not know what our set is. In that case any discussion of such a set does not make sense (essentially because, we would not know what we are discussing about!)

I have used too much English, but do not worry. We can, if challenged, precisely define what a set is. But we decided not to do it now.

Suppose A is a set. To express that an object x is in the set A, we use $x \in A$. To say x is not in the set A we use $x \notin A$.

Suppose A and B are two sets. We say $A \subset B$ (read: A is a subset of B, or A is contained in B) in case the following happens: whenever an object is in A, then it is in B also. In symbols $x \in A$ implies $x \in B$. Sometimes the same thing is expressed by writing $B \supset A$ (read: B is superset of A or B contains A or B includes A). For example

$$\frac{1}{2} \in (0,1); \quad 2 \notin (0,1); \quad (0,1) \subset [0,1]; \quad [0,1] \supset (0,1).$$

Suppose A and B are two sets. Then their union is the collection of all objects which either belong to the set A or objects which belong to the set B. This union includes all objects which are in both the sets, because such an object satisfies both the clauses. In symbols

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

The intersection of the two sets, is the collection of those objects which belong to both the sets. In symbols

$$A \cap B = \{x : x \in A \text{ and } x \in B.\}$$

The concepts of union and intersection can be defined for any collection of sets, not necessarily two. Suppose I have a collection of sets. Then their union is the set of all objects which belong to *at least one* of the sets in the collection. The intersection of the collection of sets consists of those objects which belong to *all* sets in the collection.

If we have a sequence of sets $A_1, A_2, \cdots$, one set for each $n \in N$, we denote their union by $A_1 \cup A_2 \cup A_3 \cup \cdots$ or $\cup A_n$ or $\bigcup\limits_{n \geq 1} A_n$. Similar notation is used for intersection; $A_1 \cap A_2 \cap A_3 \cap \cdots$, or $\cap A_n$ or $\bigcap\limits_{n \geq 1} A_n$.

Real Numbers.

Some of you may still not be feeling comfortable with our real numbers. After all, we are used to thinking real numbers $3, 355/576, \sqrt{29}$ as 'real' things (with flesh and blood, as if). But now I said: take a set R with the operations of addition $+$, multiplication $\cdot$, and comparision $<$, satisfying the rules we prescribed; and elements of the set are called real numbers. You might feel some sheen has disappeared. Before giving further explanation, let me remind you the following. We have immediately observed that whatever we have so far done (or are doing) with numbers are all correct with our present rules and so we can continue doing without any hesitation. The geometric picture that we have in mind representing numbers as points on a line is still correct and we continue to think of numbers placed on a line. There is absolutely no cause for alarm.

To re-inforce these things, we have also agreed to denote the multiplicative identity by the usual 1; $1 + 1$ by the usual 2, $1 + 1 + 1$ by the usual 3 and so on; the same things we have been using all along. The number which when multiplied by 5 gives 1, is still denoted by $1/5$. We did not bring any unknown notation or novel objects. Thus keep away any fears you may have.

Then why did we do this? The answer lies in the first lecture. We want to understand what we are talking about. It is important to have a clear (at least, as clear as possible) idea of real numbers. Let us not enter into philosophy, but just pause for a moment to think. Afterall, you see green coloured shirt, green coloured sari, green coloured leaf and from these you abstract (without your knowledge) the concept 'green colour' and use this word. Really speaking, you never see green colour in practice (closest to seeing green colour is experiments with prism). Again by seeing green colour, red colour, yellow colour etc you make a further abstraction and invent the concept of colour. You never see something and say 'yes, this is colour'. In the same way 'three' is an abstraction. You can have three apples, you can have three goats, but you never have just three. Afterall, what is 'three'; think about it; some write it as 3 and some as III; there are still other ways of writing it.

In a sense, we have laid down the ground rules, that is all, no more. We have made clear to ourselves, what can be assumed and what can not be. Did not our school teachers lay down the rules? Whatever they taught us is completely correct, there is nothing wrong. They did an excellent job. We have been given working knowledge, we have been working with numbers without making mistakes. Did we understand fully why certain things work. Let me give an example.

One of the simplest things we learnt is addition. To add 7 and 8 we did the following: kept eight fingers in mind, opened seven fingers, Thus we have 7 and 8. we need to put them together to add. So started counting after the eight fingers $9, 10, \cdots, 15$. Suppose you wanted to add 67 and 58. Then how do we go? Strictly speaking, we should bring 67 things, say beads, and 58 beads; put them together and count the total. But some one was smart and found out that there is a simpler way. You can carry out this procedure by adding *just* single digits at a time. Thus you add $7 + 8 = 15$, put down 5 and say 1 is *carried over*. Then you add digits at the next place $6 + 5 = 11$ and now add the carry over to this to get $11 + 1 = 12$. Put it down getting the answer 125. Did we ever ask: what is this carry over? why does this method give correct answer? It was not expected at such an immature level. Of course, we are now mature and should ask. Should we not? Similarly, if we want to multiply 67 by 58 we should bring 58 bundles of beads each bundle having 67 beads; put all of them together and count. But some smart fellow found a better way, is it not?

When you learn matrices, you find that sometimes $xy = yx$ is false! But this is one of our rules for numbers. As you learn more, you will find more and more funny things.

sequences.

After understanding numbers, there are two paths one can take. Some enter the path of continuous functions, differentiable functions; and integration and then turn to sequences and series of numbers. Others enter the path of sequences, series and then turn to functions. Both paths are fine, but I prefer the second path. This is because, I believe that sequences are simpler than functions. There is something in support of the first path too. In school, you probably never used the word sequence; but you definitely used the word function. You are familiar with sine, cosine, exponential functions. So perhaps taking the first path may appear as continuation of the high school course.

I consider 'sequences' as still understanding numbers. Let us consider real numbers. How many can we see? Well, all rational numbers 4/5 or 889/101 etc. Can we name others? Yes, for example you can show that exactly one positive number satisfies $x^2 - 2 = 0$ and you can say consider *that* number. Or you can show that there are exactly two positive numbers satisfy the equation $x^4 - 5x^2 + 6 = 0$ and you can say consider *larger of the two positive* numbers that satisfy this equation. In other words, you can name algebraic numbers.

If you are allowed geometric constructions, you can show that for any circle, the ratio of circumference to diameter, does not depend on the circle and you can ask me to consider that ratio. But at this moment let us not talk about geometric constructions.

Thus, there are very few numbers which we can name. How do we name, understand, and work with other numbers. How do we discover numbers? Afterall, there are uncountably many numbers which are not algebraic. We follow the usual procedure, an unknown thing is described in terms of known things. We show that every number has a sequence of rational numbers converging to it (and only to it), thus explaining that number in terms of rationals. Of course, such a description is too general to be of value. Better descriptions give better understanding. We shall also use sequences to discover new numbers.

What is a sequence.

Definition: A sequence is a function $f : N = \{1, 2, \cdots\} \to R$. We are talking only about sequences of real numbers. Later (when we are more mature) we talk about sequences of functions etc.

Since it is a function defined on a very simple set, one uses a better notation than the abstract symbol f. Let us denote the value $f(n)$ by x_n. The sequenec is denoted as (x_n) or $\{x_n\}$ or $(x_n : n \geq 1)$ etc. The meaning is that it is the function whose value at n is the number x_n. In other words, we specify the value at n, namely, x_n call it the n-th term of the sequence. Here are some examples:

$$x_1 = 1, x_2 = 1/2, x_3 = 1/3, \cdots, x_n = 1/n \quad \text{or} \quad \{\frac{1}{n}\}.$$

$$x_1 = -1, x_2 = 1/2, x_3 = -1/3, \cdots, x_n = (-1)^n 1/n \quad \text{or} \quad \{(-1)^n \frac{1}{n}\}.$$

$$x_1 = -1, x_2 = +1, x_3 = -1, \cdots x_n = (-1)^n \quad \text{or} \quad \{(-1)^n\}.$$

$$x_1 = (1/2), x_2 = (1/2)^2, x_3 = (1/2)^3, \cdots x_n = (1/2)^n \quad \text{or} \quad \{\frac{1}{2^n}\}.$$

Remember the word sequence has within it an implicit order on the terms. You have to distinguish it from the set which you may think of by putting all the values x_n together. For example, for the first sequence above, if you make a set by putting all the terms you get the set $\{1, 1/2, 1/3, \cdots\}$ but there is no implicit order on the members of this set. Just because you have written in a particular order, you should not be under the illusion that there is an order and that everyone would write in the same order. A set is simply a collection of objects and there is no a priori order for its elements. If you consider the third sequence above, the set consists of just two points $\{\pm 1\}$. The first sequence above is different from the sequence

$$x_1 = 1/2, x_2 = 1, x_n = 1/n \quad \text{for } n \geq 3.$$

There is something interesting with the first sequence. If you plot them on the number line, then the numbers of the sequence are getting closer and closer to zero. So is the second sequence. Sometimes it is to the left of zero and sometimes it is to the right of zero. But no matter which side it is, the numbers are getting again closer and closer to zero. On the other hand, the third sequence does not get closer to any number what-so-ever. We shall make this concept of 'getting closer to something' precise and give a name to it.

Definition: A sequence (x_n) is said to converge/ approach/tend to a number x if given any $\epsilon > 0$ there exists an integer n_0 such that $|x_n - x| < \epsilon$ for all $n \geq n_0$. That is, $x - \epsilon < x_n < x + \epsilon$ for all $n \geq n_0$. We write $x_n \to x$.

What this means is the following: No matter what the amount of error prescribed is, the terms of the sequence are for ever close to x after some stage, close meaning within the error prescribed. If the error prescribed is $\epsilon > 0$ the terms are within $x - \epsilon$ to $x + \epsilon$. The only thing to note is that we do not allow ϵ to be zero.

if you do not like error etc, you can think of $x-\epsilon$ and $x+\epsilon$ as walls built around x. Thus no matter what walls are erected around the point x, the sequence stays within the two walls eventually. Of course, you may say we have symmetrically placed walls, at $x-\epsilon$ and $x+\epsilon$. It does not matter, suppose that you give me any walls, say $x-\delta_1$ and $x+\delta_2$ around x, where $\delta_1, \delta_2 > 0$. We take $\epsilon = \min\{\delta_1, \delta_2\}$ and apply the definition to get an n_0 and observe that after the stage n_0 the sequence stays within the given walls.

Definition: We say that a sequence (x_n) is convergent if there is some number x such that $x_n \to x$.

Let us first show that a sequence can not converge to two different limits.

Fact: *Suppose that $x_n \to x$ and $x_n \to y$. Then $x = y$.*

Proof: if possible let $x < y$. Then take $\epsilon = (y-x)/2 > 0$. Then both the intervals $(x-\epsilon, x+\epsilon)$ and $(y-\epsilon, y+\epsilon)$ should contain the sequence eventually. But the intervals are disjoint and this can not happen.

Examples:
The sequence $(1/n)$ converges to zero.

In fact given any $\epsilon > 0$, we already knew that there is an integer n_0 such that $n_0 > 1/\epsilon$, that is, $1/n_0 < \epsilon$. But then $-\epsilon < 1/n < \epsilon$ for all $n \geq n_0$.

Fact: *Fix number $0 < a < 1$. The sequence (a^n) converges to zero.*
In particular. the sequence $(1/2^n)$ converges to zero.

Let $1/a = 1+h$ where $h > 0$. Remember, $(1+h)^n \geq nh$, by binomial expansion, so that

$$a^n = (\frac{1}{1+h})^n \leq \frac{1}{n}\frac{1}{h}.$$

Given $\epsilon > 0$ we can choose n_0 so that $1/n < \epsilon h$ for all $n \geq n_0$. For this n_0 we see $a^n < \epsilon$ for all $n \geq n_0$.

Sometimes careful observation of the proof gives better results.
If $0 < a < 1$, the sequence (na^n) converges to zero.

Use exactly same steps as above. But say $(1+h)^n \geq \binom{n}{2}h^2$

$$na^n = n(\frac{1}{1+h})^n \leq \frac{n}{\binom{n}{2}h^2} = \frac{1}{n-1}\frac{2}{h^2}$$

Given $\epsilon > 0$, choose n_0 so that $1/(n_0 - 1) < \epsilon h^2/2$. this will do.

If $a > 1$, then the sequence $(\sqrt[n]{a})$ converges to 1.

Since $\sqrt[n]{a} > 1$, let $\sqrt[n]{a} = 1 + h_n$ where $h_n > 0$. Thus

$$a = (1+h_n)^n \geq nh_n; \quad h_n \leq \frac{a}{n} \to 0.$$

Thus $\sqrt[n]{a} = 1 + h_n \to 1$. We have used the simple fact that $1 + h_n \to 1$ using that $h_n \to 0$.

When $a = 1$, clearly $\sqrt[n]{a} = 1$ and hence converges to 1. *This happens also for* $0 < a < 1$. This follows by a rseult we prove soon.

Example: $\sqrt[n]{n} \to 1$.

This follows from careful observation of the earlier proof. Let $\sqrt[n]{n} = 1+h_n$ where $0 < h_n < 1$ for $n > 1$. To see that $h_n < 1$ just note that $2^n > n$. Thus for $n \geq 2$ we have

$$n = (1+h_n)^n \geq \binom{n}{2}h_n^2; \quad h_n^2 \leq \frac{n}{\binom{n}{2}} = \frac{2}{n-1} \to 0.$$

Thus $h_n^2 \to 0$. From this we can deduce that $h_n \to 0$ as follows. Let $\epsilon > 0$ Choose n_0 so that $h_n^2 \leq \epsilon^2$ for $n \geq n_0$. Clearly then, $0 < h_n < \epsilon$ for $n \geq n_0$.

Definition: We say that a sequence (x_n) is increasing if $x_1 \leq x_2 \leq x_3 \leq \cdots$. Of course, if you want to avoid using dots, then you can rephrase to say: $x_n \leq x_{n+1}$ for every $n \geq 1$.

We say that a sequence (x_n) is decreasing if $x_1 \geq x_2 \geq x_3 \geq \cdots$. Of course, if you want to avoid using dots, then you can rephrase to say:

$x_n \geq x_{n+1}$ for every $n \geq 1$.

A sequence is monotone if it is either increasing or decreasing sequence. Note that in the definition of increasing sequence we did not use strict inequalities. Thus a constant sequence is both increasing and decreasing. In fact, if a sequence (x_n) is both increasing and decreasing, then there is a number a such that $x_n = a$ for every $n \geq 1$.

Definition: A sequence is bounded above if the set consisting of terms of the sequence is bounded above. Similarly, we can define the concept of bounded below and bounded.

The sequence $x_n = 1/n$ is a decreasing sequence. The sequence $y_n = (-1)^n$ is not monotone. Both sequences are bounded. Note that for the sequence (y_n) the set consisting of its terms has only two members in it, namely $+1$ and -1. If $z_n = 1/n$ when n is odd, $z_n = -n$ when n is even; then the sequence (z_n) is bounded above but not bounded below. Here is an important fact.

Fact:*Every increasing sequence which is bounded above is convergent, in fact, it converges to its supremum.*

Proof: Let (x_n) be an increasing sequence and s be its supremum. Let $\epsilon > 0$. We show n_0 such that $x_n \in (s-\epsilon, s+\epsilon)$ for all $n \geq n_0$. This is easy. Since $s - \epsilon$ is not an upper bound for the sequence, there is a term of the sequence, say $x_{n_0} > s - \epsilon$. The sequence being increasing we conclude that $x_n > s - \epsilon$ for all $n \geq n_0$. Of course, $x_n \leq s < s + \epsilon$ for all n.

Fact:*Every decreasing sequence which is bounded below is convergent, in fact, it converges to its infimum.*

Example: Let

$$x_1 = 1;\ x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}, \quad n \geq 1.$$

Then the sequence (x_n) is increasing and bounded above and hence converges.

$$x_{n+1} = x_n + \frac{1}{n!}$$

shows that the sequence is increasing. Since $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \geq 2^{n-1}$, we see using sum of finite geometric series,

$$x_n = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-2}} \leq 3.$$

The limit of this sequence is denoted by e.

Example: Let

$$y_n = (1 + \frac{1}{n})^n.$$

Then the sequence (y_n) is increasing and bounded above and hence converges.

Proof: Using binomial theorem

$$y_n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots$$
$$+ \frac{1}{n!}(1 - \frac{1}{n}) \cdots (1 - \frac{n-1}{n}).$$

Observe that y_{n+1} is obtained by replacing n by $n+1$. In particular, we see that y_{n+1} has one extra positive term than y_n. Moreover each of the preceding terms are more than the corresponding terms of y_n. Thus the sequence is increasing. We also see that $y_n \leq x_{n+1}$ and since the (x_n) sequence is bounded above, we conclude that the same bound shows that (y_n) sequence is bounded above too.

Soon we shall see that the limit of this sequence is also same as the earlier number e.

First let us ask if the notion of convergence is friendly with the operations we have. For example if two sequences converge, would their 'sum sequence' converge to the expected number? Yes.

Fact: *Suppose* $x_n \to x$ *and* $y_n \to y$.
Then $x_n + y_n \to x + y$; $-x_n \to -x$; $x_n - y_n \to x - y$
Heere by the sequence $(x_n + y_n)$ we mean the obvious thing, namely, the sequence whose n-th term is $x_n + y_n$.

Proof: Let $\epsilon > 0$. We show n_0 such that $|(x_n + y_n) - (x + y)| < \epsilon$ for all $n \geq n_0$. To do this, get m such that $|x_n - x| < \epsilon/2$ for $n \geq m$ and get k such that $|y_n - y| < \epsilon/2$ for $n \geq k$. Take $n_0 = \max\{m, k\}$. If $n \geq n_0$, then

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $|(-x_n) - (-x)| = |x_n - x|$ second statement is clear. Either you can repeat the above proof for $x_n - y_n$ or use the first two statements toi conclude the third statement.

The concept of convergence is friendly with multiplication also. First we need to observe an auxiliary fact of independent interest.

Fact: *A convergent sequence is bounded.*

Proof: Let $x_n \to x$. Taking $\epsilon = 1$, first get n_0 so that $|x_n - x| \leq 1$ for $n \geq n_0$. Thus we have $|x_n| \leq |x_n - x| + |x| \leq |x| + 1$ Thus the number

$$M = \max\{|x|_1, |x|_2, \cdots, |x|_{n_0}, |x| + 1\}$$

will show boundedness of the sequence.

Fact: *Suppose* $x_n \to x$ *and* $y_n \to x$. *Then* $x_n y_n \to xy$.
If each $x_n \neq 0$ *and* $x \neq 0$, *then* $1/x_n \to 1/x$.
As earlier $(x_n y_n)$ is the sequence whose n-th term is the product $x_n y_n$. Similarly $(1/x_n)$.

Proof: Let $\epsilon > 0$. We show n_0 such that for $|x_n y_n - xy|\epsilon$ for $n \geq n_0$. Furst observe that

$$|x_n y_n - xy| \leq |x_n y_n - x_n y + x_n y - xy| \leq |x_n||y_n - y| + |y||x_n - x|.$$

Since the convergent sequence (x_n) is bounded, fix M such that $|x_n| \leq M$ for all n. Choose m and k so that

$$|y_n - y| \leq \frac{\epsilon}{2(M+1)}, \quad n \geq m; \text{ and } |x_n - x| \leq \frac{\epsilon}{2(|y|+1)}, \quad n \geq k.$$

Choose $n_0 = \max\{m, k\}$ and verify this does.

To prove the second part, observe

$$|\frac{1}{x_n} - \frac{1}{x}| = \frac{|x_n - x|}{|x_n||x|}.$$

First we show that there is a number $a > 0$ such that $|x_n| \geq a$ for all n and $|x| \geq a$. If this is done we can argue required convergence as follows. Fix $\epsilon > 0$. Since $x_n \to x$, fix n_0 such that $|x_n - x| \leq a^2\epsilon$ for $n \geq n_0$. This will do in view of the equality displayed above and the fact that $1/(|x_n x|$ is at most $1/a^2$.

But to get such an a is easy. Let us consider the case $x > 0$. get n_0 such that $x_n \in (x/2, 3x/2)$ for $n \geq n_0$ (if you take $\epsilon = x/2$ then this interval is just $(x - \epsilon, x + \epsilon)$). Thus $x_n > x/2$ for all $n \geq n_0$. Now take a as

$$a = \min\{|x_1|, |x_2|, \cdots, |x_{n_0}|, \frac{x}{2}\}.$$

This completes the proof.

Finally, does convergence respect order relation? It respects $\leq$ and $\geq$ but not $<$ and $>$.

Fact: *Suppose $x_n \to x$ and $y_n \to y$.*
If $x_n \leq y_n$ for each n then we have $x \leq y$.
If $x_n < y_n$ for each n, we can not in general conclude that $x < y$.

Proof: The second statement is easily observed by taking $x_n = 0$ for all n and $y_n = 1/n$. To prove the first statement, it is enough to show that if $z_n \to z$ and each $z_n \leq 0$, then $z \leq 0$. If $z > 0$, then you see that the interval $(z/2, 3z/2)$ does not contain any point of our sequence, because $z/2 > 0$. Now take $z_n = x_n - y_n$and conclude the result.

Fact: Limit of the sequence $y_n = (1 + \frac{1}{n})^n$ is also e.

Proof: Recall that e is the name given to the limit of the sequence (x_n) where

$$x_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}.$$

We have seen that $y_n \leq x_{n+1}$ so that $\lim y_n \leq \lim x_{n+1} = e$. If we show that $\lim y_n \geq e$ we can conclude equality. For this it is enough to show that , for each k the following holds.

$$\lim y_n \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}.$$

Because, the right side is x_{k+1} and if it is smaller than the number $\lim y_n$ for each k then so will be the sup of the sequence x_{k+1} and this is precisely e. For every $n > k$ we have

$$\begin{aligned}
y_n &= 1 + 1 + +\frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots \\
&\quad + \frac{1}{n!}(1 - \frac{1}{n}) \cdots (1 - \frac{n-1}{n}). \\
&\geq 1 + 1 + +\frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots \\
&\quad + \frac{1}{k!}(1 - \frac{1}{n}) \cdots (1 - \frac{k-1}{n}).
\end{aligned}$$

Since k is fixed, let us denote the last expression by z_n. Thus we have $y_n \geq z_n$ for every $n > k$. Whatever be n, the number z_n is sum of a fixed number $k+1$ terms and each term converges , so

$$\lim z_n = 1 + 1 + +\frac{1}{2!}) + \frac{1}{3!} + \cdots + \frac{1}{k!}.$$

Since $y_n \geq z_n$ for all $n \geq k$ we conclude (?) that $\lim y_n \geq \lim z_n$ completing the proof.

We have only proved that if $x_n \geq y_n$ for every n, and if limits exist, then $\lim x_n \geq \lim y_n$. But is this true if we only have $x_n \geq y_n$ only for $n > k$? Yes, consider the sequences $u_n = x_{k+n}$ and $v_n = y_{k+n}$ for $n \geq 1$. then we see $u_n \geq v_n$ for every n. The question is whether these new sequences have the same limits as the original ones? Yes. This is easy.

Fact: *Limits will not change if we delete or add or alter finitely many terms.* That is Let $x_n \to x$.

Put $u_n = x_{n+10000}$ for $n \geq 1$. Then $u_n \to x$.

Let $v_n = x_{n-10000}$ for $n > 10000$ and any numbers of your choice for $n \leq 10000$. then $v_n \to x$.

Let $w_n = x_n$ for $n > 10000$ and any numbers of your choice for $n \leq 10000$ Then $w_n \to x$.

Proof: Do it.

Fact: *Given any real number x, there is a sequence of rational numbers (r_n) such that $r_n \to x$.*

You can take a rational r_n in the interval $I_n = (x - \frac{1}{n}, x + \frac{1}{n})$. This will do because, given any $\epsilon > 0$, you get n_0 such that $1/n_0 < \epsilon$ then for any $n > n_0$ the interval I_n is contained in I_{n_0} which is contained in $(x - \epsilon, x + \epsilon)$.

Alternatively you can consider the decimal expansion. For example if $x > 0$ and $x = n \cdot \epsilon_1 \epsilon_2 \cdots$ and consider the sequence of rationals

$$n, \quad n + \frac{\epsilon_1}{10}, \quad n + \frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2}, \quad \cdots.$$

Actually, each of the expansions, including the continued fraction expansion, are providing a simple sequence of rationals converging to the number x.

If you wish you can get an increasing sequence of rationals as follows. Pick a rational r_1 in $I_1 = (x - 1, x)$. If $r_1 < x - 1/2$ then choose a rational r_2 in $I_2 = (x - 1/2, x)$. If $r_1 > x - 1/2$, then choose r_2 in (r_1, x) and continue. You should be able to write down and show that the resulting sequence is increasing and converges to x.

You can also do the following. Choose a rational $s_n \in (x - 1/n, x)$. Of course, the sequence (s_n) is definitely converging to x but need not be increasing. so put $r_n = \max\{s_1, s_2, \cdots, s_n\}$. This sequence would do.

Thus every real number can be explained using appropriate sequence of rational numbers. This will be a way of discovering new numbers as happened with e.

Yu must keep in mind that the theorem above is only a reassurance that every number has rational numbers as close to it as we please. It is only theoretical, in the sense, each of them assumes that you know the number before hand. for example, the first method asks you to take a number within $x-1/n$ and $x+1/n$. This is possible only if you already knew the number x. For example how do you pick up such a rational in $(e-0.000001, e+0.000001)$.

Similarly, the decimal expansion method works if you already knew the decimal expansion. Of course we do not know. so this theorem is only a reassurance that the rational numbers we all know can be used to describe other numbers.

Cantor intersection Theorem:

One way to get numbers is to produce convergent sequences. Thus we can regard a convergent sequence as defining that point to which it converges. Here is another way of producing numbers. Any decreasing sequence of closed bounded intervals with length decreasing to zero would all have exactly one common point. Thus such a sequence of intervals can be regarded as defining that unique number which is common to all those intervals.

Fact: Let $[a_n, b_n]$ for $n \geq 1$ be a sequence of intervals, where a_n, b_n are all real numbers. Suppose that the sequence of intervals is decreasing, that is, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for $n \geq 1$. If $b_n - a_n \to 0$, then there is exactly one point which is common to all these intervals.

Proof: First note the following. If $[a, b] \supset [c, d]$, then both c and d are in the interval $[a, b]$ so that $a \leq c \leq d \leq b$. Thus we have

$$a_1 \leq a_2 \leq a_3 \cdots \cdots \leq b_3 \leq b_2 \leq b_1.$$

Thus any b_i is an upper bound for the increasing sequence (a_n). So $a_n \to s$ where s is supremum of the sequence (a_n). In particular $a_n \leq s$ for each n. Moreover, each b_i being an upper bound for the sequence (a_n) we see that $s \leq b_n$ for each n. In other words $a_n \leq s \leq b_n$ for each n, showing that the point s is in all the intervals. Let x be any other point, say $s < x$. since $b_n - a_n \to 0$, fix k large so that $b_k - a_k < (x - s)/4$. Since $s \in [a_k, b_k]$ if $x \in [a_k, b_k]$ then we would have $x - s \leq b_k - a_k$, contradiction.

Just like Cantor diagonal argument which we used to show uncountability of real number system, the above argument is also powerful and appears in several contexts.

Cauchy sequences:

How do we know if a given sequence (x_n) converges? One way is to take each real number x and see if $x_n \to x$. If the answer is yes for some number x, then stop and say that the sequence converges. If the answer is no for every x, say that the sequence does not converge. But it is difficult to test every x. Moreover, in this procedure we are looking outside the sequnece.

Is there a way to tell whether the sequence converges, by just looking at the terms of the sequence. Yes. Here it is. If a sequence converges, then the terms are getting close to something. And hence, they must be getting close among themselves. the interesting point is that if the terms of a sequence are getting closer among thmeselves, then the terms are actually getting close to some thing.

Fact: A sequence (x_n) converges iff the following holds. Given $\epsilon > 0$, there is an integer $n_0 \geq 1$ such that $|x_m - x_n| < \epsilon$ for every $n, m \geq n_0$.

Proof: Let $x_n \to x$. Let $\epsilon > 0$. Get n_0 so that $|x_n - x| < \epsilon/2$ for $n \geq n_0$. Clearly, if $m, n \geq n_0$ we have

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, suppose that the sequence satisfies the given condition. We exhibit a number x and show that $x_n \to x$.

Step 1: Take $\epsilon = 1$ and get n_1 so that $|x_m - x_n| < 1$ for $m, n \geq n_1$, in particular, we have $|x_{n_1} - x_n| < 1$ for every $n \geq n_1$. Thus if $c_1 = x_{n_1} - 1$ and $d_1 = x_{n_1} + 1$, then the interval $[c_1, d_1]$ includes all points x_n of the sequence with $n \geq n_1$.

In general, take $\epsilon = 1/k$ and get n_k so that $|x_m - x_n| < 1/k$ for $m, n \geq n_k$. Thus if $c_k = x_{n_k} - 1/k$ and $d_k = x_{n_k} + 1/k$, then the interval $[c_k, d_k]$ includes all points x_n of the sequence with $n \geq n_k$.

Observe that the interval $[c_k, d_k]$ has length $2/k$ so that lengths of thses intervals are converging to zero. However,these intervals may not be decreasing. So let us put

$$a_k = \max\{c_1, c_2, \cdots, c_k\} \quad b_k = \min\{d_1, d_2, \cdots, d_k\},$$

so that

$$[a_k, b_k] = [c_1, d_1] \cap [c_2, d_2] \cap \cdots \cap [c_k, d_k].$$

Of course, it is not clear if this is meaningful, for example, is $a_k \leq b_k$? Yes, in fact if n is larger than all of $n_1, n_2, \cdots, n_k$, then x_n is in all the intervals $[c_1, d_1], \cdots, [c_k, d_k]$ showing that $a_k \leq b_k$. Not only that, the interval $[a_k, b_k]$ contains all x_n with n larger than $p_k = \max\{n_1, n_2, \cdots, n_k\}$. By construction, these intervals are decreasing and lengths converge to zero. Thus Cantor theorem applies to provide us with a unique point x.

Step 2. The sequence (x_n) converges to x. To see this, let $\epsilon > 0$. Choose k so that $[a_k, b_k] \subset (x - \epsilon, x + \epsilon)$. For example, as soon as $b_k - a_k < \epsilon/4$ this will be true because $x \in [a_k, b_k]$. But then for all $n > p_k$ we have $x_n \in [a_k, b_k]$ and hence $x_n \in (x - \epsilon, x + \epsilon)$. This completes the proof.

The property of sequences that appeared in the above result is important enough to warrant a name.

Definition: A sequence (x_n) is Cauchy sequence if the following holds. Given $\epsilon > 0$, there is n_0 so that $|x_m - x_n| < \epsilon$ for all $m, n \geq n_0$.

This means that terms of the sequence are getting closer among themselves. Given any closeness (that is, $\epsilon > 0$) there is a stage after which any two terms of the sequence are close up to the given ϵ. With this notation, the fact observed above can be restated as follows:

Fact: *A sequence converges if and only if it is a Cauchy sequence.*

This is one way to verify the convergence of a sequence. Just show it is Cauchy sequence. Sometimes, this is much simpler than searching for a number and showing that the sequence converges to that number. This is precsiely the point. Without knowing where it converges, we will be able to conclude that it converges. sometimes the limit may be a number that we have not seen before, that is, a new number is discovered.

more on lub axiom:

We used the lub axiom to show that every bounded monotone sequence converges, which helped in the Cantor intersection theorem, which, in turn, lead to the conclusion that Cauchy sequences converge. Actually all these statements are equivalent. You may feel that the lub axiom is unnatural, or you do not like it. But you may like one of the consequences that we just mentioned or you may feel it is more natural. Then you can take that as an axiom in place of the lub axiom. We make it precsie now. You can ignore this section, its intent is exactly what I said just now, nothing more.

Fact: Let us delete the lub axiom from the set of axioms for real number system. Instead, assume the following.

If $[a_n, b_n]$ for $n \geq 1$ is a decreasing sequence of intervals
and if $b_n - a_n \leq 1/n$, then
there is exactly one point common to all these intervals

Conclusion: The lub axiom holds.

Proof:
First observe that the hypothesis implies the Archimedean property. This is because we know that the intervals $[-1/(2n), +1/(2n)]$ all contain the point zero and hence can not contain any other point. Thus, given $x > 0$ we see that the number $1/x$ is outside one of these intervals which provides you an integer $2n$ larger than x.

Let S be a non-empty set which is bounded above. Need to exhibit a number s which is an upper bound of S and any other upper bound of S is larger than s.

We take an upper bound b of S and a point $a \in S$. Thus the interval $[a, b]$ has points of S. We denote this interval as $[a_1, b_1]$. That is, set $a_1 = a$ and $b_1 = b$. Note that

$[a_1, b_1]$ contains points of S and no point of S is larger than b_1.

Consider the two intervals (left half) $[a_1, (a_1 + b_1)/2]$ and (right half) $[(a_1 + b_1)/2, b_1]$. If right half has points of S take it as $[a_2, b_2]$. Otherwise take the left half as $[a_2, b_2]$. Since $[a_1, b_1]$ has points of S, we conclude that if the right half does not contain points of S, then left half must necessarily contain points of S. Thus we have

$[a_2, b_2]$ contains points of S and no point of S is larger than b_2.

If we have obtained $[a_i, b_i]$ for $1 \leq i \leq k-1$ such that for each $i \leq k-1$

(i) The interval $[a_i, b_i]$ is either the left half or right half of $[a_{i-1}, b_{i-1}]$.
(ii) $[a_i, b_i]$ contains points of S and no point of S is larger than b_i.

Then, we consider the right half of $[a_{k-1}, b_{k-1}]$ if it contains points of S, otherwise consider the left half and designate it as $[a_k, b_k]$. Then the condition above holds for $i = k$ also showing that we can continue constructing a sequence of intervals satisfying the above two conditions for every i. Eventhough their lengths may not satisfy the required hypothesis, we can get a point common to all these as follows. The n-th interval has length $(b-a)/2^{n-1}$. Using $2^n > n$ and Archimedean property get k so that k-th interval has length smaller than 1 and consider only intervals after this stage. You see the hypothesis on lengths is satisfied.

so that we have exactly one point s common to all therse in tervals.

We shall now show that s is an upper bound of S. Let $x > s$. So fix i such that $x \notin [a_i, b_i]$. Of course, s is in this interval, x is not, $x > s$ will force $x > b_i$. But no point of S is larger than b_i. Thus $x \notin S$. Similarly, if $x < s$, then you get an i such that, $x < a_i$. Since there are points of S in this interval, x can not be an upper bound of S. Thus s is lub.

Fact: Let us delete the lub axiom from the axioms of real number system. Instead, assume the following:

Every increasing sequence bounded above converges.

Conclusion: The lub axiom holds.

Proof: Again, we start showing that Archimedean property holds. Since the sequence $(x_n = n;\ n \geq 1)$ is increasing. It can not converge because for any x, the interval $(x - 1/4, x + 1/4)$ can contain at most one integer. Thus, (x_n) is not bounded above. In other words given any $x > 0$, there is n such that $n > x$.

Now we take a non-empty set S which is bunded above. Proceed as in the earlier proof, get $[a_i, b_i]$. Observe that the sequence (a_i) is increasing and bounded above; each b_i is an upper bound. Use hypothesis to get its limit s.

You only need to note that $a_i \leq s \leq b_i$ for each i to conclude that s is lub of S.

Fact: Let us delete the lub axiom from the axioms of real number system. Instead, assume the following:

(i) Archimedean property holds and (ii) every Cauchy sequence converges.

Conclusion: The lub axiom holds.

Proof: Again start with a non-empty set S bounded above and do the construction to get intervals $[a_i b_i]$ for $i \geq 1$. We only need to show that (a_i) is a Cauchy sequence. If this is done it converges to a pont s and one can show that $a_i \leq s \leq b_i$ for all i and then conclude that s is lub of S. But length of the k-th interval is $(b-a)/2^{n-1}$ and by Archimedean property this sequence of lengths converges to zero. Note that $\{a_i : i \geq n\} \subset [a_n, b_n]$. These two comments can be used to show that (a_i) is a Cauchy sequence.

discussion of HA:

Q1: $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$. Need to show $n \sum x_i y_i \geq (\sum x_i)(\sum y_i)$.

Since $(x_i - x_j)$ and $(y_i - y_j)$ have the same sign we have

$$\sum_{i,j}(x_i - x_j)(y_i - y_j) \geq 0, \text{that is,}$$

$$\sum_{i,j} x_i y_i + \sum_{i,j} x_j y_j \geq \sum_{i,j} x_i y_j + \sum_{i,j} x_j y_i.$$

The two terms on left side are same, each equals $n \sum x_i y_i$. The two terms on right side are also equal, each equals $(\sum x_i)(\sum y_i)$.

You can also prove by induction. For $n = 1$ it is trivial. You can try $n = 2$ also just to understand. Assume for n.

$$(\sum_1^{n+1} x_i)(\sum_1^{n+1} y_i) = (\sum_1^{n} x_i)(\sum_1^{n} y_i) + x_{n+1}(\sum_1^{n} y_i) + y_{n+1}(\sum_1^{n} x_i) + x_{n+1}y_{n+1}.$$

$$\leq n\sum_1^n x_iy_i + \sum_1^n (x_iy_{n+1} + y_ix_{n+1}) + x_{n+1}y_{n+1}$$

$$\leq n\sum_1^n x_iy_i + \sum_1^n (x_iy_i + x_{n+1}y_{n+1}) + x_{n+1}y_{n+1}$$

Q 18. any two non-empty open intervals have the same number of elements.

Given (a,b) with $(a < b)$ establish bijection with $(0,1)$ by the map $F(x) = (x-a)/(b-a)$.

Q 23. To solve $x^2 - x - 1 = 0$.

We setup, $x_0 = 1$ and for $n \geq 1, x_n = 1 + (x_{n-1})^{-1}$ If $3/2 \leq x_{n-1} \leq 2$, then $1 + 1/2 \leq 1 + (x_{n-1})^{-1} \leq 1 + 2/3$.

$$|x_{n+1} - x_n| = \left|\frac{1}{x_n} - \frac{1}{x_{n-1}}\right| = \frac{|x_n - x_{n-1}|}{x_nx_{n-1}} \leq \left(\frac{2}{3}\right)^2 |x_n - x_{n-1}|.$$

Thus, letting $c = 4/9$, we have $|x_3 - x_2| \leq c|x_2 - x_1|$; $|x_4 - x_3| \leq c^2|x_2 - x_1|$ and $|x_5 - x_4| \leq c^3|x_2 - x_1|$. More generally, letting $|x_2 - x_1| = A$

$$|x_n - x_{n-1}| \leq c^{n-2}A.$$

This will help showing that (x_n) is Cauchy sequence. Indeed, take any N, then for $m > N$

$$|x_n - x_N| \leq |x_{N+1} - x_N| + |x_{N+2} - x_{N+1}| + \cdots + |x_{n-2} - x_{n-1}||x_{n-1} - x_n|$$

$$\leq \sum_{N-1}^{?} c^iA \quad \leq \sum_{N-1}^{\infty} c^iA \quad \leq \left(\frac{A}{1-c}\right)c^{N-1}.$$

If you make this last quantity small, by good choice of N, then any two terms after the N-th stage are small too.

Q 24. If you have a rational m/n, as you implement the algorithm for the expansion, each time you have smaller remainder than the previous one and sonn you will hit one and stop the process. Conversely, if the expansion is terminating, then inducting on its length, you can show it is rational.

Q 28. You already know $\sum a_i\sqrt{n} = 0$, whatever be n. Thus it suffices to show that $\sum a_i[\sqrt{n+i} - \sqrt{n}] \to 0$. Each term converges to zero and we have a fixed number of terms. You can use sum of sequences.

limit points:

it is, in general, difficult to decide whether a sequence converges or not. Even if we argue that it converge, it is difficult, in general, to find out the number to which it converges. If the sequence does not converge what can we do. Well, if it is not converging, then it is not staying close to any number; it may probably be going close to several numbers. For example consider the sequence: $+1, -1, +1, -1, \cdots$. It does not stay near any number, but it goes to $+1$ infinitely often and also goes to -1 infinitely often,. Here is another example.

$$\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{4}, \frac{1}{8}, \frac{9}{8}, \frac{1}{16}, \frac{17}{16}, \frac{1}{32}, \frac{33}{32}, \frac{1}{64}, \frac{65}{64}, \cdots\cdots$$

This sequence visits as close to zero as you please infinitely often, but does not stay there; it visits as close to one as you please infinitely often. Such numbers where the sequence goes close infinitely often are called limit points of the sequence. The interesting point is that for bounded sequences we may not be able to talk about limit unless the sequence converges; but we can always talk about limit points. The useful aspect is that we can hang on to the largest and smallest limit points to get a feel of how the sequence behaves. If both are same, that is, if there is only one limit point for a bounded sequence, then the sequence necessarily converges to that point.

Definition: Let (x_n) be a bounded sequence. A number a is a limit point of the sequence if the following holds. Given any $\epsilon > 0$, for infinitely many values of n we have $x_n \in (a-\epsilon, a+\epsilon)$. If L denotes the set of all limit points of the bounded sequence, then the following fact shows that L is a non-empty bounded set. Its supremum is called the limit superior or limsup of the sequence. The infimum of L is called the limit inferior or liminf of the sequence.

Fact: Let (x_n) be a bounded sequence. Then L, the set of its limit points is non-empty and bounded.

Proof: Since the sequence is bounded, suppose that $c \leq x_n \leq d$ for all n. If you take $a < c$, then for $\epsilon = (c-a)/4$ we see that $(a-\epsilon, a+\epsilon)$ does not contain any point of the sequence. That no number smaller than c can be a limit point. Similarly, no number larger than d can be a limit point. Thus L is bounded. We now show that $L \neq \emptyset$. This is a standard argument, imitating Cantor intersection theorem.

Denote $a_1 = c, b_1 = d$. Let $[a_2, b_2]$ be the left half or right half whichever contains x_n for infinitely many n. Since the entire sequence lives in $[a_1, b_1]$, one of these two halfs must contain infinitely many terms of the sequence. Of course, both may contain infinitely many terms of the sequence, in which case, you take the left half. If we have obtained intervals $[a_i, b_i]$ for $1 \leq i \leq k$ such that each interval is either left half or right half of the preceeding interval and each interval contains x_n for infinitely many n, then we define $[a_{k+1}, b_{k+1}]$ as the left half of $[a_k, b_k]$ if it contains infinitely many terms of the sequence; otherwise we degine it to be right half. These intervals have lengths decreasing to zero and so have exactly one number a in common.

We show that a is a limit point of the sequence. Take $\epsilon > 0$. Take k so that $[a_k, b_k] \subset (a-\epsilon, a+\epsilon)$ (this happpens if, for example, $b_k - a_k < \epsilon/4$). By construction of the intervals we see that $(a-\epsilon, a+\epsilon)$ contains infinitely many terms of the sequence. This completes the proof.

Fact: $\liminf x_n \leq \limsup x_n$.
This is easy.

Fact: If a is a limit point of (x_n), then there are integers

$$1 \leq n_1 < n_2 < n_3 < n_4 < \cdots$$

such that if we define

$$y_1 = x_{n_1}, y_2 = x_{n_2}, y_3 = x_{n_3}, \cdots$$

then $y_i \to a$.

Proof: Since infinitely many terms of the sequence are in the interval $(a-1, a+1)$, take n_1 so that $x_{n_1} \in (a-1, a+1)$. Using the same argument, pick $n_2 > n_1$ so that $x_{n_2} \in (a-1/2, a+1/2)$. Having got $n_1 < n_2 < \cdots < n_{k-1}$ so that $x_{n_i} \in (a-1/i, a+1/i)$ for $1 \leq i \leq k-1$; using the fact that infinitely many terms of the sequence are in the interval $(a-1/k, a+1/k)$ pick $n_k > n_{k-1}$ so that x_{n_k} is in this interval. This will do. Observe that all the (y_i) are in $(a-1, a+1)$; all but the first are in $(a-1/2, a+1/2)$; all but the first two terms are in $(a-1/3, a+1/3)$ etc. This shows $y_i \to a$.

on limits and convergence:

I have presented the concept of convergence as a way to discover new numbers, like e. Actually, this concept is much more basic, much more fundamental than I made it out. *Everything* that we do (well, almost everything) depends on this limit concept — continuous functions, derivative, integral etc. Even to define functions rigorously we resort to infinite sums, essentially we consider polynomials of infinite degree. Limits arise in all disciplines where maths makes its appearance.

Of course, you are having a course in physics and know how it appears. Right now, you are probably using derivatives and integrals which are defined through limits. Even if you want to discuss about the air in this lecture hall, you will not write down equations of motion for each of the paticles. Instead you will try to understand the question when there are only n particles and then take limits.

If you are doing computations and providing an algorithm for finding root of an equation or for finding minimum of a cost function over a certain region, typically your algorithm would give an iterative procedure and runs like this: start with x_1, do some thing to end up with x_2, then keep on repeating the instructions to get $x_2, x_3, \cdots$. Hope is that you will be heading to the quantity you are looking for.

Even in biological sciences it makes its appearance in an overwhelming manner. For example, you might be interested in: what happens to the species in the long run? (do they survive or become extinct) what happens to the mutant gene, in the long run, over generations (will it spread to the whole population?). Of course, most pertinent question is: what happens in the long run if we keep on using available natural resources at the current rate!

The rigorous development of Calculus was initiated by Newton and Leibnitz. But they were using infinitesimals, very small quantities etc. It was Cauchy and Bolzano who provided clear meaning of limits, though, of course, Euler, Fourier, Gauss, Poisson and several others before them already used infinite series and products. Did we not see that already Archimedes used the concept of limit, without saying so.

You should pay attention to this concept and think about it till you feel comfortable and till the concept appears *natural* to *you*.

The main difference between maths and reality is the following. In maths we can think of the following sequence: First trillion terms are zero and all later terms are equal to one. Obviously, this sequence is converging to one, but looking at the first trillion terms gives no indication of this fact. However, such a thing does not occur in nature. That is, sequences that arise in most of the practical problems do give an indication of what is happening — with its first ten thousand or so terms. This can be considered as the gift of nature to us.

series:

By virtue of the rules regarding real numbers, we can add any finitely many real numbers. But many times, we do see that infinitely many real numbers can be 'theoretically' added, that is, we feel that the sum must be a particular number.

For example, let us consider adding

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots.$$

Of course, we do not have time to add all these infinitely many numbers one by one. But, if we do start adding, we get

$$1, 2 - \frac{1}{2}, 2 - \frac{1}{2^2}, 2 - \frac{1}{2^3}, 2 - \frac{1}{2^4}, 2 - \frac{1}{2^5}, 2 - \frac{1}{2^6}, \cdots.$$

Eventhough we are unable to add all the infinitely many numbers above, we see that the successive sums are nearing 2. To put it differently, it seems reasonable to *define* the infinite sum to be 2. This is what we are going to do now.

Definition: A series is simply an expression $a_1 + a_2 + a_3 + \cdots$ or $\sum a_n$ or $\sum\limits_{n\geq 1} a_n$.

Here the a_i are numbers. Actually, as with sequences, we are only talking about series of real numbers. The number a_n is called the n-th term of the series.

There is no meaning yet for a series. Here are some examples of series:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots.$$

$$1 + 1 + 1 + 1 + \cdots.$$

$$1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots.$$

This series is usually written as

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 \cdots.$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots.$$

The first series appears meaningful and we should get the number 2, if we add *all* those terms. It is not obvious, but true and will be proved, that the last series also has a meaning.

However, the second series has the following sums if we keep on adding the terms: $1, 2, 3, 4, 5, 6, 7, \cdots$. The value of the sum, as we keep on adding, exceeds anything that you can think of — remember the Archimedean principle, namely, given any number x, after some stage all the integers are larger than x. Since there is a particular direction in which we are going — on the number line — if we keep on and on adding, it is tempting to say that the sum is infinity. Yes, it is convenient and we shall do that later. This needs introduction of funny animal $+\infty$, and for symmetry another animal $-\infty$. But remember, it is not just a matter of introducing these characters into the drama, we should also agree upon the rules as to what we can and cannot do with these new things. Otherwise, there may come a stage when we do not understand each other.

The third series is interesting. If we keep on adding we get $1, 0, 1, 0, 1, 0, \cdots$. The addition of successive numbers is not leading us away, in a particular direction or to a particular number. The sums oscillate between zero and one. In any case, if we keep on adding the sums are not approaching any number.

Thus a series is simply a sugggestion to add certain numbers. Whether we can really add or not is a different matter. Sometimes we have a feeling as to what we should get if we really add. We shall make this feeling precise (Most mathematical definitions are just precise way of expressing some feeling we have).

Definition: Let $\sum a_n$ be a series. We define its partial sums to be the sequence of numbers,

$$s_1 = a_1; \quad s_n = a_1 + a_2 + \cdots + a_n \quad n \geq 1.$$

We say that the series $\sum a_n$ converges if the sequence (s_n) converges. If $s_n \to a$, we say that the values of the series is a. We express this by writing in any of the following ways.

$$\sum a_n = a; \quad \sum_{n \geq 1} a_n = a; \quad a_1 + a_2 + a_3 + \cdots = a.$$

You should keep in mind that we said the value of the series *is* a. We did not say the series approaches a or nearly a or close to a etc. It *is* a.

You should note the distinction between series and sequence. Both have numbers in a particular order. The order is important in both of them. If you change the order, it becomes a different thing. In a sequence, we simply have numbers standing in a line or in a row. In a series, we have numbers again in a row with a suggestion to add them. This suggestion is shown by putting the plus sign between the terms. Of course, you should know, by now, that adding includes subtraction too, adding (-5) is same as subtracting 5. Whether we can execute the suggestion is a different matter and that is why we made the next definition regarding convergence or value of a series.

So, a sequence is not a series and a series is not a sequence. Of course, you may think that a series is just a sequence with plus signs between terms, instead of the commas. This is only a typographical or superficial recognition — though correct and helps you in remembering, it is not the spirit in which these concepts are to be understood.

We shall now discuss convergence of some specific series. Of course, there is no criterion to tell exactly which series converge and which do not. Instead, we have several rules that help recognize whether a particular series converges. These do not cover all possible series you cabn think of, but will include some important series we come across in practice.

Fact:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots = 2.$$

We knew this already. We are lucky, because we could actually calculate all the partial sums and see what exactly is happening.

Fact: Let $\sum a_n$ be a series where each $a_n \geq 0$. If the partial sums are bounded, then the series converges.

Since the numbers a_n are positive (non-negative), the partial sums are increasing and by hypothesis, they are bounded above. So they converge from a theorem proved earlier.

Actually, for series of positive terms, convergence holds iff the partial sums are bounded.

Fact: The series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \cdots.$$

converges. We have, of course, seen this already earlier.

Fact: Suppose that the series $\sum |a_n|$ converges. Then the series $\sum a_n$ converges.

Let s_n be the partial sums of the series $\sum a_n$. We show that the sequence (s_n) is a Cauchy sequence. Let $\epsilon > 0$. Let t_n be the partial sums of the series

$\sum |a_n|$. We know that this sequence (t_n) is a Cauchy sequence, because it converges by hypothesis. So fix n_0 so that $|t_n - t_m| < \epsilon$ for $n, m \geq n_0$. This n_0 does for the sequence (s_n) as well because of the following. Let $m > n$, then

$$|s_m - s_n| = |\sum_{n+1}^{m} a_i| \leq \sum_{n+1}^{m} |a_i| = t_m - t_n.$$

Fact: The series

$$1 \pm \frac{1}{2} \pm \frac{1}{2^2} \pm \frac{1}{2^3} \pm \frac{1}{2^4} \pm \frac{1}{2^5} \pm \frac{1}{2^6} \pm \cdots.$$

converges. Here the signs are left to your choice. You need not follow any rule, make up your mind at each term whether to put plus sign or minus sign. The resulting series converges.

This follows from the previous fact. Now you see, it is not possible to produce the number to which it converges, even if there were a rule in your putting signs. For example, if you put minus sign for all prime powers and plus for others, probably I do not know the value.

Fact: Let $0 < r < 1$. Take any *bounded* sequence of numbers (α_n). Then the series

$$\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \alpha_4 r^4 + \alpha_5 r^5 + \alpha_6 r^6 + \cdots.$$

converges.

If $|\alpha_n| \leq c$, then the series $\sum |\alpha_n| r^n$ has all partial sums bounded by $M/(1-r)$ and hence converges. Now apply an earlier fact to see that the orginal series, which has no modulus signs, also converges.

Fact: Let $(\alpha_n; n \geq 0)$ be a bounded sequence as above. Then the following series converge.

$$\alpha_0 + \alpha_1 \frac{1}{2} + \alpha_2 \frac{1}{2^2} + \alpha_3 \frac{1}{2^3} + \alpha_4 \frac{1}{2^4} + \cdots.$$

$$\alpha_0 + \alpha_1 \frac{1}{10} + \alpha_2 \frac{1}{10^2} + \alpha_3 \frac{1}{10^3} + \alpha_4 \frac{1}{10^4} + \cdots.$$

In particular, if you restrict the numbers to $\alpha_0 = 0$ and $\alpha_n \in \{0, 1, 2, 3, \cdots, 9\}$ for $n \geq 1$, in the second series, then all the resulting series converge. By taking particular choices, you can get all the numbers in the interval $[0, 1]$ and no more.

Actually when we did decimal expansion, I have already used the notion of convergence. With this discussion, it is now precise. Whatever we have done there is just showing the convergence of the expansion that we suggested.

Sometimes by observing the proof, we can get a better result.

Fact: Suppose $|a_n| \leq c_n$ for each $n \geq 1$ and the series $\sum c_n$ converges. Then, the series $\sum |a_n|$ converges and hence the series $\sum a_n$ also converges.

In fact the partial sums of the series $\sum c_n$ is bounded and hence so are the partial sums of the series $\sum |a_n|$. This can be used to complete the proof.

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series:

If you delete, or add or alter finitely many terms in a series, then convergence is unaffected.

Fact: Let $\sum a_n$ be a convergent series.
Let $b_n = a_{1000+n}$ for $n \geq 1$. Then $\sum b_n$ converges. Here we deleted the first few terms.

Let $c_n = a_{n-1000}$ for $n > 1000.$, For $n \leq 1000$ let a_n be your choice. Then $\sum c_n$ converges. Here we added a few terms at the beginning of the existing series.

Let $d_n = a_n$ for $n > 1000$ and d_n for $n \leq 1000$ be your choice. Then $\sum d_n$ converges. Here we changed the first feew terms of the series, keeping the remaining as they are.

All these three statements are proved by showing that the partial sums are Cauchy. Let (s_n) be the partial sums of the series $\sum a_n$. If (t_n) are partial sums of $\sum b_n$, then $t_n = s_{n+1000} - s_{1000}$. Others are proved in the same manner.

The last statement has the following special case. Consider the first 1000 terms of the original sequence, permute them and take as d_i. In other words, if you take a convergent series and permute the first finitely many terms, the resulting series converges; in fact, it converges to the same number as the original series.

This can be made precise as follows. Let f be a bijection of $\{1, 2, \cdots, 1000\}$ to itself. Define $d_n = a_{f(n)}$ for $n \leq 1000$ and $d_n = a_n$ for $n > 1000$. Then the series $\sum d_n$ converges. In fact if $\sum a_n = a$ then $\sum d_n = a$ as well. This is clear by noting that the partial sums for both sequences coincide as soon as $n > 1000$.

Recall that $\sum a_n$ converges if the sequence of partial sums (s_n) converges, which, in turn, is same as saying (s_n) is a Cauchy sequence. This is restated as follows.

Fact: The series $\sum a_n$ converges iff the following holds. Given $\epsilon > 0$, there is n_0 such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$ for $m > n \geq n_0$.
In particular, if $\sum a_n$ converges then $a_n \to 0$.

The first sentence follows by observing $s_m - s_n = a_{n+1} + a_{n+2} + \cdots + a_m$. The second sentence follows by taking $m = n + 1$ in the first sentence.

Fact: Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum a_n$ converges iff the series $a_1 + 2a_2 + 2^2 a_{2^2} + 2^3 a_{2^3} + \cdots$ converges.

Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_k = a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^k a_{2^k}$. Since both the series consist of non-negative terms, their convergence is equivalent to boundedness of the partial sums. Thus the stated result follows from the following two claims.
(i) For every n, there is a k such that $s_n \leq t_k$.
(ii) For every k, there is an n such that $t_k \leq 2s_n$.

We prove (i) as follows. If $2^{k-1} \leq n < 2^k$, then

$$s_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n \leq$$

$$a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k - 1})$$
$$\leq a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^{k-1} a_{2^{k-1}} = t_{k-1}.$$

where, for the inequality we used that a_n are decreasing.

We prove (ii) as follows.

$$t_k = a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^k a_{2^k} \leq$$

$$2\{a_1 + a_2 + 2a_4 + 2^2 a_8 + \cdots + 2^{k-1} a_{2^k}\} \leq$$

$$2\{a_1 + a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})\}$$

$$= 2s_{2^k}.$$

Fact: The following series converge iff $p > 1$.

$$\sum \frac{1}{n^p}; \qquad \sum_{n\geq 2} \frac{1}{n(\log n)^p}$$

For the first series, $a_n = n^{-p}$ and the terms are decreasing.

$$2^n a_{2^n} = 2^n 2^{-np} = 2^{n(1-p)}$$

Thus $\sum 2^n a_{2^n}$ is a geometric series $\sum r^n$ where $r = 2^{1-p}$.

For the second series $a_n = n^{-1}(\log n)^{-p}$. Actually, since $log 1 = 0$, it started with $n = 2$. Instead of counting that a_2 is the first term etc, to match the notation being used, take $a_1 = 0$ and a_2 is the second term etc. What we write below will be true for $n \geq 2$.

$$2^n a_{2^n} = 2^n \frac{1}{2^n (n \log 2)^p} = \frac{1}{(\log 2)^p n^p}$$

and so this series converges iff $\sum n^{-p}$ converges.

Fact: The series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

converges to e.

This follows from our discussion of the partial sum sequence earlier. We shall see it again by another method which applies not only to this series but also for several other series.

In what follows we consider series of non-zero terms, that is $a_n \neq 0$ for each n. (Is it necessary to consider series where some a_n are zero?).

Fact: Suppose that there is an α, $0 < \alpha < 1$, such that $|a_{n+1}/a_n| < \alpha$ for all large n. Then the series $\sum |a_n|$ converges and hence $\sum a_n$ converges.

Let us say that for $n \geq k$ the stated inequality holds. Then $|a_{k+1}| \leq \alpha |a_k|$, $|a_{k+2}| \leq \alpha |a_{k+1}| \leq \alpha^2 |a_k|$. In general

$$|a_{k+n}| \leq \alpha^n |a_k|, \qquad n \geq 0.$$

Since geometric series is convergent ($|\alpha| < 1$) we see, by comparison test, the series $|a_k| + |a_{k+1}| + \cdots$converges. Adding finitely many terms does not destroy convergence. Hence $\sum |a_n|$ ocnverges.

Fact: Suppose that $\lim |a_{n+1}/a_n| < 1$. Then the series $\sum |a_n|$ is convergent and hence $\sum a_n$ converges.

If this limit is denoted by l, then hypothesis says that $0 \leq l < 1$. We can fix a number α so that $l < \alpha < 1$. Then by definition of limit, the ratios are smaller than α after some stage and the previous result applies.

Fact: If $\limsup |a_{n+1}/a_n| < 1$, then the series $\sum |a_n|$ converges.

This is simply observed by looking at the earlier argument. If this limsup is denoted by s, then $0 \leq s < 1$ and you can pick $s < \alpha < 1$. Use definition (or characterization) of limsup to conclude that after some stage the ratios are smaller than α.

As an application of this we have the following.

Fact: The following series converge for every real number x.

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots.$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots.$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\cdots.$$

The sum of the first series is denoted by $e(x)$. The sum of the second series is denoted by $\sin x$. The sum of the third series is denoted by $\cos x$. Yes, these will be identified with the functions you know.

Fact: If there is a number $\alpha > 1$ such that $|a_{n+1}/a_n| > \alpha$ after some stage, then the series $\sum a_n$ does not converge.

If this happens for $n \geq k$, then by repeated application of the hypothesis we see that $|a_{n+k}| \geq \alpha^n |a_k|$ which does not converge to zero. Remember $a_k \neq 0$.

The above facts go by the name of ratio test.

Fact: If $\limsup \sqrt[n]{|a_n|} < 1$ then the series $\sum |a_n|$ converges.

Proof: If this limsup is s, then $0 \leq s < 1$ and hence we can fix $s < \alpha < 1$. Definition of limsup now says after some stage $\sqrt[n]{|a_n|} < \alpha$. That is, $|a_n| < \alpha^n$ after some stage. Now compare with geometric series $\sum \alpha^n$ as earlier.

This result has a converse too — not exactly, but nearly.

Fact: If $\limsup \sqrt[n]{|a_n|} > 1$ then the series $\sum a_n$ does not converge.

if this limsup is s then fix $1 < \alpha < s$. Again by definition of limsup, we have, after some stage $|a_n| > \alpha^n$. Since $\alpha > 1$, we conclude that $a_n \not\to 0$.

We see that $\sum 1/n$ does not converge whereas $\sum 1/n^2$ converges and in both cases the limsup equals one. This we state as follows.

Fact: If $\limsup \sqrt[n]{|a_n|} = 1$, the series may or may not converge.

At first sight the fact above may appear like a tautology. Afterall, this is true of every series, it may or may not converge; nothing else can happen. But you should keep in mind that we are not talking of a particular series. we are talking of series satisfying some condition, namely, this limsup equals 1. There are examples of series which satisfy the condition and which converge. There are also examples of series which satisfy this condition and do not converge. Thus when limsup equals one we know for sure that no conclusion can be drawn regarding convergence without any further hypothesis.

The above facts go by the name of root test.

You see that the main ingradient in all this discussion of convergence of series is just the high school geometric series. Once the new concept of convergence is understood, you can draw rather non-trivial conclusions using just what you knew already.

In all the above results, we showed actually the convergence of the series $\sum |a_n|$, not just $\sum a_n$. This notion of convergence of the series formed by absolute values is important in applications and has a name.

Definition: A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges. Of course, then $\sum a_n$ converges too. If $\sum a_n$ converges and $\sum |a_n|$ does not converge, we say that the series $\sum a_n$ is conditionally convergent.

Thus the above tests help you in testing for absolute convergence. For example, they fail to tell you if the following series converges.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\cdots.$$

The tests above do say that this series is not absolutely convergent.

Note that if the series $\sum a_n$ is to be convergent, we should at least have $a_n \to 0$. We also know that this condition alone does not imply convergence of the series $\sum a_n$; for example the series $\sum(1/n)$ shows this. Interestingly, as soon as you know that the terms a_n are decreasing, this condition is enough to ensure convergence of the alternating series.

Fact: If $a_n \downarrow 0$, the series $a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ converges.

Such series where the terms are alternatively positive and negative are called 'alternating series'. (Just to emphasize that you should not be deceived by appearance, the series above is alternating not because you see $\pm$ signs, you need to use that the numbers a_n are positive.) Of course, the series is interesting only when all a_n are different from zero. Because, as soon as one a_n equals zero, the sum is actually finite sum and convergence issue is only theoretical. In other words, the series is theoretically infinite series, but after some stage partial sums do not change.

Proof: We show the following;

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq 0 \Rightarrow 0 \leq (a_1 - a_2 + a_3 - \cdots \pm a_k) \leq a_1.$$

Let us see what happens if this is done. Returning to our series, let s_n be its partial sums. If $m > n$, then the above conclusion tells us

$$|s_m - s_n| = |a_{n+1} - a_{n+2} + a_{n+3} - \cdots \pm a_m| \leq a_{n+1}.$$

Since $a_n \downarrow 0$, given $\epsilon > 0$, we can choose p such that $a_p < \epsilon$. The above inequality tells that after p-th stage any two partial sums differ by at most ϵ. In other words, partials sums form Cauchy sequence and hence converge.

We shall now prove the inequality claimed at the beginning of the proof. The left side equals

$$(a_1 - a_2) + (a_3 - a_4) + \cdots \geq 0$$

In fact, each bracketed term is non-negative by decreasing nature of a_n; if k is even there is nothing left over and if k is odd there is a last non-bracketed term which is positive.

To see the other inequality,

$$a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots \leq a_1.$$

Each bracketed term is positive and is being subtracted from a_1. If k is even, there is an unbracketed term a_k at the end which also appears with negative sign. This completes the proof.

There is a useful generalization of this. Suppose that $\sum b_n$ is a series with bounded partial sums and $a_n \downarrow 0$, then the series $\sum a_n b_n$ converges. If we take $\sum b_n$ to be $\sum \pm 1$ we get the special case above. this is an extremely useful generalization.

Cauchy product of series:

We shall now define the concept of product of two series. For certain useful applications, we shall now consider series as $\sum\limits_{n\geq 0} a_n$ rather than $\sum\limits_{n\geq 1} a_n$. In other words we consider series $a_0+a_1+a_2+\cdots$ instead of, $a_1+a_2+a_3+\cdots$

as has been done so far. You should not get confused. Either you can set your starting point a little back and think of zeroth term, first term etc (and zeroth partial sum, first partial sum etc). Or if you have trouble thinking like that, you can think that first term is a_0, second term is a_1 and in general the n-th term is a_{n-1}. But in the long run it will help you if you get used to the first way of thinking. Thus we have partial sums $s_0, s_1, s_2 \cdots$.

So now let $\sum\limits_{n\geq 0} a_n$ and $\sum\limits_{n\geq 0} b_n$ be two series (of real numbers). We define

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0; \qquad n \geq 0.$$

Thus $c_0 = a_0 b_0$; $c_1 = a_0 b_1 + a_1 b_0$. In general, c_n is the sum of all products $a_i b_j$ where $i + j$ adds upto n. These are finite sums and you need not worry. The series $\sum\limits_{n\geq 0} c_n$ is called the Cauchy product of the two series $\sum a_n$ and $\sum b_n$.

At first sight this looks complicated. But think of multiplying two polynomials

$$A(x) = \sum_0^p a_i t^i; \qquad B(x) = \sum_0^q b_j t^j.$$

You know that product of two polynomials is again a polynomial. So let us say the product $A(t)B(t)$ is the polynomial $C(t) = \sum\limits_0^{p+q} c_i t^i$. Then what are the coeficients? You can see $c_0 = a_0 b_0$; $c_1 = a_0 b_1 + a_1 b_0$. In general, c_n is the sum of all products $a_i b_j$ where $i + j$ adds upto n exactly as above. Of course, since a polynomial is a finite sum, when you reach $p + q$ there is only one term $a^p b^q$. In the infinite series case, there is always a_0, a_1 etc and finally a_n — all appearing in c_n.

Pretend that you have two infinite degree polynomials

$$A(t) = \sum_{n\geq 0} a_n t^n; \qquad B(t) = \sum_{n\geq 0} b_n t^n.$$

Just like in the usual polynomial case, suppose you want to multiply these two infinite degree polynomials and write it again as a polynomial by collecting powers of t, then you will exactly get $\sum c_n t^n$ where c_n are as defined above.

Fact: If the series $\sum a_n$ and $\sum b_n$ are absolutely convergent and if $\sum a_n = A$ and $\sum b_n = B$, then the Cauchy product $\sum c_n$ converges and $\sum c_n = AB$.

Proof: Let s_n, t_n and u_n be the partial sums of the series $\sum a_n$, $\sum b_n$ and $\sum c_n$ respectively. Known $s_n \to A$ and $t_n \to B$. Need to show that $u_n \to AB$. Of course, we know that $s_n B \to AB$. Thus if we can show that $u_n - s_n B \to 0$ then

$$u_n = (u_n - s_n B) + s_n B \to 0 + AB$$

as wanted. But

$$u_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0).$$

Collecting terms of a_0, a_1 etc together we get

$$u_n = a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_n t_0.$$

Subtracting from the above $s_n B = a_0 B + a_1 B + a_2 B + \cdots a_n B$, we get

$$u_n - s_n B = a_0(t_n - B) + a_1(t_{n-1} - b) + a_2(t_{n-2} - B) + \cdots + a_{n-1} t_1 + a_n t_0.$$

If n is large, the first few terms on right are small because $(t_n - B)$ is small; remember that $t_n - B \to 0$. The last few terms are small because $a_n \to 0$, remember $\sum a_n$ converges. Not only the individual terms are each small but the entire sum is small, this is where absolute convergence is used.

We wish to show that $|u_n - s_n B|$ can be made small for all large values of n. Here is how. Let $\epsilon > 0$. Let $\sum |a_n| = \alpha > 0$. Note that, if $\alpha = 0$, then each $a_i = 0$ and the conclusion is easy. Choose n_0 so that $|t_n - B| < \epsilon/(2\alpha)$ for $n \geq n_0$. This is possible because $t_n \to B$. Thus as soon as $n > n_0$ we have

$$\begin{aligned}
|u_n - s_n B| &= |\sum_{k=n_0}^{n} a_{n-k}(t_k - B)| + |\sum_{k<n_0} a_{n-k}(t_k - B)| \\
&\leq \sum_{k \geq n_0} |a_{n-k}||t_k - B| + |\sum_{k<n_0} a_{n-k}(t_k - B)| \\
&\leq \frac{\epsilon}{2\alpha} \sum_{k \geq n_0} |a_{n-k}| + |\sum_{k \leq n_0} a_{n-k}(t_k - B)|
\end{aligned}$$

$$\leq \frac{\epsilon}{2} + |\sum_{k \leq n_0} a_{n-k}(t_k - B)|$$

Note that the second sum on the right side consists of n_0 many terms and each of these terms converges to zero because $a_{n-k} \to 0$ for $k = 1, 2, \cdots n_0 - 1$. Remember n_0 is fixed. Hence we can choose $n_1 > n_0$ such that the sum is smaller than $\epsilon/2$ for all $n \geq n_1$. Thus if $n > n_1$ we see that the right side is smaller than ϵ completing the proof.

As you have seen in the proof, we used that $\sum a_n$ is absolutely convergent but did not use that $\sum b_n$ is absolutely convergent. Thus the theorem is true if one of the series is absolutely convergent. If none of them is absolutely convergent, then the Cauchy product may not converge. For example if we take both the series to be the alternating series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + - \cdots$$

then the Cauchy product does not converge. This is because the n-th term c_n $(n \geq 0)$ of the Cauchy product equals $(-1)^n \sum 1/(\sqrt{k}\sqrt{n-k})$. This is sum of $n-1$ terms each at least $2/n$ by using GM $\leq$ AM. Thus c_n does not converge to zero.

This result is very useful. For example let us take two numbers x and y and consider the series for $e(x)$ and $e(y)$. Since these are absolutely convergent, we conclude that their Cauchy product converges to $e(x)e(y)$. But computation shows that the Cauchy product is the series defining the number $e(x+y)$. Thus we conclude that

$$e(x+y) = e(x) \cdot e(y).$$

Since $e(1) = e$ by definition of the number e, we see that for every natural number $e(n) = e^n$. Since $e(0) = 1$ we see that for every integer, positive or negative, $e(n) = e^n$. For an integer $q \geq 1$

$$e(1/q) \cdot e(1/q) \cdots e(1/q) \quad (q \text{ times}) = e(1) = e.$$

By definition of q-th root, it follows that $e(1/q)$ is q-th root of e. That is,

$$e(1/q) = e^{1/q}.$$

It now follows that for every rational number r, $e(r) = e^r$. Thus

$$e(x) = e^x, \qquad x \in Q.$$

We shall show later, using continuity of the functions on both sides, that the equality holds not only for rationals but for all real numbers x.

The sine and cosine functions are also defined by series and are not, at this moment, recognizable as the good old functions of high school. However, using Cauchy product, one can show

$$\sin(x+y) = \sin x \cos y + \cos x \sin y; \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Of course, the very nature of the series shows that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ as well as $\cos 0 = 1$ and $\sin 0 = 0$.

Rearrangements:

When you add finitely many numbers, you can change the order in which you add, but you still get the same answer. How do you make this precise? Let $a_1, a_2, \cdots, a_k$ be numbers. Let π be a permutation of $\{1, 2, \cdots, k\}$. Then $\sum a_i = \sum a_{\pi(i)}$. Prove this. (After you prove this once, you can say it is easy).

Is this true for infinite series? This is the question we now answer. First we need to make precise the concept of changing the order of addition. Let $\sum\limits_{n \geq 1} a_n$ be a series, Let π be a bijection of $\{1, 2, \cdots, \}$. This is called a permutation of the natural numbers. Let $b_n = a_{\pi(n)}$. The series $\sum b_n$ is called a rearrangement of the series $\sum a_n$. Thus π being a permutation, note that each a_n appears exactly once as a b_k and no others appear. If $\pi(1) = 24$ then the first term b_1 of the new series is a_{24}. If $\pi(33) = 1$ then a_1 appears as the 33-rd term of the new series.

Fact: Let $\sum a_n$ be absolutely convergent series. if $\sum a_n = A$ and if $\sum b_n$ is a rearrangement of $\sum a_n$, then the series $\sum b_n$ converges and $\sum b_n = A$.

Proof: Let (s_n) be the sequence of partial sums of $\sum a_n$ and (t_n) be partial sums of $\sum b_n$. We know $s_n \to A$. Need to show $t_n \to A$. Let $\epsilon > 0$. We exhibit n_0 so that $|t_n - A| < \epsilon$ for all $n \geq n_0$.

To do this, first observe that, since the series $\sum |a_n|$ is convergent, its partial sums are Cauchy. So given $\epsilon > 0$, we can choose n_1 so that for $m > n \geq n_1$ we have

$$|a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \epsilon/2.$$

This being true for every $m > n$ we see

$$\sum_{i>n} |a_i| \leq \epsilon/2.$$

This is true, in particular, with $n = n_1$. Thus sum of all the $|a_i|$ with $i \geq n_1$ is at most $\epsilon/2$. In particular, if you add any (not necessarily all) of the $|a_i|$ for $i \geq n_1$ the sum is at most $\epsilon/2$.

Now starting with our $\epsilon > 0$, fix n_1 as above. By taking larger value for n_1 we can assume that $|s_n - A| < \epsilon/2$ for $n \geq n_1$. This is possible because $s_n \to A$. Choose n_0 so that

$$\{\pi(1), \pi(2), \cdots, \pi(n_0)\} \supset \{1, 2, \cdots, n_1\}.$$

You only need to see that if $1 = \pi(k_1)$, $2 = \pi(k_2)$ etc upto n_1, then take n_0 as the maximum of all these finitely many k_i. We claim that for $|t_n - A| < \epsilon$ for $n \geq n_0$. This is simple. Let $n \geq n_0$,

$$t_n = \sum_1^n b_i = \sum_1^n a_{\pi(i)} = \sum_1^{n_1} a_i + \text{ remaining sum.}$$

Last equality is from the fact that among the $\pi(i)$ all the integers upto n_1 appear. Now the first sum on right side differs from A by at most $\epsilon/2$. The second sum is addition of some of the a_i where the indices i are unknown but each index i is larger than n_1. Hence this sum in modulus is at most $\epsilon/2$.

A theorem of Riemann says that if the series is conditionally convergent, then this fails in a drastic way. Whatever number a is given, you can rearrange so that the rearranged series converges to the given value a. This is what we shall do now. The reason for doing this is the following. First, it is spectacular. Second, the idea is beautiful and execution is neat. At the same time you realize that you need good vocabulary to communicate!

So in what follows $\sum a_n$ is a conditionally convergent series.

Fact: Let $\sum a_n$ be conditionally convergent. Then the sum of all the positive terms of the sequence equals $+\infty$ and sum of all negatve terms equals $-\infty$.

Let us put $x_n = a_n$ if $a_n \geq 0$; otherwise put $x_n = 0$. Similarly, put $y_n = a_n$ if $a_n < 0$; otherwise put $y_n = 0$. Clearly we have

$$x_n + y_n = a_n; \qquad x_n - y_n = |a_n|; \qquad n \geq 1.$$

If you say that $\sum x_n$ is convergent, then $\sum y_n = \sum(a_n - x_n)$ is convergent too and then $\sum |a_n| = \sum(x_n - y_n)$ is convergent too, contradicting the hypothesis that $\sum a_n$ is conditionally convergent. Note that $\sum x_n$ is a series of positive terms so that if it is not convergent then partial sums increase to ∞.

Similarly, we argue that $\sum y_n = -\infty$

Fact: If $\sum a_n$ is a conditionally convergent series and if $\alpha \in R$ then there is a rearrangement $\sum b_n$ so that the rearrangement converges and $\sum b_n = \alpha$.

The idea is simple and is as follows. start adding positive terms of the series (keep same order) so that the sum just exceeds α and then stop; now start adding negative terms so that the total sum now just falls below α and stop; now start adding positive terms (begin with where you left off earlier) so that the total sum goes just above α and stop; then start adding negative terms (begin with where you left off earlier) so that the total sum now falls just below α and continue. You can continue forever because the positive and negative terms of the series add to infinites. You will consume all a_n because in each round you are using at least one positive term and at least one negative term.

The question is whether the new series actually converges to α. Clearly partial sums are oscillating around α, but are they converging. Yes. Note that $a_n \to 0$ because the series $\sum a_n$ is convertgent. Thus in the long run whenever you exceed α or fall below α you will do so only by a small amount, not too much, because the numbers you are adding are getting smaller. We shall now make this precise.

Let $f(1)$ be the first $i \geq 1$ such that $a_i \geq 0$. Having defined $f(1) < f(2) < \cdots < f(n-1)$, we define $f(n)$ to be the least $i > f(n-1)$ such that $a_i \geq 0$. Similarly $g(1)$ is the first i such that $a_i < 0$. In general $g(n)$ is the least $i > g(n-1)$ such that $a_i < 0$. Since the positive terms and the negative terms of $\sum a_n$ add up to infinities, we deduce that $f(n)$ and $g(n)$ are defined for all $n \geq 1$.

Each integer $i \geq 1$ appears exactly once — either as an $f(n)$ or as a $g(n)$ but not as both. $(\spadesuit)$

Convergence of $\sum a_n$ combined with the facts $f(n) \uparrow \infty$, $g(n) \uparrow \infty$ gives us the following.

$$a_{f(n)} \to 0; \qquad a_{g(n)} \to 0. \tag{$\clubsuit$}$$

Let $s(n) = \sum_1^n a_{f(i)}$ and $t(n) = \sum_1^n a_{g(i)}$.

The fact observed above says

$$s_n \uparrow \infty; \qquad t_n \downarrow -\infty. \tag{$\heartsuit$}$$

We are given α and need to show a rearrangement of the series $\sum a_n$ that converges to α. We assume that $\alpha \geq \mathbf{0}$.

We pick two sequences of integers $n_1 < n_2 < \cdots$, and $m_1 < m_2 < \cdots$ such that the following hold.

(1) n_1 is the least integer with $s(n_1) > \alpha$ and m_1 is the least integer such that $s(n_1) + t(m_1) < \alpha$.

(2) For $i \geq 2$: Having selected $n_1, m_1, n_2, m_2, \cdots n_{i-1}, m_{i-1}$ we select n_i to be the least integer $> n_{i-1}$ such that $s(n_i) + t(m_{i-1}) > \alpha$ and m_i is the least integer $> m_{i-1}$ such that $s(n_i) + t(m_i) < \alpha$.

Existence of such sequences is established by induction. First choose n_1 as stated in (1) and then choose m_1 as stated in (2). Then choose n_2 as stated in (1) and then m_2 as stated in (2). That you can poceed for ever is a consequence of $(\heartsuit)$.

Claim 1:

$$0 \le s(n_1) - \alpha \le a_{f(n_1)}; \text{ for } \ k \ge 2,\ 0 \le [s(n_k) + t(m_{k-1})] - \alpha \le a_{f(n_k)}.$$

Indeed if $n_1 = 1$, then $s(n_1) = a_{f(1)} > \alpha \ge 0$ so that, $0 \le s(n_1) - \alpha \le a_{f(n_1)}$ as stated. If $n_1 \ge 2$, then by choice of n_1, $0 \le s(n_1 - 1) \le \alpha < s(n_1)$ so that $0 \le s(n_1) - \alpha \le s(n_1) - s(n_1 - 1) = a_{f(n_1)}$ as stated. Similarly, by choice of n_k, $s(n_k - 1) + t(m_{k-1}) \le \alpha < s(n_k) + t(m_{k-1})$, so that $0 \le s(n_k) + t(m_{k-1}) - \alpha \le s(n_k) - s(n_k - 1) = a_{f(n_k)}$.

Claim 2: $0 \le \alpha - [s(n_k) + t(m_k)] \le |a_{g(m_k)}|$.
This is proved exactly as above.

Claim 3: $|s(n_k) + t(m_{k-1}) - \alpha| \to 0$ and $|s(n_k) + t(m_k) - \alpha| \to 0$.
This follows from claims 1, 2 and ($\clubsuit$).

Here is the permutation (a one-to-one, onto map of natural numbers).

$$\begin{array}{rcll}
\pi(i) & = & f(i) & \text{for} \quad i \le n_1 \\
 & = & g(i - n_1) & \text{for} \quad n_1 + 1 \le i \le n_1 + m_1 \\
 & = & f(i - n_1 - m_1) & \text{for} \quad n_1 + m_1 + 1 \le i \le n_2 + m_1 \\
 & = & g(i - n_2 - m_1) & \text{for} \quad n_2 + m_1 + 1 \le i \le n_2 + m_2 \\
 & = & f(i - n_2 - m_2) & \text{for} \quad n_2 + m_2 + 1 \le i \le n_3 + m_2 \\
 & = & \cdots &
\end{array}$$

($\spadesuit$) shows that π is indeed a permutation of natural numbers. Let $u(n)$ be the n-th partial sum of the rearranged series $\sum a_{\pi(n)}$, that is, $u(n) = \sum\limits_{i=1}^{n} a_{\pi(i)}$. Shall now show that $u(n) \to \alpha$.

First note that $u(n_1) = s(n_1)$; $u(n_1 + m_1) = s(n_1) + t(m_1)$; and in general $u(n_k + m_{k-1}) = s(n_k) + t(m_{k-1})$ and $u(n_k + m_k) = s(n_k) + t(m_k)$. That these partial sums are as stated follows from the definition of permutaton π. Put $k_1 = n_1, k_2 = n_1 + m_1, k_3 = n_2 + m_1, k_4 = n_2 + m_2, k_5 = n_3 + m_2, k_6 = n_3 + m_3, \cdots$. Then Claim 3 shows that $u_{k_i} \to \alpha$. To show that the entire sequence $u_n \to \alpha$ observe that for $k_i \le j \le k_{i+1}$ we have, by construction, $u_{k_i} \le u_j \le u_{k_{i+1}}$. This completes the proof when $\alpha \ge 0$.

In case $\alpha \leq 0$, first pick m_1 (instead of n_1) such that $t(m_1) < \alpha$ and then choose n_1 etc. Same proof works. of course, even if $\alpha < 0$, the same construction as above can be used. This completes the proof.

Some of you asked if every rearrangement converges to something. the answer is no. In fact the theorem is more spectacular than the above. Let $-\infty \leq \alpha \leq \beta \leq +\infty$ are given. There is a rearrangement such that if u_n is the n-th partial sum of the rearranged series, then $\liminf u_n = \alpha$ and $\limsup u_n = \beta$.

Exactly the same construction and argument as above works when α and β are finite. You need to go beyond β and below α at each stage. If $\alpha = -\infty$ and $\beta = \infty$ you do the following. You go above n and below $-n$ in the n-th round. If $\alpha = \beta = \infty$, you proceed as follows. at the n-th round go beyond n but then take only one negative term. Other cases are similar.

infinities:

It is time to introduce the objects $+\infty$ and $-\infty$. This is only for convenience. You should keep in mind that every statement made using $\pm\infty$ can also be made, conveying the same meaning, but without using these symbols.

We start with the picture first as to how these objects fit with our picture of the real number system. We put the object $+\infty$ at the right end of the real number line and the object $-\infty$ at the left end of the real number line. So how to operate with these objects and what are the rules to which we agree upon now. First, we make a notational agreement. Just as we write 4 for $+4$ and if we want to say negative 4, we write -4, now also we do the same. Instead of writing $+\infty$, we just write ∞. So when I write ∞, you do not ask me which infinity: plus or minus? (Just as, when I write 4, you do not ask me whether I mean $+4$ or -4).

Rule 1 (order): We agree (as the picture suggests) $x < \infty$ for all $x \in R$ and $-\infty < x$ for all $x \in R$. We agree to say, as is sensible now, $-\infty < +\infty$.

Every set bounded above has a supremum. So far, a set which is not bounded above has no supremum. Now we make a definition. A set which is not bounded above also has a supremum and it is ∞. Similarly, so far a set which not bounded below has no infimum. Now we agree to say that a set which is not bounded below also has a infimum and it is $-\infty$.

Let S be a non-empty set.

(i) For a non-empty set bounded above, its supremum is the least upper bound. In symbols,

$$s = \sup S \ \leftrightarrow \ [\forall x \in S)(x \leq a)]\&[\forall x \in S, x \leq b \rightarrow a \leq b].$$

We continue to have this. Remember if S is a bounded set of real numbers the above statement is correct by definition. The symbols express just this fact, namely, supremum is an upper bound and any other upper bound is larger than this. Thus if you take the set S of all numbers which are larger than 5, we now have ∞ to be its supremum. The right side of the above statement is still correct. Unfortunately, we do not say, in words, that ∞ is least upper bound of S, though you are invited to imagine so.

The reason we do not say so is the following: we still reserve our right to say this set S, consisting of all numbers larger than 5, has no upper bound. So the question of least upper bound does not arise at all. You might ask why do we do this. Why not say, if a set is not bounded above then ∞ is its upper bound. Yes, you are invited to imagine and say so. But, even for the interval $[0, 1]$, you agree that ∞ is an upper bound, simply because for every point x of this set we have $x < \infty$. Thus saying that ∞ is an upper bound for a non-empty set is a tautology (what is a tautology?) and conveys no information. On the other hand 'S is not bounded above' conveys information. (Imagine the oxymoron, if S is no upper bound, then ∞ is its least upper bound).

(ii) In a similar manner for sets bounded below, infimum is its greatest lower bound. In symbols,

$$l = \inf S \leftrightarrow \ [\forall x \in S)(a \leq x)]\&[\forall x \in S, b \leq x \rightarrow b \leq a].$$

Commenst similar to above apply. For example, $-\infty$ is infimum of the set S of all numbers smaller than 5. Of course we do not say that it is the greatest lower bound.

Also the characterization of supremum and infimum remain correct provided, we formulate carefully. For a bounded set S, the characterization was the following:

$$s = \sup S \leftrightarrow [\forall x \in S, x \leq s] \& [\forall \epsilon > 0, \exists x \in S, x > s - \epsilon].$$

An equivalent formulation, not using ϵ, is the following

$$s = \sup S \leftrightarrow [\forall x \in S, x \leq s] \& [\forall b < s, \exists x \in S, x > b].$$

This formulation is still correct even if the supremum of the set is ∞.

Similarly,

$$l = \inf S \leftrightarrow [\forall x \in S, l \leq x] \& [\forall b > l, \exists x \in S, x < b].$$

Rule 2 (addition): $x + \infty = \infty$ for all $x \in R$ as well as for $x = \infty$. $x - \infty = -\infty$ for all $x \in R$ as well as for $x = -\infty$. We do not talk about $\infty - \infty$. The reason is simple, certain things that we know are true still remain true even with this convention. No meaning of $\infty - \infty$ will validate certain existing statements. Also $-(\infty) = -\infty$.

For example A and B are (non-empty) sets with supremums a and b, then the set $C = \{x + y : x \in A, y \in B\}$ has supremum to be $a + b$. We knew this if the numbers a and b are real. It remains true even if a and b are infinities. The sets being non-empty, the supremum can not be $-\infty$. For example, suppose $a \in R$ and $b = \infty$. This means that the set B is not bounded above. It is easy to see that C is not bounded above. similarly, if $a = \infty = b$, then C is not bounded above. similar remarks apply to infimums.

Let us make a definition. A sequence (x_n) converges to ∞, in symbols, $x_n \to \infty$ if given any number α, there is n_0 such that $x_n \geq \alpha$ for all $n \geq n_0$. This stands to reason. Afterall, ∞ is beyond all numbers. So x_n approaches ∞ should mean that x_n eventually exceeds any given number. Similarly, $x_n \to -\infty$ means that given any number α there is an n_0 such that $x_n < \alpha$ for all $n \geq n_0$.

The statement $x_n \to a$ and $y_n \to b$ implies $x_n + y_n \to a + b$ remains valid even if the limits are infinities, unless they are of infinities of opposite signs. This is simple to prove. For example if $x_n \to -4$ and $y_n \to \infty$ then given any number α , we can get n_0 such that $x_n > -5$ for $n \geq n_0$ and and an n_1 such that $y_n > \alpha + 5$ holds for $n \geq n_1$. If n is larger than both n_0 and n_1 then, $x_n + y_n > \alpha$ holds.

When $x_n \to \infty$ and $y_n \to -\infty$ holds, you can not say anything about $x_n + y_n$, in general. For example, $x_n = n$ and $y_n = -n$ tells $x_n + y_n \to 0$. $x_n = n$ and $y_n = -n^2$ says that $x_n + y_n \to -\infty$. If $x_n = n^2$ and $y_n = -n$ then $x_n + y_n \to \infty$.

Simailrly, the statement $x_n \to a$ implies $-x_n \to -a$ remains correct even if a is an infinity. Thus $x_n \to a$ and $y_n \to b$ implies $x_n - y_n \to a - b$, unless a and b are the same infinity (both ∞ or both $-\infty$).

Rule 3 (multiplication): $x \times \infty$ equals $+\infty$ if $x > 0$ or $x = \infty$; whereas it equals $-\infty$ if $x < 0$ or $x = -\infty$. Simialrly, $x \times (-\infty)$ equals $-\infty$ if $x > 0$ or $x = \infty$; whereas it equals ∞ if $x < 0$ or $x = -\infty$. What we did not define is $\infty \times 0$ and $(-\infty) \times 0$. Actually there are reasons to define them to be zero, but right now we do not define them.

The fact that $x_n \to a$ and $y_n \to b$ does imply $x_n \times y_n \to a \times b$ in all the cases when $a \times b$ is defined. The reason we did not define product of zero and infinity is the following. $x_n = 1/n$ and $y_n = n$ shows $x_n \cdot y_n \to 1$; whereas $x_n = 1/n^2$ and $y_n = n$ shows that $x_n \cdot y_n \to 0$; $x_n = 1/n$ and $y_n = n^2$ shows that $x_n \cdot y_n \to \infty$.

Finally, let us discuss limit points. For a sequence (x_n) and $a \in R$ we use the same definition as earlier to say a is a limit point of the sequence. Namely, for every $\epsilon > 0$, we have $x_n \in (a - \epsilon, a + \epsilon)$ for in finitely many values of n. We say that ∞ is a limit point if the sequence is not bounded above. This is same as saying that for every α, there are infinitely many n such that $x_n > \alpha$. Similarly, we say $-\infty$ is a limit point if the sequence is not bounded below, equivalently, given any α, there are infinitely many values of n such that $x_n < \alpha$.

Observe that the set L of limit points of a sequence (x_n) is non-empty. If it is not bounded below then $-\infty$ is a limit point. If it is not bounded above then ∞ is a limit point. If it is blounded above and also below (that is, bounded), then we already showed that there is at least one limit point. As a consequence, $\sup L$ and $\inf L$ are well defined and these are called limsup and liminf respectively.

We had the following characterization of limsup.

$$s = \limsup x_n \leftrightarrow [\forall \epsilon > 0 \;\; \exists \text{ only finitely many } \; n; \; x_n > s + \epsilon] \&$$

$$[\forall \epsilon > 0, \exists \text{ infinitely many } n, \; x_n > s - \epsilon].$$

We can restate this in an equivalent manner, without using ϵ, as follows.

$$s = \limsup x_n \leftrightarrow [\forall b > s \; \exists \text{ only finitely many } \; n; \; x_n > b] \&$$

$$[\forall a < s, \exists \text{ infinitely many } n, \; x_n > a].$$

This remains correct, irrespective of whether s is finite or infinity (an expression like a is finite is simply another way of saying $a \in R$). Similarly, the following remains correct, irrespective of whether l is finite or infinite.

$$l = \liminf x_n \leftrightarrow [\forall b < l \; \exists \text{ only finitely many } \; n; \; x_n < b] \&$$

$$[\forall a > l, \exists \text{ infinitely many } n, \; x_n < a].$$

With the concept of convergence as defined, we can say

$$\sum \frac{1}{n} = \infty$$

This is because the sum is limit of partial sums. We know that the partial sums are increasing and are not bounded above, so the sequence of partial syums converges to ∞. So far we only said that the series above does not converge, but now we are saying it is ∞. This is a better information.

Afterall, the series ± 1 also does not converge. But in the latter case the partial sums are, in a sense, oscillating. Whereas, for the sequence $\sum 1/n$, the above equation conveys the extra information that the sums eventually exceed any given value. Of course, in this case, because the series consists of non-negative terms, non-convergence of partial sums is equivalent to saying that they eventually exceed any given number. But for a general series the statement that $\sum a_n = \infty$ conveys more than simply saying that the series does not converge.

The upshot of all this is the following. Introduction of $\pm\infty$ sometimes allows us to express more than what we could do otherwise; sometimes they facilitate in succinctly expressing some statetments. You must understand why we entered this issue at all, why we introduced these objects at all. Remember that a word is invented to convey some information, which could not be conveyed without this word. In the same way we invented $\pm\infty$ to convey certain things. Think.

Some of you are, expectedly, puzzled that while talking supremum and infimum of sets I always considered non-empty sets. In practice we never have to calculate inf and sup of empty set. That is why I did not consider. But if it bothers you, here is the answer. Empty set is bounded. In fact every number is an upper bound as well as lower bound.

Let S be the empty set. If I said that $\forall x \in S, x \leq 5$, I would be correct. I only need to show that given any $x \in S$, then $x \leq 5$. This is true. Or equivalently, if you want to tell me my claim is false, you must produce $x \in S$ with $x > 5$, and you can not do this. Now comes a surprize and I would not like to waste time on it, because you are not going to learn anything by spending time on this (except worrying that infimum is strictly larger than supremum).

Here is the surprize. supremum of empty set is $-\infty$. Reason: every number being an upper bound for S, the infimum of the set of upper bounds is $-\infty$. To put it differently, given anything different from $-\infty$, I can take something smaller than that and say that is also an upper bound. Similar thought process shows that every number is a lower bound for the empty set and hence the greates lower bound is ∞.

It is better not to talk about supremum and infimum of empty set. If you ever need to calculate this, ask yourself if it is necessary at all; whether you have made life unnecessarily complicated.

discussion HA:

Q 27: To show $(n+47)^{589}/2^n \to 0$. The neumerator is a linear combination of powers of n. So if we prove that $n^k/2^n \to)0$ for each k, then we can use theorems on limits to complete the problem. As soon as $n > k$, we see

$2^n > nchoosek + 1$ so that

$$\frac{n^k}{2^n} \leq k!\,\frac{n}{n}\,\frac{n}{n-1}\,\cdots\frac{n}{n-k+1}\,\frac{1}{n-k} \to 0.$$

Note that k is fixed.

Q 32. Note that (x_n) is a sequence where each rational in $(0,1)$ appears and no others. Thus if $x > 1$, then $(x-\epsilon, x+\epsilon)$ where $\epsilon = (x-1)/4$ does not contain any x_n and hence can not be a limit point. Similarly, numbers less than zero are not limit points. Take any x with $0 \leq x \leq 1$. Consider $(x-\epsilon, x+\epsilon)$. Need to show it contains x_n for infinitely many values of n. This follows from the observation that any interval contains infinitely many rationals. Try to think.

Continuous functions:

There are several interesting things one could discuss about sequences and series. But we should stop our discussion on series to proceed to other stories. We shall now discuss continuity of functions. You are already familiar probably. We shall deal with functions defined on R or subsets of R and take values in R.

What is a function. Of course, while discussing cardinality, we did discuss functions. Thus we now deal with functions f that associate a real number with each point where it is defined. The set D of numbers for which f associates a value is called domain of the function. For $x \in D$, the value is denoted by $f(x)$, which is a real number.

Just to make you start thinking (and for nothing else), let us consider the following. For each $x \neq 0$, we associate $f(x) = x^2$. When $x = 0$ we associate $f(0)$ as follows. If there is an earthquake tomorrow, then $f(0) = 35$ and if there is no earthquake tomorrow then $f(0) = 27$. Is this a function? What is the meaning of association? You may say this is meaningless or this depends on time etc etc. But the question remains: is it a function or is it not a function?

(It is not a function today, because I do not know $f(0)$ today. Unless I know the value, I do not accept it as a function. The fact that the value is one of two numbers 27,35 is no consolation. You might then ask: what is meant by knowing. If it were $\sqrt{2}$, do you know? You may think that I do not know $\sqrt{2}$ because I, and no one, knows all its decimal places. But the point is I know that there is exactly one positive number whose square is 2 and this is *the* number we are talking about. Contents of this and previous para are intended to make you think about matters and nothing else. You are free to ignore.)

Returning to examples of funcrions, $f(x) = x^2$ is a function, it is defined on the full R and with a number x it associates the number x^2. The function $f(x) = 1/x$ is defined only for non-zero real numbers and hence this is its domain. For a number x in its domain, this function associates $1/x$. You can think of polynomials. For example

$$f(x) = 5 + 32x^3 + 99x^{55} + \sqrt{2}x^{56}.$$

This function is defined for every number x and the value is as given on the right side above.

We can think of complicated functions. For example,

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is a function. It is defined for every number x. if you take a number x, then consider the series above for that number. We already know that the series of numbers converges. The sum of this series is the value of the function for that number x. This is called exponential function $e(x)$.

Here is another function.

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - + \cdots.$$

We know that for any given real number x, the series of numbers on the right side converges and hence this function is also defined on all of R. This function is denoted $\sin x$. Yes, this is the same function, you know, explained without angles and triangles. It is periodic, but not obvious.

Here is another function.

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - + \cdots.$$

This function is denoted $\cos x$.

In olden days by a function, people understood as something given by an analytical expression, for example the above expressions. Then they realized that the function need not be given by one analytical expression, may involve more than one expression, but you should be able to draw the graph. For example, consider $f(x) = x^2$ when $x < 1$ and $f(x) = x^3$ when $x \geq 1$. Another example is the function defined on the interval $[0, 1)$ by $f(x) = 2x$ when $0 < x < 1/2$and $f(x) = 2x - 1$ when $1/2 \leq x < 1$. It takes values again in the interval $[0, 1)$.

It was realized later that, there need not be analytical expression, and we may not be able to draw the graph also. Such an example is f defined by $f(x) = 1$ when x is irrational and $f(x) = 0$ when x is rational. It is not possible to draw graph of this function in the usual x, y-coordinate system. Remember graph of f is the set G of all pairs (x, y) such that $f(x) = y$.

Whether we can draw the graph or not, the concept of graph is itself well defined and is a subset of $R \times R$. Graph has the interesting property that for every x there is at most one y such that $(x, y) \in G$. In fact if f is not defined for the point x, then there is no point in G whose first coordinate is x. On the other hand, if f is defined at the point x with, say, $f(x) = a$, then (x, a) is the only point of G with first coordinate equal to x.

Interestingly, any subset G of $R \times R$ with the property that for any x there is at most one y with $(x, y) \in G$ defines a function. This is easy. Just take domain D of the function f to be the set of all points x such that there is a y with $(x, y) \in G$. For x in D define $f(x)$ to be the unique y such that $(x, y) \in G$. (Upshot you can ignore: A function is a subset of $R \times R$, you need not use the words 'associate' etc!).

So let us start with a function f defined on all of R. When should we agree to say that it is continuous. Yes, it is continuous if it is continuous at each point a. So when shall we say that the function is continuous at a point a. Idea is the following. If you change a a little bit, then the value also should change only a little bit, not too much. Or equivalently, if x is close to a, then $f(x)$ should be close to $f(a)$. We have to make this precise. This is what we shall do now.

Continuous functions:

Thus we felt that a function is continuous at a point a if, for points near a the values should be close to the value at a. We should now make it precise. First we make an observation.

Fact: Let $f : R \to R$ and $a \in R$. Then the following two statements are equivalent.
(i) $x_n \to a$ implies $f(x_n) \to F(a)$.
(ii) Given $\epsilon > 0$, we can find a $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Proof: Let (ii) hold. Let $x_n \to \delta$. Fix $\epsilon > 0$. We exhibit n_0 so that $|f(x_n) - f(a)| < \epsilon$ for $n \geq n_0$. Go to the hypothesis (ii) and get $\delta > 0$ for this ϵ. Choose n_0 so that $|x_n - a| < \delta$ for $n \geq n_0$. But then, for these n, we have $|f(x_n) - f(a)| < \epsilon$.

Conversely, let (ii) fail. We show (i) fails by producing a sequence $x_n \to a$ but $f(x_n) \not\to f(a)$. Since (ii) is false, there is an $\epsilon > 0$ for which we can not find $\delta > 0$ satisfying the condition of (ii). Fix such an $\epsilon > 0$. Using the fact that $\delta = 1/n$ does not fulfil condition of (ii), we can fix an x_n such that $|x_n - a| < 1/n$ and yet $|f(x_n) - f(a)| \geq \epsilon$. No need to say anything more.

Definition: A function $f : R \to R$ is continuous at a point a if any one of the above conditions holds. We say that f is continuous if it is continuous at every point a.

The two conditions are interesting. Condition (i) allows us to verify discrete instances. That is, to verify this condition we fix any arbitrary sequence $x_n \to a$ and verify $f(x_n) \to f(a)$. Of course there are uncountably many sequences converging to a, but that is a different matter. Whenever you set to verify *one* instance it is a discrete sequence. On the other hand condition (ii) is a non-discrete condition. Even for one instance, you fix $\epsilon > 0$ and produce a δ; but this delta should verify something for every x with $a - \delta < x < a + \delta$, namely, for every such x we must verify $-\epsilon < f(x) - f(a) < \epsilon$. Remember this verification is to be done for every x in an interval. However, both are equivalent. Sometimes (i) and sometimes (ii) would be handy.

When we were making up our mind about continuity, some of you mentioned about limits. Yes, you are right, usually the notion of continuity is defined after defining limits of functions. One defines righthand limit, lefthand limit at a and says that the function is continuous at a if both righthand limit and lefthand limit exist and equal $f(a)$. I find it difficult to begin with these concepts, which demand understanding continuous limits. It is not to say that these concepts are unimportant. They are simple and important. After you feel comfortable with the existing concepts and assimilate them, you would have no problem discovering them yourself.

The collection of all real valued functions on a set S has a nice structure. If f and g are are such functions, you can define the function $h(x) = f(x)+g(x)$ on the set S. This function is simply denoted by $f+g$. Similarly fg denotes the function whose value at $x \in S$ is $f(x)g(x)$. The function $55f$ is the function whose value at $x \in S$ is $55f(x)$. The function $\frac{f}{g}$ or f/g has value $\frac{f(x)}{g(x)}$ at $x \in S$; but now this function may no longer be defined on all of S. It is defined only for those points $x \in S$ such that $g(x) \neq 0$. The domain of this function may not be all of S.

Thus remember sum, product etc of functions is defined by us. On the other hand, the set of real numbers R is a set with operations of addition and multiplication defined on the set satisfying certain conditions. The set of all real valued functions defined on S did not come with such operations. *We* defined those operations.

Fact: If $f : R \to$ and $g : R \to R$ are continuous functions, then so are $f+g$, fg and $29f$. The function f/g is also continuous if $g(x) \neq 0$ for all $x \in R$.

Proof is very simple, follows from definition of continuity and properties of sequences that we know.

The last statement regarding f/g is unsatisfactory, it excludes all functions g which take value zero at one point. It would be more satisfying to allow such functions too and be able to say that f/g is continuous on the set where it is defined. We shall rectify the situation soon, but let us see some examples first, of continuous functions defined ln all of R.

Fact: The following continuous from R to R.

$$f(x) \equiv 49; \quad f(x) = x; \quad f(x) = x^{100}.$$

More generally, every polynomial is a continuous function.

Definition: Let $S \subset R$ be a non-empty subset and $f : S \to R$ and $a \in S$. We say that f is continuous on S at the point a, if any one of the following two equivalent conditions holds.
(i) $x_n \to a$, $x_n \in S$ for all n implies $f(x_n) \to f(a)$.
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \in S$.
We say f is continuous on S if it is continuous on S at every point of S.

Of course, before making such a definition we should verify that the two conditions are indeed equivalent. This is very easy, if you look at the proof above you realize it. Proof of (ii) implies (i) needs no change. In the proof of '$\neg$ (ii)' implies '$\neg$ (i)' you only need to choose points x_n from S.

Fact: If f, g are real valued continuous functions on $S \subset R$, then so are $f + g$, $29f$ and fg. The function f/g is also continuous on the set $T = \{x \in S : g(x) \neq 0\}$.

Fact: Let $f(x) = (x - 55)$. Then f is continuous on R and $1/f$ is continuous on the set $\{55\}^c = R - \{55\}$.

Here we have used another notation $R - \{55\}$, but what is subtraction. It is not given to us. We define subtraction between two sets as follws: $A - B = \{x \in A : x \notin B\}$ This is same as $A \cap B^c$.

Let us see some functions which are not continuous.
The function $f(x) = 0$ for x rational and $f(x) = 1$ for x irrational is defined

on all of R and is not continuous at every $a \in R$.
The function $f(x) = x^2$ for $x \leq 0$ and $f(x) = x + 1$ for $x > 0$ is defined on all of R. It is continuous at every non-zero $a \in R$. It is not continuous at $a = 0$.
The function

$$f(x) = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-12}.$$

is defined on the set $S = \{1, 2, 12\}^c$ and is continuous on S at every point of S.

If S is the set of integers, then every real valued function on S is continuous on S. This is easy to see.
Here is a useful property of continuous functions defined on a closed bounded interval $[a, b]$.

Fact: Let f be a real valued continuus function defined an an interval $[a, b]$. Then f is bounded, that is there is a number C such that $|f(x)| \leq C$ for all x. In fact, there are points x_0 and x_1 in $[a, b]$ so that $f(x_0) \leq f(x) \leq f(x_1)$ for all x.

Thus the function attains its minimum and maximum values at the points x_0 and x_1 respectively.

Proof: Let us start with an observation. If we proved that every continuous function is bounded, then the second part follows. Indeed, let $\alpha = \inf\{f(x) : a \leq x \leq b\}$. If there is no point where the value of f equals α, then $g = 1/(f - \alpha)$ is continuous and not bounded, because for any $n \geq 1$, there is a point x with $f(x) < \alpha - (1/n)$ so that $g(x) > n$. Thus there is a point x_0 so that $f(x_0) \leq f(x)$ for all x. Similarly, there is an x_1 so that $f(x) \leq f(x_1)$ for all x.

Let us now show that f is bounded. Let $I_1 = [a, b]$. If f is not bounded, then it is not bounded either on the left half $[a, (a+b)/2]$ or on the right half $[(a+b)/2, b]$ of $[a, b]$. Let it be I_1. If f is unbounded on both halfs, take the left half. Since f is unbounded on I_1 let I_2 be the left half of I_1 if f is unbounded there or right half. In this way we get a sequence of closed bounded intervals with lengths decreasing to zero and henec by Cantor's theorem will have exactly one point, say α, in common. Clearly $\alpha \in [a, b]$. Since f is continuous on $[a, b]$ at the point α, get $\delta > 0$ so that $|f(x) - f(\alpha)| \leq 1$ for all $x \in I$ with $|x - \alpha| < \delta$. In particular, on the interval $(\alpha - \delta, \alpha + \delta)$ the function is bounded by $|f(\alpha)| + 1$. Pick k so that length (I_k) is smaller than $\delta/4$. since $\alpha \in I_k$ we see $I_k \subset (\alpha - \delta, \alpha + \delta)$. In other words f is bounded on I_k. This contradiction proves our result.

What if the function is not defined on a closed bounded interval? Clearly, you can not take your set to be unbounded. If S is unbounded, then $f(x) = x$ is a continuous function on S which is not bouinded. Of course, if it is defined on an open interval (a, b), then it need not be bounded. For instance, the function $f(x) = 1/x$ on the interval $(0, 1)$ tells you this. In this example, the function f is defined on the interval $(0, 1)$. There is a sequence in this set converging to a point outside the set, namely $1/n \to 0$. If sequences in your set do not converge to points outside the set, then this will ensure boundedness of the function.

Definition: Say that a set $C \subset R$ is closed under limits, simply **closed** if $x_n \to x$, $x_n \in C$ for all n implies $x \in C$.

For example, as seen above, the interval $(0, 1)$ is not closed but the interval $[0, 1]$ is closed. The set $\{1/n : n \geq 1\}$ is not a closed set but the set consisting of these points along with zero is a closed set.

Fact: Let f be a real valued continuous function on a closed bounded set S. Then f is bounded, in fact, there are numbers $x_0 \in S$ and $x_1 \in S$ so that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in S$.

Proof goes exactly along the same lines as earlier. The first part implies the second part by same argument. First part is proved as follows. Since the set is bounded, get real numbers $a < b$ so that $S \subset [a, b]$ let $I_1 = [a, b]$. If f is unbounded on S, then it is unbounded either on the part of S in the left half or on the part of S in the right half of I_1. Denote it by I_2. Continue always making sure that f is not bounded on the part of S in I_n and lengths are always halved. Get α common to all these intervals. Is $\alpha \in S$? Yes, because f being unbounded on the part of S in I_n, you see that in particular you can pick a point x_n of S from I_n. Obviously, $|x_n - \alpha| \leq$ length $I_n \to 0$. Since S is closed, we conclude $\alpha \in S$. Now use continuity of f at α and proceed as earlier.

Here is another property of continuous functions.

Fact: Suppose f is continuous on S and g is continuous on T. Assume that $f(x) \in T$ for each $x \in S$. Define the composition $h(x) = g(f(x))$ on S. Then h is continuous on S.

In fact, if $x_n \to x$, then $f(x_n) \to f(x)$ and so $g(f(x_n)) \to g(f(x))$.

Disucssion of HA:

Q37 $x_n \to a$ To show $a_n \to a$ where $a_n = \sum_1^n x_i$.

If you are taking average of many many numbers and if most of them are close to a then so should be their average. Here as n becomes large, you are taking average of many many numbers and by hypothesis most of them are close to a.

No loss to assume $a = 0$. This is because, you consider the sequence $y_n = x_n - a$. Then averages of y_i are just averages of x_i minus a. If the result is proved for (y_n), then you can complete the proof.

So let us assume that $x_n \to 0$. Fix $\epsilon > 0$. Shall show n_0 so that

$$|\frac{1}{n}\sum_1^n x_i| < \epsilon, \qquad n \geq n_0.$$

First fix k so that $|x_n| < \epsilon/2$ for $n \geq k$, possible since $x_n \to 0$. Having fixed k like this fix $n_0 > k$ so that

$$\frac{1}{n_0}\sum_1^k |x_i| < \epsilon/2,$$

possible because k is fixed. Now if $n > n_0$

$$\frac{1}{n}|\sum_1^n x_i| \leq \frac{1}{n}\sum_1^k |x_i| \quad + \quad \frac{1}{n}\sum_k^n |x_i| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

as promised.

Q53. To discuss convergence of $\sum \frac{P(n)}{Q(n)}$.

You are forgetting basic rule of life: If there is a complicated problem, you have no clue how to proceed, ask if you can solve a simpler problem.

Why worry about general polynomials? Ask yourself: what if $P(x) = x^l$ and $Q(x) = x^k$? Then the ratio is $\sum n^{-(k-l)}$ and it converges iff $k - l > 1$. Since we are dealing with polynomials, k, l are integers, so this amounts to saying $k \geq l + 2$.

Let now

$$P(x) = a_0+a_1x+a_2x^2+\cdots+a_lx^l; \quad Q(x) = b_0+b_1x+b_2x^2+\cdots+b_kx^k \quad a_l, b_k > 0.$$

There is no loss to assume $a_l > 0$, otherwise argue for $-P(x)$. Similarly, no loss to assume that $b_k > 0$. Observe that

$$\frac{P(n)/Q(n)}{n^l/n^k} \to \frac{a^l}{b^k} > 0$$

and the terms of the given series, namely, $P(n)/Q(n)$ as well as n^l/n_k are positive and hence either both converge or both fail to converge by Q51. But this special case we know already. Thus convergence holds iff $k \geq l + 2$.

Q51 $a_n/b_n \to 55$, all the a's and b's are strictly positive. To argue $\sum a_n$ converges iff $\sum b_n$ converges. Get k so that we have for $n \geq k$,

$$54 < \frac{a_n}{b_n} < 56 \quad (i.e.) \quad 54b_n < a_n < 56b_n.$$

Thus if $\sum a_n$ converges then $\sum\limits_{n>k} b_n \leq \sum_{n\geq k} a_n/54$ converges and hence $\sum b_n$ converges. If $\sum b_n$ converges then $\sum\limits_{n>k} a_n \leq \sum\limits_{n>k} 56b_n$ converges and hence $\sum a_n$ converges.

Return to Q 53. Since $P(n)/n^l \to a_l > 0$ we see $P(n)$ is positive after a stage. similarly $Q(n) > 0$ after some stage. Thus $P(n)/Q(n)$ is positive after some stage. But Q 51 assumes all terms a_n and b_n are positive. Sort it out.

Also hypothesis says that $Q(n)$ is never zero. But they did not say $P(n) \neq 0$. Thus it is quite likely that $P(n)/Q(n) = 0$. But Q 51 assumes all

a_n etc are strictly positive. Sort it out.

Unless you sort out these two issues, the proof is incomplete. (you may want to say: but this is easy. *Remeber*, you would know whether it is easy or difficult or even wrong, only after you argue it out once. Also moreover, if you are writing proof, you would not like to wait till someone objects and then modify your proof.)

Q55. Convergence of

$$\sum \frac{1}{n\ \log n\ (\log\log n)^p}.$$

Some of you are again forgetting basic facts of life, nothing to do with maths. I do not know if you are lazy or just afraid of the mathematical expressions and close your brain right away. You *must* attend to this, whatever it be. This is the famous phrase of Paul Erdos: keep your brains open.

Whenever you are in an unknown territory and see an animal like the above one, you should ask the natural question: did I see any similar animal earlier. there pops out the answer: yes I saw the series with terms $1/n^p$ and $1/\{n(\log n)^p\}$. You do not seem to take advantage of the fact that you have seen very few animals and it is easy to go through the list of the animals you have seen very quickly. Once the answer comes out, you should ask: how did I tackle that animal. Of course, similar thing may or may not work, but obviously you should go with your natural instincts and try out.

So coming to the problem at hand: The given series converges iff $\sum 2^n a_{2^n}$ converges, that is, iff the following series converges

$$\sum 2^n \frac{1}{2^n(n\log 2)(\log(n\log 2)^p} = \frac{1}{\log 2}\sum \frac{1}{n(\log n + \log 2)^p}.$$

Ignore the $\log 2$ factor and take the series as $\sum a_n$ and try

$$\sum b_n = \sum \frac{1}{n(\log n)^p}.$$

You see all the terms are positive and

$$\frac{b_n}{a_n} = \left[\frac{\log n + \log 2}{\log n}\right]^p \to 1.$$

Now comnplete the proof.

Remember we started with $n > 1000$ or some such thing to make sure denominator make sense. So the series you are comparing with, let it also start in a similar way. Otherwise, if you blindly compare with $\sum_{n\geq 2} b_n$ then you are not doing correctly.

Q59 Convergence of the series

$$\sum \frac{\log(n+1) - \log n}{(\log n)^2} = \sum \frac{\log(1 + \frac{1}{n})}{(\log n)^2}.$$

This is a series of positive terms and if you use the inequality $\log(1 + x) \leq x$ then this series is dominated by the series with terms $1/\{n(\log n)^2\}$ and hence converges.

One of you suggested different argument which is nice, but I forgot it at this moment.

unraveling negations;

We have employed, the method of proof by contradiction, several times.
This is how it goes. Need to show $S \Rightarrow T$.
Thus you are granted a hypothesis S.
You want to prove a sentence T.
You are unable to do so directly.
Then, you would say, alright, suppose T is false.
And work hard to show S is false.
Where are we now?
T is false $\Rightarrow$ S is false
Of course, S is true $\Rightarrow$ S is true.
Thus
S is true and T is false $\Rightarrow$ S is true and S is false.
But we can not have S is true and S is false at the same time.
So we can not have S is true and T is false at the same time.
So when S is true, T must also be true.

While running proof by contradiction, it is important to know what is the meaning of: suppose T is false. To take a specific example. in proving the equivalence of the two statements regarding continuity at a point a, we had

$$S: \quad (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon).(\spadesuit)$$

We needed to know its negation. This is not a lecture on logic, we restrict to some common sense aspects, to be able to carry on arguments.

If we have a simple sentence like $(c \leq 5)$ then its negation is easy: $(c > 5)$. If you have compound sentence like $(c < 5) \vee (c > 30)$, it says one of two things happens. So its negation is that none of them happen. Thus its negation is $(c \geq 5) \wedge (c \leq 30)$. Thus negation is $(5 \leq c \leq 30)$.

A sentence $(c < 5) \Rightarrow (d > 7)$ means when $(c < 5)$ holds then $(d > 7)$ must hold. But we always have either $(c < 5)$ or $\neg(c < 5)$. Thus we always have either $(d > 7)$ or $\neg(c < 5)$. In other words $(c < 5) \Rightarrow (d > 7)$ means $\neg(c < 5) \vee (d > 7)$. Thus sentence 1: $A \Rightarrow B$ is same as saying sentence 2: $\neg A \vee B$. What is explained just now tells you why this is so. Of course, in logic you take statement 1 as an abbreviation for statement 2. But let us not bother.

But sentences which involve quantifiers $\forall$ and $\exists$ are to be carefully analyzed for negation. If you follow logical or symbolic method of writing sentences, then making negations is easy.
S: Every student has a pen.
What is its negation: Every student has no pen?
No, it is: There are students without a pen. because even if one student has no pen, S is negated.
Thus $\forall x A(x)$ would have negation $\exists x \neg A(x)$.

S: there is a student who is sleeping.
What is its negation: there is a student who is not sleeping?
No, it is: every student is not sleeping.
Thus $\exists x A(x)$ would have negation $\forall x \neg A(x)$.

Let us return to our sentence.

$$S: \quad (\forall\epsilon > 0)(\exists\delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon).(\spadesuit)$$

Its negation is what we are interested.

$$\neg S: \quad \neg\Big\{(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)\Big\}.(\clubsuit)$$

From what was said above about $\forall$, negation of S is,

$$\neg S: \quad (\exists\epsilon > 0)\neg\Big\{(\exists\delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)\Big\}.(\clubsuit)$$

Again from what has been said about $\exists$, we see

$$\neg S: \quad (\exists\epsilon > 0)(\forall\delta > 0)\neg\Big\{(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)\Big\}.(\clubsuit)$$

Again using negation of $\forall$, this is same as

$$\neg S: \quad (\exists\epsilon > 0)(\forall\delta > 0)(\exists x)\neg\Big\{(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)\Big\}.(\clubsuit)$$

Remembering $A \Rightarrow B$ is same as saying $(\neg A) \vee B$, this is same as

$$\neg S: \quad (\exists\epsilon > 0)(\forall\delta > 0)(\exists x)\neg\Big\{[\neg|x-a| < \delta] \vee [|f(x) - f(a)| < \epsilon]\Big\}.(\clubsuit)$$

remembering negation of $A \vee B$ is $\neg A \wedge \neg B$, we see

$$\neg S: \quad (\exists\epsilon > 0)(\forall\delta > 0)(\exists x)\Big\{\neg[\neg|x-a| < \delta] \wedge \neg[|f(x) - f(a)| < \epsilon]\Big\}.(\clubsuit)$$

But of course $\neg\neg A$ is same as A. Thus we have,

$$\neg S: \quad (\exists\epsilon > 0)(\forall\delta > 0)(\exists x)\Big\{(|x-a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon)\Big\}.(\clubsuit)$$

The logical symbols are introduced to convince you how simple it is to understand negations. It is a step by step process. Let us read in words now.

The statement S reads: for every $\epsilon > 0$, there is $\delta > 0$, so that for every x either $|x-a| \geq \delta$ or $|f(x) - f(a)| < \epsilon$.

The statement $\neg S$ reads: There is an $\epsilon > 0$ so that for every $\delta > 0$, there is an x such that $|x - a| < \delta$ and also $|f(x) - f(a)| \geq \epsilon$.

This is fun and you should treat it so. In case you are getting confused, either sort it out or ignore all this and think in your own way. The issue is: you should be able to write negation of sentences without using negation for quantifiers.

Continuous functions:

We proved that a continuous function f defined on a closed bounded set S has a maximum and minimum. That is, first of all the values of the function form a bounded set. If M denotes the supremum of all the values of f and m denotes the infimum of all values of f then there are points x_0 and x_1 in S such that $f(x_0) = m$ and $f(x_1) = M$. Thus the infimum and supremum are actually attained. All other values $f(x)$ of the function are between $f(x_0)$ and $f(x_1)$.

Here is another useful property of continuous functions. Instead of closed bounded set, it should now be defined on an interval. If α and β are in the range of the function, then so is any value in between. In other words the function can not skip values. This is same as saying that the range of the function is an interval. This is known as the intermediate value property.

This is a very useful result as we see later. But for now, you can imagine the following. consider the function $f(x) = x$ for $-1 \leq x \leq 0$ and $f(x) = 1 + x$ for $0 < x \leq 1$. Imagine drawing the graph of this function. you can draw the curve from $x = -1$ to $x = 0$ without lifting your pen, however to proceed further you have to lift your pen at $x = 0$ and then continue. That is because the function after reaching vallue zero at $x = 0$ skips numbers a little above zero and starts assuming values beyond one. On the other hand imagine drawing the curve $f(x) = x$ for $-1 \leq x \leq 1$. You can do so without lifting your pen.

Of course, we have already seen that we can not draw graphs of many functions. But imagine, you are still not having the modern definition of function. You still have geometry to guide you and think of function as the graph. Then continuous function should mean a function whose graph you can draw continuously. But what is meant by being able to draw continuously? One way of interpreting is that, we should be able to draw the curve smoothly without breaks, or without lifting our pen.

Suppose that a function assumes a certain value α at a point and then immediately afterwards it starts assuming values larger than $\beta > \alpha$, missing all numbers in between. Then, you can feel that while drawing its graph, you need to necessarily lift your pen at that point. In other words, if you can draw a graph without lifting your pen, then the following happens: whenever the pen reaches a height (from x-axis) of α at some stage and later reaches a height β, the pen must have passed through all heights in between. This is, of course, an intuitive feeling. The intermediate value theorem make this precise and assures us that for a continuous function this holds good, whether you can draw the graph or not.

Fact: Let f be a continuous function defined on an interval. Suppose $f(a) = \alpha$ and $f(b) = \beta > \alpha$. Let γ be a number $\alpha < \gamma < \beta$ then there is a number c such that $a < c < b$ and $f(c) = \gamma$..

Proof is simple, but first note the following. Since we assumed that f is defined on an interval, every number between a and b is in the domain of f and hence it makes sense to talk of value of f at such points. Consider

$$c = \sup\{x : a \leq x \leq b; f(x) < \gamma\}.$$

Denote the set on right by S. Then $S \neq \emptyset$ because, $f(a) = \alpha < \gamma$; S is bounded above by b. Thus supremum makes sense.

$\underline{a < c}$. Since $f(a) < \gamma$, there is $\delta > 0$ so that $f(x) < \gamma$ for x in domain of f with $a - \delta < x < a + \delta$. To see this, just take $\epsilon = (\gamma - \alpha)/2$ in the definition of continuity. In any case for all points a little above a we have $f(x) < \gamma$. More precisely, if $\delta' = \min\{\delta, b - a\}$, then $\delta' > 0$ and we have $f(x) < \gamma$ for $a < x < a + \delta'$ showing that c must at least be $a + \delta'$.

$\underline{c < b}$. Since $f(b) > \gamma$ we argue as above to get $\delta' > 0$ so that $f(x) > \gamma$ for $b - \delta' < x \leq b$. Thus c is at most $b - \delta'$.

$\underline{\neg(f(c) < \gamma)}$. If $f(c) < \gamma$ then by continuity, we get $\delta' > 0$ so that $f(x) < \gamma$ for $c \leq x \leq \delta'$; showing that c can not be upper bound of S.

$\underline{\neg(f(c) > \gamma)}$. If $f(c) > \gamma$ then by continuity, we get $\delta' > 0$ so that $f(x) > \gamma$ for $c - \delta' \leq x \leq c$. Since c is upper bound of S, points above c are

not in S and the present inequality shows $c - \delta'$ is also an upper bound of S contradicting that c is least upper bound of S.
Thus $f(c) = \gamma$ and the proof is complete.

You should be careful. We only said that a continuous function satisfies above property. We did not say that a function which satisfies above property is continuous.

Consider the function $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Then this function has the intermediate value property, that is, the statemnent of the theorem above holds. However the function is not continuous;

$$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi} \cdots \to 0$$

but the value of f at all these points equals 1. In fact, you can take $f(0)$ to be any number in $[-1, 1]$. Then f is not continuous but has intermediate value property.

We saw that polynomials are continuous functions. Actually, polynomials of infinite degree are also continuous. We only do a special case now. Before that, we make an observation about convergent series.

Fact: Let $\sum a_n$ be a convergent series. Given $\epsilon > 0$ there is n_0 such that $|\sum\limits_{i \geq n_0} a_i| \leq \epsilon$ for $n \geq n_0$.

Just as sums of the form $\sum\limits_{i \leq n} a_i$ are called partial sums, sums of the form $\sum\limits_{i \geq n} a_i$ are called tail sums, simply because, this is sum of a 'tail' of the series. Partial sums always make sense because they are finite sums. However tail sums make sense only when the original series converges. Because then by our observation about deletion/insertion of terms, this sum also converges.

To prove the fact stated above, fix $\epsilon > 0$, get n_0 so that $|s_n - s_m| < \epsilon$ for $n, m \geq n_0$. This will do.
Now take any $n \geq n_0$. To show that $|\sum\limits_{i \geq n} a_i| \leq \epsilon$, we only need to show each of its partial sums obey this inequality. But k-th partial sum of this series is

$a_n + a_{n+1} + \cdots + a_{n+k-1}$ which is nothing but $s_{n+k-1} - s_n$.

Fact: The function $e(x)$ is a continuous function on R.

Proof: Let $x_k \to x$. Need to show $e(x_k) \to e(x)$. Let $\epsilon > 0$. We exhibit k_0 so that $|e(x_k) - e(x)| < \epsilon$ for $k \geq k_0$.

The idea is the following. The infinite sum $e(x_k)$ is close to finite sum. Since polynomials are continuous, such a finite sum is close to the corresponding finite sum of $e(x)$, which in turn is close to the infinite sum. This works out fine if we can choose 'one tail' so that the tail sum is small for all $e(x_k)$ as well as for $e(x)$. Here are the details.

Since a convergent sequence is bounded, fix C so that $|x| \leq C$ and also $|x_k| \leq C$ for all k. Since the series $\sum\limits_{i \geq 0} \frac{C^i}{i!}$ converges, choose N so that

$$\sum_{i \geq N} \frac{C^i}{i!} \leq \epsilon/4.$$

This is made possible by the previous observation. In particular we have

$$\left| \sum_{i \geq N} \frac{x^i}{i!} \right| \;\leq\; \sum_{i \geq N} \left| \frac{x^i}{i!} \right| \;\leq\; \sum_{i \geq N} \frac{C^i}{i!} \;\leq\; \epsilon/4$$

Same inequality holds for each of the x_k too. Incidentally, we used that $|\sum \alpha_i| \leq \sum |\alpha_i|$. However we have proved this only for finite sums, can we use for infinite sums? Yes, use for each partial sum and then properties of limits of sequences. Of course, you need to assume convergence of the series $\sum \alpha_i$.

We also have,

$$\sum_{i<N} \frac{x_k^i}{i!} \to \sum_{i<N} \frac{x^i}{i!}.$$

Note these are finite sums and the hypothesis $x_k \to x$ makes this true. So fix k_0 so that

$$\left| \sum_{i<N} \frac{x_k^i}{i!} - \sum_{i<N} \frac{x^i}{i!} \right| < \epsilon/4, \qquad k \geq k_0.$$

Let now $k \geq k_0$. Then

$$\begin{aligned} |e(x_k) - e(x)| &= \left| \sum_{i<N} \frac{x_k^i}{i!} + \sum_{i\geq N} \frac{x_k^i}{i!} - \sum_{i<N} \frac{x^i}{i!} - \sum_{i\geq N} \frac{x^i}{i!} \right| \\ &\leq \left| \sum_{i<N} \frac{x_k^i}{i!} - \sum_{i<N} \frac{x^i}{i!} \right| + \left| \sum_{i\geq N} \frac{x_k^i}{i!} \right| + \left| \sum_{i\geq N} \frac{x^i}{i!} \right| \;<\; 3\epsilon/4. \end{aligned}$$

We used the earlier inequalities in the last step. This completes the proof.

This is one of the nice techniques. The same type of argument shows that $\sin x$ and $\cos x$, are continuous functions too. Note that they are also defined as sum of infinite series. This technique achieves much more than what we said just now. But this we see later.

Fact: If f and g are two continuous functions on R and if $f(x) = g(x)$ for every rational number x then $f(x) = g(x)$ for every real number x.

This is easy, because if we take a real number x we can get rational numbers $r_n \to x$ and we know $f(r_n) = g(r_n)$ for every n so that continuity tells us

$$f(x) = \lim f(r_n) = \lim g(r_n) = g(x).$$

The interesting point is the following. You only need to show equality for countably many rationals. Then the equality holds for all the uncountably many irrationals too. Of course, you could have taken, instead of rationals, any other countable set D satisfying the condition: given a real number x there is a sequence d_n in the set D such that $d_n \to x$.

Discontinuity:

Let us consider $f : R \to R$. If f is not continuous at a point a, then we say that it is discontinuous at a or say that a is a point of discontinuity of f. Is there anything interesting worth studying about such points. Firstly, what exactly can happen or not happen if the function has a discontinuity at a point a. As always, it is best to look at some examples and try to get a feel.

We shall describe several functions. They are simple and you should draw graphs of all these functions.

Let $f(x) = x$ for $x \leq 0$ and $f(x) = +1$ for $x > 0$.

$g(x) = -1$ for $x < 0$ and $g(x) = x$ for $x \geq 0$.

$h(x) = x - 1$ for $x < 0$, $h(x) = x + 1$ for $x > 0$ and $h(0) = 0$.

All these functions are discontinuous at the point $a = 0$. The function f fails because to the right of zero, no matter how close you look, the values of f are not close to $f(0)$. Interestingly, to the left of zero they are close. In other words given $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for $x \in (-\delta, 0)$.

The function g fails because now values $f(x)$ to the left of zero are not close to $f(0)$. However, given $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for $x \in (0, \delta)$.

For the function h we can do neither. Values to the right or to the left, no matter how close, are not close to $f(0)$.

Let us now coonsider the following functions.

$f_1(x) = x$ for $x \leq 0$ and $f_1(x) = \sin(1/x)$ for $x > 0$.

$g_1(x) = \sin(1/x)$ for $x < 0$ and $g_1(x) = x$ for $x \geq 0$.

$h_1(x) = \sin(1/x)$ for $x \neq 0$ and $h_1(0) = 0$.

The function f_1 has exactly the same property as f. But there is a difference. Eventhough f to the right of zero takes values far from $f(0)$, those values are close to 1. In other words, given $\epsilon > 0$ we can indeed find $\delta > 0$ so that $|f(x) - 1| < \epsilon$ for $0 < x < \delta$. It just so happens that the number 1 is not $f(0)$. In other words the values to the right of zero *are* close to some thing but not $f(0)$. To put it differently, as x approaches zero, staying above zero the values approach 1, it behaves smoothly. Consider the function f_1. To the right of zero it is wiggly. As x stays above zero but goes closer and closer to zero the values of the function are not approaching any particular number.

Exactly the same kind of difference is seen between g and g_1. The function h_1 is wiggly both to the right as well as to the left of zero.

Thus when a function is not continuous at a point there are several possibilities. The values may be approaching something or wiggly when you look at the right; the values may be approaching some number or wiggly to the left. Finally even if the values on a side approach a number, it may be different from the value of the function at a.

We shall take up these issues later. Of course, we do not spend too much time on discontinuities. We spend enough time to convince that some interesting things can be said even about discontinuities.

Home Assignment:

Q: How do you show that there is no bijection between N and its power set $P(N)$. Here

$$N = \{1, 2, 3, \vdots \cdots\}, \qquad P(N) = \{A : A \subset N\}.$$

We can identify $P(N)$ with the set S of infinite sequences consisting zeros and ones. If $A \subset N$, you can identify with the sequence (x_n) where $x_n = 1$ if $n \in A$ and $x_n = 0$ if $n \notin A$. Note that given any such sequence (x_n) it corresponds to the set $\{n : x_n = 1\}$. This is a bijection between $P(N)$ and S.

Thus we need ot show that there is no bijection between N and S. But this is just Cantor diagonal argument. Suppose there is a function $f : N \to S$. Let us make a sequence (z_n) as follows. We first look at the sequence $f(n)$, then look at its n-th term. If this is 1 we take $z_n = 0$; if this is zero we take $z_n = 1$. Thus z_n is different from the n-th term of the sequence $f(n)$. In particular the sequence (z_n) can not be any of the sequences $f(1), f(2), f(3), \cdots\cdots$. Thus whatever f you take, it can not be onto S. In particular, there is no bijection.

Here is another argument without passing through sequences of zeros and ones. Suppose there is a function $f : N \to P(N)$. We make a subset of N as follows. Take an n. Thus $f(n)$ is a subset of N. There are exactly two possibilities: either $n \in f(n)$ or $n \notin f(n)$. We make a set consisting of the second kind of integers. More precisely, we define $A = \{n : n \notin f(n)\}$. Thus $A \subset N$. You may get unnecessary (irrelevant) doubts like A may be empty, it may be all of N. Do not get distracted. Consider the set A. No matter what it consists of, it is a subset of N. We say that there is no k such that $f(k) = A$.

Suppose you say that there is an $k \in N$ such that $f(k) = A$. Where is this integer k. Surely, $k \in A$ or $k \notin A$.

If you say $k \in A$. We look at the criterion for an integer to be in A. Remember $n \notin f(n)$. So if you say $k \in A$, then remembering that $A = f(k)$, you conclude that $k \notin A$.
If you say $k \notin A$, then remembering again that $A = f(k)$, you are saying $k \notin f(k)$. But then by our criterion, $k \in A$, that is, $k \in f(k)$.
In either case there is a contradiction and one of them must occur. Thus our assumptiuon that there is an k such that $f(k) = A$ is false.

Thus there is no function on N onto $P(N)$. In particular, there is no bijection.

The second proof has an advantage. it works for any set!. Let S be any set. There is no function on S onto $P(S)$ and in particular there is no bijection between S and $P(S)$. By the way, $P(S)$ is the collection of all subsets of S, that is $\{A : A \subset S\}$. the proof above applies verbatim.

Closer look tells you that, in case of N, both the above proofs are exactly (yes, exactly) the same! Decipher.

Q: Some of you have confusion regarding limit point of a set and limit point of a sequence.

For example, take the sequence: $1, 2, 1, 2, 1, 2, 1, 2, \cdots$ and the set $A = \{1, 2\}$ which consists points of the sequence. Clearly both the numbers 1 and 2 are limit points of the sequence, simply because if you take any interval around 1 or 2, there are infinitely many n such that x_n is in that interval. However the set A has no limit points, because no interval around any point has infinitely many points of the set A; afterall A is a finite set. So A has no limit point.

There are two ways you could have invited this confusion. Firstly, a is limit pont of a sequence (x_n), if for any $\epsilon > 0$, the interval $(a - \epsilon, a + \epsilon)$ contains x_n for infinitely many n. You were careless and shortened this to say $(a-\epsilon, a+\epsilon)$ contains infinitely many numbers of the sequence; naturally the sequence has only finitely many numbers (namely 1 and 2); you concluded that the sequence has no limit point. You should read the two sentences carefully, they do not convey the same meaning. You have no business to replace a definition with something which is not equivalent.

Second way you invited confusion is by thinking of the sequence as the set A and since A has no limit point, you concluded that the sequence has no limit point. This is again wrong. 'sequence' and 'set' are as different as chair and table. As I mentioned, a sequence has an order: first term of the sequence, second term of the sequence etc. On the other hand when you say set, there is no order on the elements of the set. You can say that a point is in the set and another point is not in the set. But it makes no sense to say that a point is the first point of the set! So you must not identify sequence with the set of points that the sequence consists of.

Q30: If $a_{n+1}/a_n \to L > 0$, then $\sqrt[n]{a_n} \to L$. All a_n are strictly positive.

Quick way of seeing this is to say $\log a_{n+1} - \log a_n \to \log L$ and hence their averages also converge to L.

$$\frac{1}{n+1}\log a_{n+1} = \frac{n}{n+1}\frac{\log a_{n+1} - \log a_1}{n} + \frac{\log a_1}{n+1} \to \log L.$$

Of course, you need not use logarithm etc. This is 'geometric' analogue of the Cesaro limit we considered. Obviously, one is tempted to use that idea. Hre it is.

Let $\epsilon > 0$. We show after some stage $\sqrt[n]{a_n} < L + \epsilon$. Choose k so that

$$n \geq k \Rightarrow \frac{a_{n+1}}{a_n} < L + \frac{\epsilon}{2}.$$

Now for any $n > k$

$$\sqrt[n]{a_n} = \sqrt[n]{a_k\,\frac{a_{k+1}}{a_k}\,\frac{a_{k+2}}{a_{k+1}}\,\cdots\frac{a_n}{a_{n-1}}} \leq \sqrt[n]{a_k}\;\sqrt[n]{\left(L+\frac{\epsilon}{2}\right)^{-k}}\;\left(L+\frac{\epsilon}{2}\right).$$

Using that $\sqrt[n]{\alpha} \to 1$ choose $n_0 > k$ so that for $n \geq n_0$

$$\sqrt[n]{a_k} \ \sqrt[n]{\left(L + \frac{\epsilon}{2}\right)^{-k}} \ \leq \ \frac{L+\epsilon}{L+\frac{\epsilon}{2}}.$$

This will do. Similarly, you can choose n_0 so that $L - \epsilon < \sqrt[n]{a_n}$ for $n \geq n_0$.

exponentiation. (continued).

At the expense of repetition, we shall recall exponentiation and complete that discussion. The reason I defined x^a earlier already is that it is simple and should not wait till we do sequences and continuous functions. One smart and very useful way is to say

$$x^a = e^{a \log x}.$$

This appears still worse to me because you need to wait till you learn e^x and natural logarithm etc. Generally one does not pay attention to this; worse than that, one assumes he/she knows everything — the definition and all properties. For example, did you ever understand the meaning and prove the equation,

$$(\sqrt{7})^{\sqrt{2}+\sqrt{35}} = (\sqrt{7})^{\sqrt{2}} \times (\sqrt{7})^{\sqrt{35}}$$

In the earlier discussion some details were left out because we anyway need to return for a comprehensive discussion.

Step 1: $x \neq 0$. To define x^n for $n = 1, 2, 3, \cdots$.

This is defined by induction: $x^1 = x$ and if we have defined x^n for $n = 1, 2, \cdots k$ then we put $x^{k+1} = x^k \cdot x$. Here are two facts.

$$x^{n+m} = x^n \cdot x^m; \qquad (xy)^n = x^n y^n \qquad m, n \in N$$

Usually this is mentioned but never proved in high school. It is understandable because at the high school level, concept of 'proof' is difficult. It may even be uninteresting and counter productive. Having heard it several times from your teacher, you take it as a fact that needs no proof! The old adage — familiarity breeds contempt — fits here well.

If you never saw a proof, now is the time to write a proof of this fact. Some of you felt that to prove $x^{20+30} = x^{50}$ is simple because left side is $x \times x \times x \cdots$ 50 times and the first 20 make x^{20} and the remaining make x^{30}. Do you see why this is not acceptable? Firstly, you have only restated what is to be proved, but did not prove anything. Secondly, this 'dot dot dot' is perfect in thinking but it is not the definition we adapted. Thirdly, even if someone accepts your dot dot dot, are you going to write one sentence for each pair $(20, 30)$, $(21, 33)$, $(44, 89)$, etc. Then your proof will never end.

Step 2: $x \neq 0$. To define x^n for $n \in Z$.

For $n \in N$ it is defined above. For $n = 0$, we put $x^0 = 1$. For $n < 0$ it is defined as $x^n = (1/x)^{-n}$. Prove the law of indices.

$$x^{n+m} = x^n \cdot x^m; \qquad (xy)^n = x^n y^n \qquad m, n \in Z$$

Step 3: $x > 0$. To define $x^{1/n}$ for $n = 1, 2, 3, \cdots$.

We proved in class the existence of exactly one number $y > 0$ such that $y^n = x$. We define this y as $x^{1/n}$, also denoted as $\sqrt[n]{x}$. We proved,

$$0 < x < y \Rightarrow x^{1/n} < y^{1/n}; \qquad (xy)^{1/n} = x^{1/n} y^{1/n}.$$
$$x > 1 \Rightarrow x^{1/n} > 1; \quad x < 1 \Rightarrow x^{1/n} < 1; \quad x = 1 \Rightarrow x^{1/n} = 1.$$

More precisely,

$$x > y > 1 \Rightarrow x^{1/n} > y^{1/n} > 1; \quad x < y < 1 \Rightarrow x^{1/n} < y^{1/n} < 1.$$

This last statement is expressed by saying that n-th root is monotone increasing on the set $(1, \infty)$ and monotone decreasing on the set $(0, 1)$.

Now that we know a little more about numbers and functions, we can give a smart argument. Consider the function $f(y) = y^n$ defined on $[0, \infty)$. It is a continuous function, strictly increasing, $f(0) = 0$ and the sequence $f(1), f(2), f(3), \cdots \to \infty$. You can now use the intermediate value property of continuous functions to see that range of f is indeed all of $[0, \infty)$. In other words, given $x > 0$, there is an y so that $f(y) = x$. That such a y is unique follows from the fact that the function f is strictly increasing.

Step 4: $x > 0$. To define x^r for $r \in Q$.

Let $r = m/n$ where m, n are integers and $n \geq 1$. we put $x^r = (x^m)^{1/n}$. This makes sense because $x^m > 0$ whatever be $m \in Z$. We have proved earlier that this is a good definition, in the sense, it does not depend on how you write the rational number — 2/3 or 4/6 or 6/9 etc. Prove

$$x^{r+s} = x^r x^s; \qquad (xy)^r = x^r y^r; \qquad x^r = (1/x)^{-r}.$$

For example, if r and s are two given rationals, you can write them as fractions with commmon denominators. Since the definition does not depend on how you write the rational as a fraction, let $r = m/n$ and $s = k/n$ where $n \geq 1$. Then $r + s = (m + k)/n$.

$$x^{(r+s)} = [x^{m+k}]^{1/n} = [x^m x^k]^{1/n} = [x^m]^{1/n}[x^k]^{1/n} = x^r x^s.$$

Here the first and last equalities use the definition; second equality uses law of indices proved for integers (step 2); third equality uses what was proved above (step 3).

Similarly

$$(xy)^{m/n} = [(xy)^m]^{1/n} = [x^m y^m]^{1/n} = [x^m]^{1/n}[y^m]^{1/n} = x^r y^s.$$

You should justify each of these equalities.

$$x^{m/n} = [x^m]^{1/n} = [(1/x)^{-m}]^{1/n} = (1/x)^{-m/n}.$$

$$x > 1, r < s \Rightarrow x^r < x^s; \qquad x < 1, r < s \Rightarrow x^r > x^s.$$

This follows from the fact that if $x > 1$, then $x^m > x^k$ whenever $m > k$ and property of $(1/n)$-th power (step 3) now shows $x^{m/n} > x^{k/n}$. See how we started expressing r and s with the same denominator. Similarly, we can argue for $x < 1$.

This last statement is expressed by saying that for $x > 1$, x^r is increases with r whereas for $x < 1$ it decreases as r increases.

$$r_n \to r \quad \Rightarrow \quad x^{r_n} \to x^r.$$

More precisely, if $x > 1$, the following holds. If $r_n \uparrow r$, then $x^{r_n} \uparrow x^r$, while $r_n \downarrow r$ implies that $x^{r_n} \downarrow x^r$.
If $x < 1$, the following holds. When $r_n \uparrow r$, then $x^{r_n} \downarrow x^r$, while $r_n \downarrow r$ implies that $x^{r_n} \uparrow x^r$.

The more precise statement about $\uparrow$ and $\downarrow$ follow from monotonicity observed just now. So we only need to prove convergence. Again we only need to consider the case $r = 0$. This is because $r_n - r \to 0$ and so the specail case, if proved, tells

$$x^{r_n - r} \to 1$$

so that by properties of convergence of sequences

$$x^{r_n - r} x^r \to 1 \cdot x^r$$

and the law of indices completes proof.

To prove the special case, let $r_n \to 0$. Let $\epsilon > 0$. We show n_0 so that $1 - \epsilon < x^{r_n} < 1 + \epsilon$ for $n \geq n_0$. Since $\sqrt[n]{x} \to 1$, get k_0 so that

$$1 - \epsilon < \sqrt[k]{x} < 1 + \epsilon; \quad 1 - \epsilon < \sqrt[k]{1/x} < 1 + \epsilon; \quad k \geq k_0$$

Since $r_n \to 0$, get n_0 so that

$$-\frac{1}{k_0} < r_n < \frac{1}{k_0}; \qquad n \geq n_0.$$

This n_0 will do. Check. Remember r_n may be negative or positive.

$$x > 1 \Rightarrow x^r = \sup\{x^s : s \in Q; s \leq r\} = \sup\{x^s : s \in Q; s < r\}.$$

First equality is obvious by monotonocity. For the second equality, observe that

$$x^{r-(1/n)} \uparrow x^r$$

and each of the numbers $x^{r-(1/n)}$ is in the last set.

<u>Step 5:</u> $x > 1$. To define x^a for $a \in R$.

For $x > 1$, taking a clue from the last observation of the previous step, we define $x^a = \sup\{x^r : r \in Q, r \le a\}$.

If we take any rational t, with $a - 1 < t < a$, then x^t is in the above set; if we take any rational s with $a < s < a + 1$ then x^s is an upper bound for that set. Thus supremum is sensible. Also if a happens to be rational then this definition gives the answer: x^a as defined in step 4, by monotonicity (or the last observation of step 4.

The definition is also equivalent to $x^a = \sup\{x^r : r \in Q, r < a\}$. The difference is strict inequality. This is easy. If a is irrational, there is no difference between the sets. If a is rational, this is precisely the statement proved in step 4.

$$r_n \uparrow a \quad \Rightarrow \quad x^{r_n} \uparrow x^a.$$

If a is rational, this is already done in step 4. Enough to consider a irrational. That x^{r_n} increases is by monotonicity. Let the limit be α. Let $A = \{x^r : r \text{ rational}; r < a\}$. Thus we need to show $\sup A = \alpha$. Each x^{r_n} is in A and so $x^{r_n} \le \sup A$ for each n. Hence so is their limit giving $\alpha \le \sup A$. By monotonicity, each x^r is smaller than some x^{r_n} and hence smaller than α. Thus α is an upper bound for A showing $\sup A \le \alpha$. This shows $\sup A = \alpha$ as required.

$$x^{a+b} = x^a x^b; \qquad (xy)^a = x^a y^a.$$

Note that at this moment we have defined x^a only for $x > 1$. Thus in the second equality above, it is assumed that both x and y are larger than one. Then of course $xy > 1$ too. To prove the first equality, take rationals $r_n \to a$ and $s_n \to b$. From the fact proved just now and step 4, we get

$$x^{a+b} = \lim x^{r_n + s_n} = \lim x^{r_n} x^{s_n} = x^a x^b.$$

The second equality is similar.

$$a < b \quad \Rightarrow \quad x^a < x^b.$$

Since the set whose sup defines x^a increases with a the inequality $\le$ is clear. To show strict inequality, fix rationals r, s so that $a < r < s < b$ and $x^a \le x^r < x^s \le x^b$.

$$a_n \uparrow a \quad \Rightarrow \quad x^{a_n} \uparrow x^a.$$

That x^{a_n} increases is clear. Proof that it increases to x^a is exactly as in the corresponding statement in step 4; need to prove special case $a_n \to 0$ etc. Similarly,

$$a_n \downarrow a \quad \Rightarrow \quad x^{a_n} \downarrow x^a.$$

<u>x^a for $x > 0$ and $a \in R$</u>

If $x > 1$ the above clause defines x^a. If $x = 1$, we put $x^a = 1$ whatever be a. If $0 < x < 1$, we put $x^a = (1/x)^{-a}$. This makes sense because, $1/x > 1$ and above clause applies.

If you have survived so far, you can prove the following properties. if you can not prove, return and start from step 1 and understand.

$$a_n \to a \quad \Rightarrow \quad x^{a_n} \to x^a.$$

$$x^{a+b} = x^a x^b; \qquad (xy)^a = x^a y^a.$$

$$x^a \uparrow \text{ as } a \uparrow \text{ for } x > 1; \quad x^a \downarrow \text{ as } a \uparrow \text{for } 0 < x < 1.$$

Theorem: Fix $x > 0$. Consider the function $f(a) = x^a$. Then $f : R \to (0, \infty)$.
(i) f is a continuous function satisfying two conditions: $f(a+b) = f(a)+f(b)$ and $f(1) = x$.
(ii) f is the only continuous function on R to $(0, \infty)$ satisfying the two conditions above.
(iii) If $x > 1$, then

$$f(a) \to \infty \quad \text{as} \quad a \to \infty; \qquad f(a) \to 0; \quad \text{as} \quad a \to -\infty.$$

This means the following. Given any number c, there is A so that $f(a) > c$ for all $a \geq A$. Similarly, givcen any number c there is an A so that $f(a) < c$ for all $a \leq A$.
If $x < 1$, then

$$f(a) \to 0; \quad \text{as} \quad a \to \infty; \qquad f(a) \to \infty; \quad \text{as} \quad a \to -\infty.$$

(iv) Suppose that f is any continuous function on R to R such that $f(a+b) = f(a) + f(b)$ holds for all $a, b \in R$. Then f necessarily takes values in $[0, \infty)$.

Either it is zero for all a or it is never zero. In the second case, it must be one of the functions $f(a)$ listed above, namely, $f(a) \equiv x^a$ for some $x > 0$.

Proof: (i) Continuity was already shown above. The equation is just law of indices.
(ii) Since $f(1) = x$, the conditions imply, by induction, that $f(n) = x^n$ for $n \in N$ and then for $n \in Z$. Since $(1/2) + (1/2) = 1$ we see $[f(1/2)]^2 = x$ and since $f(1/2) > 0$ we conclude that $f(1/2)$ must be $\sqrt{x}$. You can now show by induction that $[f(1/n)]^n = f(1)$ and since $f(1/n) > 0$ conclude that $f(1/n) = x^{1/n}$. Now it follows that $f(r) = x^r$ for every $r \in Q$. Since f is given to be continuous and $a \mapsto x^a$ is shown to be continuous function and since they agree at every $r \in Q$, we conclude that they agree at every $a \in R$.
(iii) If $x > 1$, then we knew, $x^n \to \infty$ and hence by (monotonocity in a) we conclude $x^a \to \infty$ in the sense described above. The part $a \to -\infty$ follows from noting $x^a = (1/x)^{-a}$ and again monotonocity. The case $0 < x < 1$ is similar.
(iv) Since for any a, $f(a) = [f(a/2)]^2$ we see that $f(a) \geq 0$ for every $a \in R$. Suppose that $f(1) = 0$. then $f(a) = f(1)f(a-1)$ shows that $f \equiv 0$. On the other hand when $f(1) > 0$ we have already discussed in part (ii) above.

We have regarded x^a as a function of a for every fixed number $x > 0$. We can also regard it as a function of x on $(0, \infty)$ for every fixed $a \in R$.

Theorem: Let $a \in R$. Define $g(x) = x^a$. Then $g : (0, \infty) \to (0, \infty)$. Then g is continuous and satisfies $g(xy) = g(x)g(y)$.

In defining x^a we have fixed x and defined for every a. So it is not clear as to what happens if you were to change x. We could not have fixed a and defined x^a for every x. So to understand properties when x is changed, we have to retrace our definition and make observations step by step.

step 1: Fix $n \in N$. Let $g(x) = x^n$ defined on $(0, \infty)$. Then g is continuous. We knew this already, in fact we knew polynomials are continuous, in fact, on all of R.

step2: Fix $n \in Z$. Let $g(x) = x^n$ defined on $(0, \infty)$. Then g is continuous. Follows from properties of sequences.

step 3: Fix $n \in N$. Let $g(x) = x^{1/n}$ defined on $(0, \infty)$. Then g is continuous. This can be seen in two ways, one method is what Uma suggested. Take $x_k \to x$. The sequence $(y_k) = (\sqrt[n]{x_k})$ is bounded, it has at least one limit point. Let z be a limit point. Take a subsequence, say $z_{k_1}, z_{k_2}, \cdots$ converging to z. their n-th powers must converge to z^n. But their n-th powers form a subsequence of (x_k) and hence must converge to x. Thus $z^n = x$. Of course $z \geq 0$, since all our numbers are non-negative, to start with. By definition of n-th root, we see that $z = x^{1/n}$. Thus the bounded sequence $(x^{1/n})$ has only one limit point and hence must converge to it, completing the proof.

Here is another way of proving the above result. It will be useful in other situations. Let us prove a theorem first and apply to the current problem. Later we see other uses too.

Theorem: Let $f : (0, \infty) \to (0, \infty)$ be a continuous strictly increasing function with $f(1/n) \to 0$ and $f(n) \to \infty$. Then for every $y \in (0, \infty)$ there is a unique $x \in (0, \infty)$, to be denoted $g(y)$ such that $f(x) = y$. In other words g is inverse of f. Moreover g is a strictly increasing continuous function on $(0, \infty)$.

Proof: The intermediate value theorem and hypothesis tell us that range of f is all of $(0, \infty)$. Since f is strictly increasing, it is one-to-one map. Thus there is an inverse map g. If $g(y_1) < g(y_2)$ then $y_1 = f(g(y_1)) < f(g(y_2)) = y_2$. So g is strictly increasing. Pause and think, we used proof by contradiction.

To see that g is continuous, let b and $\epsilon > 0$ be given. Let $g(b) = a$. Need to show $\delta > 0$ so that $|g(y) - g(b)| < \epsilon$ whenever $|y - b| < \delta$. There is no loss to assume that $0 < \epsilon < b$. Since $f(a) = b$, using f is strictly increasing, get $\delta_1, \delta_2 > 0$ so that $f(a - \delta_1) = b - \epsilon$ and $f(a + \delta_2) = b + \epsilon$. Take $\delta = \min\{\delta_1, \delta_2\}/2$. This will do and is easy from monotonicity.

If you apply the theorem above to the function $f(x) = x^n$, you see that $g(y) = y^{1/n}$ is a continuous function.

step 4: Fix rational r. Then the function $g(x) = x^r$ is continuous on

$(0, \infty)$. Indeed, if $r = m/n$ then g is composition of two continuous functions, namely, the maps $x \mapsto x^m$ and $u \mapsto u^{1/n}$.

step 5: Let $a \in R$. Then the map $g(x) = x^a$ is continuous on $(0, \infty)$. Enough to consider the case $a > 0$. In fact if $a = 0$ this is the constant function 1 and there is nothing to do. If $a < 0$, then the function $x \mapsto x^a$ is composition of two functions, namely, $x \mapsto 1/x$ and $u \mapsto u^{-a}$. Note that $-a > 0$.

So let $a > 0$. We start with an observation. Let $\beta > \alpha > 0$ and $\epsilon > 0$. Then there is an rational r so that $|x^r - x^a| < \epsilon$ for all $x \in [\alpha, \beta]$. Let us see what happens if this is done.

Let $x_n \to x$, all of them in $(0, \infty)$. Shall show that $x_n^a \to x^a$. Let $\epsilon > 0$. Propose to exhibit n_0 so that $|x_n^a - x^a| < \epsilon$ for $n \geq n_0$. Firstly, since all x_n and x are strictly positive, you can fix $0 < \alpha < \beta$ so that all of these points are in the interval $[\alpha, \beta]$. Fix rational r so that for all points x in this interval $|x^a - x^r| < \epsilon/4$. Since we know that $x \mapsto x^r$ is continuous, fix n_0 so that $|x_n^r - x^r| < \epsilon/4$ for all $n \geq n_0$. Clearly, for $n \geq n_0$,

$$|x_n^a - x^a| \leq |x_n^a - x_n^r| + |x_n^r - x^r| + |x^r - x^a| < 3\epsilon/4.$$

Returning to the proposal made at the beginning, we are given the following: $a > 0$; $0 < \alpha < \beta$ and $\epsilon > 0$. Need to locate rational r so that $|x^r - x^a| < \epsilon$ for all $x \in [\alpha, \beta]$. Note that $x^a \leq \alpha^a + \beta^a = M$ (say) for all $x \in [\alpha, \beta]$. In fact by monotonicity, if $x > 1$ then $x^a < \beta^a$ while if $x < 1$, then $x^a < \alpha^a$. Thus if we can get a rational r so that

$$|x^{r-a} - 1| < \epsilon/M; \quad \text{for all} \quad x \in [\alpha, \beta] \qquad (\spadesuit)$$

we are done, because then

$$|x^r - x^a| \leq x^a |x^{r-a} - 1| \leq M \frac{\epsilon}{M} = \epsilon.$$

Finally, to choose rational r satisfying $(\spadesuit)$ we only need to make sure

$$|\alpha^{r-a} - 1| < \epsilon/M; \quad |\beta^{r-a} - 1| < \epsilon/M.$$

But this is easy. you only need to choose r close to a. Check.

discontinuities:

We have discussed several examples of functions with a point of discontinuity illustrating several possibilities. Essentially, two phenomena can occur: the function may be well behaved getting close to something or it may behave in a wiggly manner without getting close to any particular value. Accordingly, we classify the point of discontinuity into two types.

Of course the behaviour mentioned above can happen to the right of the point under consideration or to the left of the point or on both sides. But we shall not undertake this minute classification. Before making definition, let us make an observation.

Fact: Let $f : R \to R$ be a function and $a \in R$. Let L be a real number. The following two statements are equivalent.
(i) Whenever $x_n \uparrow a$ and each $x_n < a$ we have $f(x_n) \to L$
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < a - x < \delta$ we have $|f(x) - L| < \epsilon$.

Proof is simple and similar to the corresponding statements we made while introducing continuity. Let us go through it once again.

Suppose (ii) holds. We shall prove (i). So take a sequence (x_n) where $x_n < a$ for every n and $x_n \to a$. We show $f(x_n) \to L$. So fix $\epsilon > 0$. We show a natural number n_0 such that $|f(x_n) - L| < \epsilon$ whenever $n \geq n_0$. With the given $\epsilon > 0$ in hand, use (ii) and fix $\delta > 0$ as stated. Since $x_n \to a$, fix n_0 so that for $n \geq n_0$, we have $|x_n - a| < \delta$. Now if $n \geq n_0$, using the fact that $x_n < a$ we conclude that $0 < a - x_n < \delta$ and hence choice of δ shows $|f(x_n) - L| < \epsilon$.

Suppose (ii) fails. We show that (i) fails. Fix $\epsilon > 0$ for which we can not find $\delta > 0$ as stated. Thus taking $\delta = 1$ we get an x_1 such that $0 < a - x_1 < 1$ and $|f(x_1) - L| \geq \epsilon$. Taking $\delta = 1/2$ we get an x_2 such that $0 < a - x_2 < 1/2$ and $|f(x_2) - L| \geq \epsilon$. In general, taking $\delta = 1/n$ we get an x_n such that $0 < a - x_n < 1/n$ and $|f(x_n) - L| \geq \epsilon$. Put $y_n = \max\{x_1, x_2, \cdots, x_n\}$. Then

$y_n \uparrow$; $0 < a - y_n < 1/n$ so that $y_n \uparrow a$; y_n being one of the x_i we have $|f(y_n) - L| \geq \epsilon$ for every n. This completes the proof.

Note that a number L as above may not exist at all, like in case of the examples involving $\sin(1/x)$. However if such a number exists then it is unique. There can not be two such numbers. Indeed, if L and L' are two such numbers, then $f(a - \frac{1}{n})$ converges to both L and L'. Since a sequence can not converge to two different points we conclude that $L = L'$.

If any one of the above two things happens we say that f has a left limit at the point a and the value of the left limit equals L. We express it as

$$\lim_{x<a; x\to a} f(x) = L; \quad \mathit{or} \quad f(a-) = L.$$

Note that we are not evaluating the function f at $a-$; there is no number called $a-$. It is only a notational convenience to express in that fashion.

Fact: Let $f : R \to R$ be a function and $a \in R$. Let L be a real number. The following two statements are equivalent.
(i) Whenever $x_n \downarrow a$ and each $x_n > a$, we have $f(x_n) \to L$
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < x - a < \delta$ we have $|f(x) - L| < \epsilon$.

As in the case of left limits, note that such an L, if exists, is unique. If any one of the above two things happens we say that f has a right limit at the point a and the value of the right limit equals L. We express it as

$$\lim_{x>a; x\to a} f(x) = L \quad \mathit{or} \quad f(a+) = L.$$

Note that we are not evaluating the function f at $a+$; there is no number called $a+$. It is only a notational convenience to express in that fashion.

Suppose that the point a is a discontinuity point of the function f. We say that it is a simple discontinuity if both $f(a-)$ and $f(a+)$ exist. that is, the left and right limits exist at the point a. In other words simple discontinuity is first of all a discontinuity point, but the function has right and left limits at that point. Simple discontinuity is also called discontinuity of the first kind.

If a is a point of discontinuity and if a is not a discontinuity of the first kind, we say that f has discontinuity of the second kind at the point a. Thus at a discontinuity of the second kind, either the right limit or the left limit does not exist. Of course, when one of these limits does not exist, then the function is discontinuous at that point.

Fact: f is continuous at a point a iff both $f(a-)$ and $f(a+)$ exist and equal $f(a)$.

If f is continuous at a, then whenever $x_n \to a$, we have $f(x_n) \to f(a)$. In particular this happens when $x_n \uparrow a$ or $x_n \downarrow a$. Thus $f(a-)$ and $f(a+)$ exist and equals $f(a)$.

Conversely, if both the limits exist and equal $f(a)$, then we show f is continuous at a. Let $\epsilon > 0$. We show $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x-a| < \delta$. Since both left and right limits at a equal $L = f(a)$, fix one $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta$ and also whenever $0 < x-a < \delta$. This δ will do. Note that when $x = a$ we have $f(x) - f(a) = 0$. This completes the proof.

There is one class of functions which have discontinuities of only first kind. Let us say that a function $f : R \to R$ is monotone increasing if $x < y$ implies $f(x) \leq f(y)$. Say that f is monotone decreasing if $x < y$ implies $f(x) \geq f(y)$. A function is monotone if it is either monotone increasing or monotone decreasing.

Note that the word increasing/decreasing is not used in the sense of strictly increasing or strictly decreasing. That is why sometimes a function which is monotone increasing in the above sense is also referred to as 'monotone non-decreasing'. Similarly a function which is monotone decreasing in the above sense is referred to as 'monotone non-increasing'. But we shall not do that.

Fact: Let $f : R \to R$ be monotone. Then the following are true.
(i) At every point a, both $f(a+)$ and $f(a-)$ exist.

(ii) f is continuous at a point a iff $f(a-) = f(a+)$.
(iii) f is continuous at all but countably many points.

Proof: Let us assume that f is monotone increasing.
(i) Let $a \in R$. Put
$$L = \lim f(a - \frac{1}{n}).$$
This limit exists because $f\left(a - \frac{1}{n}\right)$ is an increasing sequence, bounded above by $f(a)$. In fact $L \leq f(a)$. We now show that if we take any sequence $x_k \uparrow$, each $x_k < a$, then $f(x_k) \to L$. Since $x_k \uparrow$ we conclude that $f(x_k) \uparrow$ and hence the limit $\lim f(x_k)$ exists. Denote it by L'.

Fix any k. Since $x_k < a$ we see that for all sufficiently large n, $x_k < a - \frac{1}{n}$ so that $f(x_k) \leq f\left(a - \frac{1}{n}\right)$. remember this is true for all large n. Hence
$$f(x_k) \leq \lim f\left(a - \frac{1}{n}\right) = L.$$
Remember this is true for every k. Hence $\lim f(x_k) \leq L$. Thus $L' \leq L$.

Now fix any n. Since $x_k \uparrow a$ we have $x_k > \left(a - \frac{1}{n}\right)$ for all sufficiently large k. Hence $f(x_k) \geq f\left(a - \frac{1}{n}\right)$ for all sufficiently large k. Hence
$$L' = \lim f(x_k) \geq f\left(a - \frac{1}{n}\right).$$
Remember this is true for every n. Hence
$$L' \geq \lim f\left(a - \frac{1}{n}\right) = L.$$
Thus $L' = L$. Thus $f(a-)$ exists and $f(a-) \leq f(a)$. Similarly $f(a+)$ exists and $f(a) \leq f(a+)$. In fact $f(a+) = \lim f\left(a + \frac{1}{n}\right)$.

(ii) Since we have observed above that $f(a-) \leq f(a) \leq f(a+)$, equality of the extremes implies that they both equal $f(a)$.This, combined with earlier fact proves (ii).

(iii) Let D be the set of discontinuity points of f. Of course, if D is empty there is nothing to argue. Suppose $a \in D$. Then from (ii) $I(a) =$

$(f(a-),\ f(a+))$ is a non-empty interval. Moreover if $a < b$ are in D, then $f(a+) \leq f(\frac{a+b}{2}) \leq f(b-)$ so that the intervals $I(a)$ and $I(b)$ are disjoint. Thus we have a family of non-empty disjoint intervals of the type (a, b) with $a < b$. But any such interval contains a rational number and if D were uncountable, we would be getting uncountably many rational numbers leading to a contradiction.

Similar proof applies when f is decreasing. The proof is complete.

To define left limit, it is not necessary to take the sequence $(a-\frac{1}{n})$. We can take any sequence $a_n \uparrow a$ with each $a_n < a$ and consider $L = \lim f(a_n)$. The argument above shows precisely this, namely, limit $\lim f(x_n)$ does not depend on the particular sequence, as long as each $x_n < a$ and $x_n \uparrow a$, you get the same answer. Of course, we could also have defined $L = \sup\{f(x) : x < a\}$, thus avoiding sequences altogether.

If you consider the function $f(x)$; one or zero according as x is rational or irrational, you see that each point $a \in R$ is a point of discontinuity. Of course, this function is not monotone.

We just now saw that the set of discontinuity points of a monotone function is a countable set. In fact given any countable subset $D \subset R$, we can define $F : R \to R$ which is monotone and the given set D is precisely its set of discontinuity points.

discontinuities continued:

If you have understood the above arguments, you see that in the definition of the left limit $f(a-)$ the value of f at a, namely $f(a)$ did not play any role. In fact no value $f(x)$ for $x > a$ played any role. Thus you can talk about left limit at the point a as long as the function is defined on $(-\infty, a)$.

Also, values of f at points far below a did not play any role. What does this mean? Suppose you have two real valued functions f and g defined on $(-\infty, a)$. Suppose $f(x) = g(x)$ for all x with $a - 0.0001 < x < a$. the functions may be different outside this smll interval. It is easy to see that $f(a-)$ exists iff $g(a-)$ exists and then they are equal. This leads to the following definition. We start with an observation first and then make the definition.

Let f be a real valued function defined on some set $D_f \subset R$ and $a \in R$, Suppose that $(a_0, a) \subset D_f$ for some $a_0 < a$. Then a number L satisfies statement (i) below iff it satisfies statement (ii).
(i) $x_n \uparrow a$, each $x_n < a$, each $x_n \in D_f \Rightarrow f(x_n) \to L$.
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < a - x < \delta; \quad x \in D_f \Rightarrow |f(x) - L| < \epsilon.$$

If any one of the above two statements holds, we say L is left limit of f at a or simply $f(a-) = L$. Similarly we define right limit $f(a+)$ whenever $(a, a_1) \subset D_f$ for some $a_1 > a$. Of course these limits may not exist. If both exist and are equal, then we can define $f(a)$ to be the common value. Then f, so defined at a would be continuous at a. Of course, if $a \in D_f$ already, then f is continuous at a iff $f(a)$ equals this common value.

We can define monotone function, not necessarily taking the domain of definition to be R as we did earlier. A real valued function f with domain D_f is monotone increasing if $x, y \in D_f$ and $x < y$ imply $f(x) \leq f(y)$. Similarly we can define monotone decreasing function. Exactly the same proof given earlier shows the following. Let f be a monotone function defined on an interval I. Then at every point $a \in I$ except at the end points (if any) the left limit $f(a-)$ and $f(a+)$ exist and the set of discontinuities of f is at most countable.

At the end points one needs to be careful. For example the left end point of I is finite, say a, we can only talk about the right limit $f(a+)$ and not left limit $f(a-)$. Further the right limit $f(a+)$ is finite iff the function is bounded in an interval (a, a_1). Even if the left end point a is ∞ one can still talk about right limit at $-\infty$. But when you need this you will realize without much problem. It is unnecessary to confuse ourselves with utmost generalities at the beginning stage.

differentiation:

How to draw tangent to a curve at a point. Suppose the curve is given by $y = f(x)$ and at the point $(a, f(a))$ on the curve we are required to draw tangent. The first question is: what do you mean by tangent. well, one intuitive idea is that it is a straight line pasing through the given point and meets the curve at that point only. This is not quite correct, you can try to draw tangent to the curve $y = \sin x$ at several points and see what happens. Of course, for a circle, tangents are precisely straight lines that meet the circle at exactly one point. (why?)

Another idea is the following. Take the point $(a, f(a))$ on the curve and a nearby point $(x, f(x))$ join these two points by a straight line. This is a 'chord'. A natural question is whether this chord has a limiting form as the point x approaches a. At first sight this explanation appears too complicated because we are talking about limiting form of straight lines. But it is not difficult to understand.

Afterall, a straight line is determined by its slope — apart from a point through which it passes. In our case the point is $(a, f(a))$. Thus chords we are talking about or the tangent we are looking for , all pass through this point. So we only need to specify the slope. Thus the question amounts to asking whether the slopes of the chords have a limiting value. of course, slope of the chord we have mentioned above is nothing but $[f(x) - f(a)]/[x - a]$.. Thus we need to see if this has a limiting value as x approaches a.

To start an entirely different second line of thought, suppose a particle is starting at the origin at time zero and travelling along a path. How do we understand its velocity? If it is travelling so that at time t the particle is at $5t$, then matters are simple. At time instants $t_0 < t_1$ it is at $5t_0$ and $5t_1$ respectively, so that the distance travelled during this time duration $t_1 - t_0$ is $5t_1 - 5t_0$ and so the velocity, distance divided by time, equals 5 and does not depend on the two time points we have taken. But in practice particles accelerate and do not travel with 'uniform speed' as above.

Suppose that the particle is travelling along the curve $y = t^2$, again starting at zero. The distance travelled during the time period 1 to 2 is $4 - 1 = 3$ while the distance travelled during time period 10 to 11 equals $121 - 100 = 21$ and you can try to find out the distance travelled during time period 1000 to 1001. You see that the particle is going faster and faster as time elapses. Thus the concept of velocity does not make sense unless you specify a time point and ask: what is the velocity at this time. In other words one needs to talk about instant velocity.

Suppose a time point $t = 5$ is given. What is th velocity t time instant 5? Naturally it should depend on what is happening around this time instant and one idea is to take the distance travelled during the time period 5 to $5 + h$ and then take the ratio. In other words, if the path is given by $f(t)$, to understand the velocity at time 5 we need to calculate $[f(5 + h) - f(5)]/h$ and then see if there is any particular value to which this ratio is getting close as h gets closer to zero, that is, whether the intuitive idea of distance travelled/time approaches any aprticular value as we consider durations nearer to the time point of interest. To change notation, if the time point under consideration is a, then we need to look at the ration $[f(a) - f(x)]/[a - x]$ and see if it has a limiting value as the (time) point x approaches a.

To start a yet different, third line of thinking, let us understand complexity of functions. The simplest functions are constant functions. Suppose we are given a function f on R and a point a, Which simplest function best approximates f near a. Obviously the constant function $\varphi(x) \equiv f(a)$ is the best. When x is close to a, $\varphi(x)$ is close to $f(x)$, simply because $f(x)$ is close to the number $f(a)$ when x is close to a. In symbols, $f(x) - \varphi(x) \to 0$ as $x \to a$.This means, $f(x_n) - \varphi(x_n) \to 0$ whenever $x_n \to a$.

Suppose you are allowed to use a little more complicated functions than constant functions, namely, straight line functions. Can you do better? so what is meant by better? Earlier we simply said $f(x) - \varphi(x)$ should get close to zero as x gets close to a. We did not demand any quantitative measurement on how this quantity gos to zero. Now having allowed functions general than constant functions, we demand that not only this error gets close to zero, the ratio $[f(x) - \varphi(x)]/[x - a]$ should get closer to zero as x gets closer to a.

Note that this implies, in particular, that $f(x) - \varphi(x)$ gets closer to zero as x gets closer to a. But since both f and φ are continuous, this difference is getting closer to $f(a) - \varphi(a)$. In other words $\varphi(a) = f(a)$. Thus the straight line passes through $(a, f(a))$. If the straightline we are thinking is $\varphi(x) = mx + c$, then what we concluded just now amounts to $ma + c = f(a)$

or $c = f(a) - ma$. Thus the straight line is

$$\varphi(x) = mx + f(a) - ma = f(a) + (x - a)m.$$

Thus

$$\frac{f(x) - \varphi(x)}{x - a} = \frac{f(x) - f(a)}{x - a} - m.$$

Thus the demand that this ratio gets closer to zero as x gets closer to a, simply amounts to saying that the ratio $[f(x) - f(a)]/[x - a]$ should get closer to the number m, slope of the line we are looking for.

All these thought processes lead to one common conclusion: the rate of change of the function is an important quantity. Before defining this precisely, we make an observation.

Fact: Let $f : R \to R$ and $a \in R$. For a number $m \in R$, the following two statements are equivalent.
(i) If $x_n \to a$ and $x_n \neq a$ for each n; then

$$\frac{f(x_n) - f(a)}{x_n - a} \to m.$$

(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - m \right| < \epsilon.$$

Proof is simple and we have come across such a situation several times earlier — connection between discrete and continuous formulations of an idea.

Suppose (ii) holds. To prove (i), take $x_n \to a$ and $x_n \neq a$ for every n. Shall show n_0 such that

$$n \geq n_0 \Rightarrow \left| \frac{f(x_n) - f(a)}{x_n - a} - m \right| < \epsilon.$$

Use (ii) with the given ϵ to get a $\delta > 0$ and then use $x_n \to a$ to get n_0 so that $n \geq n_0 \Rightarrow |x_n - a| < \delta$. This will do.

Conversely, if (ii) is false, we exhibit a sequence for which (i) fails. Since (ii) is false, fix $\epsilon > 0$ for which we can not find $\delta > 0$ stisfying the stated condition. With $\delta = 1/n$ get x_n so that $0 < |x_n - a| < 1/n$ and yet

$$\left| \frac{f(x) - f(a)}{x - a} - m \right| \geq \epsilon.$$

This sequence shows failure of (i).

Definition: Let $f : R \to R$ and $a \in R$. We say that f is differentiable at the point a if there is a number l satisfying the above conditions. In such a case l is called derivative of f at a. There are several notations.

$$f'(a) = l; \quad \frac{df}{dx}(a) = l; \quad D_x f(a) = l.$$

Another way of saying the same thng is the following: Define the function $\varphi(x) = [f(x) - f(a)]/[x - a]$. Of course, this is defined on R except at the point a. If the right limit and left limit of this function exist at a and equal, then we say that the function f is differentiable at the point a and this limit is called the derivative of the function f at the point a. If the function is differentiable at every point, then we say that f is differentiable. In this case, we can define f' on all of R.

Observe that the number l, when exists, is unique.

Fact: Let $f : R \to R$. if f is diferentiable at a, then f is continuous at a.

Proof: Let $x_n \to a$. We need to show $f(x_n) \to f(a)$. If all the x_n are different from a, then

$$f(x_n) - f(a) = \frac{f(x_n) - f(a)}{x_n - a} \, (x_n - a) \to f'(a) \cdot 0 = 0.$$

Suppose that there is an n_0 such that $x_n \neq a$ for all $n \geq n_0$. Then you can write the equation above for $n \geq n_0$ obtaining the result. if there is an n_0 such that $x_n = a$ for all $n \geq n_0$, then there is nothing to do.

Finally suppose that there are infinitely many n such that $x_n \neq a$ and infinitely many n such that $x_n = a$. Then you will get two subsequences corresponding to each and the above argument applies for the two corresponding subsequences of $f(x_n)$. Using an earlier observation, we conclude that $f(x_n) - f(a) \to 0$.

Here is another way to argue. If only finitely many x_n are different from a, then after some stage $f(x_n) = f(a)$ and so we are done. If infinitely many $x_n \neq a$; let $n_1 < n_2 < \cdots$ are precisely those integers. Then $[f(x_{n_i}) - f(a)]/[x_{n_i} - a]$ converges to $f'(a)$ and hence is a bounded sequence, say bounded by C. Since $x_{n_i} - a \to 0$, there is n_k such that for all $i \geq k$ $|x_{n_i} - a| < \epsilon/C$. But then $|f(x_{n_i}) - f(a)| < \epsilon$. And of course, if $n > n_k$ and not in the subsequence then $f(x_n) = f(a)$ and hence for all $n > n_k$, we see $|f(x_n) - f(a)| < \epsilon$.

It is instructive to argue using $\epsilon - \delta$. That is, fix $\epsilon > 0$ and show $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Fact: if $f : R \to R$ and $g : R \to R$ are differentiable at a then so are $f+g$ and $39f$ and fg. In fact

$$(f+g)'(a) = f'(a) + g'(a); \qquad (39f)'(a) = 39f'(a);$$

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

If $g(x) \neq 0$ for all x, then f/g is also differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Proof: if you take any sequence $x_n \to a$, $x_n \neq a$ for all n, then

$$\frac{(f+g)(x_n) - (f+g)(a)}{x_n - a} = \frac{f(x_n) - f(a)}{x_n - a} + \frac{g(x_n) - g(a)}{x_n - a}$$

$$\to f'(a) + g'(a).$$

$$\frac{(39f)(x_n) - (39f)(a)}{x_n - a} = 39\frac{f(x_n) - f(a)}{x_n - a} \to 39f'(a).$$

$$\frac{(fg)(x_n) - (fg)(a)}{x_n - a} = \frac{f(x_n)g(x_n) - f(x_n)g(x) + f(x_n)g(x) - f(x)g(x)}{x_n - a}$$

$$\to f(a)g'(a) - f'(a)g(a).$$

Here we used continuity of f at a.
We show $1/g$ is differentiable and calculate its derivative, then the above multiplication rule can be used for the product $f \cdot (1/g)$. Take a sequence $x_n \to a$, $x_n \neq a$ for all n.Then

$$\frac{(1/g)(x_n) - (1/g)(a)}{x_n - a} = \frac{g(a) - g(x_n)}{x_n - a} \frac{1}{g(x_n)g(a)} \to \frac{g'(a)}{g^2(a)}.$$

here we have used continuity of g at a.

Fact: Let $f : R \to R$ and $g : R \to R$ and $a \in R$. Assume that f is differentiable at a and g is differentiable at the point $f(a)$. Let $h(x) = g(f(x))$, the composition. then h is differentiable at a and

$$h'(a) = g'(f(a))f'(a).$$

Proof: Take $x_n \to a$, $x_n \neq a$ for all n.

Case 1: Assume that there is an n_0 such that for all $n \geq n_0$ we have $f(x_n) = f(a)$. Then note that

$$f'(a) = \lim \frac{f(x_n) - f(a)}{x_n - a} = 0$$

so that $g'(f(a))f'(a) = 0$. Of course, $h(x_n) = h(a)$ for all $n \geq n_0$ so that $h'(a) = 0$ proving existence of limit (of $[h(x_n) - h(a)/[x_n - a])$ as well as the stated equality in the case under consideration.

Case 2: Assume that there is an n_0 such that $f(x_n) \neq f(a)$ for all $n \geq n_0$. Then for $n \geq n_0$ we have

$$\frac{h(x_n) - h(a)}{x_n - a} = \frac{g(f(x_n)) - g(f(a))}{f(x_n) - f(a)} \frac{f(x_n) - f(a)}{x_n - a}$$

$$\to g'(f(a))f'(a).$$

Here we have used that f is continuous at a, thus $f(x_n) \to f(a)$. This proves existence of limit (of $[h(x_n) - h(a)/[x_n - a])$ and also the stated formula.

Case 3. There are infinitely many n such that $f(x_n) = f(a)$ and infinitely many n such that $f(x_n) \neq f(a)$. Enumerate the integers satisfying the first condition as $n_1 < n_2 < n_3 < \cdots$ and the second kind as $l_1 < l_2 < l_3 < \cdots$ and apply the above cases for the subsequences

$$\frac{h(x_{n_i}) - h(a)}{x_{n_i} - a}; \qquad \frac{h(x_{l_i}) - h(a)}{x_{l_i} - a}$$

and conclude the result by employing earlier fact about subsequences.

So far we have not calculated derivative of any specific function.

Fact: if $f(x) = 55$ for all x then $f'(a) = 0$ for all a.
if $f(x) = x$ for all x, then $f'(a) = 1$ for every a.
if $f(x) = x^{39}$ for all x, then $f'(a) = 39a^{38}$ for every a.
Proof: First two statements are easy (should be done to say this).

Last statement follows from

$$\begin{aligned}\frac{x^{39} - a^{39}}{x - a} &= x^{38} + x^{37}a + x^{36}a62 + \cdots + a^{38} \\ &\to 39a^{38}.\end{aligned}$$

Using the rules above, you can now show that polynomials are differentiable and be able to calculate their derivatives. Also for rational functions you should be able to do. What is a rational function? a function of the form $P(x)/Q(x)$ where P and Q are polynomials and Q is not the zero polynomial. Where is this function defined? On R minus finitely many points; which is a finite unon of intervals.

if you have understood the earlier calculations, then you can do some improvements.

Why should we have functions defined on all of the real line? suppose that f is a function defined on an interval $I = (\alpha, \beta)$ where $-\infty \leq \alpha < \beta \leq +\infty$. Let $a \in I$. We say that f is differentiable at a iand $f'(a) = l$ if any one of the following two equivalent conditions hold;
(i) $x_n \in I$ for all n, $x_n \neq a$ for all $n \Rightarrow \frac{f(x_n)-f(a)}{x_n-a} \to l$.
(ii) Given $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - a| < \delta; x \in I \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - l \right| < \epsilon.$$

Of course the case $\alpha = -\infty$ and $\beta = +\infty$ corresponds to $I = R$, discussed above.

All the facts remain true. For sum, product and ratio of functions (defined on the interval I now) with exactly the same proofs. the composition rule also holds with exactly the same proof as follows: let f be real valued function defined on an interval $I = (\alpha, \beta)$. let g be a rael valued function defined on another interval $J = (\gamma, \delta)$. let us assume that range of the function f is contained in J. that is, $f(x) \in J$ for every $x \in I$. Then the composition makes sense: $h(x) = g(f(x))$ defined on I. let now $a \in I$. Assume that f is differentiable at a and g is differentiable at $f(a)$. The h is differentiable at a and $h'(a) = g'(f(a))f'(a)$.

This is not a novel. You are expected to pause and convince yourselves that what was said in the para above is true. if you have trouble, that means you have not understood the earlier calculations for functions defined on all of R. You are advised to go back and work it out taking pen and paper. then you should return to the para above and not take it for granted.

Some of you were asking about functions defined on a closed interval. Suppose that f is defined on the interval $[23, 33]$. You can define the concept of differentiability at all points in the interval $(23, 33)$. if you take a point a in this interval, there is a small $\delta > 0$ so that $(a - \delta, a + \delta) \subset (23, 33)$ — you can take $\delta = \min\{a - 23, 33 - a\}$.

Can you define derivative at the points 23 and 33? Yes, at the point 23, you can define righthand derivative, namely,limit of $[f(x) - f(23)]/[x - 23]$ as $x \downarrow 23$ as was done in discussing left and right limits of functions. of course, this limit may not exist. similarly, you can define left derivative at the point 33. some of you who are curious can think about this. However, most of you should concentrate on understanding the above discussion thoroughly and clearly.

Last week we discussed that the three problems — the geometric problem of drawing tangent to a curve at a given point on the curve, the mechanical problem of understanding the rate at which a particle in motion is travelling at a particular instant, the problem of closely approximating a function near a point by a straight line — all lead to the concept of derivative. We saw some basic properties of derivatives.

To get a feel, let us look at an instructive example. Consider the function $f(x) = \sin(1/x)$ defined for non-zero real numbers. It oscillates so badly that we can not assign a value for the function at $x = 0$ so that it is a conitnuous function on R.

Consider the function $f(x) = x\sin(1/x)$ again defined for non-zero real numbers. If we define $f(0) = 0$ then the function is a continuous function on R. However, it is not differentiable at zero. Of course, it is differentiable at all other points.

Consider the function $f(x) = x^2\sin(1/x)$ defined for $x \neq 0$. If we declare $f(0) = 0$, it is a continuous function on R. It is now differentiable also at the point $x = 0$ and in fact $f'(0) = 0$. As already noted, it is differentiable at all non-zero points. Thus f is differentiable at all points and we have the function f' on all of R given by

$$f'(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Whne x is near zero the first term oscillates and second term is near zero. Thus you can see that f' is not continuous at the point zero.

Consider the function $f(x) = x^3\sin(1/x)$ defined for $x \neq 0$ and we declare $f(0) = 0$. Then you can see, f is differentiable and f' is defined on all of R and is a continuous function.

These are pointed out so that you can see how we are achieving more and more 'smoothness'. We shall now see how derivative of f helps us to understand the function f itself better.

Definition; Let f be a function defined on a set S and $a \in S$. We say f has a local maximum at the point a, if there is an open interval (α, β) containing the point a such that $f(x) \leq f(a)$ whenever $x \in S$ and $\alpha < x < \beta$. In other words, locally, that is, in the interval (α, β) the function f has maximum at the point a. Of course, there may be points x outside this interval which are in S and $f(x) > f(a)$. If $a \in S$ is a such a pont that $f(x) \leq f(a)$ for every x in S, then we say that a is a point of global maximum. It is local maximum, but more than that.

We have similar definition regarding minimum. We say f has a local minimum at the point a, if there is an open interval (α, β) containing the point a such that $f(a) \leq f(x)$ whenever $x \in S$ and $\alpha < x < \beta$. Thus locally, that is, in the interval (α, β) the function f has minimum at the point a. There may exist points $x \in S$ such that $x \leq \alpha$ or $x \geq \beta$ where $f(x) < f(a)$. If $a \in S$ is a such a pont that $f(a) \leq f(x)$ for every x in S, then we say that a is a point of global minimum.

Fact: Let f be dfined on an interval (u, v) and $a \in (u, v)$ is a local maximum or loal minimum. If f is differentiable at the point a then $f'(a) = 0$.

Proof is simple. suppose a is a local maximum. Let us take $a \in (\alpha, \beta) \subset (u, v)$ so that $f(x) \leq f(a)$ for all $x \in (\alpha, \beta)$. if you take a sequence of points $\{x_n\}$ in this interval so that each $x_n < a$ and $x_n \uparrow a$, then we see

$$f'(a) = \lim \frac{f(x_n) - f(a)}{x_n - a} \geq 0.$$

similarly, if we take a sequence of points $y_n \downarrow a$ in this interval, each $y_n > a$, we see

$$f'(a) = \lim \frac{f(y_n) - f(a)}{y_n - a} \leq 0.$$

hence $f'(a) = 0$. similar proof applies for local minimum.

The main point is the following. if you are looking for the (local) maximum or local minimum in an open interval, just search among points where the derivative vanishes (assuming that the function is differentiable). Of course you will ask,how do I know if we have max or min? The answer has to wait till we define second derivative.

Understand carefully what we said. a is a a point of max or min implies $f'(a) = 0$. We did not say: $f'(a) = 0$ implies a is a point of max or min. if you consider $f(x) = x^3$, we see $f'(0) = 0$ but zero is neither max nor min.

We also said that the point of max or min must be inside the interval, not endpoint. For example $f(x) = x$ defined on $[3, 20]$ has minimum and maximum at end points and the derivative there exists and not zero. Of couse, you should consider left derivative at the point 3 and left derivative at the point 20.

Suppose that f is a continuous function on a closed bounded interval $[u, v]$. we do know that f is bounded and attains the bounds. suppose a is a point where f assumes largest value. If we now assume that $u < a < v$ and f is differentiable at a we can conclude $f'(a) = 0$. Gometrically it says the following. The the tangent at a is parallel to the x-axis, its slope is zero.

We shall now show that there is a tangent parallel to any given chord of the graph of f.
Fact: Let $u < v \in R$. Let f be a continuous function on an interval $[u, v]$ which is differentiable at every point $u < x < v$. Then there is a point $\theta \in (u, v)$ such that

$$f(v) - f(u) = (v - u)f'(\theta).$$

You can rewrite this as

$$f'(\theta) = \frac{f(v) - f(u)}{v - u}.$$

Observe that slope of the chord joining the two points $(u, f(u))$ and $(v, f(v))$ on the curve (graph of f) is precisely the right side above. The slope of the tangent at the point $\theta \in (u, v)$ is precisely the left side above.

Proof is simple. Shall convert the problem to one we already solved. define

$$\varphi(x) = [f(v) - f(u)]x - [v - u]f(x).$$

then $\varphi(u) = uf(v) - vf(u) = \varphi(v)$ and φ is differentiable in (u, v). Also $\varphi'(x) = [f(v) - f(u)] - [v - u]f'(x)$. If φ is constant then any point $\theta \in (u, v)$ will do the job. Otherwise φ must take values either larger than the value at the end points or values smaller than at the end points. Thus either maix-mum or minimum must be attained in the open interval (u, v). Such a point will do, by the previous fact.

This theorem is called *mean value theorem.* $[f(v) - f(u)]/[v - u]$ is the 'mean velocity' in this interval — ratio of distance travelled to the time duration. There is another interpretation. For ten numbers their mean value or average value is their sum/10. If you have a bounded function g on a finite interval, then its mean value or average value is integral/length, integral over the interval and length of the inteval. If you take $g = f'$, defined on (u, v), then its mean valus is precisely $[f(v) - f(u)]/[v - u]$. Thus the theorem says that the mean value of f' is actually value of f' at one of the points in the interval. Note that this quantity is not the mean of the values of the function f at the two end points.

Fact: Let f be defined in an open interval (u, v) differentiable at every point.

(i) If $f'(x) \geq 0$ for all x then f is increasing function in this interval.
(ii) If $f'(x) \leq 0$ for all x then f is decreasing function in this interval.
(iii) If $f'(x) = 0$ for all x then f is a constant in this interval.

Proof is simple. (i) Let $x < y$, then for some point θ,

$$f(y) - f(x) = (y - x)f'(\theta) \geq 0.$$

Same applies to prove (ii). (iii) follows from (i) and (ii).

polynomials of infinite degree:

We have been using exponential function, sine and cosine functions. We have defined exponential function as a sum of series of powers of x and identified it with the power e^x by using continuity and the functional equation $e(x)e(y) = e(x + y)$. This last equation is a consequence of Cauchy product theorem of series. We have used sine and cosine functions and in fact used their derivatives too. So the question is: are we depending on our high school definition here? If so what is it? Did we prove that it is continuous and differentiable?

Let us, once and for all, prove a general theorem. This theorem allows us to cook up functions, show their continuity, differentiability etc. This thorem also allows us to calculate their derivative by providing a formula, just like the one you have for polynomials.

The functions we are talking about are polynomials of infinite degree, sounds like an oxymoron, because when we say polynomial we mean a finite sum of powers of x and thus polynomial can not be of infinite degree. Yes, true these functions are not really clled so but are called power series.

An expression of the following form where α_i, are real numbers is called a power series.

$$P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots\cdots$$

This is only a symbolic expression. At this stage there is no meaning to it. We shall now give a meaning to it. It is nothing new. Take a real number and substitute that number for x in the above series. You get a series of numbers. This series may converge or may not. if it converges then we associate that value, namely sum of the series as the meaning of the above series at the point x ypu have taken.

Thus a value is attached for the above series whenever you take a real number and find that the series converges. To put it differently, we have a function defined on the set S of all numbers a such that the series converges when you substitute the number a for x in the above series. The important question is: what exactly is the nature of the set S and how does this function behave on that set.

For example can the set S be, say, the interval $[4, 5]$ or union of two intervals like $[1, 2] \cup [9, 10]$? the amazing thing is that such a situation can not occur. If the seris converges when you put $x = 5$, then it converges for every value $|x| < 5$. In fact whenever you take a number a with $|a| < 5$, then the series $\sum |\alpha_n a^n|$ converges, that is, the series $P(a)$ converges absolutely. Be careful, we did not say that the series defining $P(5)$ itself converges absolutely. We did not even say that the series obtained by putting $x = -5$ converges. If the series does not converge when $x = 5$, then it does not converge for any value larger than 5. Here is the main theorem about the power series.

Theorem: Let

$$P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots\cdots$$

be a power series. Let

$$r = \frac{1}{\limsup \sqrt[n]{|\alpha_n|}}$$

We take $r = 0$ in case the above limsup is ∞. We take $r = \infty$ if the above limsup is zero.

(i) The series converges absolutely for any value x with $|x| < r$. The series does not converge for any value x with $|x| > r$. When $x = \pm r$, it may or may not converge. r is called the radius of convergence of the power series.

Let us assume $r > 0$. and define the function

$$P(x) = \sum_{i \geq 0} \alpha_i x^i; \quad x \in (-r, r).$$

(ii) P is a continuous function on the interval $(-r, r)$.

Let $Q(x)$ be the power series

$$Q(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + \cdots.$$

(iii) The power series Q also has the same radius of convergence r. Thus

$$Q(x) = \sum_{i \geq 0} (i+1)\alpha_i x^i; \qquad x \in (-r, r)$$

is a well defined function.

(iv) The function P is differentiable in the interval $(-r, r)$ and $P'(x) = Q(x)$.

This theorem has a long proof. We shall develop some tools needed as we go along. But before that, a few comments.

First you should realize that the content of the theorem is trivial: you can treat this infinite degree polynomial as if it is usual polynomial you came across in high school. You can differentiate term by term, as you were doing with polynomials in high school. The only difference is that a polynomial is defined for every $x \in R$. But the power series is defined only on an symmetric interval around zero (which, of course, may turn out to be all of R in some cases depending on the numbers α).

Secondly, we have denoted the symbolic infinite series by $P(x)$. We have used the same symbol to denote the function defined on the interval $(-r, r)$. This is done just to avoid too many notational symbols. If you do not like, you can denote the function by f. Thus $P(x)$ stands for the power series $\sum \alpha_i x^i$ without any explanation as to what the symbol x is, what the meaning of the sum is, whether it exists etc. On the other hand, $f(x)$ stands for the function defined on the interval $(-r, r)$ whose value at a point a in this interval is given by the sum $\sum \alpha_i a^i$. But it is not necessary to make such a fine distinction unless you are getting confused.

Thirdly, you must be wondering about the word 'radius' in naming r as the radius of convergence. You can ignore this terminlogy and just think of the interval $(-r, r)$ as interval of convergence. But the story is different. You can think of even putting a complex number like $1 + i$ for x in the power series $P(x)$ and want to know whether the series of complex numbers so obtained converges. The answer is the following. Draw a circle of radius r in the (x, y)-plane with origin as center. if your complex number z is inside this circle, then the series $P(z)$ converges.
If your complex number z is outside the circle, then the series $P(z)$ does not converge. For points z on the circle, the series $P(z)$ may or may not converge. However, we shall not peep into complex numbers now.

Finally, we assumed that $r > 0$. Because if the series converges only for $r = 0$, then the function is not defined on an interval and the question of coninuity is simple and the question of derivative does not make sense (why?).

The theorem applies to the function

$$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

We had already shown that this function is defined on all of R and is a continuous function. In fact if we denote by e the number $P(1)$, then $P(x)$ is nothing but e^x. The extra information now you get is that this function is differentiable and you can differentiate term by term getting the derivative is again e^x.

The theorem applies to the functions

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

$$g(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

These functions are continuous on R. These are differentiable, you can differentiate term by term. You see $f'(x) = g(x)$ and $g'(x) = -f(x)$.

underlineProof of (i): Actually we have already done this. Let $\sum_n a_n$ be a series. we have proved the following:

$$\limsup \sqrt[n]{|a_n|} < 1 \Rightarrow \sum |a_n| \text{ converges;}$$

$$\limsup \sqrt[n]{|a_n|} > 1 \Rightarrow \sum a_n \text{ does not converge.}$$

Fix a number x. The series $\sum \alpha_n x^n$ converges if $\limsup \sqrt[n]{|\alpha_n} \, |x| < 1$. Thus if the limsup here is zero, then the series converges for every x. If the limsup is ∞, then this does not converge for any non-zero x. if the limsup is finite and non-zero, then the series converges if $|x| < r$. In this case the series converges absolutely.

On the other hand if $\limsup \sqrt[n]{|\alpha_n|}|x| > 1$, that is, if $|x| > r$, then the seies does not converge. This completes proof of (i).

Towards Proof of (ii):

Let $P_n(x) = \sum\limits_0^n \alpha_i x^i$. Then,by definition of sum of series, we see that for each $x \in (-r, r)$, the sequence $\{P_n(x)\}$ converges to $P(x)$. We also see that for each fixed n the function $P_n(x)$ is a polynomial and is hence continuous. Unfortunately, in general, such a point-wise a limit of a sequence of continuous functions need not be continuous.

For instance, consider the function $f_n(x) = x^n$ defined on the interval $[0, 1]$ for each $n = 1, 2, \cdots$. Let $f(x) = 0$ for $0 \le x < 1$ and $f(1) = 1$. Clearly, $f_n(x) \to f(x)$ for each $x \in [0, 1]$. Each of the functions f_n is a continuous function on $[0, 1]$ but yet f is not continuous.

When we say $f_n(x) \to f(x)$ for every x what do we mean? suppose we fix a point x, then $f_n(x) \to f(x)$, that is, given $\epsilon > 0$, there is an n_0 such that $n \ge n_0$ implies $|f_n(x) - f(x)| < \epsilon$. This n_0 depends not only on the given ϵ but also on the point x fixed. In the above example, this is what is happening. If $x = 1/2$ you see $f_n(1/2) < 1/1000$ already for $n \ge 10$. But if you want $f(0.999) < 1/1000$ then this $n_0 = 10$ will not do, you need to take much larger n_0. For each point $x < 1$, we do have $f_n(x) \to 0$, but how long should we wait to be close to zero depends on the point x.

If we can make f_n close to f irrespective of the point, then f will be continuous. Let us make it precise. suppose we have a sequence of functions $\{f_n\}$ and a function f, all defined on a set S. Let us say that $f_n \to f$ **uniformly** if given $\epsilon > 0$, there is an n_0 such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \ge n_0$ whatever be the point $x \in S$. Observe that this implies, in particular, that for every point x we have $f_n(x) \to f(x)$. Uniform convergence is more than simply saying point-wise convergence.

Temporarily take $S = [0, 1]$ so that all our functions are defined on $[0, 1]$. Imagine the graph of f and the graphs of $f + \epsilon$ and $f - \epsilon$. here $f \pm \epsilon$ means the function $f(x) \pm \epsilon$. The graph of $f + \epsilon$ is parallel to the graph of f and is above graph of f. Draw pictures. The graph of $f - \epsilon$ is again parallel to graph of f and is below graph of f. If $f_n \to f$ uniformly, then after some stage, graphs of all the functions f_n lie within this band $f - \epsilon$ to $f + \epsilon$. Just like an interval $(a - \epsilon, a + \epsilon)$ in the real line, we can imagine a band of functions, namely, all functions whose graphs lie between $f - \epsilon$ and $f + \epsilon$. Just as convergence of numbers, $x_n \to a$, demands that after some stage all numbers x_n lie within $(a-\epsilon, a+\epsilon)$; uniform convergence of functions, $f_n \to f$ uniformly, demands that after some stage all the functions should lie within the band $(f-\epsilon, f+\epsilon)$.

Convince yourself that in the above example of the sequence of functions $\{x^n\}$ on the interval $[0,1]$ the convergence is not uniform. if we were to consider the same sequence of functions on the set $S=[0,0.99]$ the convergence is indeed uniform. if you want to make $|f_n|<\epsilon$, choose k so that $(0.99)^k<\epsilon$. then whatever be $n>k$ and whatever be $x \in S$, we see that $x^n<\epsilon$.

We now make a very useful observation:

Fact: If $f_n \to f$ uniformly on S and if each f_n is continuous on S, then f is continuous on S.

Proof is simple. Suppose that $a \in S$ and $\epsilon>0$ are given. We produce $\delta>0$ so that $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$ and $x \in S$. Idea is this: $f(x)$ is close to $f_k(x)$ (if k is sufficiently large); and $f_k(x)$ is close to $f_k(a)$ (by continuity of f_k) and $f_k(a)$ is close to $f(a)$. Here is the execution.

First you choose k so that $|f_k(y)-f(y)|<\epsilon/3$ for every $y \in S$. You can do this because of uniform convergence. Actually uniform convergence tells you that you can choose an k so that this inequality is true for every $n \geq k$. But we need not bother about it now. That the inequality holds for this one k is enough. Since f_k is continuous at the point $a \in S$, choose $\delta>0$ so that $|f_k(x)-f_k(a)|<\epsilon/3$ whenever $|x-a|<\delta; x \in S$. Now if you take any $x \in S$ with $|x-a|<\delta$ we have

$$|f(x)-f(a)| \leq |f(x)-f_k(x)|+|f_k(x)-f_k(a)|+|f_k(a)-f(a)|<\epsilon.$$

returning to our power series, if only we knew that $P_n \to P$ uniformly on $(-r,r)$, we could have immediately applied the above theorem rightaway. Since each P_n, a polynomial is continuous, it follows that P is also continuous. Unfortunately, there is a little twist. In general P_n does not converge uniformly on all of $(-r,r)$. They nearly do so.

Fact: Let $0<c<r$. Then $P_n \to P$ uniformly on $[-c,c]$.

You can take $c = r - 0.00001$, but can not take $c = r$ in general.

Proof is simple. Let $\epsilon > 0$. Remember the power series converges absolutely at the point c. That is, the series $\sum |\alpha_i| c^i$ is convergent. Also remember that when a series converges, the partial sums converge to the grand sum. A consequence of this, which we observed earlier , is that tail sums converge to zero. That is if $t_n = \sum_{i>n} |\alpha_i| c^i$, then $t_n \to 0$. Let us choose k so that $t_k < \epsilon$. We claim that $|P_n(x) - P(x)| < \epsilon$ for any $n > k$ and for any $x \in [-c, c]$. In fact for any such n and x,

$$|P(x) - P_n(x)| = |\sum_{i>n} \alpha_i x^i| \leq \sum_{i>n} |\alpha_i| c^i \leq \sum_{i>k} |\alpha_i| c^i < \epsilon.$$

We now complete proof of (ii).

Let us take $a \in (-r, r)$. We show that P is continuous at a. First fix $c > 0$ so that $-r < -c < a < c < r$. The above two facts tell you that P is continuous on the interval $[-c, c]$. To show that P is continuous at a, take a sequence $x_n \to a$, $-r < x_n < r$. Then after some stage $x_n \in (-c, c)$. By continuity of P on $[-c, c]$ tell us $P(x_n) \to P(a)$.

This completes proof of (ii).

Towards proof of (iii):

Recall

$$Q(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + \cdots$$

We need ot show that Q also has radius of convergence r. Here the coefficient of x^n is $(n+1)\alpha_{n+1}$ so for the radius of convergence of Q we would be involved in $\sqrt[n]{(n+1)|\alpha_{n+1}|}$. On the other hand calculation of r involved $\sqrt[n]{|\alpha_n|}$. We first sort out this mismatch between n and $n+1$. We claim that the series $Q(x)$ converges iff the following series converges.

$$xQ(x) = \alpha_1 x + 2\alpha_2 x^2 + 3\alpha_3 x^3 + 4\alpha_4 x^4 + \cdots$$

In fact let x be any real number. if the series $Q(x)$ converges, then multiplying each term by x we get the second series and hence the second series also converges. conversely, suppose that the second series converges. If $x = 0$, then first series also converges (inspect what is the series when you put

$x = 0$). So let $x \neq 0$. But then the first series is obtained from the seond series by dividing each term by x.

As a consequence, the radius of convergence of the power series $Q(x)$ is given by

$$\frac{1}{\limsup \sqrt[n]{n|\alpha_n|}}.$$

Thus to prove (iii) we only need to show that

$$\limsup \left(\sqrt[n]{n} \ \sqrt[n]{|\alpha_n|} \right) = \limsup \sqrt[n]{|\alpha_n|}.$$

Fact: Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative numbers; $a_n \to a$ $0 < a < \infty$; $\limsup b_n = b$; $0 \leq b \leq \infty$. Then $\limsup(a_n b_n) = ab$.

if this fact is proved, then we can take $a_n = \sqrt[n]{n}$ and $b_n = \sqrt[n]{|\alpha_n|}$. we had already proved earlier that $a_n \to 1$, so that $a = 1$ and the fact applies to give us what we wanted.

Here is proof of the fact.

Case $b = \infty$. Need to show $\limsup a_n b_n = \infty$. Given any number c, we show that infinitely many of these $a_n b_n$ are larger than c. Since $\limsup b_n = \infty$, infinitely many b_n are larger than $c/(a/2)$; say $b_{n_1}, b_{n_2}, b_{n_3} \cdots$. Since $a_n \to a$, after some stage $a_n > a/2$; say for $n \geq k$. If you look at $a_{n_i} b_{n_i}$ for $n_i > k$ you see that they are all larger than c.

Case $b < \infty$.

Since $\limsup b_n = b$, there is a subsequence converging to b. Say

$$b_{n_1}, b_{n_2}, b_{n_3}, \to b.$$

Since $a_n \to a$, every sunsequence also converges to a. Hence

$$a_{n_1} b_{n_1}, a_{n_2} b_{n_2}, a_{n_3} b_{n_3}, \cdots \to ab$$

Thus ab is a limit point of the sequence $\{a_n b_n\}$ so that $\limsup a_n b_n \geq ab$. If c is any limit point of the sequence $\{a_n b_n\}$, then there is a subsequence

$$a_{n_1} b_{n_1}, a_{n_2} b_{n_2}, a_{n_3} b_{n_3}, \cdots \to c.$$

But

$$a_{n_1}, a_{n_2}, a_{n_3}, \cdots \to a.$$

Since $a \neq 0$, we conclude that

$$b_{n_1}, b_{n_2}, b_{n_3}, \cdots \to \frac{c}{a}.$$

Thus c/a is a limit point of the sequence $\{b_n\}$ and hence must be not larger than its limsup.

$$\frac{c}{a} \leq b; \qquad c \leq ab$$

Thus any limit point of the sequence $\{a_n b_n\}$ issmaller than ab. Since we have already shown that ab is a limit point, we conclude that $\limsup a_n b_n = ab$.

This completes proof of (iii)

Finally, we prove (iv):

The idea is similar to that of (ii). Fix $a \in (-r, r)$. Need to show that $[P(x) - P(a)]/[x - a]$ is close to $Q(a)$when x is close to a. We know that $[P_n(x) - P_n(a)]/[x - a]$ is close to $Q_n(a)$when x is close to a where

$$P_n(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n.$$

$$Q_n(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \cdots + n\alpha_n x^{n-1}.$$

If we can show that $[P(x) - P(a)]/[x - a]$ is close to $[P_n(x) - P_n(a)]/[x - a]$ and $Q_n(a)$ is close to $Q(a)$, we are done as earlier in (ii). this is what we execute now.

Let $\epsilon > 0$ be given. We shall exhibit $\delta > 0$ so that $(a - \delta, a + \delta) \subset (-r, r)$ and

$$0 < |x - a| < \delta \Rightarrow \left| \frac{P(x) - P(a)}{x - a} - Q(a) \right| < \epsilon.$$

This will show that P is differentiable and $P'(a) = Q(a)$.

First we fix $c > 0$ so that $-r < -c < a < c < r$. This is possible because $-r < a < r$. Since Q also has radius of convergence r and $0 < c < r$ we conclude that the series $Q(c)$ converges absolutely. That is the series

$\sum(i+1)|\alpha_{i+1}|c^i$ is convergent. So as earlier in (ii), its tail sums converge to zero. Fix N such that

$$\sum_{i\geq N}(i+1)|\alpha_{i+1}|c^i < \frac{\epsilon}{3}.$$

We have already defined

$$P_n(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n.$$

$$Q_n(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \cdots + n\alpha_n x^{n-1}.$$

Let us also put

$$\widetilde{P_n}(x) = \alpha_{n+1}x^{n+1} + \alpha_{n+2}x^{n+2} + \alpha_{n+3}x^{n+3} + \cdots.$$

$$\widetilde{Q_n}(x) = (n+1)\alpha_{n+1}x^n + (n+2)\alpha_{n+2}x^{n+1} + (n+3)\alpha_{n+3}x^{n+2} + \cdots.$$

so that for each $n \geq 1$ we have

$$P(x) = P_n(x) + \widetilde{P_n}(x); \qquad Q(x) = Q_n(x) + \widetilde{Q_n}(x); \qquad (\diamondsuit)$$

With this notation, choice of N tells us

$$|x| \leq c \Rightarrow |\widetilde{Q_N}(x)| \leq \epsilon/3. \qquad (\spadesuit)$$

Note that by mean value theorem, <u>for $x \in [-c, c]$ we have the following</u>.

$$\left|\frac{\alpha_{N+1}x^{N+1} - \alpha_{N+1}a^{N+1}}{x-a}\right| \leq (N+1)|\alpha_{N+1}|c^N \leq \frac{\epsilon}{3}.$$

$$\left|\frac{\alpha_{N+1}x^{N+1} + \alpha_{N+2}x^{N+2} - \alpha_{N+1}a^{N+1} - \alpha_{N+2}a^{N+2}}{x-a}\right|$$

$$\leq (N+1)|\alpha_{N+1}|c^N + (N+2)|\alpha_{N+2}|c^{N+1}$$
$$\leq \frac{\epsilon}{3}.$$

More generally, for every $k \geq 1$, we have

$$\left| \frac{\sum_{i=N+1}^{N+k} \alpha_i x^i - \sum_{i=N+1}^{N+k} \alpha_i a^i}{x-a} \right| \leq \frac{\epsilon}{3}.$$

Since this is true for every $k \geq 1$ we see by taking limits (over k),

$$-c \leq x \leq c \Rightarrow \left| \frac{\widetilde{P_N}(x) - \widetilde{P_N}(a)}{x-a} \right| \leq \frac{\epsilon}{3}. \qquad (\heartsuit)$$

As noted already and easy to see, derivative of the polynomial P_N is Q_N. Thus we can fix $\delta > 0$ so that $(a-\delta, a+\delta) \subset (-c, c)$ and

$$0 < |x-a| < \delta \Rightarrow \left| \frac{P_N(x) - P_N(a)}{x-a} - Q_N(a) \right| < \frac{\epsilon}{3} \qquad (\clubsuit)$$

For $0 < |x-a| < \delta$ we have the following.

In these string of inequalities below, first use $(\diamondsuit)$ and then $|t+u+v| \leq |t| + |u| + |v|$ and then use $(\clubsuit), (\heartsuit), (\spadesuit)$.

$$\left| \frac{P(x) - P(a)}{x-a} - Q(a) \right|$$

$$= \left| \frac{P_N(x) - P_N(a)}{x-a} + \frac{\widetilde{P_N}(x) - \widetilde{P_N}(a)}{x-a} - Q_N(a) - \widetilde{Q_N}(a) \right|$$

$$\leq \left| \frac{P_N(x) - P_N(a)}{x-a} - Q_N(a) \right| + \left| \frac{\widetilde{P_N}(x) - \widetilde{P_N}(a)}{x-a} \right| + |\widetilde{Q_N}(a)|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This completes the proof of the theorem.

The proof is rather long but you see that each step is simple.

We shall now discuss two generalizations of the mean value theorem and uses of these gneralizations. First let us recall the mean value theorem.

If f is a continuous function which is differentiable at every point of (a, b), then there is a point $\theta \in (a, b)$ so that

$$[f(b) - fa)] = (b - a)f'(\theta).$$

We shall complicate it now. Let g be the function $g(x) = x$. Then note that $g' \equiv 1$. Thus the above equation takes the form

$$[f(b) - f(a)]g'(\theta) = [g(b) - g(a)]f'(\theta).$$

This statement is true in general and is called the generalized mean value theorem. Here it is

gneralized MVT:

Fact: Let f and g be two continuous functions on the interval $[a, b]$, both differentiable at every point of (a, b). Then there is a number $\theta \in (a, b)$ such that

$$[f(b) - f(a)]g'(\theta) = [g(b) - g(a)]f'(\theta).$$

Proof is simple. As earlier define

$$\varphi(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Note φ is continuous on $[a, b]$ and differentiable at every point of (a, b) with

$$\varphi'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x).$$

Also $\varphi(b) = f(b)g(a) - f(a)g(b) = \varphi(a)$. Mean value theorem applies to give a point θ where $\varphi'(\theta) = 0$. This is the required conclusion.

We use this to derive a nice theorem that goes by the name of L'Hopital's rule. This theorem helps to evaluate limits of ratios of functions. Suppose

f and g are two functions and we want to evaluate $\lim_{x\to a}[f(x)/g(x)]$. If f and g converge to some finite nonzeo numbers then this is easy, the ratio of the functions converges to the ratio of the two limits. However, both f and g converge to zero we would not be able to blindly apply the theorem on limits of ratios.

L'Hopital:

Fact: Let (a,b) be a bounded interval; f and g be two differentiable functions on this interval. Assume the following.

$$\lim_{x\downarrow a} f(x) = 0; \qquad \lim_{x\downarrow a} g(x) = 0$$

$$g'(x) \neq 0 \text{ for all } x \in (a,b); \qquad \lim_{x\downarrow a} \frac{f'(x)}{g'(x)} = \alpha \in R.$$

Then $\lim_{x\downarrow a} \frac{f(x)}{g(x)} = \alpha$.

Proof is simple. Let $\epsilon > 0$ be given. we need to show a $\delta > 0$ so that

$$a < x < a+\delta \Rightarrow \left|\frac{f(x)}{g(x)} - \alpha\right| < \epsilon.$$

first note that the ratio makes sense for all numbers close to a. Indeed, there can be at most one point x with $g(x) = 0$. This is because, if there are two such points, x_1 and x_2 where g vanishes the usual mean value theorem says that in between at some point g' vanishes which is not possible in view of the hypothesis. since there is at most one number where g vanishes, we shall from now on consider all our points smaller than that number.

Let us choose, using hypothesis, a number $\delta > 0$ so that

$$a < x < a+\delta \Rightarrow \left|\frac{f'(x)}{g'(x)} - \alpha\right| < \epsilon/2.$$

To show that the same δ serves our purpose, let us take $a < x < a+\delta$. Take also some y with $a < y < x$. Then by generalized MVT, the ratio $[f(x)-f(y)]/[g(x)-g(y)]$ equals $f'(\theta)/g'(\theta)$ where θ is in between x and y;

in particular $a < \theta < a + \delta$ so that we have

$$\left| \frac{f(x) - f(y)}{g(x) - g(y)} - \alpha \right| < \epsilon/2.$$

But this is true for every $a < y < x$. If we let $y \downarrow a$ we see $f(y) \to 0$ and $g(y) \to 0$. Thus taking limits (as $y \downarrow a$) in the above inequality we get

$$a < x < a + \delta \Rightarrow \left| \frac{f(x)}{g(x)} - \alpha \right| \leq \epsilon/2.$$

This completes the proof. If you are not comfortable with the phrase $y \downarrow a$, take a sequence of points $a < y_n < x$ such that $y_n \to a$. Write the inequalities only for these y_n and take limit as $n \to \infty$.

Why did we take α to be finite? It is not necessary, we can allow $\alpha = \infty$ Thus, suppose

$$\lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = \infty$$

We show that

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \infty.$$

Let A be any given number. We shall show $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \frac{f(x)}{g(x)} > A.$$

Use hypothesis to get $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \frac{f'(x)}{g'(x)} > A.$$

As ealier this δ would do with exactly the same proof — take $a < x < a + \delta$, take any $a < y < x$, argue about the ratio $[f(x) - f(y)]/[g(x) - g(y)]$ and take $y \downarrow a$.

Can we take $\alpha = -\infty$. Yes, exactly the same argument as in the case of ∞ above would do.

Why did we take a finite? Can it be $-\infty$. Yes. But then we will be talking about lim as $x \downarrow -\infty$. So we should reformulate the argument and not use $-\infty + \delta$. For example let α be finite. Given $\epsilon > 0$ we need to show a number c so that $x < c$ implies the ratio $f(x)/g(x)$ is close to α upto ϵ. Proof is exactly the same. Same holds in case α is not finite.

Why did we take limit at a? Yes we can take limit as $x \uparrow b$ and we can allow b to be ∞ too. How about taking a point $a < c < b$; assume that f and g converge to zero as $x \to c$ but the ratio $f'(x)/g'(x) \to \alpha$ as $x \to c$, Can we still say that $f(x)/g(x)$ converges to α as $x \to c$? Yes, argue in the intervals (a, c) and (c, b).

I have stated simplest case, but all these embellishments are possible and useful too. There is one question, namely, why did you assume f and g are converging to zero? what if they converged to ∞ Instead of $0/0$ form we have ∞/∞ form. is the result true? Yes, it is true, needs different proof, shall do later.

Taylor:

We shall now generalize MVT in a different direction. But this needs the concept of higher derivatves, that is, the process of repeating differentiation. Suppose f is a function on (a, b) differentiable at each point of this interval. Thus we have a new function $f'(x)$ which associates with every point x the value of the derivative at that point. if this new function is diferentiable at a point x, we denote it by $f''(x)$, called second derivative of f at x. if the function f' is differentiable at every point x of (a, b) we can define the function f'' on (a, b) and try differentiating as long as the derivatives exist.

For example if f is a polynomial then you can talk about derivatives, they are polynomials, after some stage we get the zero function.

if $f(x) = \sin x$ and $g(x) = \cos x$, we again see that we can talk about any number of derivatives. $f' = g$; $f'' = -f$ etc.

if $f(x) = e^x$, then $f' = e^x$ again and hence differentiable any number of times.

I have introduced higher derivatives, without much ado. it is possible to motivate, at least the second derivative. Geometric picture is this. If the funcion is not a straight line, then it must be curving. How is it curving? How do you define and measure it?

Particle picture is this. If the velocity is not constant, then it is changing. What is its rate of change? Just as first derivative (velocity) is rate of change of distance travelled, second derivative is the rate of change of velocity. This is called acceleration.

Analytical picture is this. Given a function f and a point a we wanted first a constant function which approximates f near a. More precisely, wanted constant function $\varphi(x) \equiv c$ so that $f(x) - \varphi(x) \to 0$ as $x \to a$. The function is the constant function equal to the number $f(a)$. Then we allowed a little more general functions, namely straight line function φ but wanted better approximation. More precisely, wanted $\varphi(x) = \alpha + \beta x$ so that $[f(x) - \varphi(x)]/[x - a] \to 0$ as $x \to a$. of course this, in particular implies $f(x) - \varphi(x) \to 0$ as $x \to a$. We saw $\varphi(x) = f(a) + f'(a)(x - a)$.

We can allow quadratic functions but demand more better aproximation, $[f(x) - \varphi(x)]/(x - a)^2 \to 0$. Remember, this implies, in particular, $f(x) - \varphi(x) \to 0$ and also $[f(x) - \varphi(x)]/(x - a) \to 0$. All these limits are as $x \to a$. We shall not pursue the details.

Returning to our discussion on higher derivatives, you may be able to talk of only third derivative but not fourth etc. it all depends on the function. Look at the example involving $x^8 \sin(1/x)$.

If our function f is given by a power series, then the fundamental theorem on power series tells us that f' is again given by a power series and hence f'' is again given by a power series. this goes on forever; repeated application of the fundamental theorem on power series.

To denote higher derivatives f', f'', f''' is not convenient. One uses $f^{(k)}$ to denote the k-th derivative of f. Do not confuse it with k-th power of f, pay attention to the brackets. Thus for the first derivative we use $f^{(1)}$ or

f'. Sometimes we use, for notational convenience, $f^{(0)}$ for f itself, zeroth derivative.

Let us now restate MVT as follows. If f is a continuous function on $[a,b]$ which is differentiable at every point of (a,b), then there is a number $\theta \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(\theta).$$

The value of the function at b is explained in terms of $f(a)$ and derivative at some point. suppose the function has second derivative. Can we explain $f(b)$ in terms of $f(a)$, $f'(a)$ and second derivative at some point, — probably a better explanation of $f(b)$? Yes. But before doing this, let us look at proof of the MVT.

Let

$$\varphi(x) = f(x) - f(a) - C(x-a).$$

We see $\varphi(a) = 0$. the number C is so chosen that $\varphi(b) = 0$. In other words, $C(b-a) = f(b) - f(a)$. There is a number $\theta \in (a,b)$ such that $\varphi'(\theta) = 0$. If φ is constant, then any point would do, otherwise consider a point where φ has max or min. But $\varphi'(\theta) = f'(\theta) - C$. Thus there is a number θ such that

$$(b-a)f'(\theta) = (b-a)C = f(b) - f(a)$$

In other words

$$f(b) = f(a) + (b-a)f'(\theta).$$

Here is now the extension. let f be two times differentiable function on an open interval which includes two points a and b, say, $a < b$. Then there is a number $\theta \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f^{(2)}(\theta).$$

Proof is exactly as above. Put

$$\varphi(x) = f(x) - f(a) - (x-a)f'(a) - C\frac{1}{2!}(x-a)^2.$$

Note that $\varphi(a) = 0$. Choose C so that $\varphi(b) = 0$. that is

$$C\frac{1}{2!}(b-a)^2 = f(b) - f(a) - (b-a)f'(a).$$

We get $\xi \in (a, b)$ so that $\varphi'(\xi) = 0$. Note that

$$\varphi'(x) = f'(x) - f'(a) - C(x - a)$$

so that $\varphi'(a) = 0$ and MVT applied to φ' gives $\theta \in (a, \xi)$ such that $\varphi^{(2)}(\theta) = 0$. But $\varphi^{(2)} = f^{(2)} - C$. Thus $f^{(2)}(\theta) = C$. Hence,

$$\frac{1}{2!}(b - a)^2 f^{(2)}(\theta) = C\frac{1}{2!}(b - a)^2 = f(b) - f(a) - (b - a)f'(a).$$

That is

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2!}(b - a)^2 f^{(2)}(\theta).$$

Suppose that f is three times differentiable on an interval which includes $a < b$. then there is a number $\theta \in (a, b)$ such that

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2!}(b - a)^2 f^{(2)}(a) + \frac{1}{3!}(b - a)^3 f^{(3)}(\theta).$$

Proof is exactly as above. Define

$$\varphi(x) = f(x) - f(a) - (x - a)f'(a) - \frac{1}{2!}(x - a)^2 f^{(2)}(a) - C\frac{1}{3!}(x - a)^3.$$

$$\varphi'(x) = f'(x) - f'(a) - (x - a)f^{(2)}(a) - C\frac{1}{2!}(x - a)^2.$$

$$\varphi''(x) = f''(x) - f''(a) - C(x - a).$$

$$\varphi^{(3)}(x) = f^{(3)}(x) - C.$$

Thus $\varphi(a) = \varphi'(a) = \varphi''(a) = 0$. Choose C so that $\varphi(b) = 0$. Applying MVT successively for φ to get $\xi \in (a, b)$, then for φ' to get $\eta \in (a, \xi)$, then for φ'' to get $\theta \in (a, \eta)$. This will give the desired result.

If you have understood this, there is no difficulty in proving the following.

If f is n times differentiable in an interval which includes two points $a < b$, then there is a number $\theta \in (a, b)$ so that

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2!}(b - a)^2 f^{(2)}(a) + \frac{1}{3!}(b - a)^3 f^{(3)}(a) + \cdots$$

$$+\frac{1}{(n-1)!}(b-a)^{(n-1)}f^{(n-1)}(a)+\frac{1}{n!}(b-a)^n f^{(n)}(\theta).$$

In compact notation

$$f(b)=\sum_0^{n-1}\frac{(b-a)^k}{k!}f^{(k)}(a)+\frac{1}{n!}(b-a)^n f^{(n)}(\theta).$$

Recall that here we used $f^{(0)}$ for f. This is convenient notation to push it under the summation sign.

For the proof, you define

$$\varphi(x)=f(x)-\sum_0^{n-1}\frac{(x-a)^k}{k!}f^{(k)}(a)-\frac{1}{n!}(x-a)^n C.$$

Note again that $f^{(0)}(a)=f(a)$. Prove by induction that for $k\le n-1$,

$$\varphi^{(k)}(x)=f^{(k)}(x)-\sum_{i=k}^{n-1}\frac{(x-a)^{i-k}}{(i-k)!}f^{(i)}(a)+\frac{1}{(n-k)!}(x-a)^{n-k}C.$$

$$\varphi^{(n-1)}(x)=f^{(n-1)}(x)-(x-a)C;\qquad \varphi^{(n)}(x)=f^{(n)}(x)-C.$$

Observe $\varphi^{(k)}(a)=0$ for $k=0,1,2\cdots,n-1$. Now let the constant C be so chosen that $\varphi(b)=0$. That is

$$\frac{1}{n!}(b-a)^n C=f(b)-\sum_0^{n-1}\frac{(b-a)^k}{k!}f^{(k)}(a).$$

As earlier, applying MVT for φ, φ', φ'' etc you get $\theta\in(a,b)$ so that$\varphi^{(n)}(\theta)=0$. In other words $f^{(n)}(\theta)=C$. Thus

$$\frac{1}{n!}(b-a)^n f^{(n)}(\theta)=f(b)-\sum_0^{n-1}\frac{(b-a)^k}{k!}f^{(k)}(a).$$

This completes proof of the theorem.

Of course, there is nothing special in taking $a<b$, we have done it to fix ideas. We could have taken $a>b$. Exactly, the same formula is valid.

Instead of saying $\theta \in (a, b)$ we say, θ is between a and b.

Let us look at a special case. suppose that f is a function defined o $(-r, r)$ where $r > 0$, differentiable n times. Let us take $a = 0$ and b any point of this interval. we get the following. There is a number θ between zero and b so that

$$f(b) = \sum_0^{n-1} \frac{b^k}{k!} f^{(k)}(0) + \frac{1}{n!} b^n f^{(n)}(\theta).$$

In other words, for every $x \in (-r, r)$, there is a number θ between zero and x; (this θ depends on the point x) so that the following holds.

$$f(x) = \sum_0^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n)}(\theta)}{n!} x^n.$$

In long hand,

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n)}(\theta)}{n!} x^n,$$

for some θ between zeo and x. Of course, the number θ depends on the point x.

The above formula is called Taylor expansion of f around zero, with remainder R_n; or simply, Taylor formula with remainder.

Assume, for a moment that our function f has derivatives of all orders, and they are bounded by a number M. That is, $f^{(n)}$ exists for every n and $|f^{(n)}(x)| \leq M$ for all x in the interval $(-r, r)$ and also for all integers $n \geq 1$. For example if we take the function e^x or $\sin x$ or $\cos x$ and $r = 1000$, this holds. These are not the only functions.

Then we can keep on writing the Taylor formula for every n. Interestingly,

$$|R_n(x)| \leq M \frac{x^n}{n!} \to 0,$$

because the series $\sum(x^n/n!)$ converges. Where does this lead us? For every $x \in (-r, r)$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$

simply because if you consider the n-th partial sum on the right side, it differs from left side by an amount that converges to zero.

What does all this mean? Under the conditions we assumed, the unknown function is actually a power series, no more complicated! Once you understand what is going on, you can do better. For example, you can assume that there is a number M such that the n-th derivative on the interval $(-r, r)$ is bounded by M^n instead of M. Exactly the same proof works.

you can even do better. Assume that for any given c, with $0 < c < r$; there is a number M (depending on c) so that $|f^{(n)}(x)| \leq M^n$ for every $x \in [-c, c]$ and every n. Of course theses trivial statements look complicated at this stage and you need not bother. Just ignore this para, if you feel so.

Firstly, understand that there are functions which are differentiable exactly 30 times and no more. For such functions, you can not talk of Taylor series for $n > 30$. So let us consider only functions which are differentiable n times for every n. then the above thought process leads us to guess that probably, every such function is a power series.

There are two questions now. firstly, is the above guess true? Secondly, if the function is actually a power series, does Taylor series give us the same or some different series. Answer to the first question is in the negative. There are functions which are ininitely differentiable, but do not come from a power series. We shall see.

Answer to the second question is: yes. This is simple to see. suppose that we do have a function f given by a power series.

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots; \qquad -r, x < r.$$

Clearly, $f(0) = a_0$. Using the fundamental theorem on power series, we see

$$f'(0) = a_1; \quad f''(0) = 2!a_2; \cdots, \quad f^{(k)}(0) = k!a_k.$$

Thus $f^{(k)}(0)/k! = a_k$ and Taylor series agrees with the given power series.

Let us proceed to understand the first question. We give an example of a function which has all derivatives but yet the error term in Taylor expansion does not become smaller, in other words, the infinite series formula is not valid. Let us define

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

We show that f is infinitely differentiable. As you see from the formula for f, the troublesome point is zero. if $x < 0$, then the ratio $[f(x) - f(0)]/x$ is zero. if we can show that the ratio converges to zero as $x \downarrow 0$, we can conclude $f'(0) = 0$. But when $x > 0$,

$$\frac{f(x) - f(0)}{x} = \frac{1}{x} e^{-1/x^2} \to 0,$$

which is seen by recalling that $ye^{-y^2} \to 0$ as $y \to \infty$.

Unfortunately, showing that it is 30 time differentiable at zero alone, does not lead us to show that it is 31 times differentiable. we need a formula for the 30-th derivative for points near zero to calculate the 31-st derivative at zero. This is how we do.

We claim that for each $n \geq 1$, there is a polynomial $P_n(u)$ such that

$$f^{(n)}(x) = \begin{cases} P_n(\frac{1}{x}) e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

For $n = 1$, $P(u) = 2u^3$. Verify directly by calculating derivative.

Suppose it is true for n. We show for $n + 1$. For $x < 0$, since $f^{(n)} \equiv 0$ implies $f^{(n+1)} \equiv 0$. To calculate derivative at zero,

$$\lim_{x \downarrow 0} \frac{1}{x} P_n(\frac{1}{x}) e^{-1/x^2} = \lim_{y \to \infty} y P_n(y) e^{-y^2} = 0.$$

For $x > 0$, by product rule,

$$\frac{d}{dx} P_n(\frac{1}{x}) e^{-1/x^2} = P_n'(\frac{1}{x})(-\frac{1}{x^2}) e^{-1/x^2} + P_n(\frac{1}{x}) e^{-1/x^2} (\frac{2}{x^3}).$$

Thus $P_{n+1}(u) = -u^2P'_n(u) + 2u^3P_n(u)$. will satisfy our requirements.

This shows that the above function f is infinitely differentiable. However, since the value of the function as well as all derivatives at zero equal zero, Taylor expansion of any order will return only $R_n(x)$ which must equal $f(x)$.

Exponentiation again:

We have shown that $f(x) = x^n$ is a differentiable, but not yet shown that $f(x) = x^{\sqrt{2}}$ is differentiable. We shall now discuss the differentiability of the functions x^a.

Note that we have defined x^a for each $x > 0$ and each $a \in R$. Thus we have here many functions. For each fixed $a \in R$ we have the function $f(x) = x^a$ defined for $x \in (0, \infty)$. Also for each $x > 0$ we have the function $g(a) = x^a$ defined for every $a \in R$. All these functions are shown to be continuous. We shall now show that these are differentiable. First we relate these functions to the exponential function.

We know that $e(x) : (-\infty, \infty) \to (0, \infty)$ is a strictly increasing function. Hence has an inverse $L(x) : (0, \infty) \to R$. We also know from the fundamental theorem on power series, that the function e is a contiuous function. Hence L is also a continuous function. We argue that L is differentiable. Let $u > 0$. Take any sequence $v_n \to u$, each $v_n > 0$. We need to calculate limit of the ratios $[L(v_n) - L(u)]/[v_n - u]$. Let $L(u) = a$ so that $e(a) = u$. Similarly, let $L(v_n) = x_n$ so that $e(x_n) = v_n$.

$$\frac{L(v_n) - L(u)}{v_n - u} = \frac{x_n - a}{e(x_n) - e(a)} \to \frac{1}{e'(a)} = \frac{1}{e(a)} = \frac{1}{u}.$$

Thus L is differentiable at the point u and $L'(u) = 1/u$.

We claim $L(uv) = L(u) + L(v)$ for $u, v > 0$. Indeed if $e(x) = u$ and $e(y) = v$, then we know that $e(x + y) = e(x)e(y) = uv$ so that $L(uv) = x + y = L(u) + L(v)$. In particular, $L(u^2) = 2L(u)$ and by induction,we have $L(u^m) = mL(u)$ for each integer $m \geq 1$. This holds for $m = 0$ also because, $e(0) = 1$ giving us $L(1) = 0$. For any integer $n \geq 1$, we have

$L(u^{1/n}) = L(u)/n$ because

$$L(u) = L(u^{1/n}u^{1/n} \cdots n \text{ times}) = nL(u^{1/n}).$$

Thus we deduce that for any rational number $r = m/n$ with $m \geq 0, n \geq 1$, we have $L(u^r) = rL(u)$.

We claim that $L(1/u) = -L(u)$. Indeed, if $L(u) = x$, then $e(-x) = 1/e(x) = 1/u$ so that $L(1/u) = -x$. Thus if $r > 0$ is a positive rational number,

$$L(u^{-r}) = L(1/u^r) = -L(u^r) = -rL(u).$$

Tus $L(u^r) = rL(u)$ holds for every rational, positive or negative. If we fix a number $u > 0$, then $L(u^a)$ is a continuus function of a because it is composition of the two continuous functions $a \mapsto u^a$ and $u \mapsto L(u)$. Clearly, $a \mapsto aL(u)$ is a continuous function of a. Since these two functions agree at every rational, they agree at every real number. Thus $L(u^a) = aL(u)$ for every $a \in R$. This is true for every $u > 0$. Thus

$$L(u^a) = aL(u); \qquad u > 0, \;\; a \in R.$$

The function L is given the name log or logarithm or logarithm to the base e or natural logarithm. remember, this is defined only for positive numbers.

The equation deduced above can be restated as follows.

$$x^a = e(L(x^a)) = e(aL(x)); \quad i.e. \quad x^a = e^{a \log x}.$$

Now let us fix a number $a \in R$. consider the function $f(x) = x^a$. The equation above shows that this is composition of two functions, so that by chain rule

$$f'(x) = e'(aL(x))aL'(x) = x^a \; a \; \frac{1}{x} = ax^{a-1}.$$

Thus the formula: derivative of the function x^n is nx^{n-1} holds good for all numbers, not necessarily integers. But of course, the function x^n is defined on all of R as opposed to the function x^a which is defined only for $x > 0$.

to conclude this circle of ideas, let us consider the function $g(a) = x^a$ defined on R, where $x > 0$ is fixed. Again, this is composition of functions $a \mapsto aL(x) \mapsto e(aL(x))$ and so by chain rule

$$g'(a) = e'(aL(x))L(x) = x^a \log x.$$

(I am not including our discussion of Home assignment)

Bernstein Polynomials:

We saw that Taylor expansion gives us a polynomial with an error term, of course, only when the function is differentiable so many times as required there. This led us to see if, in general, functions defined on an interval around zero have power series expansion. such a hope failed because we found examples of functions which is zero at $x = 0$ and all its derivatives exist and are zero at this point. The only power series then is the zero function; but we had a non-zero function. Thus such a function, as in that example, can not be explained in terms of power series.

The question that naturally arises is, even if we can not *exactly* express it as an infinte power series, are there polynomials close to it. This question gains importance because for the above function, Taylor polynomials fail to give anything. In fact the Taylor expansion of the function (of any order) around zero will always give you $f(x) = f(x)$, simply because all derivatives are zero when $x = 0$ and thus only the 'error' term remains which then must equal $f(x)$. Yes, given any continuous function f on a closed bounded interval there are polynomials which are as close to f as you want.

We know the following. If a is a real number and $\epsilon > 0$, then there is a rational number r in the interval $(a - \epsilon, a + \epsilon)$. We shall now prove a similar theorem about continuous functions. if f is a continuous function on a closed bounded interval and $\epsilon > 0$ then there is a polynomial P whose graph lies in the band $(f - \epsilon, f + \epsilon)$. You may recall that band means the set of points $\{(x, y) : f(x) - \epsilon < y < f(x) + \epsilon)$. This is the region in the (x, y)-plane obtained by taking parallel graphs at distance ϵ above and below the graph of f. We came across the band while discussing uniform convergence of sequences of functions.

In other words, the role of real number is played by continuous function; role of interval is played by the class of functions whose graphs lie in the ϵ-band around graph of f as described above; the role of rational is played by usual high school polynomial.

To put it analytically, we can find a polynomial P such that $|f(x) - P(x)| < \epsilon$ for every x in the closed bounded interval. This is same as saying that there is a sequence of polynomials P_n which converge uniformly on the interval to f. Of course polynomials are defined on all of R, but we are not saying about the sequence of polynomials outside the interval $[0, 1]$. They may converge or may not converge.

This is very satisfying because it says that general continuous function is not too far from a polynomial. However, you should bear in mind, this does not mean any thing in terms of differentiability properties. A polynomial is differentiable at every point, whereas a continuous function need not be differentiable at any point what-so-ever. As a consequence, we can not even ask if the polynomials are related to the Taylor polynomials simply because the function, we started with, need not be differentiable.

This theorem is due to Weierstrass (as is the concept of uniform convergence of sequence of functions). But the proof we give is due to Bernstein. Usually, this theorem is not done in a first course of Calculus. The reason why I am doing is the following. first, it is satisfying and has a clear intuitive meaning devoid of complicated maths. secondly, it reassures us that the continuous functions we have dveloped are really not too far from polynomials which you have learnt in high school (as long as you are thinking of a closed bounded interval). Also the theorem has a proof using only high school algebra, again devoid of any difficult maths. And it is important from analysis point of view. It also naturally fits in with the questions raised in connection with Taylor expansion.

Theorem: Let f be a real valued continuous function on $[0, 1]$ and $\epsilon > 0$. Then there is a polynomial P such that $|f(x) - P(x)| < \epsilon$ for every $x \in [0, 1]$.

First we observe an important property of continuous functions defined on $[0, 1]$.

Fact: f is a continuous function on $[0, 1]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ and $x, y \in [0, 1]$.

When we say that a function is continuus, we just mean that it is continuous at every point a. In turn, when we say that it is continuous at a, we mean that given $\epsilon > 0$ there is a $\delta > 0$ (which could depend not only on ϵ but also on the point a), so that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. What we have said above is different. As soon as $\epsilon > 0$ is given, there is a $\delta > 0$ that works for every point a; in the sense, now if you take any point a and any x with $|x - a| < \delta$ then you will have $|f(x) - f(a)| < \epsilon$. Because of this, this property of continuous function is called *uniform continuity*.

Just to bring home the point, let us consider the function $f(x) = x^2$ on all of R. Let us try to verify continuity with $\epsilon = 1$. if our point is $a = 0$, then you can take $\delta = 1$ (you must verify this statement). if our point is $a = 10$ then $\delta = 1$ will not do, you must choose smaller δ, say $\delta = 1/21$. If our point is $a = 100$ then this δ also will not do, you need to choose a much much smaller δ. It is not difficult to see that as the point a gets larger and larger, you need to choose smaller and smaller δ which approaches zero as a gets larger and larger. in other words, you can not choose one $\delta > 0$ that works for all a. Remember $\epsilon = 1$ is fixed for all this discussion. what the fact above says is that such a thing can not happen if you had, instead of R, a closed bounded interval.

The proof of the fact is simple. If possible, fix an $\epsilon > 0$ for which we can not find a $\delta > 0$. Thus $\delta = 1/2$ or $1/2^2$ or $1/2^3$ etc will not serve our requirement. thus for each $n = 1, 2, 3 \cdots$ we ca find two points x_n, y_n so that $|x_n - y_n| < 1/2^n$ yet $|f(x_n) - f(y_n)| \geq \epsilon$. The sequence (x_n) being bounded, there is a subsequence $\{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\}$ which converges to a point a, but then $y_i = (y_i - x_i) + x_i$ tells us that $\{y_{n_1}, y_{n_2}, y_{n_3}, \cdots\}$ also converges to a, but then the sequence $f(x_{n_1}) - f(y_{n_1}), f(x_{n_2}) - f(y_{n_2}), f(x_{n_3}) - f(y_{n_1}), \cdots$ converges to $f(a) - f(a) = 0$ whereas each term of the sequence is larger than ϵ in modulus. this is a contradiction. This completes the proof.

to proceed to the proof of the polynomial approximation theorem, we make a few observations. Through out below, just for these calculations, we take $x^0 = 1$ when $x = 0$ — just for now.

$$\sum_0^n \binom{n}{k} x^k(1-x)^{n-k} = 1; \qquad n \geq 1. \qquad (\spadesuit)$$

This is simply the binomial expansion for $(x+1-x)^n$.

$$\sum_0^n k\binom{n}{k} x^k(1-x)^{n-k} = nx; \qquad n \geq 1.$$

You can sum from $k = 1$ onward; then you can write $k\binom{n}{k} = n\binom{n-1}{k-1}$; then take nx outside the sum. recognize binomial expansion of $(x+1-x)^{n-1}$.

$$\sum_0^n k(k-1)\binom{n}{k} x^k(1-x)^{n-k} = n(n-1)x^2; \qquad n \geq 1.$$

Again, it suffices to sum only for $k \geq 2$; write $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$ etc.

$$\sum_0^n (k-nx)^2\binom{n}{k} x^k(1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}; \qquad n \geq 1.$$

If we write $(k-nx)^2$ as the sum of four terms $k(k-1)+k+n^2x^2-2nxk$ and simplify the four sums using the earlier equations. The last inequality is clear becaue for $x \in [0,1]$ we have $\sqrt{x(1-x)} \leq [x+(1-x)]/2$.

We now define the Bernstein polynomials associated with a continuous function f on the interval $[0,1]$.

$$P_n(x) = \sum_0^n f(k/n)\binom{n}{k} x^k(1-x)^{n-k}; \quad x \in [0,1]; \quad n \geq 1.$$

This is clearly a polynomial in x. The values of the function f at certain points appear in the definition of these polynomials.

The motivation for looking at such polynomials occurs from Probability theory. Roughly, the reason why this polynomial should be close to f is the following. The binomial coefficients $\binom{k}{n}x^k(1-x)^{n-k}$ increase as k increases from zero to near nx, reaching a maximum value near nx and then start decreasing. Also these coefficients are nearly zero at the tails. Thus in the above sum, $f(k/n)$ gets maximum weight whenever k is near nx. In other words values of f near x gets high weight in the above averaging and values away from x get weight very close to zero. Thus the average is close to $f(x)$.

To prove the theorem, fix $\epsilon > 0$. We show an N such that P_N satisfies the requirement. This is done as follows. First fix $\delta > 0$ so that $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| < \delta$, possible by uniform continuity of f. Fix a number C so that $|f(x)| < C$ for every $x \in [0, 1]$, possible because continuous function on a closed bounded interval is bounded. Finally, fix integer $N > (C/\delta^2\epsilon)$. This will do. Actually, we show, $|f(x) - P_n(x)| < \epsilon$ for every $n \geq N$ and every $x \in [0, 1]$.

From (♠) we see

$$f(x) = \sum_0^n \binom{n}{k} f(x) x^k (1-x)^{n-k},$$

so that

$$|f(x) - P_n(x)| = |\sum_k [f(x) - f(k/n)] \binom{n}{k} x^k (1-x)^{n-k}|$$

$$\leq \sum_k |f(x) - f(k/n)| \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum\nolimits' |f(x) - f(k/n)| \binom{n}{k} x^k (1-x)^{n-k}$$

$$+ \sum\nolimits'' |f(x) - f(k/n)| \binom{n}{k} x^k (1-x)^{n-k}$$

Where $\sum'$ is sum over $\{k : |x - \frac{k}{n}| \leq \delta\}$. But by choice of δ this sum is at most $(\epsilon/2)$ times the sum of the binomial coefficients. Hence $\sum' \leq \epsilon/2$.

The sum $\sum''$ is over $\{k : |x - \frac{k}{n}| > \delta\}$. But by choice of C this sum is at most $2C$ times sum of the binomial coefficients. For every k in this sum, we have $|nx - k| > n\delta$ so that

$$\sum\nolimits'' |f(x) - f(k/n)| \binom{n}{k} x^k (1-x)^{n-k}$$

$$\le 2C{\sum}'' \binom{n}{k} x^k(1-x)^{n-k}$$

$$\le 2C{\sum}'' \frac{(k-nx)^2}{n^2\delta^2}\binom{n}{k} x^k(1-x)^{n-k}$$

$$\le \frac{2C}{n^2\delta^2}\frac{n}{4} = \frac{1}{2}\frac{C}{n\delta^2} = \frac{\epsilon}{2}.$$

This completes the proof of the theorem.

Why did we take the interval $[0,1]$? Just to conveniently describe the polynomials. Any closed bounded interval is as good. More precisely, let f be a continuous function on a closed bounded interval $[a,b]$ and $\epsilon > 0$. Then there is a polynomial P such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon.$$

This is observed by changing the action to the unit interval and coming back as follows. define $g(x) = f(a+[b-a]x)$ on $[0,1]$. This is continuous, get a polynomial Q so that $|g(x) - Q(x)| < \epsilon$ for all $x \in [0,1]$. Then $P(x) = Q([x-a]/[b-a])$ is a polynomial and serves our purpose. in fact $f(x) - P(x) = g([x-a]/[b-a]) - Q([x-a]/[b-a])$.

Can we do on the real line? That is, given a continuous function f on R and $\epsilon > 0$, can we find a polynomial P so that $|f(x) - P(x)| < \epsilon$ for all $x \in R$. This is false in general. In fact take $f(x) = \sin x$ and $\epsilon = 1/4$. if you take a constant polynomial, it will not do because f takes values zero as well as one. If you take a non-constant polynomial, then it is not bounded.

L'Hopital revisited:

We shall discuss two issues connected with L'Hopital rule. Recall, it tells us that if $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, and if $f'(x)/g'(x) \to \alpha$, then $f(x)/g(x) \to \alpha$. This is the $0/0$ form. We shall now show that the same result holds even for ∞/∞ form.

L'Hopital's rule: Let f and g be two differentiable functions on (a,b) such that $\lim f(x)$ as well as $\lim g(x)$ are ∞ as $x \to a$; and $g'(x) \neq 0$ on (a,b) and

$\lim f'(x)/g'(x) \to \alpha$ as $x \to a$. Then $\lim f(x)/g(x) \to \alpha$ as $x \to a$.

The proof proceeds along the same lines as the 0/0 case, but a little more involved.

We treat the case $\alpha \in R$. Other cases, namely, $\alpha = \infty$ and $\alpha = -\infty$ are similar. So let $\epsilon > 0$ be given. Need to show a number $\delta > 0$ so that

$$x \in (a, a+\delta) \Rightarrow |\frac{f(x)}{g(x)} - \alpha| < \epsilon.$$

As earlier, using the hypothesis, fix $\delta_1 > 0$

$$x \in (a, a+\delta_1) \Rightarrow \alpha - \frac{\epsilon}{4} < \frac{f'(x)}{g'(x)} < \alpha + \frac{\epsilon}{4}.$$

Let us fix a number y with $a < y < a + \delta_1$ (sort of a reference point and will not be changed in our calculations from now). The generalised MVT implies

$$a < x < y \Rightarrow \alpha - \frac{\epsilon}{4} < \frac{f(x) - f(y)}{g(x) - g(y)} < \alpha + \frac{\epsilon}{4}.$$

Take δ_2 so that $0 < \delta_2 < \delta_1$ and $g(x) > \max\{g(y), 0\}$ for $x \in (a, a+\delta_2)$. This is possible because $g(x) \to \infty$ as $x \to a$. Now let us take any $x \in (a, a+\delta_2)$ and multiply the above inequality by the positive number $[g(x) - g(y)]/g(x)$. Thus for $a < x < a + \delta_2$ we have

$$\left(\alpha - \frac{\epsilon}{4}\right) \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)} < \frac{f(x)}{g(x)} < \left(\alpha + \frac{\epsilon}{4}\right) \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)}.$$

Let us observe that $[g(x) - g(y)]/g(x) \to 1$ as $x \to a$ because y is fixed and $g(x) \to \infty$. Thus we can choose δ_3 so that $0 < \delta_3 < \delta_2$ and for $a < x < a+\delta_3$

$$\left(\alpha - \frac{\epsilon}{4}\right) \frac{g(x) - g(y)}{g(x)} > \alpha - \frac{\epsilon}{2}; \qquad \left(\alpha + \frac{\epsilon}{4}\right) \frac{g(x) - g(y)}{g(x)} < \alpha + \frac{\epsilon}{2}.$$

Thus for

$$a < x < a + \delta_3 \Rightarrow \left(\alpha - \frac{\epsilon}{2}\right) + \frac{f(y)}{g(x)} < \frac{f(x)}{g(x)} < \left(\alpha + \frac{\epsilon}{2}\right) + \frac{f(y)}{g(x)}.$$

Since $f(y)/g(x) \to 0$ as $x \to a$, choose δ_4 so that $0 < \delta_4 < \delta_3$ and for $x \in (a, a + \delta_4)$ this ratio is between $-\epsilon/2$ and $+\epsilon/2$. Thus we have

$$a < x < a + \delta_4 \Rightarrow \alpha - \epsilon < \frac{f(x)}{g(x)} < \alpha + \epsilon.$$

This completes proof of the rule.

The second issue related to L'Hopital's rule is the following. Let us again consider the 0/0 case. What if f' and g' also converge to zero at a? The answer is that we can try second derivatives and so on.

Fact: Suppose f and g are $(n-1)$-times continuously differentiable on an interval $[a, b]$ and f as well as g and all their first $(n-1)$ derivatives are zero at a. Suppose $f^{(n)}$ and $g^{(n)}$ exist in the open interval; $g^{(n)}(x) \neq 0$ for all $x \in (a, b)$ and $f^{(n)}(x)/g^{(n)}(x) \to \alpha$ as $x \downarrow a$. then $f(x)/g(x) \to \alpha$ as $x \downarrow a$.

As suggested by Pranav, you can use the earlier case repeatedly as follows. First observe that $g^{(n)}$ never takes the value zero, so that $g^{(n-1)}(x)$ can be zero for at most one value of x in (a, b) — use mean value theorem. Say it is nonzero in (a, b_1). Repeat this argument to see $g^{(n-2)}$ is non-zero in (a, b_2) and so on; finally getting an interval (a, β) where all these derivatives of g are non-zero and consider only this interval in what follows. This is alright because we are interested as $x \to a$.

Applying earlier version of L'Hopital successively to $f^{(k)}; g^{(k)}$ with $k = n-1, n-2, \cdots 1$ deduce

$$\lim_{x \downarrow a} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x \downarrow a} \frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} = \lim_{x \downarrow a} \frac{f^{(n-2)}(x)}{g^{(n-2)}(x)} = \cdots = \lim_{x \downarrow a} \frac{f(x)}{g(x)}.$$

Here is another method when α is finite. This depends on a generalization of MVT with higher order derivatives or Taylor for two functions.

Fact: f, g are $n-1$ times continuously differentiable in the interval $[a, b]$ and $f^{(n)}$, $g^{(n)}$ exist in the interval (a, b). There is a number $\theta \in (a, b)$ such that

$$\left[f(b) - \sum_{0}^{n-1} f^{(k)}(a) \frac{(b-a)^k}{k!} \right] g^{(n)}(\theta) = \left[g(b) - \sum_{0}^{n-1} g^{(k)}(a) \frac{(b-a)^k}{k!} \right] f^{(n)}(\theta).$$

Proof consists of applying the earlier version to the following functions.

$$F(x) = f(x) + \sum_{1}^{n-1} \frac{f^{(k)}(x)}{k!}(b-x)^k.$$

$$G(x) = g(x) + \sum_{1}^{n-1} \frac{g^{(k)}(x)}{k!}(b-x)^k.$$

These are continuous on $[a, b]$ and differentiable in (a, b) so there is $\theta \in (a, b)$ such that

$$[F(b) - F(a)]G'(\theta) = [G(b) - G(a)F'(\theta). \qquad (\clubsuit)$$

$$F(b) - F(a) = f(b) + 0 - f(a) - \sum_{1}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k$$

$$= f(b) - \sum_{0}^{n-1} f^{(k)}(a)\frac{(b-a)^k}{k!} \qquad (*)$$

For $x \in (a, b)$

$$G'(x) = g'(x) + \sum_{1}^{n-1} \frac{g^{(k+1)}(x)}{k!}(b-x)^k - \sum_{1}^{n-1} \frac{g^{(k)}(x)}{(k-1)!}(b-x)^{k-1}$$

$$= g^{(n)}(x)\frac{(b-a)^{n-1}}{(n-1)!}. \qquad (**)$$

Similarly,

$$G(b) - G(a) = g(b) - \sum_{0}^{n-1} g^{(k)}(a)\frac{(b-a)^k}{k!} \qquad (\dagger)$$

and

$$F'(x) = f^{(n)}(x)\frac{(b-a)^{n-1}}{(n-1)!}. \qquad (\dagger\dagger)$$

Substituting $(*), (**), (\dagger), (\dagger\dagger)$ in $(\clubsuit)$ we get the result.

Now the proof of L'Hopital goes exactly as in the earlier case. Fix $\epsilon > 0$. Need to show $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \alpha - \epsilon < \frac{f(x)}{g(x)} < \alpha + \epsilon.$$

Fix, using hypothesis, $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \alpha - \epsilon < \frac{f^{(n)}(x)}{g^{(n)}(x)} < \alpha + \epsilon.$$

The same δ would do. take any $a < y < x < a + \delta$ apply the above MVT on the interval $[y, x]$ and let $y \downarrow a$. This completes the alternative proof.

Newton's algorithm for zero:

We shall discuss just one inteersting computational application of the Taylor formula. It is to find a zero of a function. Shall present the most primitive version of an algorithm Newton discovered.

Suppose we have a function $f : R \to R$. Let z be a zero of f, that is, $f(z) = 0$. Find an algorithm to calculate z with required degree of accuracy, if we know roughly where it is located.

Start with a point z_0. This is your initial or starting approximation to z (and hence, in general, you need to start close to z). Draw tangent to the curve, graph of f, at $(z_0, f(z_0))$. Suppose it cuts the x-axis at z_1 (so you need to assume that tangent is not parallel to the x-axis). This is your first approximation to z. Then draw tangent to the curve at $(z_1, f(z_1))$ Suppose that it cuts the x-axis at z_2. This is your second approximation. Continue the process. The hope is you are heading towards to z, the actual zero.

What makes us hope so? Well, as is always the case, look at some examples. For instance consider the curve $f(x) = x^2 - 2$ and take $z_0 = 1$ and try. Take the curve $f(x) = x^3 - 2$ and try again. of course, the fact that ti works in the examples we have seen is not good enough to believe that this is always true. In fact it is not always true that these numbers z_n so obtained converge to the actual zero. But, under fairly general conditions they do converge.

Let us see what the algiorithm says. The tangent at $(z_0, f(z_0))$ to the curve is given by

$$y - f(z_0) = f'(z_0)(x - z_0).$$

To find the point of intersection with the x-axis, set $y = 0$ to see

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)}.$$

To get z_2 you repeat the same formula above. Here is the precise fact.

Suppose that f is a twice differentiable function in an open interval I. Suppose that $f'(x)$ is never zero in this interval. Assume that

$$\frac{\sup |f''(x)|}{\inf |f'(x)|} = \alpha < \infty.$$

Let z be a point in I with $f(z) = 0$.
Start with a point $z_0 \in I$ with $\alpha|z - z_0| < 1$. Define recursively,

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}.$$

Then each of the points z_n is in the interval I and $z_n \to z$.

Here is the proof. Suppose $z_n \in I$. By Taylor expansion around z_n, there is a point θ between z_n and z such that

$$f(z) = f(z_n) + f'(z_n)(z - z_n) + \frac{1}{2}f''(\theta)(z - z_n)^2.$$

Since $f(z) = 0$ we gets

$$f(z_n) = -f'(z_n)(z - z_n) - \frac{1}{2}f''(\theta)(z - z_n)^2.$$

Hence

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} = z_n + (z - z_n) + \frac{1}{2}\frac{f''(\theta)}{f'(z_n)}(z - z_n)^2.$$

$$|z_{n+1} - z| = |\frac{1}{2}\frac{f''(\theta)}{f'(z_n)}(z - z_n)^2.| \le \frac{1}{2}\alpha|z_n - z|^2$$

In particular

$$|z_1 - z| \le \frac{1}{2}\{\alpha|z_0 - z|\}|z_0 - z| < \frac{1}{2}|z_0 - z|.$$

showing that $z_1 \in I$ and in fact closer to z than z_0. In particular $\alpha|z_1 - z| < 1$. Thus

$$|z_2 - z| \le \frac{1}{2}\{\alpha|z_1 - z|\}|z_1 - z| < \frac{1}{2}|z_1 - z|. \le \frac{1}{2^2}|z_0 - z|.$$

By induction, you can conclude the following for each n:

$$z_n \in I; \qquad \alpha|z_{n-1} - z| < 1; \qquad |z_n - z| \le \left(\frac{1}{2}\right)^n |z_0 - z|.$$

This completes proof of the assertion made. In fact the convergence of the approximations is 'geometric'.

Integration:

Here is the problem, familiar from high school. How do you calculate areas. of course, if we have a rectangular region, we some how seem sure and agreed upon that its area is product of lengths of sides. Thus areas of rectangles are taken as known. Of course, from this we some how built up several other areas, for example area of a circle of radius r equals πr^2. We also know how to calculate areas of triangles and some other figures like parallelogram, trapezium etc.

Let us consider a function f defined on the interval $[0, 1]$. As a first step, let us assume that the function takes only non-negative values, so that the graph of the function is above the x-axis. We also assume that the function is bounded. Consider the region under the curve, bounded below by the x-axis. On the sides it is bounded by the vertical lines at $x = 0$ and $x = 1$. Thus to the left, the region is bounded by the y-axis and to the right, it is bounded by the vertical line at $x = 1$.

Analytically, it is the region $\{(x, y) : 0 \le x \le 1, 0 \le y \le f(x)\}$.

How nice if the function is a constant $f \equiv c > 0$. Then the graph of f is just a flat horizontal line at height c and thus the region is just a rectangle with sides of lengths one and c. Its area is c. Suppose the function is piece-wise constant, say, on $[0, 1/4]$ value of f is α; on $(1/4, 1/3)$ value of f is β and finally on $[1/3, 1]$ value of f is γ. Then the region under consideration consists of three rectangles and so the 'total' area of the region equals

$\alpha(1/4) + \beta(1/12) + \gamma(2/3)$.

If the function is not piece-wise constant, then the region is, of course, not made up of finitely many rectangles. Let us denote the area by A. The plan is to find a number which bounds A from below and to find a number which bounds A from above. For instance, suppose that there are reasons to believe that the unknown area A must at least be 3 and also can be at most 3. Then you must agree that the area must equal 3, neither more nor less. Of course, if you can only conclude that the area must at least be 3 and can not exceed 4, then we are sure that it must be a number between 3 and 4; but not yet sure what exactly it should equal.

If you are trying to see a lower bound for the area, the best way is to fit in non-overlaping rectangles in our region and take their total area. this can be done in several ways. what is the most efficient way of doing it? here is what Riemann's idea is. Break up the interval $[0, 1]$ into finitely many intervals, say $[0, a_1]$, $[a_1, a_2]$ $\cdots$ $[a_{55}, 1]$ — 56 pieces. On each piece, let us calculate the infimum of the function. Let them be $m_0, m_1, \cdots, m_{55}$. Consider the rectangles with base $[a_i, a_{i+1}]$ and height m_i for $i = 0, 1, 2, \cdots, 55$. Clearly these rectangles are non-overlapping (have common sides) and are all within the region of our interest. Thus the area A must be at least sum of the areas of theses rectangles.

For every break up of the interval $[0, 1]$ we calculate a number, namely, sum of the areas of the rectangles constructed as above, by taking infimum of the function in each of the intervals. Let L be the set of numbers so obtained. Thus, we must agree to the following. Whatever be the area A, we must have $\alpha \leq A$ for each $\alpha \in L$. ln other words A is an upper bound for the set L. Let l be the least upper bound of the set L, that is, supremum of L. Thus we have $l \leq A$.

Let us use the same notation as in the earlier para, and now let us consider the supremum of the function in each of the intervals, say, $M_0, M_1, \cdots, M_{55}$. Consider the rectangles with base $[a_i, a_{i+1}]$ and height M_i for $i = 0, 1, 2, \cdots, 55$. Clearly these rectangles are non-overlapping (have common sides) and all the rectangles put together include the region of our interest. Thus the area A can at most be sum of the areas of these rectangles.

For every break up of the interval $[0, 1]$ we calculate a number, namely, sum of the areas of the rectangles constructed as above, by taking supremum of the function in each of the intervals. Let U be the set of numbers so obtained. Thus, we must agree to the following. Whatever be the area A, we must have $A \leq \alpha$ for each $\alpha \in U$. ln other words A is a lower bound for the set U. Let u be the infimum or the glb of the set U. Thus we have $A \leq u$.

Thus we have calculated two numbers l and u and we are sure that $l \leq A \leq u$. In particular, if we are lucky and it so turns out that $l = u$ then this *must* be the area of the region and there is nothing for us to decide. What is amazing is that this equality holds in many cases. Thus this intuitive algorithm leads to an answer for the concept of area in many situations.

Of course, it does not lead to an answer in many other cases. For example, if our function f is given by: $f(x)$ equals one or zero according as x is rational or not. Then L consists of one number, namely 0; whereas U consists of one number 1. thus $l = 0$ and $u = 1$, unequal.

We shall now start with definitions execute the above idea. In what follows $[a, b]$ is a closed bounded interval. By a partition of this interval we mean a finite set of points $a = a_0 < a_1 < a_2 < \cdots, a_k = b$. We denote partition by P. Thus partition is just a finite subset of $[a, b]$ and $a \in P$ and $b \in P$. Of course a set consists of only points and there is nothing like first or second element in the set. When we think of a partition, we keep the order also in mind when we picturize it. A partition as above breaks the interval into finitely many intervals, namely, $[a, a_1]$, $[a_1, a_2]$, $\cdots$, $[a_{k-1}, b]$. Sometimes we refer to these intervals as intervals of the partition.

If P_1 and P_2 are two partitions, we say P_2 is finer than P_1 if every point of P_1 is also in P_2, that is, $P_1 \subset P_2$. We write $P_1 \leq P_2$. Thus, for example if the interval is $[0, 1]$ then

$$\{0, 1/2, 1\}; \quad \{0, 1/4, 1/3, 1/2, 1\}; \quad \{0, 1/8, 3/8, 1\}$$

are all partitions. The second one is a refinement of the first one. The third one is not comparable to the other two. However we can think of a partition that refines both the second and third, namely

$$\{0, 1/8, 1/4, 1/3, 3/8, 1/2, 1\}.$$

In general refinement of two partitions P_1 and P_2 is just $P_1 \cup P_2$, of course, you need to arrange the points in increasing order when you picturize. This is denoted by $P_1 \vee P_2$

Alternatively, you can define a partition as a finite increasing subset of $[a, b]$ by bringing in the order also as part of the definition. Then of course common refinement would consist of increasing arrangement of the points of both taken together.

Now let f be a bounded real valued function on $[a, b]$. For a partition $P = \{a = a_0 < a_1 < a_2 < \cdots < a_k = b\}$, we define

$$U(P, f) = \sum_{0}^{k-1} M_i(a_{i+1} - a_i); \qquad M_i = \sup\{f(x) : a_i \leq x \leq a_{i+1}\}.$$

$$L(P, f) = \sum_{0}^{k-1} m_i(a_{i+1} - a_i); \qquad m_i = \inf\{f(x) : a_i \leq x \leq a_{i+1}\}.$$

$U(P, f)$ is the upper Riemann sum for the partition P and $L(P, f)$ is the lower Riemann sum for the partition P. We put

$$U(f) = \inf_P U(P, f); \qquad L(f) = \sup_P L(P, f).$$

In the above, the inf and sup is over all the partitions of the interval $[a, b]$.

We say that f is *Riemann integrable* if $U(f) = L(f)$ and then define integral of f as this common value $U(f) = L(f)$. The integral is denoted $\int_a^b f(x)dx$. Convince yourself that this is precisely what we thought at the beginning.

Here are some facts regarding the Riemann sums and integrals.

1: For any partition P, $L(P,f) \leq U(P,f)$
This follows from the fact $m_i \leq M_i$ for each i.

2. If $P_1 \leq P_2$, then $L(P_1,f) \leq L(P_2,f)$ and $U(P_1,f) \geq U(P_2,f)$.

First suppose that P_2 has only one extra point than P_1; say

$$P_1 = \{a = a_0 < a_1 < \cdots < a_k = b\}.$$

Suppose P_2 has one extra point α; $a_j < \alpha < a_{j+1}$. Let

$$m' = \inf\{f(x) : a_j \leq x \leq \alpha\}; \quad m'' = \inf\{f(x) : \alpha \leq x \leq a_{j+1}\}.$$

Then m_j being $\inf\{f(x) : a_j \leq x \leq a_{j+1}\}$ we see $m_j \leq m'$ and $m_j \leq m''$; as the set gets larger, inf gets smaller. Thus

$$\begin{aligned} m_j(a_{j+1} - a_j) &= m_j(\alpha - a_j) + m_j(a_{j+1} - \alpha) \\ &\leq m'(\alpha - a_j) + m''(a_{j+1} - \alpha). \end{aligned}$$

Observe that the only difference between $L(P_1,f)$ and $L(P_2,f)$ is the following. The term $m_j(a_{j+1}-a_j)$ appearing in $L(P_1,f)$ is replaced by the right side above in $L(P_2,f)$. Thus the above inequality shows that $L(P_1,f) \leq L(P_2,f)$. Similarly if

$$M' = \sup\{f(x) : a_j \leq x \leq \alpha\}; \quad M'' = \sup\{f(x) : \alpha \leq x \leq a_{j+1}\}.$$

then we see $M' \leq M_j$ and $M'' \leq M_j$ leading to the inequality $U(P_2,f) \leq U(P_1,f)$.

Clearly, by induction, the inequalities follow if P_2 has k extra points than P_1; $k = 1, 2, \cdots$ This completes the proof.

3. For any two partitions P_1 and P_2, we have $L(P_1,f) \leq U(P_2,f)$.
In fact if $P = P_1 \vee P_2$, their refinement, we see using the two facts above

$$L(P_1,f) \leq L(P,f) \leq U(P,f) \leq U(P_2,f).$$

4. $L(f) \leq U(f)$.
The earlier fact says that every number $L(P,f)$ is a lower bound for the set

$S = \{U(Q,f) : Q \text{ partition of } [a,b]\}$. Thus $L(f)$ is a lower bound for S. But $U(f)$ is the glb of S. Hence $L(f) \le U(f)$.

5. If $\alpha \le f(x) \le \beta$ for all $x \in [a,b]$, then for each partition P, $\alpha(b-a) \le L(P,f)$ and $U(P,f) \le \beta(b-a)$.

Just note that $\alpha \le m_j \le M_j \le \beta$ for each j.

6. If $\alpha \le f(x) \le \beta$ for all $x \in [a,b]$, then

$$\alpha(b-a) \le L(f) \le U(f) \le \beta(b-a).$$

This follows from the above.

7. Every continuous function on $[a,b]$ is integrable.
We need to show that $L(f) = U(f)$. Since $L(f) \le U(f)$ always, we only need to show $U(f) \le L(f)$. Fix $\epsilon > 0$. We show a partition P so that $U(P,f) - L(P,f) < \epsilon$ Then it follows that

$$U(f) - L(f) \le U(P,f) - L(P,f) \le \epsilon$$

Since $\epsilon > 0$ is arbitrary, it follows that $U(f) \le L(f)$ as required.

Use uniform continuity of f on $[a,b]$ to get $\delta > 0$ so that

$$x, y \in [a,b]; |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Now take any partition

$$P = \{a = a_0 < a_1 < \cdots < a_k = b\}$$

such that each interval of the partition has length smaller than δ. There are such partitions. For example, $a, a+(\delta/2), a+2(\delta/2), \cdots$. Of course, when $k(\delta/2)$ exceeds b, you stop and take the last point as b. If x and y are in $[a_j, a_{j+1}]$ we see $|f(x) - f(y)| < \epsilon/(b-a)$ and hence $M_j - m_j < \epsilon/(b-a)$. In fact, f being continuous, you can take x and y to be the points in the interval $[a_j, a_{j+1}]$ where the max and min are attained. Thus

$$U(P,f) - L(P,f) = \sum (M_j - m_j)(a_{j+1} - a_j)$$

$$\leq \frac{\epsilon}{b-a} \sum (a_{j+1} - a_j) = \epsilon.$$

Thus we see that a large class of functions, namely continuous functions, are integrable.

We shall continue our discussion of properties of integrals and how to calculate integrals. We are only considering bounded functions on a closed bounded interval. We showed that every continuous function is integrable.This we did by showing that for every $\epsilon > 0$, there is a partition P such that $U(P, f) - L(P, f) < \epsilon$. Actually we observed a more precise property.

8. Let f be a continuous function on $[a, b]$. Given $\epsilon > 0$, there is a $\delta > 0$ with the following property: whenever we take any partition with difference between successive points smaller than δ, then $U(P, f) - L(P, f) < \epsilon$.

Let us introduce a word that reduces our writing. Given a partition $P = \{a = a_0 < a_1 < a_2 < \cdots < a_k = b\}$, the maximum distance between successive points, $\max_i(a_{i+1} - a_i)$ is denoted by $||P||$, *norm* of P. A *selection* for a partition is simply a selection of points from each partition interval; more precisely, a (finite) sequence of points $s = \{x_0 \leq x_1 \leq x_2 \leq \cdots x_{k-1}\}$ such that $x_i \in [a_i, a_{i+1}]$ for $i = 0, 1, 2, \cdots, k - 1$. Given any interval, there are several possible partitions of the interval. Given one partition, there are several possible selections for the partition; we can pick any one point from each interval of the partition.

Given a partition P and a selection s for the partition, we define

$$R(P, f, s) = \sum_{i=0}^{k-1} f(x_i)[a_{i+1} - a_i].$$

This is called the Riemann sum for the partition and selection. Recall that, instead of value of the function at the selected point, if we used the infimum and supremum in each partition interval we get the lower and upper Riemann sums.

We denote integral of f over the interval $[a, b]$ by $\int\limits_a^b f(x)dx$ or $\int\limits_a^b f$ or simply $\int f$ if the interval is clear from the context.

9. Let f be a continuous function on $[a, b]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $|R(P, f, s) - \int f| < \epsilon$ for any partition P with $||P|| < \delta$ and for any selection s for the partition.

We only need to observe that both $\int f$ and $R(P, f, s)$ are between $L(P, f)$ and $U(P, f)$. So the same δ as above would do.

10. Let $\{P_n\}$ be a sequence of partitions of $[a, b]$ with $||P_n|| \to 0$ and for each n, let s_n be a selection for the partition P_n. Then for any continuous function f on $[a, b]$ we have:

$$U(P_n, f) \to \int f; \qquad L(P_n, f) \to \int f.; \qquad R(P_n, f, s_n) \to \int f$$

This is clear from the previous statement. Thus even though the appearance of selection appears an extra complication, you should keep in mind that it is one more choice at our disposal and some times some one (like mean value theorem) may already make a choice for us. You will see this in the fundamental theorem of integral calculus.

Though we are concentrating on continuous functions now, one naturally wonders whether there are functions which are not continuous but integrable. the answer is yes. The evidence is also easy to get. Just consider the function $f(x) = 1$ for $0 < x < 1$; and $f(0) = 54, f(1) = 1/2$. This function is integrable. In fact lower sums and upper sums are easily calculable and they yield $U(f) = 1 = L(f)$.

But what is not easy to answer is the following question: what precisely are the functions which are integrable? The answer roughly is that f is integrable when and only when its set of discontinuities is 'small'. This is an important issue but but shall not enter this discussion now. It is more important and basic to see how to calculate integrals and how to use integrals to our benefit. However, some of the observations we made above are true without assuming that we have a continuous function. Here is an example whose proof is easy.

11. Let f be bounded function on $[a, b]$. Then f is integrable iff for every $\epsilon > 0$, there is a partition P such that $U(p, f) - L(P, f) < \epsilon$; or equivalently,

for every $\epsilon > 0$ there is a partition P such that $|R(P, f, s) - R(P, f, s')| < \epsilon$ for any two selections s and s'.

This is immediate from the following simple fact. Suppose we have two sets S_1 and S_2. Suppose that $a \le b$ for every $a \in S_1$ and every $b \in S_2$. Then $\sup S_1 = \inf S_2$ iff for every $\epsilon > 0$, there are $a \in S_1$ and $b \in S_2$ with $b - a < \epsilon$.

For some of the statements below, continuity of the functions is not needed, integrability is enough; of course proofs have to be done with more care. But, as mentioned earlier, we shall not complicate life now.

11. if f and g are continuous, then

$$\int (f+g) = \int f + \int g; \qquad \int (39f) = 39 \int f.$$

if $f \equiv 28$ on $[a, b]$, then $\int\limits_a^b f = 28(b-a)$.

12. If f is continuous on $[a, b]$ and $a < c < b$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Just take a sequence of partitions P_n such that $||P_n|| \to 0$ and each P_n includes the point c. Then the set $Q_n = \{x \in P_n : x \le c\}$ and the set $R_n = \{x \in P_n : x \ge c\}$ will constitute partitions of $[a, c]$ and $[c, b]$ respectively. Clearly

$$||Q_n|| \to 0; \qquad ||R_n|| \to 0; \qquad U(P_n f) = U(Q_n, f) + U(R_n, f)$$

and proof is completed by noting

$$U(P_n, f) \to \int_a^b f; \quad U(Q_n, f) \to \int_a^c f; \quad U(R_n, f) \to \int_c^b f.$$

13. for any continuous function $|\int f| \le \int |f|$.
Note that $|f|$ is again a continuous function and hence its integral makes sense. proof is simple because for any partition

$$|U(P, f)| \le U(P, |f|).$$

14. **Fundamental Theorem of Integral Calculus:**
let f be a continuous function on $[a, b]$.
(i) Define $F(a) = 0$ and $F(x) = \int\limits_a^x f$ for $a < x \leq b$.
Then F is continuous on $[a, b]$; it is differentiable; $F' = f$.

(ii) Let G be any continuous function on $[a, b]$ which is differentiable on (a, b) and $G'(x) = f(x)$ for $a < x < b$. Then $\int\limits_a^b f = G(b) - G(a)$.

(iii) If G_1 and G_2 are two such functions as in (ii), then there is a number α such that $G_1(x) = \alpha + G_2(x)$ for every $x \in [a, b]$.

This is an extremely useful and powerful theorem. Firstly, it relates integration to derivatives. Secondly, it reduces our job of calculating integrals to finding functions whose derivative is the given function. This is easier than struggling with partitions, sups, infs, and limits.

Any function G as in (ii) is called a *primitive* for f. Part (i) says F is always a primitive, unfortunately, its definition depends on integration again. So it can not be used as a way to evaluate integrals. It is important and assures us of the existence of a primitive.

Sometimes only part (ii) is called fundamental theorem. Usually it is stated for integrable functions rather than only for continuous functions as we did above. For example if you take the function f which is one on $(0, 1)$ and our choice of numbers at $x = 0$ and $x = 1$; then f need not be continuous on $[0, 1]$ but the function $G(x) = x$ fits the bill.

Before proving this theorem, let us see two useful applications. The fundamental theorem makes it possible to translate theorems on derivatives to theorems on integrals, which help in calcualting integrals. We 'translate' the chain rule' and 'product rule' of differentiation.

15. Let φ be a strictly increasing continuously differentiable function on

$[a, b]$ onto $[c, d]$. Let f be a continuous function on $[c, d]$. then

$$\int_a^b f(\varphi(x))\varphi'(x)dx = \int_c^d f(y)dy.$$

since φ' is assumed to be continuous, the integrand on the left side is continuous and hence integral makes sense.

Thus integrating f on the interval $[c, d]$ is not simply integrating the composed function on $[a, b]$, but you need to multiply this composed function with φ'. the reason is that when you calculate Riemann sums on $[a, b]$ you multiply with length of partition interval. If $x_1 < x_2$ then length of the interval $[x_1, x_2]$ is $(x_2 - x_1)$. If $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$ then length of the image interval $[y_1, y_2]$ is $\varphi(x_2) - \varphi(x_1)$ which is $\varphi'(\theta)[x_2 - x_1]$ for some number θ between x_1 and x_2. Thus when an interval in domain is shifted to the range, the length gets magnified by a factor (the factor could be less than one).

In practice, the method above is implemented as follows. You need to evaluate an integral which you recognize as the left side. you say put $y = \varphi(x)$ so that $dy = \varphi'(x)dx$ and the left side becomes the right side after noting that $y = c$ when $x = a$ and $y = d$ when $x = b$.

You can state a similar theorem when φ is decreasing. We need to multiply with $|\varphi'(x)| = -\varphi'(x)$ instead of simply $\varphi'(x)$.

Proof of the formula is simple. Let F be a primitive for f on $[c, d]$. Thus it is differentiable and $F'(y) = f(y)$ for $c < y < d$. Set $G(x) = F(\varphi(x))$ on $[a, b]$. Clearly, G is continuous on $[a, b]$, differentiable on (a, b), and by chain rule,

$$G'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x).$$

Thus by fundamental theorem

$$\int_a^b f(\varphi(x))\varphi'(x)dx = G(b) - G(a) = F(d) - F(c) = \int_c^d f(y)dy.$$

We have taken a simple route by using the extra concept of primitive assured by the fundamental theorem. You should remember that, such a method of taking easy way out will not help us when life does get complicated. But can we prove it starting from definition of integral? Yes. Here is another proof, you may ignore if you wish.

For any partition P of $[a, b]$, let $\varphi(P)$ denote the partition of $[c, d]$ obtained by taking images of points in P. First observation is this: If $||P_n|| \to 0$, then $||\varphi(P_n)|| \to 0$. This is a simple consequence of uniform continuity of φ.

Let us temporarily name $g(x) = f(\varphi(x))\varphi'(x)$ defined on $[a, b]$. The second observation is the following. Given any partition P of $[a, b]$, there is a selection s for the partition P of $[a, b]$ so that

$$R(P, g, s) = R(\varphi(P), f, \varphi(s)).$$

First you should note that image of a selector is a selector for the image partition. To show the stated selector, if $[\alpha, \beta]$ is an partition interval for P, then by MVT, there is a γ in this interval so that

$$[\varphi(\beta) - \varphi(\alpha)] = \varphi'(\gamma)[\beta - \alpha].$$

Let s be the selection for the partition P so that from any partition interval $[\alpha, \beta]$ it picks a point γ satisfying the above equation from this interval. MVT assures there is at least one such point. Then

$$g(\gamma)[\beta - \alpha] = f(\varphi(\gamma))[\varphi(\beta) - \varphi(\alpha)]$$

It is clear that this selector will do the job.

Now to complete the proof is simple. Take a sequence of partitions P_n with $||P_n|| \to 0$. Then $||\varphi(P_n)|| \to 0$. For each n select s_n as described above. Then

$$R(P_n, g, s_n) = R(\varphi(P_n), f, \varphi(s_n)).$$

Here the left side converges to $\int g$ and right side to $\int f$ over the appropriate intervals.

This completes the alternative proof.

This is the most commonly used form of method of substitution. However, the fundamenttal theorem tells us better. Do not assume φ is strictly increasing. Just assume that it is a continuously differentiable on $[a, b]$ onto $[c, d]$ with $\varphi(a) = c$ and $\varphi(b) = d$. Then the proof using the fundamental theorem still holds good and hence the formula is still true.

Here is translation of product rule.

16. Let F and G be two continuously differentiable functions on $[a, b]$ with derivatives f and g. Then

$$\int_a^b F(x)g(x)dx = \{F(b)G(b) - F(a)G(a)\} - \int_a^b f(x)G(x)dx.$$

Proof is trivial. The function FG is continuous and its derivative equals $Fg + fG$ and so is a primitive for the later. By Fundamental theorem

$$F(b)G(b) - F(a)G(a) = \int (Fg + gF) = \int Fg + \int fG.$$

Let us now return to the proof of the fundamental theorem. Interestingly, the proof is straight forward.

(i) Shall show uniform continuity of F on $[a, b]$. Fix $\epsilon > 0$. Since f is continuous, it is bounded, say $|f(x)| < M$ for all x. Take $\delta = \epsilon/M$ Now take x, y with $|x - y| < \delta$. No loss to assume $x < y$, If $x = a$ then $F(a) = 0$ and so

$$|F(y) - F(a)| = |F(y)| = |\int_a^y f| \leq \int_a^y |f| \leq M(y - a) < \epsilon.$$

If $a < x < y$ our earlier observation tells

$$\int_a^y f = \int_a^x f + \int_x^y f; \quad i.e, \quad F(y) = F(x) + \int_x^y f;$$

so that

$$|F(y) - F(x)| = |\int_x^y f| \leq \int |f| \leq M(y - x) \leq \epsilon.$$

We show that F is differentiable and $F'(x) = f(x)$. Fix x and let us denote $f(x) = \alpha$. In what follows when we integrate α over an interval, it is understood that we are talking about the constant function identically equal to α on that interval. Also all the points appearing below are in the interval $[a.b]$

Let $\epsilon > 0$. We exhibit $\delta > 0$ so that

$$0 < y - x < \delta \Rightarrow |\frac{F(y) - F(x)}{y - x} - \alpha| < \epsilon.$$

and

$$0 < x - y < \delta \Rightarrow |\frac{F(y) - F(x)}{y - x} - \alpha| < \epsilon.$$

This will prove the stated result. Do not worry, if your point $x = b$ then first implication can not arise and when $x = a$ the second can not. We take $\delta > 0$ so that

$$y \in [a, b]; |y - x| < \delta \Rightarrow |f(y) - \alpha| < \epsilon.$$

Note that x is fixed and $f(x)$ is named α. So the above is possible by continuity of f. For any $y > x$, we have

$$F(y) - F(x) = \int_x^y f(t)dt; \quad \int_x^y \alpha dt = \alpha(y - x).$$

$$|\frac{F(y) - F(x)}{y - x} - \alpha| = |\int \frac{1}{y - x} \int_x^y [f(t) - \alpha]dt| \leq \frac{1}{y - x} \int_x^y |f(t) - \alpha|dt.$$

If $|y - x| < \delta$ then for every $t \in [x, y]$ we have $|t - x| < \delta$ so that the integrand above is at most ϵ and so the integral is at most $\epsilon(y - x)$ showing what we wanted.

Similar computation yields the result for $0 < x - y < \delta$.

This completes proof of (i).

Proof of (ii). First we observe the following. Given any partition P, there is a selection s so that $G(b) - G(a) = R(P, f, s)$. This will complete proof as follows. Take a sequence of partitions P_n with $||P_n|| \to 0$. For each n, get selector s_n for P_n as claimed above. Then proof is completed by noting

$$R(P_n, f, s_n) \to \int f; \qquad R(P_n, f, s_n) = F(b) - F(a) \quad \text{for all } n.$$

So let

$$P = \{a = a_0 < a_1 < a_2 < \cdots < a_k = b\}$$

and let us get selection s as claimed. For each i let $x_i \in (a_i, a_{i+1})$ be given by the MVT to satisfy

$$G(a_{i+1}) - G(a_i) = f(x_i)(a_{i+1} - a_i).$$

This is the selection s. then

$$G(b) - G(a) = \sum_{i=0}^{k-1} [G(a_{i+1}) - G(a_i)] = \sum f(x_i)(a_{i+1} - a_i) = R(P, f, s).$$

This completes proof of (ii)

Proof of (iii): suppose that there are two functions G_1 and G_2 having the same derivative f. Then $G_1 - G_2$ has zero derivative and hence is a constant in the interval (a, b). Since $G_1 - G_2$ is continuous on the interval $[a, b]$ and is a constant in the interval (a, b) it must be constant in the interval $[a, b]$.

This completes proof of the fundamental theorem.

We have excellent tools before us to evaluate integrals and also put them to use. sometimes we use a notation as follows: $\int f = F$ without mentioning any interval $[a.b]$. We use the same variable x for both f and F. This is to be interpreted as saying that F is a primitive for f. It simply means that over a 'meaningful' interval $F' = f$. Thus if you have two numbers $a < b$ in this interval then $\int\limits_a^b f(x)dx = F(b) - F(a)$.

I assume that you have come across the following in high school and so go over them fast to reach new and interesting things. If you did not go through them in high school or if you do not remember, then you should convince yourself about their truth. Do not take anything for granted.

$$\int x^n = \frac{x^{n+1}}{n+1}. \qquad \int x^a dx = \frac{x^{a+1}}{a+1} \quad \text{if} \quad a \neq -1.$$

We have explained the calculation of Archimedes for calculating $\int\limits_0^1 x^2 dx$. It follows exactly the upper and lower sums. It would be a nice exercise to calculate $\int x^n$ for positive integers n, following the same idea.

$$\int e^x dx = e^x, \quad x \in R. \qquad \int \frac{1}{x} dx = \log x, \quad x > 0.$$

$$\int \cos x dx = \sin x. \qquad \int \sin x dx = -\cos x.$$

$$\int \sinh x dx = \cosh x; \qquad \int \cosh x dx = \sinh x.$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \log(x + \sqrt{1+x^2}).$$

$$\int \frac{1}{\sqrt{x^2-1}}dx = \log(x+\sqrt{x^2-1}) \qquad |x|>1.$$

$$\int \frac{1}{-\sqrt{x^2-1}}dx = \log(x-\sqrt{x^2-1}) \qquad |x|>1.$$

$$\int \frac{1}{1-x^2}dx = \frac{1}{2}\log\frac{1+x}{1-x} \qquad |x|<1.$$

$$\int \frac{1}{1-x^2}dx = \frac{1}{2}\log\frac{x+1}{x-1} \qquad |x|>1.$$

If we write $\int f = \varphi - \int g$ it simply means

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a) - \int_a^b g(x)dx.$$

Walli's product:

$$\int_0^{\pi/2} \sin^0 x dx = \frac{\pi}{2}; \qquad \int_0^{\pi/2} \sin x dx = \cos 0 - \cos(\pi/2) = 1.$$

If $m > 1$, then integration by parts gives

$$\int_0^{\pi/2} \sin^m x dx = \int_0^{\pi/2} \sin^{m-1} x(-\cos x)'dx$$

$$= \int_0^{\pi/2} \cos x \ \ (m-1)\sin^{m-2} x \cos x dx$$

$$= (m-1)\int_0^{\pi/2} \sin^{m-2} x dx - (m-1)\int_0^{\pi/2} \sin^m x dx$$

so that

$$\int_0^{\pi/2} \sin^m x dx = \frac{m-1}{m}\int_0^{\pi/2} \sin^{m-2} x dx.$$

Thus

$$\int_0^{\pi/2} \sin^{2m} x dx = \frac{2m-1}{2m}\ \frac{2m-3}{2m-2}\ \frac{2m-5}{2m-4}\ \cdots \frac{3}{4}\ \frac{1}{2}\ \frac{\pi}{2}.$$

$$\int_0^{\pi/2} \sin^{2m+1} x dx = \frac{2m}{2m+1}\ \frac{2m-2}{2m-1}\ \frac{2m-4}{2m-3}\ \cdots \frac{4}{5}\ \frac{2}{3}\ 1.$$

dividing the first equation by the second

$$\frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx} = \frac{(2m-1)(2m+1)}{(2m)^2} \; \frac{(2m-3)(2m-1)}{(2m-2)^2} \; \cdots$$

$$\cdots \frac{3 \times 5}{4^2} \; \frac{1 \times 3}{2^2} \; \frac{\pi}{2}.$$

$$\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \; \frac{4^2}{3 \cdot 5} \; \frac{6^2}{5 \cdot 7} \cdots \frac{(2m-2)^2}{(2m-3)(2m-1)} \; \frac{(2m)^2}{(2m-1)(2m+1)}$$

$$\times \frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx}.$$

We shall now show that as $m \to \infty$;

$$\frac{\int_0^{\pi/2} \sin^{2m} x dx}{\int_0^{\pi/2} \sin^{2m+1} x dx} \to 1. \qquad (\spadesuit)$$

It will then follow that

$$\frac{\pi}{2} = \lim_{m \to \infty} \frac{2^2}{1 \cdot 3} \; \frac{4^2}{3 \cdot 5} \; \frac{6^2}{5 \cdot 7} \cdots \frac{(2m-2)^2}{(2m-3)(2m-1)} \; \frac{(2m)^2}{(2m-1)(2m+1)}.$$

This is called wall's product.

$$\frac{\pi}{2} = \lim_{m \to \infty} \; \frac{2^{2m}(m!)^2}{3^2 \cdot 5^2 \cdots (2m-1)^2(2m+1)} = \lim_{m \to \infty} \; \frac{2^{4m}(m!)^4}{[(2m)!]^2(2m+1)}.$$

Or

$$\sqrt{\frac{\pi}{2}} = \lim_{m \to \infty} \frac{2^{2m}(m!)^2}{(2m)!\sqrt{(2m+1)}}$$

Or

$$\sqrt{\pi} = \lim_{m \to \infty} \; \frac{2^{2m}(m!)^2}{(2m)!\sqrt{(m+1/2)}}$$

Since $\sqrt{m}/\sqrt{m+1/2} \to 1.$ we get

$$\sqrt{\pi} = \lim_{m \to \infty} \; \frac{2^{2m}(m!)^2}{(2m)!\sqrt{m}}$$

This is called Walli's formula for $\sqrt{\pi}$.

Euler's constant:

We know that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n}$$

increases to infinity as n becomes large. But how large is the above quantity, in other words, how fast is the above sequence increasing towards infinity?. Knowledge of integration helps to answer this question. The above quantity is like $\log n$. We shall show this now. Let

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} - \log n$$

Observe,

$$\log(k+1) - \log k = \int_k^{k+1} \frac{1}{x} dx$$

so that

$$\frac{1}{k+1} \leq \log(k+1) - \log k \leq \frac{1}{k}$$

So that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} \geq \sum_1^n [\log(k+1) - \log k] = \log(n+1).$$

$$a_n \geq \log(n+1) - \log n \geq 0.$$

Also the same inequality above shows

$$a_n - a_{n+1} = \log(n+1) - \log n - \frac{1}{n+1} \geq 0.$$

Thus (a_n) is a decreasing sequence of non-begative numbers and hence converges. The limit is usually denoted by γ, called Euler's constant.

There is another (essentially same argument as above) argument to show convergence of (a_n) and also to see that it is strictly positive. The above inequalities show

$$1 \geq [\log 2 - \log 1] \geq \frac{1}{2} \geq [\log 3 - \log 2] \geq \frac{1}{3} \geq [\log 4 - \log 3] \geq \frac{1}{4} \cdots \cdots.$$

Leibnitz's theorem on alternating series tells that the alternating series with above terms converges. The sequence a_n we have is nothing but its partial sums (not all, a subsequence) and hence converges and this also shows that the sum is at least $1 - \log 2$.

Incidentally, no nice alternate description seems to be known to decide whether γ is rational or not.

some more friends:

There are certain functions which are important and we have not yet met them. We saw the exponential function, e^x and some trigonometric functions; $\sin x$ and $\cos x$. We have also calculated their derivatives.

$(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ and hence $(\cos^2 x + \sin^2 x)' = 0$. Since this sum of squares equals one at $x = 0$ we see $\sin^2 x + \cos^x = 1$. This also shows

$$-1 \leq \sin x \leq 1; \qquad -1 \leq \cos x \leq +1.$$

Not to interrupt our plan of meeting more friends, we assume the following fact and prove it later. This shows that the functions sin and cos are just as you know in high school.

These functions, $\sin x$ and $\cos x$, are periodic of period 2π,

where π is the area of the circle of radius one. $(\spadesuit)$.

and also $\pi/2$ is the least positive number where $\cos x = 0$.

Thus the function $\sin x$ is one-to one strictly increasing function on the interval $[-\pi/2, \ \ +\pi/2]$ onto $[-1, +1]$. general theory regarding continuous functions tells us that the inverse function g is continuous on the interval $[-1, +1]$ onto $[-\pi/2, \ \ +\pi/2]$ Also it is differentiable at every point in $(-1, +1)$ with derivative given by the general formula as follows. temporarily denote by f the sin function on the interval $[-\pi/2 \ \ + \pi/2]$ we see $f(g(y)) = y$ for each $y \in [-1, +1]$. Since the derivative of f is non zero at every point in $(-1, 1)$ general theory tells us g is differentiable in $(-1, 1)$ and

$$f'(g(y))g'(y) = 1;$$

so that
$$g'(y) = \frac{1}{\cos g(y)} = \frac{1}{\sqrt{1 - \sin^2(g(y))}} = \frac{1}{\sqrt{1 - y^2}}.$$
where in the last equality we used that g is inverse of sine function.

Usually $g(y)$ is denoted by $\sin^{-1}(y)$ (inverse of the sine function) or $\arcsin y$ (arc whose sine is y, here arc refers to the angle subtended at the centre by the arc). Since cos is positive in the interval $[-\pi/2, +\pi/2]$ we have taken positive root in the second equality above.

Of course, the sine function is on-to-one in the interval $[+\pi/2, \quad 3\pi/2]$ onto $[-1, +1]$. we could have defined the inverse sine function so that it takes values in this interval. Nothing wrong with it, it would also be differentiable on $(-1, +1)$. Of course it would then be decreasing and derivative will be negative — cosine function is negative in the interval $[+\pi/2, \quad 3\pi/2]$.

Thus several 'branches' are possible for the inverse function. We settled on one branch, that is all. This is the branch usually one takes.

The cosine function is not one-to-one on the interval $[-\pi/2, \quad +\pi/2]$. But it is one to one, strictly decreasing on $[0, \pi]$ onto $[-1, 1]$. Its inverse h is defined on the interval $[-1, +1]$; takes values in $[0, \pi]$,; strictly decreasing and continuous; differentiable on $(-1, 1)$ with derivative
$$h'(y) = \frac{1}{-\sin(h(y))} = \frac{-1}{\sqrt{1 - \cos^2 h(y)}} = \frac{-1}{\sqrt{1 - y^2}}.$$
Usually, $h(y)$ is denoted by $\cos^{-1}(y)$ (inverse of the cosine function) or $\arccos x$ (arc whose cosine is y).

Again as in the case with sine function, several branches of the inverse function are possible for the cosine function too.

The function $\tan x$ is defined as $\sin x / \cos x$. Of course, this is not defined on all of R. This is not defined precisely at those points where $\cos x = 0$. It is defined at all other points. It is one-to-one on $(-\pi/2, \quad +\pi/2)$ onto $(-\infty, +\infty)$, strictly increasing and differentiable. If $-\pi/2 < x < 0$, sine is negative and since $\cos x$ approaches zero as x approaches $-\pi/2$ we see that tan approaches $-\infty$ as x approaches $-\pi/2$. similarly it approaches $+\infty$ as x approaches $+\pi/2$.

$$(\tan x)' = \frac{1}{\cos^2 x}.$$

Thus inverse g of the tangent function is defined on all of R, continuous, increasing, takes values in $(-\pi/2, \ \ +\pi/2)$, it is differentiable and

$$g'(y) = \frac{1}{\cos^2(h(y))} = \frac{1}{1+\tan^2(h(y))} = \frac{1}{1+y^2}.$$

Usually $g(y)$ is denoted $\tan^{-1}(y)$ or $\arctan y$.

We can define $\cot x$, $\sec x$ and cosec x. Since there is nothing we can add to what you know from high school, we shall not continue in this direction. But you should recollect them, use composition rule for differentiation to calculate their derivatives.

The trigonometric functions are also called circular functions because $(\cos x, \sin x)$ form points on circle. we shall now introduce hyperbolic functions,

$$\cosh x = \frac{e^x + e^{-x}}{2}. \qquad \sinh x = \frac{e^x - e^{-x}}{2} \quad x \in R.$$

These are called hyperbolic cosine and hyperbolic sine respectively, because the points $(\cosh x, \sinh x)$ lie on the hyperbola $y^2 - x^2 = 1$.

$$\cosh 0 = 1; \qquad \cosh x \geq 1;$$

$$\cosh(-x) = \cosh x; \qquad \lim_{x\to\pm\infty} \cosh x = \infty.$$

$$\sinh 0 = 0; \qquad \sinh(-x) = -\sinh x;$$

$$\lim_{x\to-\infty} \sinh x = -\infty.; \qquad \lim_{x\to\infty} \sinh x = \infty.$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

$$(\sinh x)' = \cosh x; \qquad (\cosh x)' = \sinh x.$$

The hyperbolic sine is strictly increasing and has range all of R. Hence its inverse map is defined on all of R, is differentiable. Here we can calculate explicitly the inverse map.

Similarly, hyperbolic cosine is strictly increasing on $[0, \infty)$ and has range $[1, \infty)$. Thus its inverse is defined on $[1, \infty)$ with values in $[0, \infty)$ and is continuous. It is differentiable on $(1, \infty)$. We can explicitly solve for the inverse. Of course, cosh is one to one on $(-\infty, 0]$ onto $[1, \infty)$ and so we can think of another branch of its inverse. We shall not take it for inverse.

$$\cosh^{-1}(y) = \log\left(y + \sqrt{y^2 - 1}\right); \qquad \sinh^{-1}(y) = \log\left(y + \sqrt{y^2 + 1}\right).$$

By chain rule you can calculate their derivatives too.

Fine tuning of integral:

(i) After defining upper sums; lower sums; integrability and after showing that very continuous function on a closed bounded interval is integrable, we have been specializing to continuous functions. However, one does come across functions which are not continuous or functions which are bounded continuous but defined only on an open interval.

For example the function $\sin(1/x)$ defined on the open interval $(0, 1)$, continuous and bounded. We, at this moment, are unable to talk about integrability because we did everything on a closed bounded interval. This was only done to fix ideas and have concrete picture in mind. Discussing open intervals poses no serious problems. This we do first.

(ii) so let us take a bounded interval (a, b) and a bounded function f on this interval. As earlier, partition is a finite sequence of points

$$P = \{a = a_0 < a_1 < a_2 < \cdots < a_k = b\}.$$

Given a partition P we define $U(P, f)$ and $L(P, f)$ the upper and lower sums as earlier; just that for the first and last interval we take sup and inf only over $(a, a_1]$ and $[a_{k-1}, b)$. As earlier, we say that f is Riemann integrable if Sup of all lower sums equals inf of all upper sums and in that case, the common value is called integral of f and is denoted

$$\int\limits_a^b f; \qquad \int_a^b f(x)dx.$$

The fact that every lower sum is smaller than every upper sum is obvious and also the fact that upper sums decrease whereas lower sums increase as the partition becomes finer. It is also easy to show that f is integrable iff for any given $\epsilon > 0$, we can get a partition P so that $U(P, f) - L(P, f) < \epsilon$..

(iii) We can show that a bounded continuous function is integrable. Earlier we used uniform continuity, but now this is no longer immediately possible because continuous function on an open interval need not be uniformly continuous. However, we can take advantage of the fact that the function is bounded.

Let $|f(x)| \leq M$ for all $x \in (a, b)$. Let $\epsilon > 0$. Let us choose $\delta > 0$ so that $2M\delta < \epsilon/4$. Note that on any subinterval, sup minus inf of f is at most $2M$. Thus in particular, if you consider the interval $(a, a + \delta]$ or the interval $(b - \delta, b]$ you see that sup minus inf over that interval times delta is smaller than $\epsilon/4$.

The function being uniformly continuous on $[a + \delta, b - \delta]$, get a partition P_1 of this $[a + \delta, a - \delta]$ so that $U(P_1, f) - L(P_1, f) < \epsilon/4$. The partition P for (a, b) is simply the points in P_1 along with a at the beginning and b at the end. Thus the first interval of this partition is $(a, a + \delta]$. and the last interval of this partition is $(b - \delta, b]$. It is easy to see that

$$U(P, f) - L(P, f) < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.$$

(iv) it is interesting to see that in the notation for integral, $\int\limits_a^b$ there is nothing to show wether we have open interval or closed interval. Since area of a line is zero, it makes no difference whether you include the lines at the end points or not, in calculating the area under the curve.

More precisely, suppose f were actually continuous on $[a, b]$ and you ignore it and consider the function only on (a, b) and calculate the integral. You get the same answer as you would get when you calculated for the closed interval. The proof is very simple, for each partition whether you calculate the sums over (a, b) or $[a, b]$ you get the same answer. After all, the only difference is the end intervals and there continuity of the function tells you whether you include end point or not you get the same value for the sup or inf.

(v) You can also show that selections (given a partition, select points from each partition interval) and Riemann sums also lead to the value of the integral, even on an open interval. In fact, whenever $||P_n|| \to 0$ the sums $R(P_n, f, s_n)$ converge to the integral whatever be the selection s_n for P_n.

(vi) The idea in the above argument has something more to offer. suppose that f is a bounded function on $[a, b]$ but only known to be continuous on (a, b). Is it integrable? Yes, the argument above show that it is integrable and equals the integral of f over the open interval (a, b).

(vii) In fact one can go further. Suppose f is a bounded function defined on (a, b) and is continuous at all but finitely many points. then f is integrable and the integral equals integral of f on the complement of this finite set. Note that complement of this finite set is made up of finitely many disjoint open intervals, so from the above, f is integrable on each of these intervals, so calculate them and add them up. This is the meaning of integral of f over the complement of the finite set.

Carefully understand the statement above. Firstly, we said that f is integrable. Secondly, we said to calculate the integral, you do on the complement of the finite set. the first statement is easy to see by improvising the above idea, enclose each of the finitely many discontinuity points in small intervals and get partitions of the remaining parts carefully and put all together to get a partition P of (a, b) to see $U(P, f) - L(P, f) < \epsilon$. The second statement also follows from this, but its importance is that it gives us a method of calculating the integral.

($viii$) Actually all this is a reflection of the fact that finitely many lines have area zero. of course, countably many lines also have area zero and even if f has countably many discontinuity points the result must be true. Yes, it is indeed so. But we shall not discuss. But are they the only functions which are integrable.? No, there are more.

(ix) The properties of integrals that we verified for continuous functions hold good for integrable functions. For example, if f and g are bounded integrable functions on (a, b) (or $[a, b]$) then so is their sum $f + g$ and $\int(f + g) = \int f + \int g$. Also $7f$ is integrable and $\int(7f) = 7\int f$.

In other words a fine tuning is possible and would make the theory better and complete. But if you understand the story of functions with finitely many discontinuities and how to calculate integrals, ti would suffice for a first course.

You must keep in mind that all this story we developed is for *bounded functions* on *bounded intervals.* if the function is unbounded, you can immediately see that this procedure is useless. There one interval of the partition where the sup is $+\infty$ or there is a partition interval where the inf is $-\infty$. In the first two cases we get $\pm\infty$ for each partial sum. However when the last case occurs, we can not even define Riemann sum we will be involved in $\infty - \infty$ for which we have not given any meaning.

When the interval is infinite, then also we enter a similar situation.

Most of the integrals are of this kind, that is, either the function is unbounded or the interval of integration is unbounded. We deal with such situations next.

Stirling formula for $n!$:

it says that $n!$ is like $\sqrt{2\pi}e^{-n}n^{n+1/2}$.

This is to be interpreted in the folioing sense. Their ratio converges to one. When we say that a sequence (a_n) is like another sequence (b_n) (both are sequences of strictly positive numbers) there are two ways of understanding. Either $a_n - b_n \to 0$ or $a_n/b_n \to 1$ Of course when the first happens, then the second also happens. However the other way is not in general true.

For example (n) is like $(n + 1/n)$ in the first sense. the sequence (n^2) is like $(n^2 + n)$ in the second sense. they are not so in the first sense. In fact the difference between the two sequences is n which becomes larger and larger. But then in what sense are they like each other? well, Both numbers are becoming large, when you replace one by the other, the relative error is going to zero.

If you are measuring length of this room, if you are off by a mile then the error is indeed very huge. On the other hand if you are measuring distance (of earth) to sun, if you are off by a mile or even hundred miles, the error is very very small. So the absolute error is many times unimportant and it is the relative error that matters.

We shall try to approximate the area of the function $f(x) = \log x$ from $x = 1$ to $x = n$ Let us first make a few observations which depend on the fact that

$$f'' = -1/x^2 \leq 0.$$

Let g be a twice differentiable function on an interval (a, b) with $g'' \leq 0$.

Consider any two points $u < v$ in the interval (a, b). We claim that the chord (or secant) joining the two points $(u, f(u))$ and $(v, f(v))$ lies below the graph of f.

To see this, first observe that the equation of the chord is

$$y - f(u) = \frac{f(v) - f(u)}{v - u}(x - u).$$

Consider any point $w \in [u, v]$. We need to show

$$f(u) + \frac{f(v) - f(u)}{v - u}(w - u) \leq f(w).$$

That is,

$$f(u) - f(w) + \frac{f(v) - f(u)}{v - u}(w - u) \leq 0.$$

Or

$$[f(u) - f(w)](v - u) + [f(v) - f(u)](w - u) \leq 0.$$

$$[f(u) - f(w)](v - u) + [f(v) - f(w) + f(w) - f(u)](w - u) \leq 0.$$

$$[f(v) - f(w)](w - u) - [f(w) - f(u)](v - w) \leq 0.$$

By MVT, there are points $\theta \in (u, w)$ and $\eta \in (w, v)$ such that $f(w) - f(u) = f'(\theta)(w - u)$ and $f(v) - f(w) = f'(\eta)(v - w)$. So we need to show

$$f'(\eta)(v - w)(w - u) - f'(\theta)(v - w)(w - u) \leq 0.$$

That is,

$$[f'(\eta) - f'(\theta)](v - w)(w - u) \leq 0.$$

Since $\theta < \eta$ and $f'' \leq 0$ we conclude that the first factor above is negative. Since $u, w < v$, the other two factors are positive and hence the inequality is true.

Consider any point u in the interval (a, b). We claim that the tangent (to the graph of f) at u lies above the graph.

The equation of the tangent is

$$y - f(u) = f'(u)(x - u).$$

Let us take any other point $w \in (a, b)$. We need to show

$$f(w) \leq f(u) + f'(u)(w - u).$$

That is,

$$\frac{f(w) - f(u)}{w - u} \le f'(u).$$

But the left side is $f'(\theta)$ for some $u < \theta < w$ and since f' is decreasing, the inequality is verified.

Thus, for $k \ge 1$, the area under the curve $y = \log x$ is in between the area under the chord joining $(k, \log k)$, $(k+1, \log(k+1))$ and area under the tangent at $x + 1/2$. Thus

$$\frac{1}{2}\log(k+1) + \frac{1}{2}\log k \le \int_k^{k+1} \log x \, dx \le \log(k + 1/2).$$

Adding these for $k = 1, 2, \cdots, n-1$ and remembering that $x \log x - x$ is a primitive for $\log x$ we get

$$\log(n!) - \frac{1}{2}\log n \le n \log n - n + 1 \le \sum_1^{n-1} \log(k + 1/2).$$

Let

$$a_n = n \log n - n + 1 - [\log(n!) - \frac{1}{2}\log n] = \log\left\{\frac{e^{-n} n^{n+1/2}}{n!}\right\} + 1.$$

Then a_n is the area between the curve $y = \log x$ and the chords explained above, from $x = 1$ to $x = n$ Thus we see

$$a_n \ge 0; \qquad a_n \uparrow . \qquad\qquad (\spadesuit)$$

Also

$$a_n \le \sum_1^{n-1} \{\log(k + 1/2) - \frac{1}{2}\log(k+1) - \frac{1}{2}\log k\}.$$

$$= \frac{1}{2}\sum_1^{n-1} \left\{\log\frac{(k+1/2)}{k} - \log\frac{(k+1)}{k+1/2}\right\}$$

$$\le \frac{1}{2}\sum_1^{n-1} \left\{\log(1 + \frac{1}{2k}) - \log(1 + \frac{1}{2(k+1/2)})\right\}$$

$$\le \frac{1}{2}\sum_1^{n-1} \left\{\log(1 + \frac{1}{2k}) - \log(1 + \frac{1}{2(k+1)})\right\}$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\log(1+\frac{1}{2n}).$$

As a consequence a_n is bounded above . So (♠) implies that a_n converges. Say $a_n \uparrow c$

$$\log\left\{\frac{e^{-n}n^{n+1/2}}{n!}\right\} = a_n - 1 \uparrow c - 1.$$

Or

$$\log\left\{\frac{n!}{e^{-n}n^{n+1/2}}\right\} \to k$$

for some constant k. Or

$$\frac{n!}{ke^{-n}n^{n+1/2}} \to 1.$$

We shall now evaluate the constant k by using the above limit in a known case, namely, Walli's product. We know

$$\frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}} \to \sqrt{\pi}.$$

Suppose that we have strictly positive numbers a_n and b_n and $a_n/b_n \to 1$. If $\alpha_n \times a_n \to c$ then $\alpha_n \times b_n \to c$. This is because

$$\alpha_n \times b_n = \alpha_n \times a_n \times \frac{b_n}{a_n} \to c \times 1.$$

Similarly if $\alpha_n/a_n \to c$ then $\alpha_n/b_n \to c$. In other words we can replace a_n by b_n. As a consequence the above result of Walli implies

$$\frac{2^{2n}k^2e^{-2n}n^{2n+1}}{ke^{-2n}(2n)^{2n+1/2}\sqrt{n}} \to \sqrt{\pi}.$$

That is,

$$k = \sqrt{2\pi}.$$

Thus

$$\frac{n!}{\sqrt{2\pi}e^{-n}n^{n+1/2}} \to 1.$$

Improper integrals:

We have discussed integrals of bounded functions over bounded intervals, both open as well as closed. Integrals where either the function is unbounded or the interval is unbounded are called improper integrals. There is nothing improper about them, just that you can not use Riemann sums blindly. Just as infinite sums are defined as limits of finite (partial) sums, so are these integrals. We shall first discuss unbounded functions over bounded interval.

Of course there are several possibilities. For example, you can consider the function

$$f(x) = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x-1}} + \frac{1}{\sqrt{x-2}}; \quad x \in (0,3), x \neq 1; \quad f(1) = 0.$$

Then f is unbounded near three points, namely, zero, one and two. We start with simple situations.

Let us consider function f on an open interval (a, b) which is unbounded only near a, that is for any $\epsilon > 0$, it is bounded on $(a+\epsilon, b)$ Then for every $\epsilon > 0$, the integral $\int\limits_{a+\epsilon}^{b} f$ makes sense and is defined by earlier considerations. We say that $\int\limits_{a}^{b} f$ exists if

$$\lim_{\epsilon \downarrow 0} \int_{a+\epsilon}^{b} f$$

exists and then we define this limit as $\int\limits_{a}^{b} f$. You must recall the meaning of limit $\epsilon \downarrow 0$.

Thus $\int\limits_{a}^{b} f$ exists if there is a number α such that for every sequence $a_n \downarrow a$, $a_n > a$ for all n we have

$$\int_{a_n}^{b} f \to \alpha.$$

In that case this number α is defined to be the value of the integral of f over the interval (a, b).

For example consider

$$\int_{a}^{b} \frac{1}{x} dx.$$

This does not exist because for integral over $(\epsilon, 1)$, you get $\log \epsilon$ and does not converge to a finite limit as $\epsilon \downarrow$. Actually the same situation occurs and the integral

$$\int_0^1 \frac{1}{x^\alpha} dx$$

does not exist for $\alpha \geq 1$. On the other hand the integral exists for $\alpha < 1$. Of course if $\alpha \leq 0$, then the function is bounded and actually continuous on the closed interval $[0, 1]$ and we need not discuss. So let $0 < \alpha < 1$.

$$\int_\epsilon^1 \frac{1}{x^\alpha} dx = \frac{1}{1-\alpha} \left(1 - \epsilon^{-\alpha+1}\right) \to \frac{1}{1-\alpha},$$

as $\epsilon \downarrow 0$.

In a similar way, if the function f defined on an open interval (a, b) is unbounded only near b, that is for any $\epsilon > 0$, it is bounded on $(a, b - \epsilon)$ then for every $\epsilon > 0$, the integral $\int\limits_a^{b-\epsilon} f$ makes sense and is defined by earlier considerations. We say that $\int\limits_a^b f$ exists if

$$\lim_{\epsilon \downarrow 0} \int_a^{b-\epsilon} f$$

exists and then we define this limit as $\int\limits_a^b f$. You must recall the meaning of limit $\epsilon \downarrow 0$.

We can, in a fashion analogous to the above, show that the integral

$$\int_a^b \frac{1}{(b-x)^\alpha} dx$$

exists if $\alpha < 0$ and does not exist if $\alpha \geq 1$.

The substitution rule and integration by parts hold good for these integrals. For example suppose φ is a continuously differentiable strictly increasing function on a bounded interval (a, b) onto the bounded interval (c, d). Suppose that f is a function integrable on (a, b), possibly unbounded. Then integral of $f(\varphi(x))\varphi'(x)$ exists over (a, b) and we have

$$\int_a^b f(\varphi(x))\varphi'(x) dx = \int_c^d f(y) dy.$$

Suppose that f is unbounded only near a, then verify that the equality holds when you consider $a+\epsilon$ to b on left side and from $\varphi(a+\epsilon)$ to d on the right side and then take limits. Similar remark applies if the infinity occurs only near b.

Similar argument holds for integration by parts, do it over appropriate interval and take limits.

Let us denote by π the area of the region enclosed by circle of radius one. Thus $\pi/4$ is the area of the quarter circle. More precisely,

$$\int_0^1 \sqrt{1-u^2}du = \frac{\pi}{4}.$$

Note that the integrand on the left side is precisely the function describing the quarter circle in the first quadrant. We now show, without using trigonometric functions,

$$\int_0^1 \frac{1}{\sqrt{1-u^2}}du = \frac{\pi}{2}, \qquad (\clubsuit)$$

To prove the claim regarding the improper integral first do integrate by parts to see that for $0 < a < 1$

$$\begin{aligned}\int_0^a \sqrt{1-u^2}\ \ du &= a\sqrt{1-a^2} - \int_0^a u\left[\sqrt{1-u^2}\right]' du \\ &= a\sqrt{1-a^2} + \int_0^a \frac{u^2}{\sqrt{1-u^2}}du \\ &= a\sqrt{1-a^2} + \int_0^a \frac{1}{\sqrt{1-u^2}}du - \int_0^a \sqrt{1-u^2}du.\end{aligned}$$

Thus

$$\int_0^a \frac{1}{\sqrt{1-u^2}}du = a\sqrt{1-a^2} + 2\int_0^a \sqrt{1-u^2}du.$$

Now taking limit as $a \uparrow 1$ we get ($\clubsuit$).

Final look at trigonometric functions:

Let us recall the functions

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots;$$

$$g(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots;$$

We have seen, as a consequence of the fundamental theorem on power series, the following.

(1^0). f, g are defined on all of R, differentiable,

$$f' = g; \quad g' = -f; \quad f(0) = 0; \quad g(0) = 1.$$

This implies $(f^2 + g^2)' = ff' + gg' = 0$. Hence $f^2 + g^2$ must be a constant function; since at zero its value is one, we conclude

(2). $f^2 + g^2 \equiv 1$.

As a consequence we have the following.

(3). $-1 \leq f \leq 1; \quad -1 \leq g \leq 1$. and also that when one of the functions assumes the value zero, then the other must take value ± 1.

We now show the following.

(4). $g(x) = 0$ for some $x > 0$.

Since $g(0) = 1$, and g is continuous, we can fix a $\delta > 0$ so that $g(x) > 0$ for all $x \in [0, \delta]$. Since $f(0) = 0$ and its derivative is strictly positive in $[0, \delta]$ we conclude that f is strictly increasing in this interval and $f(\delta) = c > 0$. We claim that g must assume value zero in the interval $[\delta, \delta + \frac{2}{c}$ Otherwise, $g(\delta) > 0$ implies that g is positive through this interval. But then f must be strictly increasing and hence $f(x) > c$ throughout this interval. But then by MVT, there is a θ in this interval such that

$$\left| g(\delta + \frac{2}{c}) - g(\delta) \right| = \frac{2}{c} |f(\theta)| > 2.$$

But from (3) we have

$$\left| g(\delta + \frac{2}{c}) - g(\delta) \right| \leq 2.$$

This contradiction proves the claim.

(5). Let $\alpha = \min\{x > 0 : g(x) = 0\}$. Then $\alpha > 0$.

In other words, α is the smallest positive zero of g. Indeed from (4), we see that the set on right side is non-empty. By continuity of g it follows that $g(\alpha) = 0$, recall that there is a sequence of points in the set decreasing to its infimum. Hence α is again in the set and hence it is minimum. Since $g(0) = 1$ we see that $\alpha > 0$.

(6). $f(\alpha) = +1$

In fact as noticed in (3), $f(\alpha) = \pm 1$. But f is strictly increasing through out in $[0, \alpha]$ with $f(0) = 0$. Hence $f(\alpha) == 1$.

(7). $\alpha = \pi/2$ where π is the area enclosed by the unit circle.

As noted above f is strictly increasing differentiable function on $(0, \alpha)$ onto $(0, 1)$. If $\varphi(y) = 1/\sqrt{1 - y^2}$ for $0 < y < 1$, then as discussed in the improper integrals,

$$\int_0^1 \varphi(y)dy = \pi/2.$$

But by substitution rule this integral equals

$$\int_0^\alpha \varphi(f(x))f'(x)dx = \int_0^\alpha g(x)/\sqrt{1 - f^2(x)} = \int_0^\alpha 1dx = \alpha.$$

Thus $\alpha = \pi/2$.

(8) The functions $\varphi = f$ and $\psi = g$ are the only solutions of

$$\varphi' = \psi; \quad \psi' = -\varphi; \quad \varphi(0) = 0; \quad \psi(0) = 1.$$

from (1), we know that f and g are indeed solutions. If there is another pair; $\varphi = \tilde{f}$ and $\psi = \tilde{g}$, then the functions

$$\eta = f - \tilde{f}; \qquad \zeta = g - \tilde{g}$$

satisfy

$$\eta' = \zeta; \quad \zeta' = -\eta; \quad \eta(0) = 0 = \zeta(0).$$

But then the argument leading to (2) applies to show that $\eta^2 + \zeta^2 \equiv 0$ showing that $f = \tilde{f}$ and $g = \tilde{g}$.

(9). $f(x + \alpha) = g(x); \qquad g(x + \alpha) = -f(x).$

Define the functions $f_1(x) = -g(x + \alpha)$ and $g_1(x) = f(x + \alpha)$. Then by definition of α we see $f_1(0) = 0$ and by (6) we see $g(0) = 1$. Moreover, by chain rule fro derivatives we see

$$f_1'(x) = -g'(x + \alpha) = -[-f(x + \alpha)] = f(x + \alpha) = g_1(x).$$

similarly, $g_1'(x) = -f_1(x)$.. But then (8) shows that these functions f_1 and g_1 are same as the functions f and g proving the claim.

(10). $f(x + 2\alpha) = -f(x); \qquad g(x + 2\alpha) = -g(x).$

Repeated application of (9) proves this. Again a repeated application of this proves

(11). $f(x + 4\alpha) = f(x); \qquad g(x + 4\alpha) = g(x).$

Thus the functions f and g are periodic of period 2π. We have also seen as an application of the Cauchy product of power series the following.

(12). $f(x + y) = f(x)g(y) + g(x)f(y); \quad g(x + y) = g(x)g(y) - f(x)f(y).$

This concludes our discussion of the trigonometric functions. This identifies the functions we introduced as power series with the sine and cosine functions that you learnt in high school. Of course, the uniqueness result (8) is enough for such an identification. But this discussion also clarifies the role of π.

return to improper integrals:

We have discussed the definition of $\int\limits_a^b f(x)dx$ if f is unbounded at one of the points a or b. What if f is unbounded at both ends. For example

$$\int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx$$

Suppose that f is unbounded at both end points, but is bounded otherwise, that is if $a < u < v < b$ then f is bounded on the interval $[u, v]$. We say that the integral $\int\limits_a^b f$ exists if there is a number α such that

$$\int_{a_n}^{b_n} f \to \alpha$$

whenever we have $a < a_n < b_n < b$ and $a_n \to a$ and $b_n \to b$. In that case we write

$$\int_a^b f(x)dx = \alpha.$$

Note that this is not same as saying that the limit

$$\lim_{\epsilon \to 0, \epsilon > 0} \int_{a+\epsilon}^{b-\epsilon} f$$

exists.

Let us consider the case of unbounded interval. Suppose that f is a function defined on the interval $(0, \infty)$ which is bounded on every interval $(0, a)$. We say that the integral

$$\int_0^\infty f$$

exists if the limit

$$\lim_{a_n \to \infty} \int_0^{a_n} f$$

exists and then we define this limit as the value of the integral.

for example

$$\int_0^\infty e^{-x}dx$$

exists and equals one.

$$\int_0^\infty x^n e^{-x}dx$$

exists and equals $n!$ if n is a positive integer. you prove it buy induction on n and integration by parts over $(0, a)$. You need to use $a^k e^{-a} \to 0$ as $a \to \infty$.

improper integrals:

We shall continue our discussion of improper integrals. Let us again recall that there is nothing improper about them. It so happens that the recipe of making Riemann sums and taking limits is not done.

if the function is unbounded, we take integral over a smaller interval where the function is bounded and let the smaller interval grow to the large interval of interest. Similarly, if the function is bounded but we are integrating over an infinite interval, we calculate integrals over a smaller finite interval and take limit as the smaller interval grows to the large interval of interest.

Of course if the function is both unbounded and interval of integration is infinite, we take smaller bounded intervals where the function is bounded and let the interval grow to the large interval of integration. Of course, there are several other possibilities, we may not always integrate over intervals, we may have to integrate over union of intervals. For example, this happens for functions like

$$f(x) = \frac{1}{\sqrt{x(1-x)}} \quad 0 < x < 1; \qquad f(1) = 0;$$

$$f(x) = \frac{1}{\sqrt{(x-1)(2-x)}} \quad 1 < x < 2$$

defined on the interval $(0, 2)$. Or

$$f(x) = \frac{1}{\sqrt{1-x}} \quad 0 < x < 1; \qquad f(1) = 0;$$

$$f(x) = \frac{1}{\sqrt{(x-1)}} \quad 1 < x < 2$$

We shall not discuss all these possibilities partly because we do not need them. And also because, if you understood these simple cases you know how to define and calculate these integrals too.

(i) *Consistency:*

The first questions that arises is whether for bounded functions on bounded intervals this recipe agrees with the earlier definition. This we have already seen earlier. If you have a bounded function on a bounded interval (a, b), you can follow two procedures and they both lead to the same answer.

The first method is to take partitions of the open interval (a, b), calculate the lower and upper Riemann sums; say that the function is integrable if sup of lower sums agrees with inf of upper sums and declare the common value as integral.

Second method is to take integral over $[a + \epsilon, b - \epsilon]$ and take the limit as $\epsilon \downarrow 0$. Both lead to the same answer.

We can also take $a < \alpha < \beta < b$, calculate integral over $[\alpha, \beta]$ and then take limit as $\alpha \downarrow a$ and $\beta \uparrow b$. These are called double limits and we have not discussed. so you should carefully understand what such things mean. This means, there is a number c such that the following happens: Whenever $\epsilon > 0$ is given, there is a $\delta > 0$ so that

$$a < \alpha < a + \delta; b - \delta < \beta < b \Rightarrow |\int_{\alpha}^{\beta} f - c| < \epsilon.$$

This also leads to the same answer as above. This is how we defined, in case f were unbounded at both end points.

(ii) *usual rules:*

The second question that needs to be looked into is whether the usual simple rules —for sum, constant multiple, etc — apply. Yes, it is just a matter of using them for the case we know and applying limits. We discuss some examples, rather than stating general theorems.

(a) If f and g are unbounded but the improper integrals $\int\limits_a^b f$ and $\int\limits_a^b g$ exist then so does $\int\limits_a^b (f+g)$ and

$$\int\limits_a^b (f+g) = \int_a^b f + \int_a^b g.$$

This is easy.

$$a < \alpha < \beta < b \Rightarrow \int_\alpha^\beta (f+g) = \int_\alpha^\beta f + \int_\alpha^\beta g.$$

Now use the fact that limit of sum is sum of limits.

You must carefully notice that in the above statement, it is quite possible that f is unbounded and hence $\int f$ is an improper integral, but g is a bounded integrable function.

(b) similarly

$$\int_a^b (10f) = 10 \int_a^b f.$$

This means that if the integral on one side of the equation above exists, then so does the integral on the other side and the equality holds.

Of course, similar argument applies for integrals over infinite intervals.

(c) If $a < c < b$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

This means if both integrals on the right side exist then so does the integral on the left side; if the integral on the left side exists then so do both integrals on the right side; and then equality holds. Remember that we are assuming f is unbounded only near the end points, either both end points or only one. In other words, if we take any $a < \alpha < \beta < b$, then f is bounded on $[\alpha, \beta]$.

you can prove again by taking $a < \alpha < c < \beta < b$ and using known result over the interval $[\alpha, \beta]$ and then taking limits.

Same argument applies over infinite intervals too.

(3) *integration by parts:*

The integration by parts formula is valid even for improper integrals. Suppose that (a, b) is a bounded interval. Let F and G be two C^1 functions on this intervals. Assume that F and G have limits at a and b. That is,

$$\lim_{x \downarrow a} F(x); \qquad \lim_{x \uparrow b} F(x); \qquad \lim_{x \downarrow a} G(x); \qquad \lim_{x \uparrow b} G(x)$$

all exist. Denote these lints by $F(a)$, $F(b)$, $G(a)$, $G(b)$ respectively. Then

$$\int_a^b fG = [F(b)G(b) - F(a)G(a)] - \int_a^b Fg.$$

again this means: if the integral on one side exists, then so does the other and equality holds.

Again, if left side exists, write the equation with a, b replaced by α, β and then take limits. Similar argument applies if the right side exists.

The same argument applies even if both a , b or one of them is infinite. Of course you need to assume as above that the limits at the end points exist for the functions.

(4) *substitution:*

Recall that the method of substitution says the following.

$$\varphi : (a, b) \to (c, d)$$

strictly increasing C^1 function with bounded derivative and f is a continuous function on (c, d). Then

$$\int_a^b f(\varphi(x))\varphi'(x) = \int_c^d f(y).$$

The same result holds for improper integrals too.

There are several possibilities. For example the intervals may be bounded and f may also be bounded but φ' may be unbounded. For instance, both the intervals are $(0,1)$; $\varphi(x) = \sqrt{x}$, $f(x) = 1$. Then of course, the right side is usual integral and the left side is improper integral.

The function f may be unbounded and both sides could be improper integrals.

If you have understood the spirit of earlier arguments, there is nothing new in proving this.

example:

$$\int_0^\infty \frac{1}{1+x^2} = \frac{\pi}{2}$$

Consider the functions

$$\varphi(x) = \tan x : [0, \pi/2) \to [0, \infty); \quad f(x) = \frac{1}{1+x^2} : [0, \infty) \to R$$

Then $\varphi'(x) = 1 + \tan^2 x$ and $f(\varphi(x))\varphi'(x) = 1$.

To calculate $\int\limits_{-\infty}^{\infty}$, either you can use the same argument. Or, you can also write this as sum of two integrals, over $(-\infty, 0)$ and $(0, \infty)$ and add. To calculate the integral over $(-\infty, 0)$ substitute $y = -x$ to see this integral is same s the integral over $(0, \infty)$.

gamma integral:

The integral

$$\int_0^\infty e^{-x} x^{a-1} dx$$

is called Gamma integral. It converges for $a > 0$ and does not converge for $a \leq 0$. For $a > 0$, the value of the integral is denoted by $\Gamma(a)$. We ahve already seen that for $a = 1$, the integrand is just e^{-x} and the integral converges and has value one.

First note that the integrand is positive. If $a \geq 1$ the integrand is bounded at zero. It is improper only because range of integration is infinite interval. We show that

$$\lim_{A\to\infty} \int_0^A e^{-x} x^{a-1}$$

exists. Denote this integral by I_A. Since integrand is positive, we conclude that I_A increases with A. It suffices to show that $\{I_A : A > 0\}$ is bounded.

If this is done, then I_A converges to the supremum. Indeed let

$$c = \sup\{I_n : n = 1, 2, 3, \cdots\}.$$

Then $I_n \uparrow c$. We argue that $I_A \uparrow c$ as follows. Let $\epsilon > 0$ be given. Choose m so that $I_m > c - \epsilon$. Then for any $A > m$, we have $I_A \geq I_m > c - \epsilon$, more precisely, $c - \epsilon \leq I_A \leq c$ for $A > m$.

Let $k > a - 1$ be any integer. We know that $e^{-x/2} x^k \to 0$ as $x \to \infty$ and hence

$$e^{-x/2} x^{a-1} \to 0; \qquad \text{as} \quad x \to \infty.$$

Say, it is smaller than one for $x \geq \alpha$. On the interval $[0, \alpha]$ the function $e^{-x/2} x^{a-1}$ is continuous and hence bounded; remember that $a \geq 1$. Thus there is a number M so that

$$e^{-x/2} x^{a-1} \leq M; \qquad \forall x \geq 0.$$

Thus

$$e^{-x} x^{a-1} \leq e^{-x/2} M; \quad \forall x \geq 0.$$

So for any $A > 0$

$$\int_0^A e^{-x} x^{a-1} \leq \int_0^A M e^{-x/2} = M2[1 - e^{-A/2}] \leq 2M.$$

showing that the set $\{I_A : A > 0\}$ is bounded.

Let $0 < a < 1$. In this case the integrand is unbounded at zero and also the range of integration is unbounded. For any $0 < \alpha < 1$, we have

$$\int_\alpha^1 e^{-x} x^{a-1} \leq \int_\alpha^1 x^{a-1} = \frac{1}{a} - \frac{\alpha^a}{a} \leq \frac{1}{a}.$$

Thus the integral converges over the interval $(0, 1)$. More precsiely, there is a number c_1 such that

$$\int_{\alpha}^{1} e^{-x} x^{a-1} \to c_1; \quad \text{as} \quad \alpha \to 0.$$

Also for any $\alpha > 1$, noting that $a - 1 < 0$ and hence $x^{a-1} < 1$ for $x > 1$, we have

$$\int_{1}^{\alpha} e^{-x} x^{a-1} \leq \int_{1}^{\alpha} e^{-x} = e^{-1} - e^{-\alpha} \leq 1.$$

Thus the integral converges over the interval $(1, \infty)$. More precsiely, there is a number c_2 such that

$$\int_{1}^{\alpha} e^{-x} x^{a-1} \to c_2; \quad \text{as} \quad \alpha \to \infty.$$

Thus the integral converges over $(0, \infty)$. In fact, if we take $\alpha_n \to 0$ and $\beta_n \to \infty$, then after some stage $\alpha_n < 1 < \beta_n$. Thus after this stage,

$$\int_{\alpha_n}^{\beta_n} e^{-x} x^{a-1} = \int_{\alpha_n}^{1} e^{-x} x^{a-1} + \int_{1}^{\beta_n} e^{-x} x^{a-1} \to c_1 + c_2.$$

An integration by parts (and induction) shows that for integers $n = 1, 2, \cdots$

$$\Gamma(n) = (n-1)!.$$

More generally, integartion by parts shows that for $a > 0$,

$$\Gamma(a+1) = a\Gamma(a).$$

This simple equation has interesting consequences. For example, when $a = -1/2$, which we are not supposed to take because the equation above holds for $a > 0$, since $\Gamma(-\frac{1}{2}+1) = \Gamma(1/2)$ is defiined we can take the above equation to define

$$\Gamma(-1/2) = -2\Gamma(1/2).$$

Now we can define $\Gamma(-3/2)$ etc. In other words the equation above which is a theorem for $a > 0$ can be taken used for defining $\Gamma(a)$ for negative values of a. Thus one can define the Gamma function for all real values *except* for $a = 0, -1, -2, \cdots$, that is, except for non-positive integers. In fact it can be extended for complex numbers a also. Such an extension is not only fun, but also has interesting consequences.

beta integral:

The integral
$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$
is cvalled beta integral. The integral converges for $a > 0$ and $b > 0$. It does not converge fi either $a \leq 0$ or $b \leq 0$. The value of the integral is denoted by $\beta(a, b)$. Thus this is defined only when both a and b are strictly positive.

When $a = 1, b = 1$, then the integrand is one and hence so is the value of the integral. For $a \geq 1, b \geq 1$ the integrand is a nice continuous function on the closed interval $[0, 1]$ and the integral is the usual one and is not improper. You may say it is a proper integral.

If $b \geq 1$ but $a < 1$ then the integrand is unbounded at zero, and the integral is improper. Let now $0 < a < 1$. Then for any $0 < \alpha < a$ we have
$$\int_\alpha^1 x^{a-1}(1-x)^{b-1} \leq \int_\alpha^1 x^{a-1} \leq \frac{1}{a} - \frac{\alpha^a}{a} \leq \frac{1}{a}.$$
and hence the integral converges. the same arguments as in the case of gamma integral would do. In fact the set
$$\{\int_\alpha^1 x^{a-1}(1-x)^{b-1} : 0 < \alpha < 1$$
is a bounded set and its supremum is the value of the integral from zero to one.

Let us continue assuming that $b \geq 1$ but now $a \leq 0$. Thsn
$$\int_\alpha^1 x^{a-1}(1-x)^{b-1} \geq \int_\alpha^{1/2} x^{a-1}(1-x)^{b-1} \geq (1/2)^{b-1} \int_\alpha^1 x^{a-1}$$
$$= (1/2)^{b-1}\frac{1}{a} - \frac{\alpha^a}{a} \leq \frac{1}{a}.$$
Since $a < 0$ we see that this last quantity increases to infinity as $\alpha \to 0$. Thus the integral does not converge.

Exactly similar arguments apply when $a \geq 1$ and $0 < b < 1$.

When $0 < a < 1$ and $0 < b < 1$, the integrand is unbounded at both end points of the interval $(0, 1)$. We argue that integral over $(0, 1/2)$ converges and integral over $(1/2, 1)$ converges and argue as earlier (see gamma function discussion) that the integral over $(0, 1)$ convergs and actually it converges to the sum of the above two, namely, integral over $(0, 1/2)$ and integral over $(1/2, 1)$.

Finally when both $a < 0$ and $b < 0$, the integral does not converge. Ideas needed are already present in the above argument.

There is a close relation between the beta and gamma integrals. These are two important integrals that arise in practice. As you see in both cases we have positive integrands. here is one interesting specific improper integral where the integrand takes negative and positive values. This integral also appears in several discussions, especially Fourier series and integrals.

$\int \frac{\sin \mathbf{x}}{\mathbf{x}} = \frac{\pi}{\mathbf{2}}$:

1. We first show that the integral is convergent. That is,

$$I_A = \int_0^A \frac{\sin x}{x}$$

has a finite limit as $A \to \infty$. Note that the integral is improper only at infinity, th integrand is bounded near 0.

We first show that given $\epsilon > 0$, there is a number A_0 such that $\int\limits_A^B \frac{\sin x}{x} < \epsilon$ for $B > A > A_0$. Indeed take $A_0 > 4/\epsilon$. Let $B > A > A_0$. Then

$$\int_A^B \frac{\sin x}{x} = -\frac{\cos B}{B} + \frac{\cos A}{A} + \int_A^B \frac{\cos x}{x^2}$$

so that

$$\left| \int_A^B \frac{\sin x}{x} \right| \leq \frac{2}{A} + \int_A^B \frac{1}{x^2} \leq \frac{4}{A} < \epsilon.$$

Just as Cauchy sequences have limits, such functions have limits and is argued as follows. Take any sequence increasing to ∞, say $\{n\}$. Clearly, the above inequality shows that $\{I_n\}$ is a Cauchy sequence and hence ha s a finite limit, say c. We show that $I_A \to c$. For this, fix $\epsilon > 0$. First choose A_0 so that

$$B > A > A_0 \Rightarrow \left| \int_A^B \frac{\sin x}{x} \right| < \frac{\epsilon}{2}.$$

Choose $k > A_0$ so that

$$n \geq k \Rightarrow |I_n - c| < \frac{\epsilon}{2}.$$

Now

$$A > n_0 \Rightarrow |I_A - c| \leq |I_A - I_k| + |I_k - c| < \epsilon.$$

We now need to show that this limit equals $\pi/2$.

2. Did we evaluate any integral involving sine functions? Yes $\sin nx$ etc. But is there anything with denominator? Yes

$$\int_0^\pi \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} = \frac{\pi}{2}.$$

This is very simple. The sine and cosine formulae give

$$2\sin\frac{x}{2}\cos kx = \sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x.$$

Adding from $k = 1$ to $k = n$, we see

$$2\sin\frac{x}{2}\sum_1^n \cos kx = \sin(n+\frac{1}{2})x - \frac{\sin}{x}2.$$

or

$$\frac{1}{2} + \sum_1^n \cos kx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

integrating and noting that cosine integrals vanish, we get

$$\int_0^\pi \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} = \frac{\pi}{2}.$$

3. It is sufficient, in view of (1), to show that

$$\int_0^{(n+\frac{1}{2})\pi} \frac{\sin x}{x} \to \frac{\pi}{2}.$$

that is, should show

$$\int_0^{\pi} \frac{\sin(n+\frac{1}{2})x}{x} \to \frac{\pi}{2}.$$

4. It is sufficient to show, in view of (2) and (3), that

$$\int_0^{\pi} \left(\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right) \sin(n+\frac{1}{2})x \to 0$$

This follows from the following two claims:

$$\int_a^b \varphi(x)\sin\lambda x \to 0 \quad \text{for any } C^1 \text{ function } \varphi\,. \quad (*)$$

$$\left(\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right) \quad \text{a } C^1 \text{ function on } [0,\pi]. \quad (**)$$

Proof of (*) is again integration by parts as in (1).

$$\int_a^b \varphi(x)\sin\lambda x = \frac{\varphi(a)\cos\lambda a - \varphi(b)\cos\lambda b}{\lambda} + \int_a^b \varphi' \frac{\cos\lambda x}{\lambda}$$

So

$$|\int_a^b \varphi(x)\sin\lambda x| \leq \frac{2M}{\lambda} + \frac{M(b-a)}{\lambda} \to 0.$$

as $\lambda \to 0$. Here M is a bound for φ and φ' on the closed bounded interval $[a,b]$.

Proof of (**) is L'Hospital's rule. Of course as it stands value of the function at zero is not defined but one shows that the function converges to zero as $x \to 0$ and hence by defining the value at zero to be ero we see it is a continuous function on $[0,\pi]$. To see that the limit at zero is zero, differentiate numerator and denominator of

$$\frac{2\sin\frac{x}{2} - x}{2x\sin\frac{x}{2}}$$

twice and see.

Similarly, one shows that the function has derivative at every point of $[0, \pi]$ and it is continuous. This is again by L'Hospital's rule.

This completes the proof.

While “pleasure” and “enjoyment” are often used to characterize one’s efforts in science, failures, frustrations, and disappointments are equally, if not the more, common ingradeints of scientific experience. Overcoming difficulties, undoubtedly, contributes to one’s final enjoyment of success.

S. Chandrasekhar.

“If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.”

John von Neumann

Following is expected from you.

Reach the class room on time.

If you have to be absent, find out what was done — before you come to the next class. Also let me know why you were absent.

Every week there will be home assignment, please go through, think, work out, pen solutions and Finally read what you have written.

If you have trouble understanding an exercise, keep reading again and again. You are sure to succeed.

If an exercise asks you to do something complicated, see if you can do a simpler thing and build on your success.

Regarding discipline and hard work, stick to your school routine.
Regarding Math, breathe an air of freedom,
start thinking and questioning.
What is a proof? How to communicate your proof to others?

Make a habit to consult books in the library, for example,
Tom Apostol; Robert Bartle; Bartle and Donald Sherbert

Richard Courant and David Hilbert; Courant and Fritz John; Walter Rudin; George Simmons etc etc, unlimited!

* * * * * *

1. (a) Show

$$\begin{aligned} xy &\le \frac{1}{2}(x^2+y^2) \qquad x, y \in R. \\ 2\sqrt{xy} &\le x+y \qquad x>0, y>0. \\ |x_1+x_2+\cdots+x_{97}| &\le |x_1|+|x_2|+\cdots+|x_{97}|. \end{aligned}$$

   (b) If $x_1 \ge x_2 \ge \cdots \ge x_n \ge 0$ and $y_1 \ge y_2 \ge \cdots \ge y_n \ge 0$, show that

$$n\sum x_i y_i \ge \left(\sum x_i\right)\left(\sum y_i\right).$$

   (c) If x, y, z are non-negative numbers, show that

$$x^2+y^2+z^2 \ge xy+yz+zx,$$

$$(x+y)(y+z)(z+x) \ge 8xyz,$$
$$x^2y^2+y^2z^2+z^2x^2 \ge xyz(x+y+z).$$

2. I have a finite set S. Anand tells me that S has 13 elements, that is, he has a function $f : S \to \{1, 2, \cdots, 13\}$; one-to-one and onto. Bhakta tells me that S has 11 elements, that is, he has a function $g : S \to \{1, 2, \cdots, 11\}$; one-to-one and onto. We all believe that one of them (at least) must be wrong. They refuse to show us their functions. How do you convince them that one of them must be wrong?

3. Given a real number $x \in [0, 1]$ show that there exists a sequence of integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ each ϵ_i being one of $0, 1, \cdots, 9$ such that

$$x = \frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2} + \cdots + \frac{\epsilon_n}{10^n} + \cdots.$$

   This sequence $(\epsilon_1, \epsilon_2, \cdots)$ is called the decimal expansion of x. It is usually written as $\cdot\epsilon_1\epsilon_2\ldots$.

   Show that if a number x has two expansions, then one expansion must 'end' with all zeros (called terminating expansion) and the other must end with all 9 (called non-terminating expansion). Further, a number can not have more than two expansions.

4. We have described decimal expansions of numbers in the interval $[0,1]$. To complete the picture, show that every non-negative integer can be expressed as a finite sum $\eta_k(10)^k + \eta_{k-1}(10)^{k-1} + \cdots + \eta_1(10) + \eta_0$ where each η_i is one of the numbers $\{0,1,2\cdots,9\}$. Moreover such an expression is unique subject to $\eta_k \neq 0$, in case $k \geq 1$. We consider only such expressions.

   Conclude that if $x \in R$ and $x \geq 0$, then we can express

   $$x = \eta_k(10)^k + \eta_{k-1}(10)^{k-1} + \ldots + \eta_0 + \epsilon_1(\frac{1}{10}) + \ldots + \epsilon_j(\frac{1}{10})^j + \ldots,$$

   where each of the numbers ϵ and η are among $\{0,1,\cdots,9\}$ (with $\eta_k \neq 0$ in case $k \geq 1$). If we were to *denote* $1/10$ *by* z, and η_i by ϵ_{-i} then this expression takes the pleasing form

   $$x = \frac{\epsilon_{-k}}{z^k} + \cdots + \frac{\epsilon_{-1}}{z} + \epsilon_0 + \epsilon_1\ z + \epsilon_2 z^2 + \cdots.$$

   $$\text{symbolically,}\quad x = \epsilon_{-k}\epsilon_{-k+1}\ldots\epsilon_0\ \cdot\ \epsilon_1\epsilon_2\ldots\epsilon_j\ldots.$$

5. To conclude this circle of ideas about decimal representation, let us consider once again decimal expansion of numbers $x \in (0,1)$. Say that a decimal expansion $\cdot\epsilon_1\epsilon_2\ldots$ is recurring if (a block repeats after some stage) there are integers $k \geq 0$ and $l \geq 1$ such that

   $$(\epsilon_{k+1}\cdots\epsilon_{k+l}) \ = \ (\epsilon_{k+l+1}\cdots\epsilon_{k+2l}) \ = \ (\epsilon_{k+2l+1}\cdots\epsilon_{k+3l}) \ = \ \cdots.$$

   Show that x is rational iff it has a recurring expansion.

6. Recall the axioms for real number system $(R,+,\cdot,<)$ which we have adapted.

   Axiom set I: for addition $(+)$. Additive identity is denoted 0 and additive inverse of x is denoted $-x$.

   Axiom set II: for multiplication $(\cdot)$. Multiplicative identity is denoted $1(\neq 0)$, multiplicative inverse for $x \neq 0$ is denoted $1/x$.

   Axiom set III: says $(+,\cdot)$ are friendly. $x\cdot(y+z) = x\cdot y + x\cdot z$.

   Axiom set IV: $(<)$. For any x, y exactly one of $x < y$, $x = y$, $y < x$ holds. If $x < y$ and $y < z$ then $x < z$.
   We use $x \leq y$ as abbreviation for '$x < y$ or $x = y$'.

Axiom set V: says $(<)$ is friendly with $(+, \cdot)$, namely,
$(y < z$ implies $x + y < x + z$) and $(0 < x,\ 0 < y \Rightarrow 0 < x \cdot y)$.

Final Axiom set VI: is the least upper bound axiom. Let $S \subset R$ be non-empty. Suppose S has an upper bound — $(\exists y)(\forall x \in S)(x \leq y)$. Then S has a least upper bound —
$(\exists z)\ \{[\forall x \in S,\ x \leq z]\ \&\ [(\forall x \in S,\ x \leq y) \Rightarrow z \leq y]\ \}$.
Sometimes this axiom is also called 'continuity axiom' or 'completemess axiom', because it tells us that our geometric picture of real numbers as a line without breaks/gaps is justified.

Fix such a system $(R, +, \cdot, <)$ once and for all. Elements of R are called real numbers.

Here are some questions that need to be answered *immediately*. (1) Did we not know real numbers already? (2) What is this business of axioms? (3) Is there such a system at all? (4) How many such systems are there? (5) What is the relation of such a system to the real numbers we have been using all along, in particular, in school?

In a nutshell here are the answers. (1) Yes, we have working knowledge of real numbers, afterall we have been working with real numbers. We know real numbers just as we know 'colour' or 'green colour' and so on. Think about it. (2) The axioms are the rules we accept once and for all about numbers. You can be assured that when we have a question (about real numbers) to be answered, we only use these few rules and would not make new rules. (3) Yes, such a system exists. We postpone construction to the end of the course, not because it is difficult,but because it is boring/dull. We need to collect some bricks, arrange them, throw in some cement, water and so on. (4) Such a system is unique, in the sense, if there are two such systems then there is an isomorphism between them that preserves the operations $(+)$ and $(\cdot)$ as well as the relation $\leq$. (5) What we have been using all along is just such a system, no more and no less. You need not panic. To justfy this, we need to show that everything we used so far about real numbers can be *deduced* from the few axioms listed above. This is interesting, though tiresome at times. We see some examples in the class, just to convince ourselves.

Show sum and product of rational numbers is rational. Show that sum and product of two algebraic numbers is algebraic. Do you think this will be true if algebraic (through out the sentence) is replaced by irrational?

Is there a rational number whose square is 42? Write a rigorous argument for your answer so that others are convinced after reading it. (You will not be there to explain *what you meant*, they only read *what you have written*).

Let me remind you that having square roots and cube roots (for positive numbers) is not our birth-right. Think of the set of rational numbers. It does not have a number whose square is 3. The set of reals R can provide a number whose square is 3, but is unable to provide a number whose square is (-3). Later, you will know that the set of complex numbers provides this also. Think about these matters *till* you are convinced that what we are doing in class *needs* to be done.

I would also like you to appreciate the subtle point in showing that *there is no one-to-one map on* $\{1, 2, \cdots, 11\}$ *onto* $\{1, 2, \cdots, 13\}$. You may start thinking: if I associate with 1, then with 2 etc, I run out of numbers in the domain but numbers remain in the range etc etc. It is fine, but does not solve our problem. It is important and necessary to feel what goes wrong, but you have to support the feeling with argument that no-one can refute. For example, some one says, I have such a function, but I will not show you. How do you convince him: you can not fool me; I know, for sure, you are wrong?

I am calling your attention to tricky points in arguments so that you get a hang of 'what is a proof' and 'how to write a proof.'.

You have to keep two things separate: abstraction and clarity. We are not trying to be abstract. We want to be clear about what we are saying. The purpose of saying something is to get across that thing to another person. If he/she did not understand what you said, the purpose is lost. That is why clarity is important. For this, you should first know what you are saying.

One important thing to keep in mind is that, in the middle of all this, we should not loose track of the inherent beauty of ideas and arguments.

7. Given a real number $x \in [0,1]$ show that there exists a sequence of integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ each ϵ_i is either zero or one such that

$$x = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots + \frac{\epsilon_n}{2^n} + \cdots.$$

   This sequence $(\epsilon_1, \epsilon_2, \cdots)$ is called binary expansion of x. It is usually written as $\cdot\epsilon_1\epsilon_2 \ldots$.

   Show that if a number x has two expansions, then one expansion must 'end' with all zeros (called terminating expansion) and the other must end with all ones (called non-terminating expansion). Further, a number can not have more than two expansions.

8. Let us now fix an integer $r \geq 2$. Let me repeat that this integer is fixed. Given a real number $x \in [0,1]$ show that there exists a sequence of integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ each ϵ_i being one of $0, 1, \cdots, r-1$ such that

$$x = \frac{\epsilon_1}{r} + \frac{\epsilon_2}{r^2} + \cdots + \frac{\epsilon_n}{r^n} + \cdots.$$

   This sequence $(\epsilon_1, \epsilon_2, \cdots)$ is called expansion of x to the base r. It is usually written as $\cdot\epsilon_1\epsilon_2 \ldots$.

   Show that if a number x has two expansions, then one expansion must 'end' with all digits being zero after some stage (called terminating expansion) and the other must end with all digits being $r-1$ after some stage (called non-terminating expansion). Further, a number can not have more than two expansions.

   When $r = 2$ you get binary expansion; $r = 10$ gives decimal expansion; $r = 3$ gives ternary expansion for the number x.

   Why are you boring us with all these expansions? Well, if there is a main center in a town, it is good /important to know several roads that lead to the place — later you can choose whatever road is convenient for you.

9. Let S be a non-empty subset of R.

   Say that S is bounded below if there is an $a \in R$ such that $a \leq x$ holds for every $x \in S$. Any such number is called a lower bound for the set

S. Greatest lower bound (glb), if exists, is a number l such that it is a lower bound for S and $m \leq l$ holds for every lower bound m of S. glb of S is also called infimum of the set S.

Say that S is bounded above if there is an $b \in R$ such that $x \leq b$ holds for every $x \in S$. Any such number is called an upper bound for the set S. lub of S is also called supremum of the set S.

Say that S is bounded if there is an $a \in R$ and an $b \in R$ such that $a \leq x \leq b$ holds for every $x \in S$.

Show that the lub axiom for R, which we have assumed, implies the following:
(i) Every non-empty subset of R which is bounded below has a glb.
(ii) Every non-empty subset of R which is bounded has a glb
(iii) Every non-empty subset of R which is bounded has a lub.
Show conversely the following: If you did not assume the lub axiom for R, but instead assumed any one of the above three statements then you can prove lub axiom as a theorem.

Moral: We did not show any partiality in assuming lub axiom, ti is same as glb axiom.

10. It will be good to have a criterion to recognize lub of a set. Let S be a non-empty set bounded above. Show the following. A number s is lub of S iff the following two conditions hold: (i) If $x \in S$ then $x \leq s$ and (ii) if $\epsilon > 0$ then there is at least one element $x \in S$ such that $x > s - \epsilon$.

    Let S be a non-empty subset of R which is bounded below. A number m is glb of S iff following two conditions hold: (i) If $x \in S$ then $m \leq x$ and (ii) if $\epsilon > 0$, there is at least one element $x \in S$ such that $x < m + \epsilon$.

11. I have an interval $[a, b]$. Can there be two numbers x, y in this interval such that $x - y > b - a$?. Suppose I have two intervals $[a, b]$ and $[c, d]$. Suppose every element of $[a, b]$ belongs to $[c, d]$. In other words $[a, b] \subset [c, d]$. Show that $c \leq a \leq b \leq d$.

12. Let S be a non-empty subset of R. Define a new set, $T = \{-x : x \in S\}$. If S is bounded above then show that T is bounded below. Also show that, if s is lub of S, then $-s$ is glb of T.

13. Let A and B be two non-empty sets of positive real numbers. Let us make a new set, $C = \{xy : x \in A, y \in B\}$ If s is lub of A and t is lub of B, show that st is lub of C.
    Do you think this will be true if A and B are arbitrary (not necessarily positive) non-empty subsets?

14. Let A and B be two non-empty sets of real numbers. Let us make a new set, $C = \{x + y : x \in A, y \in B\}$ If s is lub of A and t is lub of B, show that $s + t$ is lub of C.

15. Let $x \in R, x \neq 0, x \neq 1$. Consider the sequence $a_n = x^n$ for $n = 1, 2, 3, \cdots$. Show that this is an increasing sequence iff $x > 1$. Show that this is a decreasing sequence iff $0 < x < 1$. Show that this is a monotone sequence iff $x > 0$.

16. Let $x \neq 0$. Find conditions on x so that the set $\{x^n : n \in Z\}$ is bounded. Find conditions on x so that the set $\{x^n : n \in N\}$ is bounded.

17. Show that the set P, of polynomials in one variable x with integer coefficients, is a countable set. Show that the set of algebraic numbers is a countable set.

    Numbers which are not algebraic are said to be transcendental.

18. Show that every interval (a, b) with $a < b$ is uncountable. In fact show that every such interval has the same number of elements as the interval $(0, 1)$.

    Show that between any two distinct real numbers there is a transcendental number.

We shall discuss some of these problems next week.

I hope you are paying attention to the first page of the first assignment. In particular, solve and write the solution and equally important, read what you have written. This last instruction is to be executed sincerely.

Sometimes I do not understand what I have written. Obviously, I can not expect others to understand.

Sometimes I understand, but it is incorrect.

Sometimes I understand, it is correct, but there are some steps for which I have not provided justification. If I did not justify, then the proof is incomplete (and also the reader might think that I am bluffing my way through).

Remember, you will not be sitting next to the reader explaining what you meant! Writing a proof needs practice and you have plenty of time (unless, you want to convert exam as practice session!).

If you have any doubts, you are free to discuss with other students or meet me.

19. Draw the number line and plot the following set.

    (i) $\{x : x^2 - 5x + 6 > 0\}$ (ii) $\{x : x^2 - 5x + 6 \leq 0\}$

    (iii) $\{x : -5 < x^4 < 16\}$.

20. I am sure you all know the following. Prove them.

$$\sum_1^n i = \frac{n^2}{2} + \frac{n}{2}; \qquad \sum_1^n i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

$$\sum_1^n i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}; \qquad \sum_1^n i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

21. For each of the intervals I: $(-23, +85)$ $[-23, +85)$ $(-23, +85]$ $[-23, +85]$ show that $\sup A = +85$ and $\inf A = -23$.

22. For a number $x > 0$ we defined $\sqrt[n]{x} = \sup\{z > 0 : z^n < x\}$. Show that $\sqrt[n]{x} = \inf\{z > 0 : z^n > x\}$.

23. Solve: $x > 0$, $x^2 - x - 1 = 0$.

We are taught in school how to solve quadratic equations. We get $x = \frac{1 \pm \sqrt{5}}{2}$. Since we want positive solution we take $(1+\sqrt{5})/2$. Solving quadratics with the formula $[-b \pm \sqrt{b^2 - 4ac}]/2a$ is very important. It is such an excellent answer, we stopped thinking further about the problem. Let us think afresh. First of all x can not be zero.

Want $x^2 = x+1$. That is, $x = 1 + \dfrac{1}{x}$. This is very interesting equation. It does not give us the value of x, but explains x in terms of x. We can use this information for 'self improvement'. Use this value of x on right side,

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{x}}; \quad x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{x}}}$$

It is natural to believe that the solution is

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Non-sense or meaningful? We are the masters to give a meaning.

Put $x_0 = 1$, put inductively for $n \geq 1$, $x_n = 1 + \dfrac{1}{x_{n-1}}$. Shall show the sequence (x_n), so defined, converges to the solution.

If $\frac{3}{2} \leq x_{n-1} \leq 2$ then show that the same holds for x_n too. Show x_1 satisfies these inequalities and hence all x_n for $n \geq 1$, satisfy. Show

$$|x_{n+1} - x_n| \leq (\frac{2}{3})^2 |x_n - x_{n-1}|; \qquad n \geq 3.$$

Use this to show (x_n) converges. If the limit is a, show $\frac{3}{2} \leq a \leq 2$ and $a = 1 + \frac{1}{a}$ and hence $a = (1+\sqrt{5})/2$.

24. We have seen that a real number has several dresses — binary, decimal and so on — in which it can appear.

For example, consider a, the multiplicative inverse of $1+1+1$, you can write as $1/3$ or $\frac{1}{3}$ or 3^{-1}. When it wears decimal dress it appears to you as $0.33333333\cdots\cdots$. When it wears binary dress, it appears as $0.0101010101\cdots\cdots$. In ternary dress it has two different styles of appearance $0.100000\cdots\cdots$ and $0.0222222222222\cdots\cdots$..

We discuss one more colourful dress possessed by numbers.

For any non-negative number a, let $\langle a\rangle$ and (a) denote the largest integer not exceeding a and the fractional part of a respectively. Thus, $(a) = a - \langle a\rangle$. It is customary to denote $\langle x\rangle$ by $[x]$. Unfortunately, in the present context, the brackets [ ] are reserved for something else.

Fix $x \in (0, 1)$, we define a sequence (finite or infinite) of integers $(n_1, n_2, \cdots,)$, where each $n_i \geq 1$, as follows. Set

$$n_1 = \langle 1/x\rangle; \quad r_1 = (\frac{1}{x}) = \frac{1}{x} - n_1.$$

In general, having defined n_i and r_i for $1 \leq i \leq k$; put

$$n_{k+1} = \langle 1/r_k\rangle; \qquad r_{k+1} = (\frac{1}{r_k}) = \frac{1}{r_k} - n_{k+1}.$$

If at some stage we find $r_k = 0$, we stop and say that $[n_1, n_2, \ldots, n_k]$ is the *continued fraction expansion* of x. If this process continues for ever, then we say that the infinite sequence $[n_1, n_2, \ldots]$ is the continued fraction expansion of x. In the first case, we say that x has a terminating expansion and in the second case, we say it has a non-terminating expansion.

We write $x = [n_1, n_2, n_3, \cdots]$ or also as (which occupies more space)

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}}}.$$

What is the meaning of left side above? If each $a_i > 0$, we define $[a_1, \cdots, a_k]$ by induction on k as follows. (pause/think)

$$[a_1] = 1/a_1; \quad \text{for } k > 1, \quad [a_1, \cdots, a_k] = [a_1, \cdots, a_{k-2}, a_{k-1} + \frac{1}{a_k}].$$

We define $[n_1, n_2, n_3, \cdots] = \lim_k [n_1, \cdots, n_k]$. Does this limit exist? Yes, and equals x. We postpone these matters, but do the following.

Show that $x \in (0,1)$ is rational iff the expansion is terminating. What is the expansion for $4/5$?, $144/89$? What number is $[1,2,3,4]$?

If $x > 1$ and not an integer, then its continued fraction expansion is given by $[n_0; n_1, n_2, \cdots]$ where $n_0 = \langle x \rangle$ and $[n_1, n_2, \cdots]$ is the expansion of (x). Notice that n_0 is separated from the rest by a semi-colon, so there is no confusion. If $x \geq 1$ is an integer, we simply say $[x;]$ is its continued fraction expansion (funny, semicolon followed by bracket?).

25. If $\{x_n\}$ converges, show that $\{|x_n|\}$ converges. Is the converse true?

26. For which real numbers x does the sequence $\{x^n\}$ converge. In such a case, what is the limit?

27. Show
$$\frac{(n+47)^{589}}{2^n} \to 0; \qquad \sqrt[n]{n^{43}} \to 1; \qquad \lim \sqrt[2n+1]{n^2+n} = 1.$$

28. If $\sum_1^{59} a_i = 0$, show that $\lim \sum_1^{59} a_i \sqrt{n+i} = 0$.

29. Show $\sqrt{2}, \quad \sqrt{2+\sqrt{2}}, \quad \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots \to 2.$

30. If $a_n > 0$ and $\lim(a_{n+1}/a_n) = L > 0$, then show that $\sqrt[n]{a_n} \to L$. Use this to evaluate limits of $\sqrt[n]{n}; \quad \sqrt[n]{n^5+n^4}, \quad \sqrt[n]{n!/n^n}$

    Use the last limit to show that $n! = n^n e^{-n} a_n$ where $\sqrt[n]{a_n} \to 1$.

    (Much later, you will learn something called Stirling's formula which gives a better understanding of $n!$).

31. Purba defines: a sequence (x_n) converges to a number x iff *given* any integer $m = 1, 2, \cdots$ *there is* an integer n_0 (possibly depending on m) such that $|x_n - x| < 9^{-m}$ for all $n \geq n_0$.

    Uma defines: a sequence (x_n) converges to a number x iff *given* any integer $m = 1, 2, \cdots$ *there is* an integer n_0 (possibly depending on m) such that $|x_n - x| < 2^{-m}$ for all $n \geq n_0$.

Do you think they are related to our definition? What if they replace $<$ by $\leq$?

32. Let $f : \{1, 2, 3, \cdots\} \to Q \cap (0, 1)$ be a bijection. Let $x_n = f(n)$. Calculate $\liminf x_n$ and $\limsup x_n$. Find the set of all limit points of the sequence.

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For R we have adopted six axiom sets. A system $(R, +, \cdot)$ satisfying the first three sets is called an *algebraic field* or simply, a field. A system $(R, +, \cdot, <)$ satisfying the first five sets of axioms is called an *ordered field*. A system satisfying all the six sets is called a *complete ordered field* or *complete Archimedean ordered field*. Thus Real number system is nothing but a complete ordered field.

These terms or useful when you communicate with others. You need not pay much attention to these technical terms, especially because all your teachers would probably be introducing many new words and it becomes hard for you. As far as our course is concerned, it is important to remember what the rule or axiom says and what can be done with those; rather than remembering just these technical names.

I am not yet sure if you have started writing solutions for exercises. If you could not solve an exercise and we did it in class, then you should, after the class, spend a few minutes analyzing why you could not solve and what is it that you missed. This is important.

33. Suppose that a sequence (x_n) converges. show that the following sequence also converges.

$$x_1, x_1, x_2, x_2, x_3, x_3, x_4, x_4, x_5, x_5, \cdots\cdots.$$

What is the n-th term of this sequence?

What if I repeated each term ten times.

What if I repeated n-th term n times, like

$$x_1, x_2, x_2, x_3, x_3, x_3, x_4, x_4, x_4, x_4, x_5, \cdots\cdots.$$

You may, at first sight, think that these are trivial and I am showing you the same sequence again and again. But please do recall the definition of sequence and convince yourself that these are all different sequences.

What if I deleted all odd terms, that is, consider the sequence,

$$x_2, x_4, x_6, x_8, x_{10}, x_{12} \cdots\cdots.$$

What if I take integers $1 \leq n_1 < n_2 < n_3 < n_4 < \cdots$ and defined a sequence $y_k = x_{n_k}$ for $k = 1, 2, 3, 4, \cdots$.

34. I have a sequence (x_n). I know that the sequence $y_n = x_{2n}$ $(n \geq 1)$ converges. I know that the sequence $z_n = x_{2n-1}$ $(n \geq 1)$ converges. That is, the sequence of even terms converges and the sequence of odd terms converges.

    Do you think the sequence (x_n) converges. Under what conditions on the limits of these two sequences can you conclude that the sequence (x_n) converges?

    Can you think of a generalization of the above.

35. Suppose a sequence (a_n) of real numbers converges to a number a. I have a polynomial of one variable $P(x)$. Show that the sequence of numbers $\{P(a_n)\}$ converges to the number $P(a)$.

36. Later we shall define the sine function rigorously. But starting with your knowledge, discuss the following. For which real numbers x does the sequence $\{(\sin x)^n\}$ converge. In case the sequence does not converge, explain what are all its limit points.

37. (not easy) If your sequence of numbers are getting close to a then successive averages of your numbers also get close to a.

    If a sequence $\{x_n\}$ converges, then show that the sequence $\{a_n\}$ where $a_n = \frac{1}{n}\sum_1^n x_k$ also converges. Actually, these averages also converge to the same limit as the original sequence.

    If a seqeunce $\{x_n\}$ is such that the sequence of successive averages converges, then the sequence $\{x_n\}$ is said to converge in the sense of *Cesaro*. Thus any convergent sequence is also convergent in the sense of Cesaro.

    Show that the converse is not true by considering the ± 1 sequence.

38. Show, without using the fact that Cauchy sequences converge, that sum and product of Cauchy sequences is again Cauchy. That is, if (x_n) and (y_n) are Cauchy sequences, then so are the sequences $(x_n + y_n)$ and $(x_n y_n)$. should not use the difficult fact that Cauchy sequences converge.

    Suppose that (x_n) is a Cauchy sequence and $x_n \neq 0$ for every n. do you think $(1/x_n)$ is a Cauchy sequence?

    The reason, I did not want you to use convergence of Cauchy sequences is the following. Afterall, if your world consists of the set of rationals (and no more) then also the definition of Cauchy sequence makes sense; but of course in this world now there are Cauchy sequences that do not converge.

39. Let (x_n) be a sequence and $a \leq x_n \leq b$ for each n. If $x_n \to x$, then show that $a \leq x \leq b$.

    More generally, show that the above inequality holds for any limit point of the sequence.

40. Write complete proof of the fact explained in class: The limsup and liminf of a bounded sequence are indeed limit points of the sequence. Remember these are defined as the supremum and infimum of the set of limit points. (We showed that this set is non-empty).

41. Let (x_n) be a bounded sequence.

    A number s is limsup of the sequence if and only if the following two conditions hold:

    (i) for any $\epsilon > 0$, there are only finitely many n such that $x_n > s + \epsilon$.

    (ii) for any $\epsilon > 0$, there are infinitely many n such that $x_n > s - \epsilon$.

    A number l is liminf of the sequence if and only if the following two conditions hold:

    (i) for any $\epsilon > 0$, there are only finitely many n such that $x_n < l - \epsilon$.

    (ii) for any $\epsilon > 0$, there are infinitely many n such that $x_n < l + \epsilon$.

42. Let (x_n) and (y_n) be bounded sequences. Show that

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

$$\liminf x_n = -\limsup(-x_n).$$

$$\limsup x_n = -\liminf(-x_n).$$

$$\limsup(29x_n) = 29\limsup x_n.$$

$$\liminf(29x_n) = 29\liminf x_n.$$

43. Let (x_n) be a bounded sequence.

    Define $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \cdots\}$. Show that (y_n) is decreasing and bounded below and hence converges to a limit. Show that this limit is indeed $\limsup x_n$. Thus limsup is actually limit of the supremums of the 'tails'(?) of the sequence (not only supremum of limit points).

    Define $z_n = \inf\{x_n, x_{n+1}, x_{n+2}, \cdots\}$. Show that (y_n) is increasing and bounded above and hence converges to a limit. Show that this limit is indeed $\liminf x_n$. Thus liminf is actually limit of the infimums of the 'tails' of the sequence (not only infimum of limit points).

44. A sequence (x_n) converges if and only if it is bounded and $\limsup x_n \leq \liminf x_n$.

    This last inequality is same as saying that the bounded sequence has excatly one limit point.

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45. Fix a number h. Let us define $(x)_n$ for $n \geq 0$ by

$$(x)_0 = 1, ; \quad (x)_n = x(x-h)(x-2h)\cdots(x-[n-1]h).$$

Show the following for $n = 1, 2, 3 \cdots$.

$$(x+y)_n = (x)_n + \binom{n}{1}(x)_{n-1}(y)_1 + \binom{n}{2}(x)_{n-2}(y)_2 + \cdots + (y)_n.$$

How does this read if $h = 0$?

46. Do you think a convergent sequence (a_n) must have a maximum, that is, an k such that $a_k \geq a_n$ for all n?

Do you think a convergent sequence (a_n) must have a minimum, that is, an k such that $a_k \leq a_n$ for all n?

Show that a convergent sequence (a_n) must have either a maximum or a minimum.

47. Suppose that $a_n > 0$ for each n and $\liminf a_n = 0$. Show that for infinitely many values of n the following happens:
a_n is strictly smaller than $a_1, a_2, a_3, \cdots, a_{n-1}$.

48. Suppose that $a_n > 0$ for each n and the sequence (a_n) converges to zero. Show that for infinitely many values of n the following happens:
a_n is strictly larger than $a_{n+1}, a_{n+2}, a_{n+3}, \cdots\cdots$.

49. Suppose $a_n \leq b_n \leq c_n$. if $\lim a_n = \alpha = \lim c_n$, then show that $\lim b_n$ exists and equals α. If you are only told that $\lim a_n$ and $\lim c_n$ exist, then can you conclude that $\lim b_n$ exists?

50. Test for convergence of $\sum a_n$ where,

$$a_n = \frac{2n}{2n+1} - \frac{2n-1}{2n}, \qquad a_n = (-1)^n \frac{n}{n+1}.$$

51. If $\sum a_n$ and $\sum b_n$ are series of strictly positive terms and $\lim \frac{a_n}{b_n} \to 1$, show that that the series $\sum a_n$ converges iff the series $\sum b_n$ converges. What if the limit were 1000 instead of one.

52. Test the following for convergence.

$$\sum \frac{2000}{\sqrt[3]{29n^4 - 35}}; \qquad \sum \frac{2000}{\sqrt[4]{29n^3 - 35}};$$

$$\sum \frac{1}{n(1.01)^n}; \qquad \sum \frac{\log n}{n^{1.0001}}; \qquad \sum \frac{n^{100000}}{n!}; \qquad \sum \frac{(100000)^n}{n!}.$$

You may need to use facts about log function, that we have not yet discussed.

53. Let $P(x)$and $Q(x)$ be two polynomials in one variable x. Assume that $Q(n) \neq 0$ for $n = 1, 2, 3, \cdots$. Discuss convergence of $\sum \frac{P(n)}{Q(n)}$.

54. If $\sum a_n^2$ converges, show that $\sum \frac{a_n}{n}$ converges.

    Sometimes a more general formulation will help solve the problem, because it gives an idea. If $\sum a_n^2$ and $\sum b_n^2$ converge, then show that the series $\sum a_n b_n$ converges absolutely.

    If $\sum |a_n|$ converges, show that $\sum a_n^2$ converges. If $\sum a_n$ converges, do you think that $\sum a_n^2$ converges?

55. Show that the series below converge iff $p > 1$.

$$\sum_{n \geq 100} \frac{1}{n \log n \, (\log \log n)^p} \qquad \sum_{n \geq 10000} \frac{1}{n \log n \log \log n \, (\log \log \log n)^p}$$

56. If $0 < r < 1$, you already know that $\sum r^n$ converges. Show that $\sum n^{100} r^n$ converges. More generally, if $P(x)$ is any polynomial in one variable x, show that $\sum P(n) r^n$ converges.

57. Show that for $0 < x < 1$, the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots$$

    converges.

58. Show that the series

$$\sum (-1)^n \frac{\sin(19/n)}{\log \log(100 + n)}. \qquad \sum \frac{1}{n} \sin \frac{1}{n}.$$

converge. You can proceed with your knowledge of sine function, though we develop this function rigorously later. Do you think this series is absolutely convergent?

59. discuss convergence of the following two series.

$$\sum_{n\geq 2} \frac{\log(n+1) - \log n}{(\log n)^2}; \qquad \sum_{n\geq 1} \frac{1\cdot 2\cdot 3\cdot \cdots \cdot n}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}.$$

For the first series, you need to know a little more than the definition of log function.

60. Let $0 < a < b < c < 1$. Show that the following series converges.

$$a + b + c + a^2 + b^2 + c^2 + a^3 + b^3 + c^3 + \cdots\cdots.$$

Show that the series $\sum x_n$ also converges, where $x_n = a^n + b^n + c^n$. Note that this series is different from the series above (just ask yourself: what is the first term, what is the second term). Do these two series have the same limit? Justify your answer.

61. Sometimes the statement of the problem is long and probably frightening, but the solution is immediate. Here are two such problems.

Let $\sum a_n$ be a series of non-zero numbers.

Suppose that there is an $\epsilon > 0$ and and an integer n_0 such that for every $n \geq n_0$

$$\frac{\log \frac{1}{|a_n|}}{\log n} > 1 + \epsilon.$$

Then show that the series $\sum a_n$ converges.

Suppose that there is an $\epsilon > 0$ and and an integer n_0 such that for every $n \geq n_0$

$$\frac{\log \frac{1}{|a_n|}}{\log n} < 1 - \epsilon.$$

Then show that the series $\sum a_n$ does not converge.

62. One suggestion to show convergence of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is to write it as

$$(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots$$

and show that this series converges. This needs justification, because the two series are different (ask yourself: what is the first term etc). Justify by doing the following. Show that the second series converges. So partial sums are Cauchy. Then argue that for the first series also partial sums are Cauchy.

63. The theorem on alternating series said the following: If $a_n \downarrow 0$, then the series

$$a_1 - a_2 + a_3 - a_4 + - \cdots$$

converges.

Here is a generalization. Let $\sum b_n$ be a series with bounded partial sums. Let $a_n \downarrow 0$. Then the series $\sum a_n b_n$ converges. Note that by taking the series $\sum b_n$ to be $\sum \pm 1$ series, you get the theorem on alternating series (remember, we are not demanding convergence of $\sum b_n$).

To prove this generalization, procees ad follows. Let (s_n) be the partial sums of the series $\sum b_n$. Show for $m > n$

$$a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + a_{n+3}b_{n+3} + \cdots + a_{m-1}b_{m-1} + a_m b_m =$$

$$-s_n a_{n+1} + s_{n+1}(a_{n+1} - a_{n+2}) + s_{n+2}(a_{n+2} - a_{n+3})$$
$$+ \cdots + s_{m-1}(a_{m-1} - a_m) + s_m a_m.$$

Use this to show convergence of the series $\sum a_n b_n$.

This generalization comes to our rescue in difiicult situations.

Using your understanding of sine and cosine functions, show that the series $\sum \sin nx$ has bounded partial sums. (Hint: multiply and divide partial sum by $\sin(x/2)$ and see.) Conclude that the series $\sum \frac{\sin nx}{n}$ converges.

64. Using Cauchy product of series, show
$\sin(x+y) = \sin x \cos y + \cos x \sin y$, and
$\cos(x+y) = \cos x \cos y - \sin x \sin y$.

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65. Suppose that (a_n) is a Cauchy sequence. Suppose that a subsequence converges to a. This means, there are $n_1 < n_2 < \cdots\cdots$ and if $b_k = a_{n_k}$ then the sequence (b_k) converges to a. Show that $a_n \to a$.

66. Suppose $x_n \to a$. Let π be a permutation of $\{1, 2, \cdots\}$. Put $y_n = x_{\pi(n)}$. Show that $y_n \to a$. (This is just to make sure that you do not confuse between sequences and series).

67. We have shown that Cauchy product of two absolutely convergent series is convergent. Show that it is absolutely convergent.

68. $\sum a_n$ is a convergent series of strictly positive numbers, show $\sum \frac{\sqrt{a_n}}{n}$ converges.

69. $\sum a_n$ is a series of positive numbers which does not converge. Show that $\sum \frac{a_n}{1+a_n}$ does not converge.

    If $s_n = a_1 + \cdots + a_n$, show that

$$\frac{a_{n+1}}{s_{n+1}} + \cdots + \frac{a_{n+k}}{s_{n+k}} \geq 1 - \frac{s_n}{s_{n+k}}$$

    Deduce that $\sum \frac{a_n}{s_n}$ does not converge.

    Show that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

    Deduce that $\sum \frac{a_n}{s_n^2}$ converges.

    Discuss convergence of

$$\sum \frac{a_n}{1+na_n} \qquad \sum \frac{a_n}{1+n^2a_n}.$$

70. $\sum a_n$ is a convergent series of strictly positive numbers. Put $r_n = \sum\limits_{m \geq n} a_m$. If $m < n$ show that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}.$$

Deduce that $\sum \frac{a_n}{r_n}$ does not converge.

Show that
$$\frac{a_n}{\sqrt{r_n}} \le 2(\sqrt{r_n} - \sqrt{r_{n+1}}).$$
Deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

71. Let $a > 0$. Choose a number $x_1 > \sqrt{a}$. Define recursively
$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}), \qquad n \ge 2$$
Show $x_n \downarrow \sqrt{a}$.

If $\epsilon_n = x_n - \sqrt{a}$, show
$$\epsilon_{n+1} = \frac{\epsilon_n^2}{x_n} \le \frac{\epsilon_n^2}{\sqrt{a}}.$$

72. Fix $a > 1$. Choose a number $x_1 > \sqrt{a}$. Define recursively
$$x_{n+1} = \frac{a + x_n}{1 + x_n} = x_n + \frac{a - x_n^2}{1 + x_n}.$$
Show $x_1 > x_3 > x_5 \cdots\cdots$.

Show $x_2 < x_4 < x_6 < \cdots\cdots$.

Show $\lim x_n = \sqrt{a}$.

73. Let $f : R \to R$.
Verify: : f is continuous at a is same as
$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)\Big\{[|x - a| \ge \delta] \bigvee [|f(x) - f(a)| < \epsilon]\Big\}.$$
understand the symbols as well as its meaning in words.
How do you understand its negation: f is not continuous at a. Remember, it is not simply putting a negation symbol $\neg$ before this formula. You need to write positive statement, that is, where negations do not appear for quantifiers.
Explain in symbols as well as words. (I repeat: expressing in words is

not simply saying: f is not continuous at a. It means, first express in symbols as mentioned above without negation symbols for quantifiers and then write it in words).

Verify: f is continuous on R is same as

$$(\forall a \in R)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)\Big\{[|x-a| \geq \delta] \bigvee [|f(x) - f(a)| < \epsilon]\Big\}.$$

understand the symbols as well as its meaning in words.
How do you understand its negation: f is not continuous on R. Explain in symbols as well as words.

74. Verfiy: $x_n \to x$ is same as

$$(\forall \epsilon > 0)(\exists k)(\forall n \geq k)(|x_n - x| < \epsilon).$$

Understand the symbols as well as the meaning in words.
Express its negation, $x_n \not\to x$ both in symbols as well as words.

75. Verify: x is a limit point of (x_n) is same as

$$(\forall \epsilon > 0)(\forall k)(\exists n > k)(|x_n - x| < \epsilon).$$

Express its negation: x is not a limit point of (x_n) both in symbols as well as words.

76. x is limsup of (x_n) is same as

$$(\forall \epsilon > 0)\Big\{\Big[(\forall k)(\exists n > k)(x_n > x - \epsilon)\Big] \bigwedge \Big[(\exists k)(\forall n > k)(x_n \leq x + \epsilon)\Big]\Big\}.$$

77. Let $f : R \to R$. We say that f is uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that $f(x) - f(y) < \epsilon$ whenever $|x - y| < \delta$. Verify the following formula expresses this.

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y)\Big\{[|x-y| \geq \delta] \bigvee [|f(x) - f(y)| < \epsilon]\Big\}.$$

What is its negation (in symbols as well as words).

78. Just like limit point of a sequence, you can define limit poiint of a set. Let $A \subset R$. Say that a number a is a limit point of the set A if there are infinitely many points of A close to a, more precisely, given $\epsilon > 0$, there are infinitely many points of A in $(a-\epsilon, a+\epsilon)$. Verify that the following formula expresses the same thing.

$$(\forall \epsilon > 0)(\forall k)\left(\text{cardinality}\left\{(a-\epsilon, a+\epsilon)\bigcap A\right\} \geq k\right).$$

What does it mean to say that a is not a limit point of the set A.?

79. Define $f : R \to R$ by $f(x) = 0$ if x is rational and $f(x) = 1$ if x is irrational.

Describe the set of all points a such that f is continous at a.

Do the same thing for the following functions:

(a) $f(x) = 0$ if x is an integer and $f(x) = 1$ when x is not an integer.

(b) $f(x) = 1$ if $x \geq 0$ and $f(x) = 55$ if $x < 0$.

(c) $f(x) = 1$ if $x > 0$; $f(x) = 55$ if $x < 0$ and $f(0) = 44$.

(d) $f(x) = [x]$, the greatest integer not exceeding x.

(e) $f(x) = x - [x]$, the fractional part of x.

80. You know that the function $f(x) = x^2$ is continuous. If I give $a = 2$ and $\epsilon = 1$ what will be your δ? What if $a = 20$ and same ϵ. What if $a = 200$ and same ϵ?

Let $f(x) = 1/x$ defined on the interval $(0, 33)$. Take $\epsilon = 0.1$ and $a = 5$, find δ. With the same ϵ and $a = 1$, $a = 1/5$ and $a = 1/100$ find δ.

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Most of you have done reasonably well in the midsem.

Some of you (those who got at least 30) have done very well, and should maintain this level. Remember maintaining current level also needs effort.

Some of you (those who got at least 20, but below 30) have done well. You can improve if you work/think slightly harder. Remember, you should not try to stay where you are.

Some of you (those who got less than 20) are not doing well at this stage. However, you are definitely capable, but you need to put in your best efforts. You can recover and do well. Remember, if you can learn to stand, it is not difficult to walk and then it is not difficult to run!

I hope you have all realized that the story of calculus is continuation of high school story with essential differences — concept of proof, clarity regarding what you can use and what you can not, understanding of the ideas, the freedom to question things (and not to blindly reproduce what teacher says), accept responsibility to what we write (and not to blame the book or someone else), developing the ability to communicate what you want to say, etc etc.

Some exercises in this set need thinking on your part, they are not routine.

81. Show that
$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$
is an integer. Here n is a natural number.

82. Let $a_1, a_2, \cdots, a_n$ be strictly positive numbers.

    Show that $\sqrt[2]{a_1 a_2} \leq (a_1 + a_2)/2$.

When n is an integer of the form $2, 2^2, 2^4, 2^8, 2^{16}, \cdots$ (the exponent of 2 is itself a power of 2), show that

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Actually this is true for all integers $n \geq 2$. You need not do.

However, let me explain how to do it for $n = 3$, if you know for $n = 2^2$. Start with a_1, a_2, a_3 and take $a_4 = \sqrt[3]{a_1 a_2 a_3}$ and try your luck.

83. For each natural number n, show that there is a natural number k (of course, depending on the given n) such that

$$(\sqrt{2} - 1)^n = \sqrt{k} - \sqrt{k-1}.$$

84. In introducing the number e we found that it is limit of a series of numbers and also it is limit of the *increasing* sequence $\left(1 + \frac{1}{n}\right)^n$.

    Show that the sequence $\left(1 + \frac{1}{n}\right)^{n+1}$ is *decreasing*. What is its limit?

85. If f is a montone function defined on an interval and has the intermediate value property, show that f is a continuous function.

    Intermediate value property means the following: If $a < b$; and u is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = u$.

    Do you think that intermediate value property without monotonicity will imply continuity?

    Remember continuous function defined on an interval has the intermediate value property.

86. Let $f(x)$ be defined on the real line as follows: if x is irrational number then $f(x) = 0$. If x is a rational number p/q in lowest terms, then $f(x) = 1/q$. Show that f is continuous at a iff a is irrational number.

    p/q in lowest terms means p and q are integers without common factor. This means: if x is a natural number that divides both p and q, then $|x| = 1$.

87. Let ABC be a triangle in the plane (A, B, C not on a line) . Suppose that a line L in the plane is given. Show that there is a line parallel to L that bisects the triangle into two parts of equal area.

88. Consider the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots .$$

If the n-th number is p_n/q_n, then $p_{n+1} = p_n + 2q_n$ and $q_{n+1} = p_n + q_n$.

Show that all the fractions are in lowest terms.

Show that p_n/q_n converges to $\sqrt{2}$.

89. Let $0 < a < b$ be given. Define

$$a_1 = \sqrt{ab} \qquad b_1 = \frac{a+b}{2}.$$

$$a_2 = \sqrt{a_1 b_1} \qquad b_2 = \frac{a_1+b_1}{2}.$$

and in general,

$$a_{n+1} = \sqrt{a_n b_n} \qquad b_{n+1} = \frac{a_n+b_n}{2}.$$

Show that the sequence (a_n) converges. Show that the sequence (b_n) converges. Show that these two limits are equal.

This is called arithmetic-geometric mean or arithmetico-geometric mean of the two given numbers.

90. For $x, y > 0$ show that

$$\frac{x^n + y^n}{2} \geq \left(\frac{x+y}{2}\right)^n .$$

What does this mean geometrically?

91. Let $0 \leq x \leq 1$. Put

$$s_1 = x; \qquad s_{n+1} = \frac{1}{2}\left(s_n + \frac{x}{s_n}\right) \qquad n \geq 1.$$

Show $|x - \sqrt{x}| \leq 1/4$ and $|s_{n+1} - \sqrt{2}| \leq |s_n - \sqrt{2}|/2$ and deduce

$$|s_n - \sqrt{x}| \leq \left(\frac{1}{2}\right)^{n+1}.$$

Thus $r_n = s_n/\sqrt{x} \to 1$. How fast does it converge? Show

$$r_{n+1} - 1 = \frac{1}{2r_n}(r_n - 1)^2; \qquad s_{n+1} - \sqrt{x} = \frac{1}{2s_n}(s_n - \sqrt{x})^2$$

Deduce that

$$0 \leq r_{n+1} - 1 \leq \frac{1}{2}(r_n - 1)^2.$$

92. Suppose $f : R \to R$ is differentiable and f' is a polynomial of degree $n-1$. Show that f must be a polynomial of degree n.

93. Let $f(x) = x^2 + bx + c$. Show that f is increasing for $x > -b/2$ and decreasing for $x < -b/2$.

94. Let $f : R \to R$ be a function. Then f is said to be an even function if $f(-x) = f(x)$ for every $x \in R$. f is said to be an odd function if $f(-x) = -f(x)$.

    Give two examples of even functions and two examples of odd functions.

    If f is differentiable and is even, show that f' is an odd function.

    If f is differentiable and is odd, then show that f' is an even function.

    If f' is odd show that f is even function.

    If f' is even and $f(0) = 0$, show that f is odd.

95. Find the point of intersection of: tangent to the curve $y = x^2 - x$ at the point $(2, 0)$ and tangent to the curve $y = 1 - x^2$ at the point $(1, 0)$.

96. Show that the functions

    $f(x) = \sqrt{x^2 - 1}$ defined on the interval $(1, \infty)$ and

    $g(y) = \sqrt{y^2 + 1}$ defined on the interval $(0, \infty)$

    are inverses of each other. Verify

$$f'(g(y))g'(y) = 1; \qquad g'(f(x))f'(x) = 1.$$

97. Show that the function

$f(x) = x^2 + 3x + 1$ defined on the interval $(1, \infty)$ is invertible.

Find its inverse g. Verify

$$f'(g(y))g'(y) = 1; \qquad g'(f(x))f'(x) = 1.$$

98. (difficult to understand but trivial to solve) Let $r_1, r_2, \cdots$ be an enumeration of the set fo rationals on the real line. For each $x \in R$, let us put

$$f(x) = \sum_{n: r_n \leq x} \frac{1}{2^n}.$$

That is, given a number x do the following: see if $r_1 \leq x$, if so put $1/2$ in your bag, if not do not put;see if $r_2 \leq x$, if so put $1/2^2$ in your bag, if not do not put. Continue this way. Now add all the numbers you have put in your bag. It is meaningful. What you get is declared as $f(x)$.

Show f is continuous at a point a iff a is irrational. for every $x \in (0, 1)$

99. Continuing the previous exercise, show the following: Let $D \subset R$ be a countable set given to you. Find a monotone function on R whose set of discontinuity points is exactly the given set D.

    (The idea is not to complicate life, but to see if you understood the previous exercise.)

100. Let f be a strictly increasing continuous function on a closed bounded interval $[a, b]$. Let $c = f(a)$ and $d = f(b)$. Show that range of f is exactly $[c, d]$. Show that the inverse function g, defined on $[c, d]$ is strictly increasing and is again continuous.

     Let f be as above but is moreover differentiable at every point in (a, b). Show that the inverse function g defined on $[c, d]$ is differentiable at every point in (c, d). show that

$$g'(y) = \frac{1}{f'(g(y))}, \quad c < y < d; \quad \text{and} \quad f'(x) = \frac{1}{g'(f(x))}, \quad a < x < b.$$

     Prove a similar statement for strictly decreasing continuous function on $[0, 1]$.

     Suppose that f is a continuous function on $[0, 1]$. Assume that it is one-to-one. That is, if $x \neq y$ then $f(x) \neq f(y)$. Then show that f must either be strictly increasing or strictly decreasing.

     (Idea is to see if you can unravel the meaning of: suppose f is neither increasing nor decreasing $\cdots$.)

101. Can you define a continuous function on the interval $[0, 1]$ whose range is the set of natural numbers.

     Let S be the set of irrational numbers on the interval $[0, 1]$. Is there a continuous function defined on S having its range the set of natural numbers.

(The idea is *not* to understand continuous functions defined on arbitrary subset S. Who cares about this in a first course of calculus. The idea is to see if you made friends with real numbers.)

102. Let $f(x) = x^{230}e^{-x}$. Show that $f(x) \to 0$ as $x \to \infty$. This means the following: given any $\epsilon > 0$, we can find x_0 so that $|f(x)| \leq \epsilon$ for all $x \geq x_0$.

     (Idea is to see if you are comfortable with numbers. Do not complicate life.)

     Use the above to calcuate

$$\lim_{x\to 0} \frac{1}{x} e^{-1/x^2}.$$

     Let $P(x)$ be any polynomial. Show that $P(x)e^{-x} \to 0$ as $x \to \infty$.

103. We have defined two functions f and g in the class and named them as $\sin x$ and $\cos x$ — without any evidence that they are indeed sine and cosine functions, in fact you may find it confusing. By looking at the series, it is not even clear that they take values in the interval $[-1, +1]$.

     In any case, $f(-x) = -f(x)$ and $g(-x) = g(x)$ follow from definition.

     You saw one evidence in the class: $f' = g$ and $g' = -f$.

     Here is another evidence: Show that $f^2(x) + g^2(x) = 1$ for every x. Deduce that these functions indeed take values in the interval $[-1, +1]$. Also when one of them takes the value zero, then the other one must take a value ± 1.

     Hre is another evidence. Using power series, argue

$$\frac{f(x)}{x} \to 1 \quad \text{as} \quad x \to 0.$$

     Here is another evidence, this is only for fun because we have no plans, now, of doing complex numbers. However let us see what can be achieved if we knew complex numbers. Write the series definition of exponential function, with ix instead of x, let us name it e^{ix}. Calculate its real part and imaginary parts. What do you see? Remember $i^2 = -1$.

Do you know something called DeMoivre's formula? do you see any glimpse of it?

104. I have a function f defined on the interval $[0, 1]$. I do not know what exactly is the function, but I know the following:

$$|f(x) - f(y)| \leq 589 \sin(|x - y|^{235}); \qquad x, y \in [-1, 1].$$

Show that f is a continuous function. Show that f is differentiable. Show that f is a constant function.

105. Let f be a continuous function on the interval $[0, 1]$ which is differentiable at every point of $(0, 1)$. Assume that $|f'(x)| \leq 33$ for every $x \in (0, 1)$. Show that

$$|f(x) - f(y)| \leq 33|x - y|; \qquad x, y \in [0, 1].$$

106. Consider the two power series:

$$P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots.$$

$$Q(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \cdots.$$

Show that for any $x \in R$, the first series converges iff the second series converges. Conclude that both have the same radius of convergence. Thus

$$\limsup \sqrt[n]{|\alpha_n|} = \limsup \sqrt[n]{|\alpha_{n+1}|}.$$

Let (a_n) and (b_n) be sequences of positive numbers. Suppose that the sequence $\{a_n\}$ converges to a finite non-zero number a. Then $\limsup a_n b_n = a \limsup b_n$.
Use this to show that the following power series also has the same radius of convergence as the earlier two.

$$T(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + \cdots.$$

107. Let $f(x) = \sin(1/x)$ defined for $x \neq 0$. Can you prescribe a number $f(0)$ so that f is continuous on R?

Let $f(x) = x\sin(1/x)$ defined for $x \neq 0$. Can you prescribe a number $f(0)$ so that f is continuous on R. Then will your function be differentiable at zero?

Let $f(x) = x^2\sin(1/x)$ defined for $x \neq 0$. Can you prescribe a value $f(0)$ so that f is continuous on R. Then will your function be differentiable at zero? Is the functuion $f'(x)$ continuous at zero?

Answer the same questions if $f(x) = x^3\sin(1/x)$.

This story can go on — why not higher powers? why only integer powers of x? why not powers for $1/x$ too within the sine function?

108. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

Show f is differentiable function on R.

More generally, Let P be a polynomial in one variable.

$$f(x) = \begin{cases} P(1/x)\, e^{-1/x^2} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

Show f is differentiable function on R.

109. Since you are learning linear algebra, here is an interesting formula. suppose I have nine differentiable functions $\{f_{ij}; 1 \leq i, j \leq 3\}$ on R. Define a function φ on R by

$$\varphi(x) = \begin{vmatrix} f_{11}(x) & f_{12}(x) & f_{13}(x) \\ f_{21}(x) & f_{22}(x) & f_{23}(x) \\ f_{31}(x) & f_{32}(x) & f_{33}(x) \end{vmatrix}$$

Here $|A|$ is determinant of A. Show that φ is differentiable and

$$\varphi' = \begin{vmatrix} f'_{11} & f'_{12} & f'_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f'_{21} & f'_{22} & f'_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f'_{31} & f'_{32} & f'_{33} \end{vmatrix}$$

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Some exercises below are computational, you need to think how large or how small are things. Some are theoretical, but they are easy. If you open your pen without understanding the problem or try random paths blindly, you will get frustrated. Please do not do so.

110. Find derivatives of the following functions.

$$e^{(x^{33})}. \qquad (e^x)^{33}. \qquad \sin(\cos x). \qquad \frac{e^{\cos x}}{1+x^2}.$$

111. Let $f(x) = -x\log x - (1-x)\log(1-x)$ $(0 < x < 1)$. Is it possible to define this function at the points zero and one so that it is continuous on the closed interval $[0,1]$? Where is the function increasing, where is it decreasing, Where is its maximum value, what is it? Sketch its graph.

112. Prove the following formula, called Leibnitz's formula.

If f and g are each differentiable n times, then so is the product fg and

$$(fg)^{(n)} = f^{(n)} + \binom{n}{1} f^{(n-1)}g^{(1)} + \binom{n}{2} f^{(n-2)}g^{(2)} + \cdots + g^{(n)}.$$

113. For any real number α let us define

$$\binom{\alpha}{0} = 1; \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\times 2\times 3\times\cdots\times n}.$$

If α is a positive integer, does this agree with what you know from high school?

Eventhough there is a formula for radius of convergence of power series, many times it is convenient to use ratio test. Let us fix any $\alpha \neq 0$. Consider the power series

$$\sum_0^\infty \binom{\alpha}{n} x^n.$$

If $\alpha \geq 1$ is an integer, this is actually a finite sum and so has radius of convergence ∞ and equals $(1+x)^n$. Prove it.

If α is not an positive integer, show that its radius of convergence is one. Show that for any $|x| < 1$, sum of the above series is $(1+x)^\alpha$. Ask Taylor for help.

Here are special cases.

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{1 \times 3}{2!}\left(\frac{x}{2}\right)^2 + \frac{1 \times 3 \times 5}{3!}\left(\frac{x}{2}\right)^3 + \cdots$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{1 \times 3}{2!}\frac{x^4}{2^2} + \frac{1 \times 3 \times 5}{3!}\frac{x^6}{2^3} + \frac{1 \times 3 \times 5 \times 7}{4!}\frac{x^8}{2^4} + \cdots.$$

114. Find the radius of convergence of the following power series (ratio test is better than the formula). All are easy, serve to test if you made friends with numbers.

$$\sum \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}\, x^n$$

$$\sum \frac{(2n)!}{n!n!}\, x^n. \qquad \sum \frac{(n+1)! - n!}{2^n}\, x^n. \qquad \sum \frac{(n!)^2}{(n^2)!}\, x^n. \qquad \sum \frac{n!}{n^n}\, x^n.$$

$$\sum \frac{e^{n^2}}{n!}\, x^n. \qquad \sum \frac{n^3}{e^n}\, x^n. \qquad \sum \frac{e^n}{n!}\, x^n. \qquad \sum \frac{2^n}{(n!)^{\sqrt{2}}}\, x^n.$$

115. Evaluate

$\lim\limits_{x \to 0^+} x^{1/3}(\log x)^3. \qquad \lim\limits_{x \to 0^+} x(\log x)^2. \qquad \lim\limits_{x \to 0}(\sin x)^x.$

Taking help of L'Hopital, evaluate

$\lim\limits_{x \to 0^+} \frac{\log \sin x}{\log(e^x - 1)}. \qquad \lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right).$

116. You know that a sequence of real numbers converges iff it is a Cauchy sequence. Similar result is true for uniform convergence of functions too. Let (f_n) be a seqence of functions on a set S. Say that the sequence is uniformly Cauchy if the following holds: given $\epsilon > 0$, there is n_0 such that $|f_n(x) - f_m(x)| < \epsilon$ for every $n, m \geq n_0$ and every $x \in S$.

Show that (f_n) converges uniformly on S to some function iff it is uniformly Cauchy sequence.

117. There are several important ideas in proof of the main theorem on power series.

Suppose, I have a series of functions $\sum f_n$ on a set S. Suppose there are numbers M_n such that $\sum M_n$ converges and for each n, the function f_n is bounded by M_n, That is, $|f_n(x)| \leq M_n$ for every n and every $x \in S$. Show that the series $\sum f_n$ converges uniformly. This is called Weierstrass M-test.

Suppose I have a sequence (f_n) of differentiable functions on $(-30, +30)$ and $f_n(0) \to 17$. Suppose that $f_n' \to g$ uniformly on $(-30, +30)$. Then show that the sequence $\{f_n\}$ converges uniformly to a function f; f is continuous, f is differentiable; $f' = g$.

118. Show that if

$$f(x) = \sum_1^\infty \frac{\cos nx}{n^{7/3}}$$

then

$$f'(x) = -\sum \frac{\sin nx}{n^{4/3}}.$$

119. find if the following sequences/series converge unformly on the intervals menioned.

(a) Sequence (f_n) where $\quad f_n(x) = \dfrac{n^2 x}{1 + n^3 x^2}, \qquad [-1, 1]? \qquad [0, \infty)?$

(b) $\sum_1^\infty (x \log x)^n$ on $(0, 1)$.

(c) $\sum n^{77} e^{-nx}$ on $(0.001, \infty)$? on $(0, \infty)$?.

(d) The series $\sum_1^\infty [1 - \cos(x/n)]$ converges uniformly on the interval $[-10^{10}, +10^{10}]$ but does not converge uniformly on R.

120. Similar to series which are made up of powers of x, there are important class of series which are made up of powers of n.

Let $(a_n : n \geq 1)$ be real numbers. A series of the form $\sum \dfrac{a_n}{n^x}$ is called Dirichlet series. We shall not complicate our life and be content with a simple instance.

Show that the series, $f(x) = \sum \frac{1}{n^x}$, converges uniformly on the interval $[1+\epsilon, \infty)$, whatever be $\epsilon > 0$.

Show that the series $g(x) = -\sum \frac{\log n}{n^x}$ also converges uniformly on the interval $[1+\epsilon, \infty)$ for every $\epsilon > 0$.

Argue that f is a continuous function on $(1, \infty)$; indeed f is differentiable on $(0, \infty)$; indeed $f' = g$.

121. A man in a boat on one shore of a river wishes to reach a point on the other shore that is 4 miles down the stream. The river is 2 miles wide and has negligible current. He can row at 4 mph and run along the opposite bank at 8 mph. If he rows along a straight line path to a point on the opposite bank and then runs along the opposite bank to his destination, where should he land to minimize time taken to reach destination.

122. A window in the form of a rectangle surmounted by an isosceles triangle (with top side of the rectangle as base), the altitude of the triangle being (3/8)-th of its base. If the perimeter of the window is 30 ft, find the dimensions of the window for admiting maximum light.

123. A wire bent in the form of a circle of radius a exerts an attractive force upon a particle on the axis of the circle (that is on the line through center and perpendicular to the plane of the circle). From the theory of attraction (let us believe it), this force is proportional to $h/(a^2+h^2)^{3/2}$ where h is the height of the particle above the plane of the circle. At what height is the attraction maximum?

{Birkbeck college once announced an evening lecture by John Buchan, with the title 'Margins of life' and I expected the speaker to talk about some scientific field, perhaps about viruses or very large molecules on the border line between inorganic matter and living organisms. Not a bit: what he spoke about, to my growing astonishment, was the importance for a student not to work too hard! A student should not devote his entire time to the study of his subject; he should leave a margin on which he could scribble notes on what went around him. I was quite amazed that such advice should be regarded as necessary; I felt that students were generally a scatter-brained lot and in my view ought to be encouraged to stick to their books. But the lecturer obviously thought that the opposite advice was necessary to prevent them from becoming narrow-minded. (Otto Frisch)}

The problems in this set are of two kinds — Profound looking statements which are trivial; not too complicated applications of the machinery we developed.

124. Show

$$\sum_1^\infty \frac{1}{(n+x)(n+x+1)(n+x+2)} = \frac{1}{2(x+1)(x+2)}.$$

Here x is a number which is not negative integer.

$$\sum_2^\infty \frac{\log[(1+\frac{1}{n})^n(1+n)]}{\log n^n \log(n+1)^{n+1}} = -\log 4.$$

$$\sum_2^\infty \frac{1}{n^2-1} = \frac{3}{4}; \qquad \sum \frac{n^2x^n}{n!} = (x^2+x)e^x.$$

$$\sum_1^\infty nx^n = \frac{x}{(1-x)^2}; \qquad \sum (n+1)x^n = \frac{1}{(1-x)^2}. \quad |x|<1.$$

$$\text{Simplify } \sum \frac{(n-1)(n+1)}{n!}.$$

125. Test the following series for convergence.

$$\sum \frac{\log n}{n\sqrt{n+1}}; \qquad \sum \frac{n!}{(n+2)!}; \qquad \sum \frac{1}{(\log n)^{1000}}; \qquad \sum \frac{1+\sqrt{n}}{(n+1)^3-1}.$$

$$\sum \frac{\sin(1/n)}{n}; \qquad \sum [1-n\sin(1/n)]; \qquad \sum \log(n\sin(1/n)).$$

$\sum a_n$ where a_n equals $1/n$ if n is a square and equals $1/n^2$ if n is not a square.

$\sum a_n$ where a_n equals $1/n^2$ if n is odd and $-1/n$ if n is even.

126. if $\sum a_n$ is a convergent series of positive terms, show that the series $\sum \sqrt{a_n a_{n+1}}$converges.

127. If the series $\sum a_n$ is absolutely convergent, show that the following series are also convergent. For the second eeries below we assume that all the a's are different from -1.

$$\sum a_n^2; \qquad \sum \frac{a_n}{1+a_n}; \qquad \sum \frac{a_n^2}{1+a_n^2}.$$

128. Evaluate the following limits using L'Hopital when needed.

$$\lim_{x\to 0} \frac{\log(\cos ax)}{\log(\cos bx)}. \qquad \lim_{x\to 1} \frac{\sum_1^n x^k - n}{x-1}. \qquad \lim_{x\to 0+} \frac{x - \sin x}{(x \sin x)^{3/2}}.$$

$$\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}. \qquad \lim_{x\to 1+} \frac{x^x - x}{1 - x + \log x} \qquad \lim_{x\to\infty} x^{1/x}.$$

129. Taking the help of Taylor, when needed (sometimes you do not need), prove the following.

(i) For $|x| < 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

$$\log\frac{1}{1-x} = \sum \frac{x^n}{n}.$$

$$\frac{1}{2}\log\frac{1+x}{1-x} = \sum_1^\infty \frac{x^{2n-1}}{2n-1}.$$

$$(1+x)\log(1+x) = x + \sum_2^\infty (-1)^n \frac{x^n}{n(n-1)}.$$

$$\log(x + \sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \cdots$$

$$+(-1)^{n+1}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!}\ \frac{x^{2n+1}}{2n+1}+\cdots.$$

(ii) For $x \in R$,

$$(1+x)e^{-x}=1+\sum_{2}^{\infty}(-1)^{n-1}\frac{n-1}{n!}x^n$$

$$\frac{e^x-1}{x}=1+\frac{x}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{n!}+\cdots$$

$$\sin^2 x=\frac{2x^2}{2!}-\frac{2^3x^4}{4!}+\frac{2^5x^6}{6!}-\cdots+(-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots.$$

$$\cos^2 x=1+\frac{1}{2}\sum_{1}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}.$$

$$e^x\sin x=x+x^2+\frac{2x^3}{3!}-\frac{4x^5}{5!}+\cdots$$

I do not think the general term is $\pm(n-1)x^n/n!$, have not checked.

130. Starting from $z_0 = 2$ use newton's approximation for finding $\sqrt{2}$. Calculate the first ten approximations using acalculator. Square them and see.

Do the same thing for $\sqrt[3]{2}$ starting from 2 again.

131. We defined uniform convergence of a sequence of functions, uniform continuity of a function. Here is another 'uniformity' phenomenon.

Let f be a function differentiable on $[0,1]$. Thus at zero and one the left and right derivatives exist Asssume that f' is continuous. Show the following. Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$x, a \in [0,1];\ x \neq a;\ |x-a| < \delta\ \Rightarrow \left|\frac{f(x)-f(a)}{x-a}-f'(a)\right| < \epsilon.$$

This verifies the ratio $[f(x)-f(a)]/[x-a]$ converges to $f'(a)$. However given $\epsilon > 0$, we can find δ that works for all point intervals in this interval.

132. Is the function $f(x) = \sin(1/x)$ uniformly continuous on the interval $(0, 1)$. What about the function $g(x) = x\sin(1/x)$ on the same interval $(0, 1)$. Is the function $f(x) = \sin(1/x)$ uniformly continuous on the interval $(1, \infty)$?

133. This exercise is not difficult. However, if you do not want to do, it is fine.

(i) For $0 < a \leq 1$ put

$$\zeta(s, a) = \sum_0^\infty (n + a)^{-s}.$$

Show that the series converges absolutely for $s > 1$. the function $\zeta(s) = \zeta(s, 1)$ is the Riemann zeta function. Prove

$$\sum_{h=1}^k \zeta\left(s, \frac{h}{k}\right) = k^s \zeta(s).$$

Prove

$$\sum_1^\infty (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s})\zeta(s); \qquad s > 1.$$

(ii) If $\sum a_n$ is a convergent series of positive numbers show that the series $\sum \sqrt{a_n}\, n^{-p}$ converges for $p > 1/2$.

(iii) Just like Cauchy product of power series, we can talk about product of Dirichlet series. Suppose that

$$A(s) = \sum_1^\infty \frac{a_n}{n^s}; \qquad B(s) = \sum_1^\infty \frac{b_n}{n^s}$$

are two absolutely convergent Dirichlet series. Show that

$$\sum_1^\infty \frac{c_n}{n^s} = A(s)B(s); \qquad c_n = \sum_{d|n} a_d b_{n|d}.$$

(iv) Show that for the Riemann zeta function $\zeta(s)$ satisfies,

$$\zeta^2(s) = \sum_1^\infty \frac{d(n)}{n^s},$$

where $d(n)$ is the number of divisors of n — including 1 and n.

134. Show

$$\int_{-\pi}^{+\pi} \sin mx \cos nx = 0$$

$$\int_{-\pi}^{+\pi} \sin mx \sin nx = 0 \quad m \neq n; \quad = \pi \quad m = n.$$

$$\int_{-\pi}^{+\pi} \sin mx \cos nx = 0 \quad m \neq n; \quad = \pi \quad m = n.$$

$$\int \frac{1}{x} \log x = \frac{1}{2}(\log x)^2.$$

Do not differentiate right side, try to go from left to right.

$$\int e^{ax} \sin bx = \frac{1}{a^2 + b^2} e^{ax}[a \sin bx - b \cos bx]$$

$$\int e^{ax} \cos bx = \frac{1}{a^2 + b^2} e^{ax}[a \cos bx + b \sin bx]$$

135. Define $f(x)$ as follows. equals $1 + x^2$ for $x \leq 0$; equals $1 + x^3$ for $0 < x < 2$; equals $(1 + x)^2$ for $x \geq 2$. Discuss continuity and differentiability of the function at the points $0, 1, 2$.

136. Let L, M, B, C be fixed real numbers. Define a function

$$U(x) = \frac{Lx + M}{x^2 - 2Bx + C}$$

Let U_n be the n-th derivative of U. Show

$$\frac{x^2 - 2Bx + C}{(n+1)(n+2)} U_{n+2} + \frac{2(x - B)}{n + 1} U_{n+1} + U_n \equiv 0.$$

137. Let

$$u_n = \int_0^{\pi/4} \tan^n x dx.$$

Show

$$u_{n+2} + u_n = \frac{1}{n+1}.$$

138. Suppose that φ and f are continuous functions defined on a bounded interval $[a, b]$. Put $f_0 = f$ and for $n \geq 1$

$$f_n(x) = \alpha + \int_a^x \varphi(t) f_{n-1}(t) dt; \qquad a \leq x \leq b.$$

Show that these functions converge uniformly to a function g on $[a, b]$. Show $g'(x) = \varphi(x) f(x)$ and $g(a) = \alpha$. Deduce

$$g(x) = \alpha e^{\int\limits_a^x \varphi(t) dt}.$$

139. If f is a continuous function on $[0, \infty)$ and for each x

$$f(x) = \int_0^x f(t) dt$$

show that $f \equiv 0$.

140. Let $a > 1$. Recall that $x \mapsto a^x$ is defined on $(-\infty, \infty)$ onto $(0, \infty)$ and its inverse map is the map defined on $(0, \infty)$ onto $(-\infty, \infty)$ denoted $y \mapsto \log_a y$. Calculate its derivative.

141. $D^n(e^{x^2/2}) = u_n(x) e^{x^2/2}$ where u_n is a polynomial of degree n.
$u_{n+1} = x u_n + u_n'$.
Show $D(e^{x^2/2}) = x e^{x^2/2}$ apply Leibnitz and show

$$u_{n+1} = x u_n + n u_{n-1}$$

Use these two equations to show $u_n'' + x u_n' - n u_n = 0$ Find a polynomial solution $x^n + a_1 x^{n-1} + a^2 x^{n-2} + \cdots + a_n = 0$ of this equation.

142. Show

$$(-1)^n \int_{-1}^1 (x^2 - 1)^n = \frac{2^{2n+1} (n!)^2}{(2n+1)!}.$$

143.
$$\binom{n}{k} = \frac{1}{(n+1)\int_0^1 x^k(1-x)^{n-k}dx}.$$

144. Calculate
$$\int \frac{x}{(x^2+1)^n}; \qquad n = 1, 2, 3, \cdots.$$

If
$$I_n = \int \frac{1}{(x^2+1)^n}dx$$
reduce I_n to I_{n-1} by writing
$$\frac{1}{(x^2+1)^n} = \frac{1}{(x^2+1)^{n-1}} - \frac{x^2}{(x^2+1)^n}.$$

145. f is a continuous function on $[a, b]$ and $\int_a^x f = 0$ for $a < x < b$. Show that $f \equiv 0$.

f is a continuous function on $[0, \infty)$ and $f(x) = \int_0^x f(t)dt$ for all $x \geq 0$. Show that $f \equiv 0$.

146. Let f be a non-negative continuous function on $[a, b]$ with maximum value M. Show
$$\left[\int_a^b f^n(x)dx\right]^{1/n} \to M.$$

147. Show
$$\lim \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \frac{\pi}{4}. \qquad \lim \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}} = \log(1+\sqrt{2}).$$

148. Calculate
$$\int \frac{\sqrt{1+\sqrt{y}}}{\sqrt{y}}dy; \quad \int_0^9 \sqrt{1+\sqrt{y}}dy.$$

149. First sketch the two curves below and then fnd the area enclosed between the two curves
$$y = x^3 - 12x; \qquad y = x^2.$$

150. Same as above for the following two curves: $y = x^3 - x$ and the tangent to the curve $y = x^3 - x$ at $x = -1$.

151. Same as above for the following curves:

$$x + y = 1; \quad x + y = -1; \quad x - y = 1; \quad x - y = -1.$$

152. All the functions f_n and f below are assumed to be continuous. suppose that $f_n \to f$ uniformly on $[0, 1]$. show that $\int f_n \to \int f$.

Suppose that all the functions are bounded by one, say, but the convergence is only pointwise. do you think integrals converge.

You will not be asked to prove Stirling's formula/Walli's product in the exam. so enjoy these and understand proofs. Appreciate how simple ideas lead to interesting results. Pause and think if you found something difficult.

153. I have a sequence (x_n) of strictly positive numbers with $\limsup x_n = a$. You do not need pen to show the following..

 (i) $\limsup \sqrt{x_n} = \sqrt{a}$. (ii) $\limsup \frac{x_n}{1+x_n} = \frac{a}{1+a}$.

 Assuming $a \neq 0$, do you think $(1/x_n)$ has lim sup equal to $1/a$?

154. suppose that (a_n) and (b_n) are strictly positive sequences of numbers. and $\frac{a_n}{b_n} \to 1$. show that the series $\sum a_n$ converges iff the series $\sum b_n$ converges. (need pen?).

 show that the series $\sum \binom{2n}{n} \frac{1}{2^{2n}}$. does not converge.

155. Show $\lim \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$.

156. Show that th coefficient of x^n in the binomial expansion of $\frac{1}{\sqrt{1-x}}$ is asymptotically (?) equal to $1/\sqrt{\pi n}$.

157. Let r be a non-negative number. Show that the series $\sum_1^\infty \left(\dfrac{nr}{n+1}\right)^n$ converges if $0 \leq r < 1$ and does not converge if $r \geq 1$.

158. Let f be a continuous function on R such that $\lim\limits_{x\to\pm\infty} f(x) = 0$. Show that f is uniformly continuous on R. Show that the same holds if $\lim\limits_{x\to\pm\infty} f(x) = 59$. (pen needed?)

159. Suppose that f is a one-to-one continuous function on an interval $[a, b]$. Show that f must be monotone. Show that the same is true even if you have an open interval (a, b).

 Do you think this is true without continuity?

160. I have a continuous function on $(1,3)$. I know it is differentiable at every point except 2. I also know that $f'(x) \to 97$ as $x \to 2, x \neq 2$. This means: given $\epsilon > 0$, there is a $\delta > 0$ such that $|f'(x) - 97| < \epsilon$ when $0 < |x-2| < \delta$.

Show f is differentiable at 2 and $f'(2) = 97$. (pen needed?)

161. A function f on an interval I is said to be convex if

$$x \in I, y \in I, 0 < \lambda < 1 \Rightarrow f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Let $I = [0,1]$ and f be defined as follows: $f(0) = f(1) = 1$ and $f(x) = 0$ if $0 < x < 1$. Show that f is convex but not continuous. Thus a convex function need not be continuous.

162. However if we have a convex function defined on an open interval then it is continuous. Argument follows. Let I be an open interval and f a convex function on I. All points x, y etc below are from I.

Show
$$a < b < c \Rightarrow \frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(b)}{c-b}.$$

Now close your pen. Show

$$a < b < c < d \Rightarrow \frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(b)}{c-b} \leq \frac{f(d)-f(c)}{d-c}.$$

Now take points $a < b < c < d$ Show that for any two points $x < y$ in (b,c)
$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(d)-f(c)}{d-c}.$$

If we have three numbers $p < q < r$ show that $|q| \leq \max\{|p|, |r|\}$.

Show that there is a number M such that

$$x, y \in (b,c) \Rightarrow |f(x) - f(y)| \leq M|x-y|.$$

Show that f is continuous in the interval (b,c). Deduce f is continuous on I. Remember I is open interval.

163. Let $f(x) = |x|$ for $x \in R$. Show that f is convex. Thus a convex function need not be differentiable.

However, it has left and right derivatives. Argument follows. So f is a convex function defined on an open interval I.

Show that

$$x < y < z \Rightarrow \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Now close your pen.

Take any point $a \in I$. Let $x_n \uparrow a, x_n < a$ for all n. show that $[f(a) - f(x_n)]/[a - x_n]$ increases and is bounded above (Remember I is an open interval). So it has a finite limit, say α.

Show that

$$\alpha = \sup \left\{ \frac{f(a) - f(x)}{a - x} : x < a \right\}.$$

Thus the number α does not depend on the sequence we have taken above.

show that for any sequence $a_n \uparrow a$, and $a_n < a$ for all n; we have

$$\frac{f(a) - f(a_n)}{a - a_n} \to \alpha.$$

The number α is called the left derivative at the point a.

Similarly, show that f has a finite right derivative at the point a

Show that the left derivative is not larger than the right derivative at every point.

164. Let f be a convex function on an open interval. Assume that f is differentiable. Show that f' is an increasing function, that is, $x < y$ implies $f'(x) \leq f'(y)$.

Let f be a convex function on an open interval which is two times differentiable. Show that $f''(x) \geq 0$ for all x.

165. A function f defined on an interval I is said to be concave if

$$x, y \in I; 0 < \lambda < 1 \Rightarrow f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

Show that f is concave iff $(-f)$ is convex. How do the above results read for concave functions?

166. Can you give a continuous function on R which is differentiable at all but one point?

Can you give a continuous function on R which is differentiable at all but two points? at all but 100 points? At all but countably infinitely many points?

Can you give a continuous function on $[0,1]$ which is differentiable at all but countably infinitely many points?

There are continuous functions on R which are nowhere differentiable, that is, at each point, $f'(x)$ does not exist. Sorry, I did not present such examples..

If $f_n \to f$ uniformly and each f_n is differentiable, do you think f must be differentiable?

167. Using product of power series show for $|x| < 1$,

$$\frac{\log(1+x)}{1+x} = x - (1+\frac{1}{2})x^2 + (1+\frac{1}{2}+\frac{1}{3})x^3 - \cdots$$
$$+(-1)^n(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})x^n.$$

168. Show that for $|x| < 1$,

$$\int_0^x \log\frac{1+t}{1-t}dt = 2\sum_1^\infty \frac{x^{2n-1}}{(2n-1)^2}.$$

169. Let $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$.

$$\int_{-1}^1 P_n(x)P_m(x)dx = 0; \qquad m \neq n. \quad \int_{-1}^1 P_n^2(x)dx = \frac{2}{2n+1}.$$
$$\int_{-1}^1 x^m P_n(x)dx = 0; \quad m < n; \qquad \int_{-1}^1 x^n P_n(x)dx = ?.$$

170. Which of the functions are integrable on $(0, \infty)$.

$$\frac{x}{\sqrt{x}(1+x)} \qquad \frac{x}{1+x^2} \qquad \frac{\sqrt{x+1}-\sqrt{x}}{1+x} \qquad \frac{1}{x+x^4}.$$

171. Show the existence of the following integrals and evaluate them

$$\int_1^\infty \frac{dx}{x^{1.00001}} \qquad \int_0^1 \frac{dx}{x^{0.999999}} \qquad \int_{-1}^1 \frac{dx}{x^{2/3}}.$$

172. Evaluate $\int\limits_0^1 \sqrt{1+\sqrt{x}}\, dx$.

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173. Let v and u be differentiable functions on an interval and are strictly positive and $v > 1$. Let $f(x) = \log_{v(x)} u(x)$, that is $\log u(x)$ to the base $v(x)$. Calculate its derivative.

174. Here is an expressive vocabulary.

The concept of 'limit' is a *tail phenomenon*. This means the following, Suppose you have two sequences (a_n) and (b_n) and suppose they agree on a tail, that is, there is a k such that $a_n = b_n$ for all $n \geq k$. Then either both sequences converge or none converges. If they do converge, then they have same limit.

Similarly, 'convergence of series' is a *tail phenomenon*. if you have two series $\sum a_n$ and $\sum b_n$ and if they agree on a tail, then either both series converge or none converges. Of course, if they converge, you can not conclude their value is same. Sum depends on *all* terms, not just the tail, of the series. Thus value of the series is not a tail phenomenon whereas convergence is.

'Continuity at a point a' and 'differentiability at a point a' for a function f are *local phenomena*. This means the following. Suppose you have two functions f and g and suppose they agree locally around a, that is, $f(x) = g(x)$ for every $x \in (a - 0.00001, a + 0.00000001)$. Then either both functions are continuous at a or none of them is continuous at a. Same applies for differentiability at a.

Prove these statements (pen not needed).

175. Discuss convergence of the following integrals.

$$\int_1^\infty \sin(1/x)dx; \qquad \int_1^\infty \sin(1/\sqrt{x})dx; \qquad \int_1^\infty \sin(1/x^2)dx.$$

$$\int_0^1 \sin(1/x)dx; \qquad \int_0^1 \sin(1/\sqrt{x})dx; \qquad \int_0^1 \sin(1/x^2)dx.$$

$$\int_1^\infty \sin(1/x^\alpha)dx. \qquad \int_1^\infty \frac{\sin x}{x^4}dx; \qquad \int_1^\infty \frac{\cos x}{x^{56}}dx.$$

$$\int_0^\infty e^{-x}\frac{\sin x}{x}dx; \qquad \int_1^\infty e^{-2x}\cos(4x)dx; \qquad \int_0^1 \frac{\cos 4x - \cos 5x}{x}dx.$$

$$\int_0^1 x^{n-1}\log x\; dx; \qquad \int_0^\infty e^{-x^2}dx; \qquad \int_{-\infty}^\infty e^{(x-5)^2/33}dx.$$

176. For the following, try giving rigorous arguments without using pen.

First recall the function f we have described in class: $f(x) = 0$ for $x \leq 0$ and $f(x) = \exp\{-1/x^2\}$ for $x > 0$. We showed that it is differentiable any number of times.

if a function is differentiable k times and the k-th derivative is also continuous then f is called a C^k-function. Thus for a C^1 function it is not enough if it is differentiable. The derivative should be continuous. So C^0 is just the collection of continuous functions.

If a function is C^k for every k then it is called infinitely differentiable and is named C^∞ function.

Show that the function $g(x) = f(x+1)$ is also C^∞. The function $h(x) = g(-x)$ is also C^∞. Show that product of two C^∞ functions is C^∞.

Show that there is a C^∞ function on R which is strictly positive in $(-1,1)$ and zero outside this interval.

If ψ is a function as above, then show that $\varphi(x) = \int\limits_{-\infty}^{x} \psi(t)dt$ is a C^∞ function. If $a = \int\limits_{-1}^{1} \psi(t)dt$ then the function φ/a is a C^∞ function which is zero upto some point, then increases to the value one and remains one from then on.

Show that $L(x) = \varphi(-x)$ is also C^∞.

Using these bricks show the following: if $a < b < c < d$ are given numbers, then there is a C^∞ function f which is zero below a; and increases from zero to one during $[a,b]$; and remains at 1 during $[b,c]$; and decreases from one to zero during $[c,d]$; and remains zero beyond d.

In particular $0 \leq f \leq 1$.

A function $f : R \to R$, is said to have compact support if $f(x) = 0$ for all x outside a bounded interval. Such functions are called compactly supported functions.

Thus the function constructed above is compactly supported.

177. Let f be a decreasing function on the interval $[1, \infty)$ which is strictly positive. Let $\sum_1^\infty a_n$ be a series.

If $\int_1^\infty f$ is finite and $|a_n| \leq f(n)$ for all n then show that the series $\sum a_n$ converges. [Discussion of Euler constant helps.]

if the series $\sum a_n$ converges and $f(n) \leq a_n$ for all n, then show that $\int_1^\infty f$ converges.

The above two statements are named 'integral test'. Use the integral test to show (i) the series $\sum n^{-p}$ converges for $p > 1$ and (ii) the series $\sum(1/n)$ does not converge.

178. Show

$$\int_1^\infty \frac{x}{1+x^3}\, dx = \int_0^1 \frac{1}{1+x^3}\, dx.$$

Left side is an improper integral and right side is not. So there is nothing improper about improper integrals.

179. Tricky problems.

$$\int_0^{\pi/2} \log \sin x dx = -\frac{\pi}{2} \log 2.$$

[One idea to struggle with: same as with cosine and hence same as their average.]

$$\int_0^\pi \log(1 - \cos x)dx = -\pi \log 2 = \int_0^\pi \log(1 + \cos x)dx.$$

$$\int_0^{\pi/2} \cos 2nx \quad \log \sin x dx = -\frac{\pi}{4n}; \quad n = 1, 2, 3, \cdots.$$

$$\int_0^{\pi/2} \cos 2nx \quad \log \cos x dx = -\frac{\pi}{4n} \cos n\pi; \quad n = 1, 2, 3, \cdots.$$

$$\int_0^\pi \cos nx \log[2(1-\cos x)]dx = -\frac{\pi}{n}; \quad n = 1, 2, 3, \cdots.$$
$$\int_0^\pi \cos nx \log[2(1+\cos x)]dx = -\frac{\pi}{n}\cos n\pi; \quad n = 1, 2, 3, \cdots.$$

180. show that the series $\sum\limits_{-\infty}^{\infty} e^{-n^2}$ converges.

181. Show that
$$0 < x \leq 1 \Rightarrow |x \log x| \leq 1/e.$$
Let $a > 0$. Prove that the series
$$1 + a(x\log x) + \frac{a^2}{2!}(x\log x)^2 + \frac{a^3}{3!}(x\log x)^3 + \cdots$$
uniformly converges to x^{ax} on $(0, 1]$. Deduce
$$\int_0^1 x^{ax}dx = \sum_0^\infty (-1)^n \frac{a^n}{(n+1)^{n+1}}.$$

182. If $I_{m,n} = \int x^m(\log x)^n$ then
$$(m+1)I_{m,n} = x^{m+1}(\log x)^n - nI_{m,n-1}.$$
If $I_{m,n} = \int \cos^m x \sin^n x$ then check (not sure)
$$\begin{aligned}(m+1)I_{m,n} &= -\cos^{m+1} x \sin^{n-1} x + (n-1)I_{m,n-2} \\ &= \cos^{m-1} x \sin^{n+1} x + (m-1)I_{m-2,n}.\end{aligned}$$
If $I_{m,n} = \int x^m(1+x)^n$ then check (not sure)
$$(m+1)I_{m,n} = x^{m+1}(1+x)^n - nI_{n+1,m-1}.$$
If $I_{m,n} = \int \frac{x^m}{(1+x^2)^n}$ then check (not sure)
$$2(n-1)I_{m.n} = -x^{m-1}(1+x^2)^{-(n+1)} + (m-1)I_{m-2,n-1}.$$
These are recursion realtions that help in calculating the integrals. Do not waste your time, last three formulae may not be correct, do integration by parts and see what you get.

Good Luck

BSc I/Calculus Chennai Mathematical Institute 24-09-2013

Total Marks: 40 Midsemester Examination Time: Two hours

Instructions: Justify your steps. Marks are shown in brackets.

Results from class-work can be used, but quote precisely.

Results from Home-Asgnmts, if used, must be proved.

(a) (a) Expansion of a number to the base 9 equals $0.11111111\cdots$ (all digits equal one). Calculate decimal expansion of that number.[3]

(b) Decimal expansion of a number equals 0.625. Calculate its continued fraction expansion. [3]

(b) Fix a number h. Define $(x)_n$ for $n = 0, 1, 2, 3\cdots$ by
$(x)_0 = 1$; $\quad (x)_n = x(x-h)(x-2h)\cdots(x-[n-1]h)$. Prove:
$(x+y)_n = (x)_n + \binom{n}{1}(x)_{n-1}(y)_1 + \binom{n}{2}(x)_{n-2}(y)_2 + \cdots + (y)_n$. [5]

(c) For every real number x, consider the following series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Compute precisely the set of all numbers x for which the series above

(a) is absolutely convergent.

(b) is convergent but not absolutely convergent.

(c) is not convergent, [10]

(d) (a) Let (x_n) be a sequence of strictly positive numbers and $x_n \to 43$. Let π be a permutation of $\{1, 2, 3, \cdots\}$ and $y_n = x_{\pi(n)}$. Does the sequence x_n/y_n converge. If so find its limit. [4]

(b) Suppose $f : R \to R$ is a function such that for every $x, y \in R$; $|f(x) - f(y)| \leq 31|x-y|^{1/19}$. Prove f is a continuous function.[4]

(e) (a) Consider the function $f(x) = x + x^2$. Exhibit a number $\delta > 0$ such that $|f(x) - f(1)| < 0.1$ whenever $|x-1| < \delta$. [5]

(b) Define f on $(0, 1)$ as follows: $f(x)$ is the second decimal digit in the non-terminating decimal expansion of x. For example if

$x = 0.1$, then its nonterminating expansion is $0.09999\cdots$ and hence $f(0.1) = 9$. Compute precisely set of points x in $(0, 1)$ at which f is discontinuous.

[6]