Chennai Mathematical Institute

Masters Thesis

Random Spanning Trees

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Declaration of Authorship

I, Bhishmaraj S, declare that this thesis titled, “Random Spanning Trees” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
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CHENNAI MATHEMATICAL INSTITUTE

Abstract

Faculty Name
Computer Science at CMI

Master of Science

Random Spanning Trees

by Bhishmaraj S

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too. . . . I see

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

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LAH	**L**ist **A**bbreviations **H**ere
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4.2.1 Techniques used

Naive chain rule algorithm

Input: $G = (V, E)$ and L_G^+
Output: Set of edges corresponding to a random spanning tree
for $e = (u, v) \in E$ **do**
  $R_e^{\text{eff}} = (\chi_u - \chi_v)^T \; L_G^+ \; (\chi_u - \chi_v)$;
  if $(X \sim Bernoulli(R_e^{\text{eff}})) = 1$ **then**
    Add edge e to the spanning tree;
    $G = G/e$;
  else
    $G = G \setminus e$;
  end
  Update L_G^+ ;
end

Algorithm 1: Sampling uniform spanning tree using chain rule

4.2.2 Facts used

Fact 1 (Woodbury matrix identity)**.** *Let $M \in \mathbb{M}_{n\times n}, U \in \mathbb{M}_{n\times k}, V \in \mathbb{M}_{n\times k}$. Suppose M is non-singular then $M+UV^T$ is non-singular $\iff$ $I+V^TM^{-1}U$ is non-singular. If $M + UV^T$ is non-singular, then*

$$(M + UV^T)^{-1} = M^{-1} - \left(M^{-1} \cdot U \cdot (I + V^TM^{-1}U)^{-1} \cdot V^T \cdot M^{-1}\right)$$

Proof. TODO ■

Fact 2. *For any $L \in \mathbb{M}_{n\times n}$ with $ker(L) = span(\mathbf{1})$, we have $LL^+ = I - \frac{\mathbf{1}\cdot\mathbf{1}^T}{n}$ and $P := I - \frac{\mathbf{1}\cdot\mathbf{1}^T}{n}$ is called the **projection matrix**.*

Fact 3 (Sub-matrices)**.** *For all the results below, S denotes a index set and S^c denotes it's complement.*

1. *For any $A, B \in \mathbb{M}_{n\times n}$, $(A + B)_{S,S} = A_{S,S} + B_{S,S}$*

2. *If $C = D \cdot E \cdot F$ then $C_{S,S} = D_{S,*} \cdot E \cdot F_{*,S}$*

3. *For $A \in \mathbb{M}_{m\times n}, B \in \mathbb{M}_{n\times l}$, If $A_{S^c,S^c} = 0$ or $B_{S^c,S^c} = 0$ then $(AB)_{S,S} = A_{S,S} \cdot B_{S,S}$*

4. *For any matrix C where $C = D \cdot E \cdot F$. If $D_{*,S^c} = 0$ and $F_{S^c,*} = 0$, then $C = D_{*,S} \cdot E_{S,S} \cdot F_{S,*}$*

5. $D = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, *and* $E = \begin{bmatrix} A & B \\ X & Y \end{bmatrix}$ *where $M, A \in \mathbb{M}_{n\times n}$ and If $(MA - I)$ is invertible Then,*

$$(DE - I)^{-1} = \begin{bmatrix} (MA - I)^{-1} & (MA - I)^{-1} \cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

Proof. $(DE - I)^{-1}$ can be computed using Shur's Complement(Wikipedia contributors, 2020) .

Suppse $N = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ and Shur's complement of block S and P is

$$N/S := P - QS^{-1}R \qquad N/P := S - RP^{-1}Q$$

Then

$$N^{-1} = \begin{bmatrix} P^{-1} + (P^{-1}Q(N/P)^{-1}RP^{-1}) & -(P^{-1}Q(N/P)^{-1}) \\ -((N/P)^{-1}RP^{-1}) & (N/P)^{-1} \end{bmatrix}$$

In our case $N = \begin{bmatrix} MA - I & MB \\ 0 & -I \end{bmatrix}$ and $(N/P) = -I$. From this it follows that

$$N^{-1} = (DE - I)^{-1} = \begin{bmatrix} (MA - I)^{-1} & (MA - I)^{-1} \cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

■

Fact 4. *Let $A, B \in \mathbb{M}_{n\times n}$ with B being symmetric PSD. Suppose x is an eigenvector of AB corresponding to eigenvalue λ. Then $\sqrt{B}x$ is an eigenvector of $\sqrt{B}A\sqrt{B}$ corresponding to eigenvalue λ*

Fact 5 (Laplacian and graph connectivity)**.** *Let G be a graph with n vertices. Suppose $(\lambda_1, \lambda_2 \cdots \lambda_n)$ be the eigenvalues corresponding to the eigenvectors $(v_1, v_2 \cdots v_n)$ of the Laplacian of G denoted as L_G. L_G is symmetric PSD with $\lambda_1 = 0$ and $v_1 = \mathbf{1}$. The following properties relate the eigenvalues of L_G with the connectivity of G :*

1. *$\lambda_2 > 0 \iff G$ is connected*
2. *G is disconnected $\iff \exists z$ with $z^T\mathbf{1} = 0$ and $z^T L_G z = 0$*

The above is true for L_G^+ also

Definition 1 (χ_u)**.** *χ_u is a vector of size $|V|$*

$$\chi_u(i) = \begin{cases} 1, & \text{if } i = u \\ 0, & \text{otherwise} \end{cases}$$

Definition 2 (Uniform random spanning tree)**.** *Let $\hat{T}$ be the random variable denoting a uniformly random spanning tree, then $\mathbb{P}(\hat{T} = T) = \frac{1}{|\mathcal{T}|}$, where $\mathcal{T}$ is the set of all spanning trees of G.*

Fact 6. *Given a graph $G = (V, E)$ with laplacian L_G, the effective resistance of an edge $e = \{u, v\} \in E$ is*

$$R_e^{\mathit{eff}} = (\chi_u - \chi_v)^T \; L_G^+ \; (\chi_u - \chi_v)$$

Then for any $e \in E$ we have

$$\mathbb{P}(e \in \hat{T}) = R_e^{\mathit{eff}}$$

4.2.3 Recursive Algorithm with lazy updates

Deletion

Lemma 1 (Formulas in **Theorem 1** are well defined)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq E$ then*
$\left(I - L_D L_G^+\right)$ *is invertible* $\iff$ $G \setminus D$ *is connected*

Proof. First let's show that If $(I - L_D L_G^+)$ is singular then $G \setminus D$ is disconnected

- Since $(I - L_D L_G^+)$ is singular $\exists x \neq 0$ s.t. $(I - L_D L_G^+)x = 0$

$$\implies L_D L_G^+ \; x = x \tag{4.1}$$
$$\implies 1 \in eigenvalues(L_D L_G^+) \tag{4.2}$$
$$\implies 1 \in eigenvalues((L_G - L_{G \setminus D}) L_G^+) \tag{4.3}$$

- Let $x \perp \mathbf{1}$ be an eigenvector of $(L_G - L_{G \setminus D}) L_G^+$ with eigenvalue 1.
- By **Fact 4**, $y = \frac{\sqrt{L_G^+} x}{\left\| \sqrt{L_G^+} x \right\|}$ is an eigenvector of $\sqrt{L_G^+}(L_G - L_{G \setminus D})\sqrt{L_G^+}$

$$= y^T \cdot \sqrt{L_G^+}(L_G - L_{G \setminus D})\sqrt{L_G^+} \cdot y = 1 \tag{4.4}$$
$$= y^T \sqrt{L_G^+} L_G \sqrt{L_G^+} y = (HOW) y^T L_G^+ L_G y = y^T P y \tag{4.5}$$
$$= y^T \left(I - \frac{\mathbf{1}^T \mathbf{1}}{n} \right) y = y^T y - \left(\frac{y^T \mathbf{1}^T \mathbf{1} y}{n} \right) = (HOW) y^T y = 1 \tag{4.6}$$

- $\therefore y^T \sqrt{L_G^+} L_{G \setminus D} \sqrt{L_G^+} y = 0$ now if we consider $z = \sqrt{L_G^+} y$ and show that $z^T \mathbf{1} = 0$ then we can use **Fact 5** to complete the proof

$$y^T \sqrt{L_G^+} \mathbf{1} = x^T \sqrt{L_G^+} \sqrt{L_G^+} \mathbf{1} = 0 (\text{HOW is 1 in kernel of } L_G^+ \tag{4.7}$$
$$G \setminus D \text{ is disconnected} \tag{4.8}$$

Now to prove the converse, If $G \setminus D$ is disconnected then $I - L_D L_G^+$ is singular

- If $G \setminus D$ is disconnected then $\exists y \perp \mathbf{1}, ||y|| = 1$ we have
  1. $y^T \cdot \sqrt{L_G^+} \cdot L_{G \setminus D} \cdot \sqrt{L_G^+} \cdot y = 0 (HOW)$
  2. $y^T \cdot \sqrt{L_G^+} \cdot L_G \cdot \sqrt{L_G^+} \cdot y = y^T y = 1$

- From (1) and (2) we get $y^T\sqrt{L_G^+}(L_G - L_{G\setminus D})\sqrt{L_G^+}y = 1$

$$\implies y^T \cdot \sqrt{L_G^+} \cdot L_D \cdot \sqrt{L_G^+} \cdot y = 1 \tag{4.9}$$

$$\implies 1 \in \text{eigenvalues}(L_D L_G^+)(HOW) \tag{4.10}$$

$$\implies (I - L_D L_G^+) \text{ is singular} \tag{4.11}$$

■

Theorem 1 (Update identity for Deletion)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq E$. If $G \setminus D$ is connected then*

$$(L_G - L_D)^+ = L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)$$

Proof. If R.H.S is indeed true then it should satisfy the property of $LL^+ = P$

$$(L_G - L_D) \cdot (L_G - L_D)^+$$

$$(L_G - L_D) \cdot \left(L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)\right)$$

$$\left[P - L_D L_G^+\right] - \left[(L_G L_G^+ - L_D L_G^+) \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right]$$

$$\left[P - L_D L_G^+\right] + \left[\left((L_D L_G^+ - I) + \frac{\mathbf{1}\mathbf{1}^T}{n}\right) \cdot \left((L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)\right]$$

$$\left[P - L_D L_G^+\right] + \left[(L_D L_G^+) + \left(\frac{\mathbf{1}\mathbf{1}^T}{n} \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)\right]$$

We can see that $-\mathbf{1}^T(L_D L_G^+ - I) = -\mathbf{1}^T L_D L_G^+ + (I\mathbf{1})^T = 0 + \mathbf{1}^T = \mathbf{1}^T$. Hence $\mathbf{1}^T(L_D L_G^+ - I)^{-1} = -\mathbf{1}^T$. And also $\mathbf{1}^T L_D = 0$. Hence,

$$P - L_D L_G^+ + L_D L_G^+ + \mathbf{1}^T L_D L_G^+ = P$$

■

Definition 3 (Submatrix)**.**

Corollary 1 (Improved **Theorem 1** for submatrix)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq G$. For $S \subseteq V$ define $E[S]$ as $(S \times S) \cap E$. Suppose $E_D \subseteq E[S]$ and $G \setminus D$ is connected then*

$$(L_G - L_D)^+_{S,S} = (L_G^+)_{S,S} - \left((L_G^+)_{S,S} \cdot ((L_D)_{S,S}\ (L_G^+)_{S,S}\ -\ I)^{-1} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S}\right)$$

Proof. From **Theorem 1** we know that $(L_G - L_D)^+ = L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)$. If we apply **Fact 3.1** to $(L_G - L_D)^+$ we get

$$(L_G^+)_{S,S} - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)_{S,S}$$

Applying **Fact 3.3** we get (HOW)

$$(L_G^+)_{S,S} - \left((L_G^+)_{S,S} \cdot (L_D L_G^+ - I)^{-1}_{S,S} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S}\right)$$

Now applying **Fact 3.5** to $(L_D L_G^+ - I)^{-1}_{S,S}$

Fact 3.5 states that If $D = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, and $E = \begin{bmatrix} A & B \\ X & Y \end{bmatrix}$where $M, A \in \mathbb{M}_{n \times n}$ and If $(MA - I)$ is invertible Then,

$$(DE - I)^{-1} = \begin{bmatrix} (MA - I)^{-1} & (MA - I)^{-1} \cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

Here we have $L_D = \begin{bmatrix} (L_D)_{S,S} & 0 \\ 0 & 0 \end{bmatrix}$, (HOW can it always be this way) and $L_G = \begin{bmatrix} (L_G)_{S,S} & (L_G)_{S,S^c} \\ (L_G)_{S^c,S} & (L_G)_{S^c,S^c} \end{bmatrix}$

$$\therefore\ (L_D L_G - I)^{-1} = \begin{bmatrix} ((L_D)_{S,S}(L_G)_{S,S} - I)^{-1} & (L_D)_{S,S}(L_G)_{S,S^c} \\ 0 & -I \end{bmatrix}$$

Hence we get the required result

$$(L_G - L_D)^+_{S,S} = (L_G^+)_{S,S} - \Big((L_G^+)_{S,S} \cdot ((L_D)_{S,S}\ (L_G^+)_{S,S}\ -\ I)^{-1} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S} \Big)$$

■

Contraction

Definition 4 (Incidence Matrix). *Let $G = (V, E)$, given an edge $e = u, v \in E$ the incidence vector of e is defined as $v_e = (\chi_u - \chi_v)$. Given a set of edges $D = \{e_1, e_2 \cdots e_m\} \subseteq E$, the incidence matrix of D is defined as $B_D = [v_{e_1} | v_{e_2} \cdots | v_{e_m}]$*

Definition 5 ($G+ke$). *$G+ke$ is the weighted graph obtained by increasing e's weight by k*

Lemma 2 (Formulas in **Theorem 2** are well defined). *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ with $|F| = r$ and let B_F be the incidence matrix of F.*

$$B_F^T\ L_G^+\ B_F \text{ is invertible} \iff F \text{ is a forest}$$

Proof. For any $x \in \mathbb{R}^r, x \neq 0,$ let $y = B_F x$. We first prove the following claim

Claim 1. *The incidence matrix of a acyclic graph has full column rank*

Proof. TODO ■

Therefore $y \neq 0$

■

Lemma 3 (Formulas in **Theorem 2** are well defined). *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$\text{If } F \text{ is a forest then } \left(\frac{I}{k} + B_F^T\ L_G^+\ B_F \right) \text{ is invertible for any } k > 0$$

Proof. TODO ■

Theorem 2 (Contraction update formula for finite k). *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$(L_G + k\ L_F)^+ = L_G^+ - \left(L_G^+ \cdot B_F \cdot (\frac{I}{k} + B_F^T\ L_G^+\ B_F)^{-1} \cdot B_F^T \cdot L_G^+ \right)$$

Proof. TODO ∎

Corollary 2 (Improves **Theorem 2** for sub-matrices)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. Suppose $F \subseteq E[S]$, where $S \subseteq V$. For any $k > 0$,*

$$(L_G + k\ L_F)^+_{S,S} = (L_G^+)_{S,S} - \Big((L_G^+)_{S,S}\ (B_F)_{S,*}\ (\frac{I}{k} + (B_F^T)_{S,*}\ (L_G^+)_{S,S}\ (B_F)_{S,*})^{-1}\ (B_F^T)_{S,*}\ (L_G^+)_{S,S} \Big)$$

Proof. TODO ∎

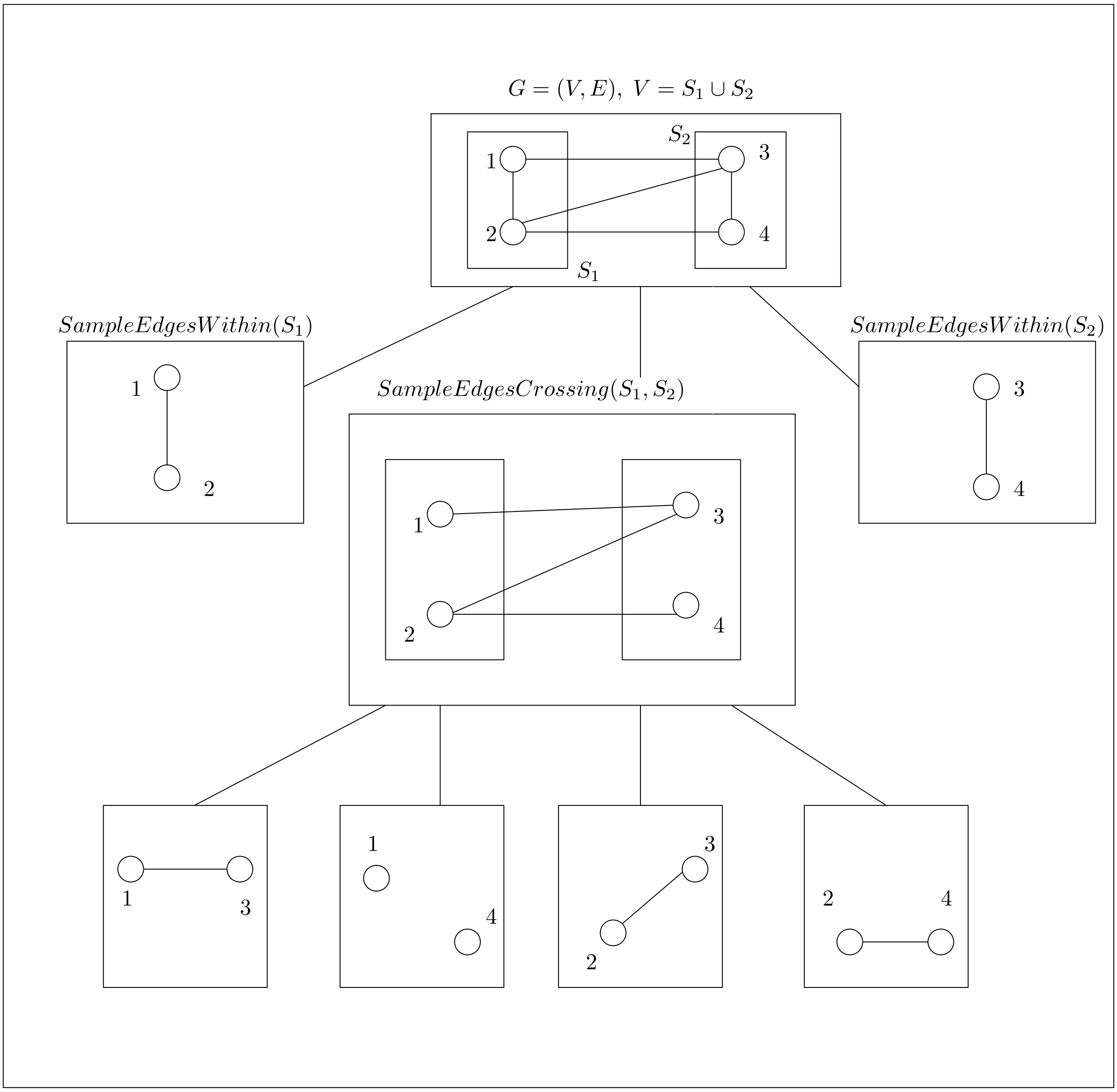
Theorem 3 (Extends **Theorem 2** to $k \to \infty$ case)**.** *For a forest $F_1 \subseteq E$, let $G(k) = G + k\ F_1$ as defined in **Definition 3** . Let $F_2 \subseteq E$ be disjoint from F_1 such that $F_1 \cup F_2$ is a forest. Let B_{F_2} be the incidence matrix of F_2. For $k > 0$ define $N = \lim_{k\to\infty} L^+_{G(k)}$*

$$\lim_{k\to\infty} L^+_{G(k)+kF_2} = N - \Big(N \cdot B_{F_2} \cdot (B^T_{F_2}\ N\ B_{F_2}) \cdot B^T_{F_2} \cdot N \Big)$$

Also $ker \Big(\lim_{k\to\infty} L^+_{G(k)+kF_2} \Big) = span\ (B_{F_1 \cup F_2} \cup \boldsymbol{1})$

Proof. TODO ∎

4.2.4 The updated algorithm



5 Laplacian Paradigm

5.1 Kelner, Madry

6 Conclusion

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