Chennai Mathematical Institute

Masters Thesis

Random Spanning Trees

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Declaration of Authorship

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Abstract

Faculty Name
Computer Science at CMI

Master of Science

Random Spanning Trees

by Bhishmaraj S

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too. . . .
I see

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

Contents

List of Figures

List of Tables

List of Abbreviations

LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

1 Introduction

2 Background

2.1 Markov Chains

2.1.1 Fundamental theorem of Markov chain

2.1.2 Markov chain tree theorem

2.2 Results from Spectral Graph Theory

2.2.1 Kirchoff Matrix Tree Theorem

2.2.2 Some properties of Laplacian

2.3 Electric Networks

3 Random Walk Approach

3.1 Aldous, Broder

3.2 Wilson

4 Matrix Approach

4.1 Colbourn, Day, Nel

4.2 Harvey, Xu

Harvey and Xu, 2016 proposed a $\mathcal{O}(N^{\omega})$ algorithm for sampling a uniform spanning tree which is much simpler compared to the one proposed earlier by CMN which also has the same running time. The initial starting point for the algorithm is using the relationship between effective resistance and the probability that an edge belongs to a spanning tree. And also the fact that sampling an edge corresponds to contracting it and discarding the edge corresponds to deleting it. The following naive chain rule algorithm works on the same principle.

Naive chain rule algorithm

Input: $G = (V, E)$ and L_G^+
Output: Set of edges corresponding to a random spanning tree
for $e = (u, v) \in E$ **do**
  $R_e^{\text{eff}} = (\chi_u - \chi_v)^T \; L_G^+ \; (\chi_u - \chi_v)$;
  if $(X \sim \mathit{Bernoulli}(R_e^{\text{eff}})) = 1$ **then**
    Add edge e to the spanning tree;
    $G = G/e$;
  else
    $G = G \setminus e$;
  end
  Update L_G^+ ;
end

Algorithm 1: Sampling uniform spanning tree using chain rule

Computing L_G^+ takes $\mathcal{O}(N^3)$ hence the overall running time is $\mathcal{O}(MN^3)$

4.2.1 Summary of the paper

On a high level the main ideas used in the paper are

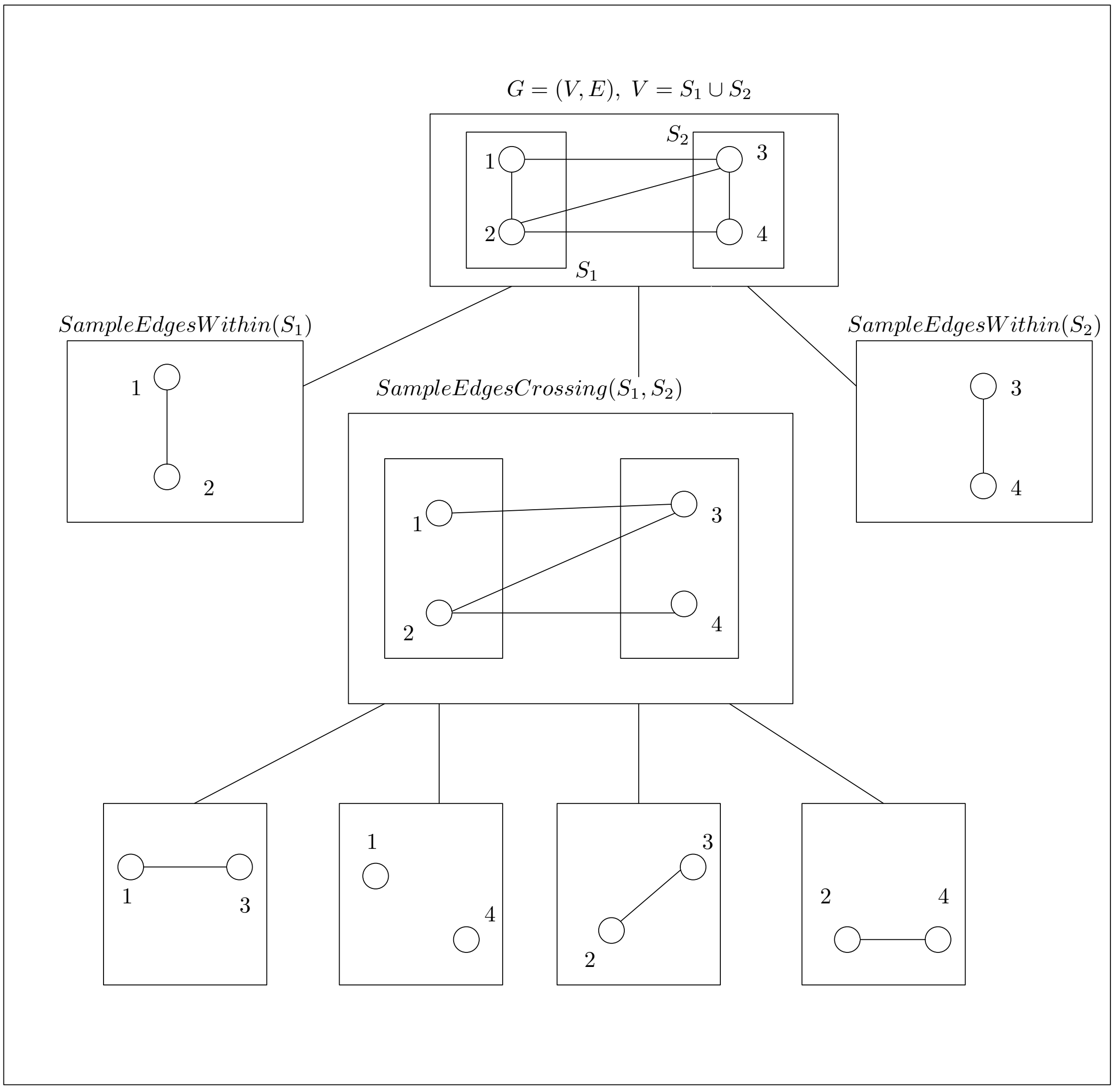
1. Use a divide and conquer algorithm to break the graph into smaller parts and sample on each parts seperately and update the pseudoinverse of the laplacian lazily only on the subgraph when needed
2. The important insight here is that the sampling probability of an edge depends only on 4 entries of the pseudoinverse of the laplacian. Hence we don't need to update all the entries of the matrix when the graph is modified
3. A well known method to compute inverse of a matrix with updates is to use the Sherman-Morrison-Woodbury formula. But in this case the formula has to be modified to work for the case where only a submatrix is modified.

4. Since while contracting an edge the number of vertices decreases it would get cumbersome to modify the dimension of the matrix evertime. So they overcome this issue by considering the formula on the limit case. When the graph is considered as a electric network then increasing the weight of an edge corresponds to shorting that link, hence in the limit case we get the same result as contracting the edge

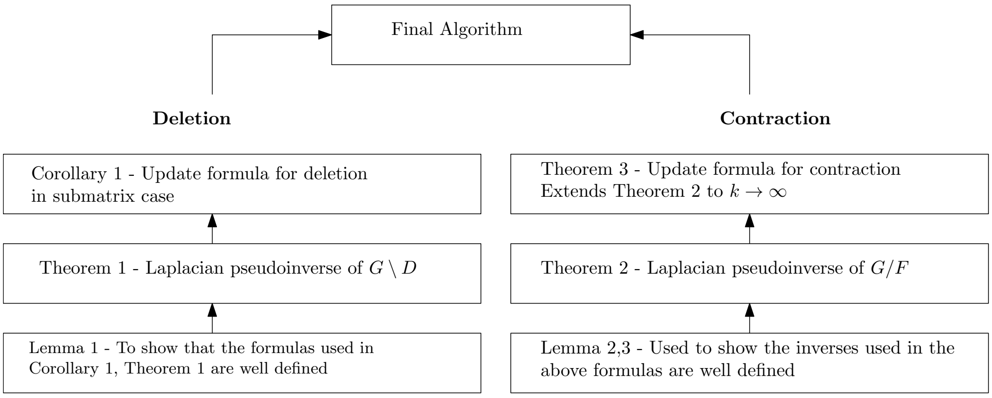
5. Also one of the main improvements over the previous algorithms (Colbourn, Myrvold, and Neufeld, 1996) is that the intricacies of LU decomposition is avoided since the current algorithm uses only matrix inversion as black box.

4.2.2 Harvey, Xu Algorithm

(TODO) Insert algorithm here and explain how it works



4.2.3 Structure of the paper



4.2.4 Facts used

Fact 1 (Woodbury matrix identity)**.** *Let $M \in \mathbb{M}_{n\times n}, U \in \mathbb{M}_{n\times k}, V \in \mathbb{M}_{n\times k}$. Suppose M is non-singular then $M+UV^T$ is non-singular $\iff$ $I+V^TM^{-1}U$ is non-singular. If $M+UV^T$ is non-singular, then*

$$(M+UV^T)^{-1} = M^{-1} - \left(M^{-1}\cdot U \cdot (I+V^TM^{-1}U)^{-1}\cdot V^T \cdot M^{-1}\right)$$

Proof. TODO ∎

Fact 2. *For any $L \in \mathbb{M}_{n\times n}$ with $ker(L) = span(\mathbf{1})$, we have $LL^+ = I - \frac{\mathbf{1}\cdot\mathbf{1}^T}{n}$ and $P := I - \frac{\mathbf{1}\cdot\mathbf{1}^T}{n}$ is called the **projection matrix**.*

The folowing set of facts are about the properties of matrix operations (addition, multiplication, etc.) on sub-matrices. The first 4 are easy to see, so I haven't derived them. For the last one I have written a derivation using Shur's complement from Wikipedia

Fact 3 (Sub-matrices)**.** *For all the results below, S denotes a index set and S^c denotes it's complement.*

1. *For any $A, B \in \mathbb{M}_{n\times n}$, $(A+B)_{S,S} = A_{S,S} + B_{S,S}$*
2. *If $C = D\cdot E\cdot F$ then $C_{S,S} = D_{S,*}\cdot E \cdot F_{*,S}$*
3. *For $A \in \mathbb{M}_{m\times n}, B \in \mathbb{M}_{n\times l}$, If $A_{S^c,S^c} = 0$ or $B_{S^c,S^c} = 0$ then $(AB)_{S,S} = A_{S,S}\cdot B_{S,S}$*
4. *For any matrix C where $C = D\cdot E\cdot F$. If $D_{*,S^c} = 0$ and $F_{S^c,*} = 0$, then $C = D_{*,S}\cdot E_{S,S}\cdot F_{S,*}$*
5. *$D = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, and $E = \begin{bmatrix} A & B \\ X & Y \end{bmatrix}$ where $M, A \in \mathbb{M}_{n\times n}$ and If $(MA - I)$ is invertible Then,*

$$(DE - I)^{-1} = \begin{bmatrix} (MA-I)^{-1} & (MA-I)^{-1}\cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

Proof. $(DE - I)^{-1}$ can be computed using Shur's Complement(Wikipedia contributors, 2020) .

Suppse $N = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ and Shur's complement of block S and P is

$$N/S := P - QS^{-1}R \qquad N/P := S - RP^{-1}Q$$

Then

$$N^{-1} = \begin{bmatrix} P^{-1} + (P^{-1}Q(N/P)^{-1}RP^{-1}) & -(P^{-1}Q(N/P)^{-1}) \\ -((N/P)^{-1}RP^{-1}) & (N/P)^{-1} \end{bmatrix}$$

In our case $N = \begin{bmatrix} MA - I & MB \\ 0 & -I \end{bmatrix}$ and $(N/P) = -I$. From this it follows that

$$N^{-1} = (DE - I)^{-1} = \begin{bmatrix} (MA - I)^{-1} & (MA - I)^{-1} \cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

■

Fact 4. *Let $A, B \in \mathbb{M}_{n\times n}$ with B being symmetric PSD. Suppose x is an eigenvector of AB corresponding to eigenvalue λ. Then $\sqrt{B}x$ is an eigenvector of $\sqrt{B}A\sqrt{B}$ corresponding to eigenvalue λ*

Fact 5 (Laplacian and graph connectivity (Fiedler value))**.** *Let G be a graph with n vertices. Suppose $(\lambda_1, \lambda_2 \cdots \lambda_n)$ be the eigenvalues corresponding to the eigenvectors $(v_1, v_2 \cdots v_n)$ of the Laplacian of G denoted as L_G. L_G is symmetric PSD with $\lambda_1 = 0$ and $v_1 = \mathbf{1}$. The following properties relate the eigenvalues of L_G with the connectivity of G :*

1. *$\lambda_2 > 0 \iff G$ is connected*
2. *G is disconnected $\iff \exists z$ with $z^T \mathbf{1} = 0$ and $z^T L_G z = 0$*

The above is true for L_G^+ also

Definition 1 (χ_u)**.** *χ_u is a vector of size $|V|$*

$$\chi_u(i) = \begin{cases} 1, & \text{if } i = u \\ 0, & \text{otherwise} \end{cases}$$

Definition 2 (Uniform random spanning tree)**.** *Let $\hat{T}$ be the random variable denoting a uniformly random spanning tree, then $\mathbb{P}(\hat{T} = T) = \frac{1}{|\mathcal{T}|}$, where $\mathcal{T}$ is the set of all spanning trees of G.*

Fact 6. *Given a graph $G = (V, E)$ with laplacian L_G, the effective resistance of an edge $e = \{u, v\} \in E$ is*

$$R_e^{\mathit{eff}} = (\chi_u - \chi_v)^T \; L_G^+ \; (\chi_u - \chi_v)$$

Then for any $e \in E$ we have

$$\mathbb{P}(e \in \hat{T}) = R_e^{\mathit{eff}}$$

4.2.5 Technical details of the results

Deletion

The first step in obtaining a updation formula for deletion is to make sure $(I - L_D L_G^+)$ is invertible. As the inverse of this term would be used in the expansion of $(L_G - L_D)^+$

For the first direction of Lemma 1, the main result used is **Fact 5** (G is disconnected $\iff \exists z$ with $z^T \mathbf{1} = 0$ and $z^T L_G^+ z = 0$). So if we can show this for a suitable z we are done. Now using the hypothesis that $(I - L_D L_G^+)$ is singular and **Fact 4** they derive the following $y^T \cdot \sqrt{L_G^+} \cdot L_{G \setminus D} \cdot \sqrt{L_G^+} \cdot y = 0$. As we an see the remaining part is to show $(z = \sqrt{L_G^+} y) \perp \mathbf{1}$

Lemma 1 (Formulas in **Theorem 1** are well defined)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq E$ then*
$\left(I - L_D L_G^+\right)$ *is invertible* $\iff$ $G \setminus D$ *is connected*

Proof. First let's show that If $(I - L_D L_G^+)$ is singular then $G \setminus D$ is disconnected

- Since $(I - L_D L_G^+)$ is singular $\exists x \neq 0$ s.t. $(I - L_D L_G^+)x = 0$

$$\implies L_D L_G^+ \, x = x \tag{4.1}$$
$$\implies 1 \in eigenvalues(L_D L_G^+) \tag{4.2}$$
$$\implies 1 \in eigenvalues((L_G - L_{G \setminus D}) L_G^+) \tag{4.3}$$

- Let $x \perp \mathbf{1}$ be an eigenvector of $(L_G - L_{G \setminus D}) L_G^+$ with eigenvalue 1.

- By **Fact 4**, $y = \frac{\sqrt{L_G^+} x}{\left\| \sqrt{L_G^+} x \right\|}$ is an eigenvector of $\sqrt{L_G^+}(L_G - L_{G \setminus D})\sqrt{L_G^+}$

$$= y^T \cdot \sqrt{L_G^+}(L_G - L_{G \setminus D})\sqrt{L_G^+} \cdot y = 1 \tag{4.4}$$
$$= y^T \sqrt{L_G^+} L_G \sqrt{L_G^+} y = (HOW) y^T L_G^+ L_G y = y^T P y \tag{4.5}$$
$$= y^T \left(I - \frac{\mathbf{1}^T \mathbf{1}}{n}\right) y = y^T y - \left(\frac{y^T \mathbf{1}^T \mathbf{1} y}{n}\right) = (HOW) y^T y = 1 \tag{4.6}$$

- $\therefore y^T \sqrt{L_G^+} L_{G \setminus D} \sqrt{L_G^+} y = 0$ now if we consider $z = \sqrt{L_G^+} y$ and show that $z^T \mathbf{1} = 0$ then we can use **Fact 5** to complete the proof

$$y^T \sqrt{L_G^+} \mathbf{1} = x^T \sqrt{L_G^+} \sqrt{L_G^+} \mathbf{1} = 0 (\text{HOW is 1 in kernel of } L_G^+ \tag{4.7}$$
$$G \setminus D \text{ is disconnected} \tag{4.8}$$

Now to prove the converse, If $G \setminus D$ is disconnected then $I - L_D L_G^+$ is singular

- If $G \setminus D$ is disconnected then $\exists y \perp \mathbf{1}, ||y|| = 1$ we have

    1. $y^T \cdot \sqrt{L_G^+} \cdot L_{G \setminus D} \cdot \sqrt{L_G^+} \cdot y = 0 (HOW)$
    2. $y^T \cdot \sqrt{L_G^+} \cdot L_G \cdot \sqrt{L_G^+} \cdot y = y^T y = 1$

- From (1) and (2) we get $y^T\sqrt{L_G^+}(L_G - L_{G\setminus D})\sqrt{L_G^+}y = 1$

$$\implies y^T \cdot \sqrt{L_G^+} \cdot L_D \cdot \sqrt{L_G^+} \cdot y = 1 \tag{4.9}$$
$$\implies 1 \in \text{eigenvalues}(L_D L_G^+)(HOW) \tag{4.10}$$
$$\implies (I - L_D L_G^+) \text{ is singular} \tag{4.11}$$

■

In **Theorem 1** they just show that the formula for the updated pseudoinverse is indeed true. This is shown using the following identity $LL^+ = P$

Theorem 1 (Update identity for Deletion)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq E$. If $G \setminus D$ is connected then*

$$(L_G - L_D)^+ = L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)$$

Proof. If R.H.S is indeed true then it should satisfy the property of $LL^+ = P$

$$(L_G - L_D) \cdot (L_G - L_D)^+$$

$$(L_G - L_D) \cdot \left(L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)\right)$$

$$\left[P - L_D L_G^+\right] - \left[(L_G L_G^+ - L_D L_G^+) \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right]$$

$$\left[P - L_D L_G^+\right] + \left[\left((L_D L_G^+ - I) + \frac{\mathbf{1}\mathbf{1}^T}{n}\right) \cdot \left((L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)\right]$$

$$\left[P - L_D L_G^+\right] + \left[(L_D L_G^+) + \left(\frac{\mathbf{1}\mathbf{1}^T}{n} \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)\right]$$

We can see that $-\mathbf{1}^T(L_D L_G^+ - I) = -\mathbf{1}^T L_D L_G^+ + (I\mathbf{1})^T = 0 + \mathbf{1}^T = \mathbf{1}^T$. Hence $\mathbf{1}^T(L_D L_G^+ - I)^{-1} = -\mathbf{1}^T$. And also $\mathbf{1}^T L_D = 0$. Hence,

$$P - L_D L_G^+ + L_D L_G^+ + \mathbf{1}^T L_D L_G^+ = P$$

■

Definition 3 (Submatrix)**.** *A submatrix of a martix A containing rows S and columns T is denoted as $A_{S,T}$*

Corollary 1 modifies the update formula in **Theorem 1** to work for submatrices and hence reduce the compliexity to $\mathcal{O}(|S|^\omega)$. They do this by first applying the facts related to submatrices **Fact 3.3, 3.5**.

Corollary 1 (Improved **Theorem 1** for submatrix)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq G$. For $S \subseteq V$ define $E[S]$ as $(S \times S) \cap E$. Suppose $E_D \subseteq E[S]$ and $G \setminus D$ is connected then*

$$(L_G - L_D)^+_{S,S} = (L_G^+)_{S,S} - \left((L_G^+)_{S,S} \cdot ((L_D)_{S,S}\ (L_G^+)_{S,S}\ -\ I)^{-1} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S}\right)$$

Proof. From **Theorem 1** we know that $(L_G - L_D)^+ = L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)$. If we apply **Fact 3.1** to $(L_G - L_D)^+$ we get

$$(L_G^+)_{S,S} - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)_{S,S}$$

Applying **Fact 3.3** we get (HOW)

$$(L_G^+)_{S,S} - \left((L_G^+)_{S,S} \cdot (L_D L_G^+ - I)^{-1}_{S,S} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S}\right)$$

Now applying **Fact 3.5** to $(L_D L_G^+ - I)^{-1}_{S,S}$

Fact 3.5 states that If $D = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, and $E = \begin{bmatrix} A & B \\ X & Y \end{bmatrix}$ where $M, A \in \mathbb{M}_{n\times n}$ and If $(MA - I)$ is invertible Then,

$$(DE - I)^{-1} = \begin{bmatrix} (MA-I)^{-1} & (MA-I)^{-1} \cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

Here we have $L_D = \begin{bmatrix} (L_D)_{S,S} & 0 \\ 0 & 0 \end{bmatrix}$, (HOW can it always be this way) and $L_G = \begin{bmatrix} (L_G)_{S,S} & (L_G)_{S,S^c} \\ (L_G)_{S^c,S} & (L_G)_{S^c,S^c} \end{bmatrix}$

$$\therefore\ (L_D L_G - I)^{-1} = \begin{bmatrix} ((L_D)_{S,S}(L_G)_{S,S} - I)^{-1} & (L_D)_{S,S}(L_G)_{S,S^c} \\ 0 & -I \end{bmatrix}$$

Hence we get the required result

$$(L_G - L_D)^+_{S,S} = (L_G^+)_{S,S} - \left((L_G^+)_{S,S} \cdot ((L_D)_{S,S}\ (L_G^+)_{S,S}\ -\ I)^{-1} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S}\right)$$

■

Contraction

Definition 4 (Incidence Matrix)**.** *Let $G = (V, E)$, given an edge $e = u, v \in E$ the incidence vector of e is defined as $v_e = (\chi_u - \chi_v)$. Given a set of edges $D = \{e_1, e_2 \cdots e_m\} \subseteq E$, the incidence matrix of D is defined as $B_D = [v_{e_1}|v_{e_2} \cdots |v_{e_m}]$*

Definition 5 ($G+ke$)**.** *$G+ke$ is the weighted graph obtained by increasing e's weight by k*

Lemma 2 (Formulas in **Theorem 2** are well defined)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ with $|F| = r$ and let B_F be the incidence matrix of F.*

$$B_F^T\ L_G^+\ B_F \textit{ is invertible} \iff F \textit{ is a forest}$$

Proof. For any $x \in \mathbb{R}^r, x \neq 0,$ let $y = B_F x$. We first prove the following claim

Claim 1. *The incidence matrix of a acyclic graph has full column rank*

Proof. TODO ■

Therefore $y \neq 0$

■

Lemma 3 (Formulas in **Theorem 2** are well defined)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$\textit{If } F \textit{ is a forest then } \left(\frac{I}{k} + B_F^T\ L_G^+\ B_F\right) \textit{ is invertible for any } k > 0$$

Proof. TODO ■

Theorem 2 (Contraction update formula for finite k)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$(L_G + k\ L_F)^+ = L_G^+ - \left(L_G^+ \cdot B_F \cdot (\frac{I}{k} + B_F^T\ L_G^+\ B_F)^{-1} \cdot B_F^T \cdot L_G^+ \right)$$

Proof. TODO ■

Corollary 2 (Improves **Theorem 2** for sub-matrices)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. Suppose $F \subseteq E[S]$, where $S \subseteq V$. For any $k > 0$,*

$$(L_G + k\ L_F)^+_{S,S} = (L_G^+)_{S,S} - \left((L_G^+)_{S,S}\ (B_F)_{S,*}\ (\frac{I}{k} + (B_F^T)_{S,*}\ (L_G^+)_{S,S}\ (B_F)_{S,*})^{-1}\ (B_F^T)_{S,*}\ (L_G^+)_{S,S} \right)$$

Proof. TODO ■

Theorem 3 (Extends **Theorem 2** to $k \to \infty$ case)**.** *For a forest $F_1 \subseteq E$, let $G(k) = G + k\ F_1$ as defined in **Definition 3** . Let $F_2 \subseteq E$ be disjoint from F_1 such that $F_1 \cup F_2$ is a forest. Let B_{F_2} be the incidence matrix of F_2. For $k > 0$ define $N = \lim_{k\to\infty} L^+_{G(k)}$*

$$\lim_{k\to\infty} L^+_{G(k)+kF_2} = N - \left(N \cdot B_{F_2} \cdot (B_{F_2}^T\ N\ B_{F_2}) \cdot B_{F_2}^T \cdot N \right)$$

Also $ker\left(\lim_{k\to\infty} L^+_{G(k)+kF_2}\right) = span\ (B_{F_1 \cup F_2} \cup \boldsymbol{1})$

Proof. TODO ■

5 Laplacian Paradigm

5.1 Kelner, Madry

6 Conclusion

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