Chennai Mathematical Institute

Masters Thesis

Random Spanning Trees

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A thesis submitted in fulfillment of the requirements for the degree of Master of Science

in the

Computer Science at CMI

June 1, 2020

Declaration of Authorship

I, Bhishmaraj S, declare that this thesis titled, “Random Spanning Trees” and the work presented in it are my own. I confirm that:

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CHENNAI MATHEMATICAL INSTITUTE

Abstract

Faculty Name
Computer Science at CMI

Master of Science

Random Spanning Trees

by Bhishmaraj S

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too. . . . I see

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

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LAH	**L**ist **A**bbreviations **H**ere
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4.2.1 Techniques used

Naive chain rule algorithm

Input: $G = (V, E)$ and L_G^+
Output: Set of edges corresponding to a random spanning tree
for $e = (u, v) \in E$ **do**
  $R_e^{\text{eff}} = (\chi_u - \chi_v)^T \; L_G^+ \; (\chi_u - \chi_v)$;
  if $(X \sim Bernoulli(R_e^{\text{eff}})) = 1$ **then**
    Add edge e to the spanning tree;
    $G = G/e$;
  else
    $G = G \setminus e$;
  end
  Update L_G^+ ;
end

Algorithm 1: Sampling uniform spanning tree using chain rule

4.2.2 Facts used

Fact 1 (Woodbury matrix identity)**.** *Let $M \in \mathbb{M}_{n\times n}, U \in \mathbb{M}_{n\times k}, V \in \mathbb{M}_{n\times k}$. Suppose M is non-singular then $M + UV^T$ is non-singular $\iff$ $I + V^T M^{-1} U$ is non-singular. If $M + UV^T$ is non-singular, then*

$$(M + UV^T)^{-1} = M^{-1} - \left(M^{-1} \cdot U \cdot (I + V^T M^{-1} U)^{-1} \cdot V^T \cdot M^{-1}\right)$$

Proof. TODO ■

Fact 2. *For any $L \in \mathbb{M}_{n\times n}$ with $ker(L) = span(\mathbf{1})$, we have $LL^+ = I - \frac{\mathbf{1}\cdot\mathbf{1}^T}{n}$ and $P := I - \frac{\mathbf{1}\cdot\mathbf{1}^T}{n}$ is called the **projection matrix**.*

Fact 3 (Sub-matrices)**.** *For all the results below, S denotes a index set and S^c denotes it's complement.*

1. *For any $A, B \in \mathbb{M}_{n\times n}, (A + B)_{S,S} = A_{S,S} + B_{S,S}$*

2. *If $C = D \cdot E \cdot F$ then $C_{S,S} = D_{S,*} \cdot E \cdot F_{*,S}$*

3. *For $A \in \mathbb{M}_{m\times n}, B \in \mathbb{M}_{n\times l}$, If $A_{S^c,S^c} = 0$ or $B_{S^c,S^c} = 0$ then $(AB)_{S,S} = A_{S,S} \cdot B_{S,S}$*

4. *For any matrix C where $C = D \cdot E \cdot F$. If $D_{*,S^c} = 0$ and $F_{S^c,*} = 0$, then* $C = D_{*,S} \cdot E_{S,S} \cdot F_{S,*}$

5. $D = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, *and* $E = \begin{bmatrix} A & B \\ X & Y \end{bmatrix}$ *where $M, A \in \mathbb{M}_{n\times n}$ and If $(MA - I)$ is invertible Then,*

$$(DE - I)^{-1} = \begin{bmatrix} (MA - I)^{-1} & (MA - I)^{-1} \cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

Proof. $(DE - I)^{-1}$ can be computed using Shur's Complement(Wikipedia contributors, 2020) .

Suppse $N = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ and Shur's complement of block S and P is

$$N/S := P - QS^{-1}R \qquad N/P := S - RP^{-1}Q$$

Then

$$N^{-1} = \begin{bmatrix} P^{-1} + (P^{-1}Q(N/P)^{-1}RP^{-1}) & -(P^{-1}Q(N/P)^{-1}) \\ -((N/P)^{-1}RP^{-1}) & (N/P)^{-1} \end{bmatrix}$$

In our case $N = \begin{bmatrix} MA - I & MB \\ 0 & -I \end{bmatrix}$ and $(N/P) = -I$. From this it follows that

$$N^{-1} = (DE - I)^{-1} = \begin{bmatrix} (MA - I)^{-1} & (MA - I)^{-1} \cdot M \cdot B \\ 0 & -I \end{bmatrix}$$

■

Fact 4. *Let $A, B \in \mathbb{M}_{n\times n}$ with B being symmetric PSD. Suppose x is an eigenvector of AB corresponding to eigenvalue λ. Then $\sqrt{B}x$ is an eigenvector of $\sqrt{B}A\sqrt{B}$ corresponding to eigenvalue λ*

Fact 5 (Laplacian and graph connectivity)**.** *Let G be a graph with n vertices. Suppose $(\lambda_1, \lambda_2 \cdots \lambda_n)$ be the eigenvalues corresponding to the eigenvectors $(v_1, v_2 \cdots v_n)$ of the Laplacian of G denoted as L_G. L_G is symmetric PSD with $\lambda_1 = 0$ and $v_1 = \mathbf{1}$. The following properties relate the eigenvalues of L_G with the connectivity of G :*

1. $\lambda_2 > 0 \iff$ *G is connected*
2. *G is disconnected* $\iff \exists z$ *with* $z^T \mathbf{1} = 0$ *and* $z^T L_G z = 0$

The above is true for L_G^+ also

Definition 1 (χ_u)**.** *χ_u is a vector of size $|V|$*

$$\chi_u(i) = \begin{cases} 1, & \text{if } i = u \\ 0, & \text{otherwise} \end{cases}$$

Definition 2 (Uniform random spanning tree)**.** *Let $\hat{T}$ be the random variable denoting a uniformly random spanning tree, then $\mathbb{P}(\hat{T} = T) = \frac{1}{|\mathcal{T}|}$, where $\mathcal{T}$ is the set of all spanning trees of G.*

Fact 6. *Given a graph $G = (V, E)$ with laplacian L_G, the effective resistance of an edge $e = \{u, v\} \in E$ is*

$$R_e^{\mathit{eff}} = (\chi_u - \chi_v)^T \; L_G^+ \; (\chi_u - \chi_v)$$

Then for any $e \in E$ we have

$$\mathbb{P}(e \in \hat{T}) = R_e^{\mathit{eff}}$$

4.2.3 Recursive Algorithm with lazy updates

Deletion

Lemma 1 (Formulas in **Theorem 1** are well defined)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq E$ then*
$\left(I - L_D L_G^+\right)$ *is invertible* $\iff$ $G \setminus D$ *is connected*

Proof. First let's show that If $(I - L_D L_G^+)$ is singular then $G \setminus D$ is disconnected

- Since $(I - L_D L_G^+)$ is singular $\exists x \neq 0$ s.t. $(I - L_D L_G^+)x = 0$

$$\implies L_D L_G^+ \; x = x \tag{4.1}$$
$$\implies 1 \in \mathit{eigenvalues}(L_D L_G^+) \tag{4.2}$$
$$\implies 1 \in \mathit{eigenvalues}((L_G - L_{G\setminus D})L_G^+) \tag{4.3}$$

- Let $x \perp \mathbf{1}$ be an eigenvector of $(L_G - L_{G\setminus D})L_G^+$ with eigenvalue 1.

- By **Fact 4**, $y = \frac{\sqrt{L_G^+}x}{\left\|\sqrt{L_G^+}x\right\|}$ is an eigenvector of $\sqrt{L_G^+}(L_G - L_{G\setminus D})\sqrt{L_G^+}$

$$= y^T \cdot \sqrt{L_G^+}(L_G - L_{G\setminus D})\sqrt{L_G^+} \cdot y = 1 \tag{4.4}$$
$$= y^T \sqrt{L_G^+} L_G \sqrt{L_G^+} y = (HOW) y^T L_G^+ L_G y = y^T P y \tag{4.5}$$
$$= y^T \left(I - \frac{\mathbf{1}^T\mathbf{1}}{n}\right) y = y^T y - \left(\frac{y^T\mathbf{1}^T\mathbf{1}y}{n}\right) = (HOW) y^T y = 1 \tag{4.6}$$

- $\therefore y^T \sqrt{L_G^+} L_{G\setminus D} \sqrt{L_G^+} y = 0$ now if we consider $z = \sqrt{L_G^+} y$ and show that $z^T\mathbf{1} = 0$ then we can use **Fact 5** to complete the proof

$$y^T \sqrt{L_G^+}\mathbf{1} = x^T \sqrt{L_G^+}\sqrt{L_G^+}\mathbf{1} = 0 (\text{HOW is 1 in kernel of } L_G^+ \tag{4.7}$$
$$G \setminus D \text{ is disconnected} \tag{4.8}$$

Now to prove the converse, If $G \setminus D$ is disconnected then $I - L_D L_G^+$ is singular

- If $G \setminus D$ is disconnected then $\exists y \perp \mathbf{1}, ||y|| = 1$ we have

  1. $y^T \cdot \sqrt{L_G^+} \cdot L_{G\setminus D} \cdot \sqrt{L_G^+} \cdot y = 0(HOW)$
  2. $y^T \cdot \sqrt{L_G^+} \cdot L_G \cdot \sqrt{L_G^+} \cdot y = y^T y = 1$

- From (1) and (2) we get $y^T\sqrt{L_G^+}(L_G - L_{G\setminus D})\sqrt{L_G^+}y = 1$

$$\implies y^T \cdot \sqrt{L_G^+} \cdot L_D \cdot \sqrt{L_G^+} \cdot y = 1 \tag{4.9}$$
$$\implies 1 \in \text{eigenvalues}(L_D L_G^+)(HOW) \tag{4.10}$$
$$\implies (I - L_D L_G^+) \text{ is singular} \tag{4.11}$$

■

Theorem 1 (Update identity for Deletion). *Let $G = (V, E)$ be a connected graph and $D \subseteq E$. If $G \setminus D$ is connected then*

$$(L_G - L_D)^+ = L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)$$

Proof. TODO ■

Definition 3 (Submatrix).

Corollary 1 (Improved **Theorem 1** for submatrix). *Let $G = (V, E)$ be a connected graph and $D \subseteq G$. For $S \subseteq V$ define $E[S]$ as $(S \times S) \cap E$. Suppose $E_D \subseteq E[S]$ and $G \setminus D$ is connected then*

$$(L_G - L_D)_{S,S}^+ = (L_G^+)_{S,S} - \left((L_G^+)_{S,S} \cdot ((L_D)_{S,S}\ (L_G^+)_{S,S}\ -\ I)^{-1} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S}\right)$$

Proof. TODO ■

Definition 4 (Incidence Matrix). *Let $G = (V, E)$, given an edge $e = u, v \in E$ the incidence vector of e is defined as $v_e = (\chi_u - \chi_v)$. Given a set of edges $D = \{e_1, e_2 \cdots e_m\} \subseteq E$, the incidence matrix of D is defined as $B_D = [v_{e_1}|v_{e_2} \cdots |v_{e_m}]$*

Definition 5 ($G+ke$). *$G+ke$ is the weighted graph obtained by increasing e's weight by k*

Contraction

Lemma 2 (Formulas in **Theorem 2** are well defined). *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ with $|F| = r$ and let B_F be the incidence matrix of F.*

$$B_F^T\ L_G^+\ B_F \textit{ is invertible} \iff F \textit{ is a forest}$$

Proof. TODO ■

Lemma 3 (Formulas in **Theorem 2** are well defined). *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$\textit{If } F \textit{ is a forest then } \left(\frac{I}{k} + B_F^T\ L_G^+\ B_F\right) \textit{ is invertible for any } k > 0$$

Proof. TODO ■

Theorem 2 (Contraction update formula for finite k). *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$(L_G + k\ L_F)^+ = L_G^+ - \left(L_G^+ \cdot B_F \cdot (\frac{I}{k} + B_F^T\ L_G^+\ B_F)^{-1} \cdot B_F^T \cdot L_G^+\right)$$

Proof. TODO ∎

Corollary 2 (Improves **Theorem 2** for sub-matrices)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. Suppose $F \subseteq E[S]$, where $S \subseteq V$. For any $k > 0$,*

$$(L_G + k\ L_F)^+_{S,S} = (L_G^+)_{S,S} - \left((L_G^+)_{S,S}\ (B_F)_{S,*}\ (\frac{I}{k} + (B_F^T)_{S,*}\ (L_G^+)_{S,S}\ (B_F)_{S,*})^{-1}\ (B_F^T)_{S,*}\ (L_G^+)_{S,S}\right)$$

Proof. TODO ∎

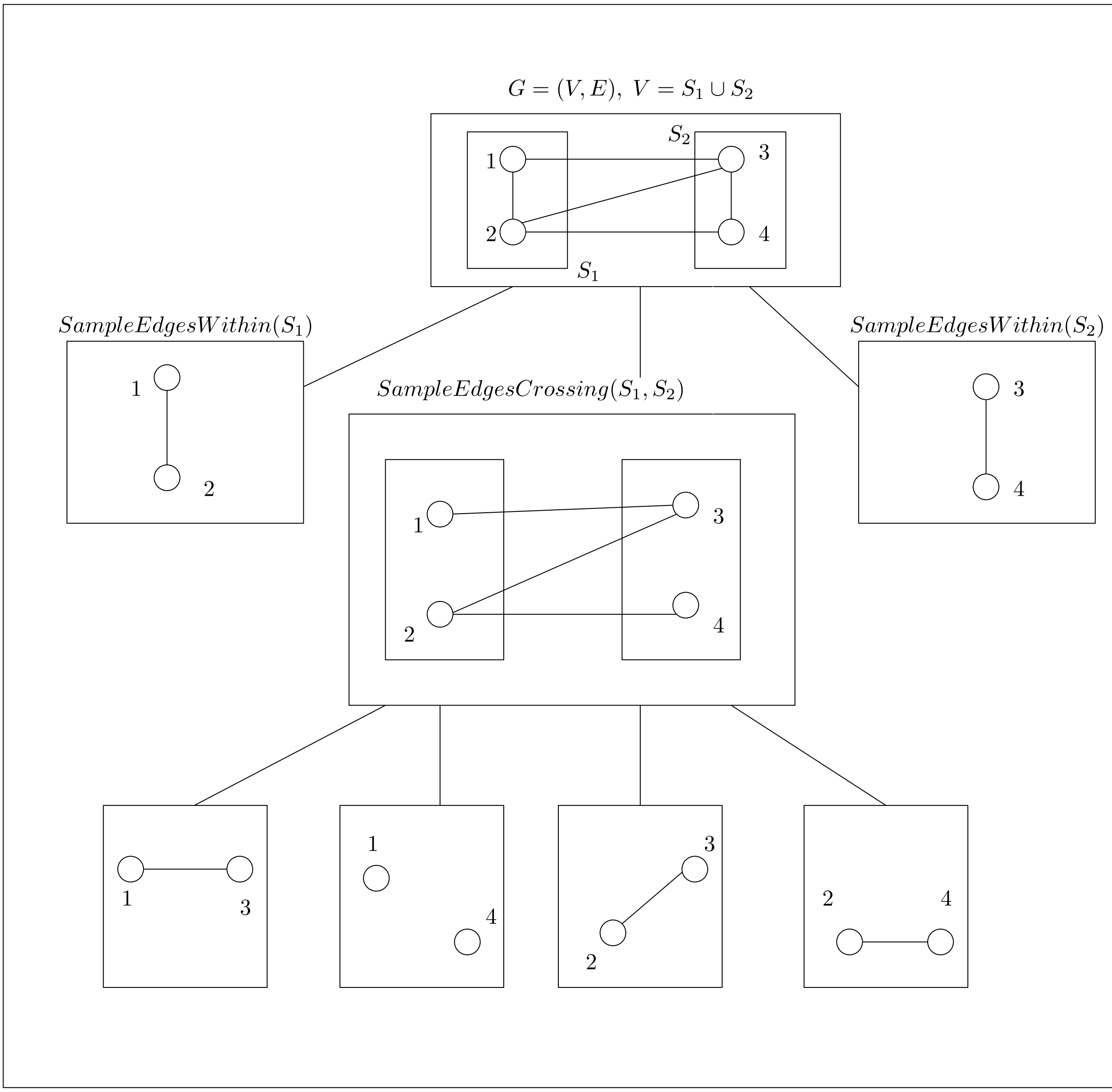
Theorem 3 (Extends **Theorem 2** to $k \to \infty$ case)**.** *For a forest $F_1 \subseteq E$, let $G(k) = G + k\ F_1$ as defined in **Definition 3** . Let $F_2 \subseteq E$ be disjoint from F_1 such that $F_1 \cup F_2$ is a forest. Let B_{F_2} be the incidence matrix of F_2. For $k > 0$ define $N = \lim_{k\to\infty} L^+_{G(k)}$*

$$\lim_{k\to\infty} L^+_{G(k)+kF_2} = N - \left(N \cdot B_{F_2} \cdot (B_{F_2}^T\ N\ B_{F_2}) \cdot B_{F_2}^T \cdot N\right)$$

Also $ker\left(\lim_{k\to\infty} L^+_{G(k)+kF_2}\right) = span\ (B_{F_1\cup F_2} \cup \boldsymbol{1})$

Proof. TODO ∎

4.2.4 The updated algorithm



5 Laplacian Paradigm

5.1 Kelner, Madry

6 Conclusion

Bibliography

Wikipedia contributors (2020). *Schur complement — Wikipedia, The Free Encyclopedia*. [Online; accessed 28-May-2020]. URL: https://en.wikipedia.org/w/index.php?title=Schur_complement&oldid=947233275.