Chennai Mathematical Institute

Masters Thesis

Random Spanning Trees

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in the

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Declaration of Authorship

I, Bhishmaraj S, declare that this thesis titled, “Random Spanning Trees” and the work presented in it are my own. I confirm that:

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- I have acknowledged all main sources of help.
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CHENNAI MATHEMATICAL INSTITUTE

Abstract

Faculty Name
Computer Science at CMI

Master of Science

Random Spanning Trees

by Bhishmaraj S

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too. . . .
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Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

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LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

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Naive chain rule algorithm

Input: $G = (V, E)$ and L_G^+
Output: Set of edges corresponding to a random spanning tree

for $e = (u, v) \in E$ **do**
  $R_e^{\text{eff}} = (\chi_u - \chi_v)^T \; L_G^+ \; (\chi_u - \chi_v)$;
  if $(X \sim \mathit{Bernoulli}(R_e^{\text{eff}})) = 1$ **then**
    Add edge e to the spanning tree;
    $G = G/e$;
  else
    $G = G \setminus e$;
  end
  Update L_G^+ ;
end

Algorithm 1: Sampling uniform spanning tree using chain rule

4.2.2 Facts used

Fact 1 (Woodbury matrix identity)**.** $\mathbb{M}$

4.2.3 Recursive Algorithm with lazy updates

Deletion

Lemma 1 (Formulas in **Theorem 1** are well defined)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq E$ then*
$\left(I - L_D L_G^+\right)$ *is invertible* $\iff$ $G \setminus D$ *is connected*

Proof. TODO ∎

Theorem 1 (Update identity for Deletion)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq E$. If $G \setminus D$ is connected then*

$$(L_G - L_D)^+ = L_G^+ - \left(L_G^+ \cdot (L_D L_G^+ - I)^{-1} \cdot L_D \cdot L_G^+\right)$$

Proof. TODO ∎

Definition 1 (Submatrix)**.**

Corollary 1 (Improved **Theorem 1** for submatrix)**.** *Let $G = (V, E)$ be a connected graph and $D \subseteq G$. For $S \subseteq V$ define $E[S]$ as $(S \times S) \cap E$. Suppose $E_D \subseteq E[S]$ and $G \setminus D$ is connected then*

$$(L_G - L_D)^+_{S,S} = (L_G^+)_{S,S} - \Big((L_G^+)_{S,S} \cdot ((L_D)_{S,S}\ (L_G^+)_{S,S}\ -\ I)^{-1} \cdot (L_D)_{S,S} \cdot (L_G^+)_{S,S}\Big)$$

Proof. TODO ■

Definition 2 (Incidence Matrix)**.** *Let $G = (V, E)$, given an edge $e = u, v \in E$ the incidence vector of e is defined as $v_e = (\chi_u - \chi_v)$. Given a set of edges $D = \{e_1, e_2 \cdots e_m\} \subseteq E$, the incidence matrix of D is defined as $B_D = [v_{e_1}|v_{e_2} \cdots |v_{e_m}]$*

Definition 3 ($G+ke$)**.** *$G+ke$ is the weighted graph obtained by increasing e's weight by k*

Contraction

Lemma 2 (Formulas in **Theorem 2** are well defined)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ with $|F| = r$ and let B_F be the incidence matrix of F.*

$$B_F^T\ L_G^+\ B_F \text{ is invertible} \iff F \text{ is a forest}$$

Proof. TODO ■

Lemma 3 (Formulas in **Theorem 2** are well defined)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$\text{If } F \text{ is a forest then } \left(\frac{I}{k} + B_F^T\ L_G^+\ B_F\right) \text{ is invertible for any } k > 0$$

Proof. TODO ■

Theorem 2 (Contraction update formula for finite k)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. For any $k > 0$,*

$$(L_G + k\ L_F)^+ = L_G^+ - \left(L_G^+ \cdot B_F \cdot (\frac{I}{k} + B_F^T\ L_G^+\ B_F)^{-1} \cdot B_F^T \cdot L_G^+\right)$$

Proof. TODO ■

Corollary 2 (Improves **Theorem 2** for sub-matrices)**.** *Let $G = (V, E)$ be a connected graph. Given $F \subseteq E$ and let B_F be the incidence matrix of F. Suppose $F \subseteq E[S]$, where $S \subseteq V$. For any $k > 0$,*

$$(L_G + k\ L_F)^+_{S,S} = (L_G^+)_{S,S} - \left((L_G^+)_{S,S} \cdot (B_F)_{S,*} \cdot (\frac{I}{k} + (B_F^T)_{S,*}\ (L_G^+)_{S,S}\ (B_F)_{S,*})^{-1} \cdot (B_F^T)_{S,*} \cdot (L_G^+)_{S,S}\right)$$

Proof. TODO ■

Theorem 3 (Extends **Theorem 2** to $k \to \infty$ case)**.** *For a forest $F_1 \subseteq E$, let $G(k) = G + k\ F_1$ as defined in **Definition 3** . Let $F_2 \subseteq E$ be disjoint from F_1 such that $F_1 \cup F_2$ is a forest. Let B_{F_2} be the incidence matrix of F_2. For $k > 0$ define $N = \lim_{k \to \infty} L^+_{G(k)}$*

$$\lim_{k \to \infty} L^+_{G(k)+kF_2} = N - \left(N \cdot B_{F_2} \cdot (B_{F_2}^T\ N\ B_{F_2}) \cdot B_{F_2}^T \cdot N\right)$$

Also $ker\left(\lim_{k\to\infty} L^{+}_{G(k)+kF_2}\right) = span\left(B_{F_1\cup F_2} \cup \boldsymbol{1}\right)$

Proof. TODO ∎

5 Laplacian Paradigm

5.1 Kelner, Madry

6 Conclusion