Question-2

Table 1: Comparison of Integration Methods for Two Integrals

Integral	Result (Romberg)	Result (Trapezoidal)	Iterations	Function Evaluations
$\int_0^1 x^1 3\, dx$	0.7499968521860756	0.7499917199899571	11	2049
$\int_0^1 x^2 e^x\, dx$	0.16060280012980105	0.16107989607963955	3	9

Romberg Integration achieves exceptional accuracy with a relative error of 0.000684%, outperforming the Trapezoidal Rule's 0.298%. However, it requires more computational effort due to higher iterations and function evaluations.

The Trapezoidal Rule, while less precise, is simpler and computationally efficient, making it suitable for lower precision needs. For high-accuracy applications, Romberg Integration remains the preferred method despite its higher cost.

Question-3

Results for Integral (a): $\int_0^1 x^{1/3}\, dx$

Method	n	Result	Function Evaluations	Relative Error (vs. Romberg)
Romberg Integration	11	0.7499968521860756	2049	N/A
Gaussian Quadrature	2	0.7597776742687159	2	1.304112×10^{-2}
	3	0.7538551970905132	3	5.144481×10^{-3}
	4	0.75194624763378	4	2.599205×10^{-3}
	5	0.7511321130643606	5	1.513688×10^{-3}

Table 2: Comparison of Romberg Integration and Gaussian Quadrature for Integral (a).

Results for Integral (b): $\int_0^1 x^2 e^{-x}\, dx$

Method	n	Result	Function Evaluations	Relative Error (vs. Romberg)
Romberg Integration	3	0.16060280012980105	9	N/A
Gaussian Quadrature	2	0.15941043096637894	2	7.424336×10^{-3}
	3	0.16059538680891922	3	4.615935×10^{-5}
	4	0.16060277751468477	4	1.408140×10^{-7}
	5	0.16060279412343753	5	3.739887×10^{-8}

Table 3: Comparison of Romberg Integration and Gaussian Quadrature for Integral (b).

For the function $f(x) = x^{1/3}$, Gaussian quadrature produces larger error values compared to Romberg Integration. However, Romberg Integration demands a significantly higher number of function evaluations.

For the function $f(x) = x^2 e^{-x}$, Gaussian quadrature achieves a smaller error for $n = 5$ compared to Romberg Integration. This indicates that Gaussian quadrature can deliver more accurate results with fewer function evaluations for this specific function.

Question-1

The integral of $f(x)$ over the interval $[x_0, x_2]$ is expressed as:

$$\int_{x_0}^{x_2} f(x)\,dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + K f^{(4)}(\xi)$$

where $\xi \in [x_0, x_2]$.

Step 1: Applying Simpson's Rule

Let x_1 be the midpoint of $[x_0, x_2]$:

$$x_1 = \frac{x_0 + x_2}{2}.$$

Using the assumption that Simpson's rule is exact for $f(x) = x^n$ when $n = 0, 1, 2, 3$, we calculate a_0, a_1, and a_2 by substituting these powers of x into the equation.

Step 2: Finding a_0, a_1, a_2

For $f(x) = 1$:

$$\int_{x_0}^{x_2} 1\,dx = x_2 - x_0.$$

Substitute into the approximation:

$$a_0 + a_1 + a_2 = x_2 - x_0.$$

For $f(x) = x$:

$$\int_{x_0}^{x_2} x\,dx = \frac{x_2^2 - x_0^2}{2}.$$

Substitute into the approximation:

$$a_0 x_0 + a_1 x_1 + a_2 x_2 = \frac{x_2^2 - x_0^2}{2}.$$

For $f(x) = x^2$:

$$\int_{x_0}^{x_2} x^2\,dx = \frac{x_2^3 - x_0^3}{3}.$$

Substitute into the approximation:

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 = \frac{x_2^3 - x_0^3}{3}.$$

For $f(x) = x^3$:

$$\int_{x_0}^{x_2} x^3\,dx = \frac{x_2^4 - x_0^4}{4}.$$

Substitute into the approximation:

$$a_0 x_0^3 + a_1 x_1^3 + a_2 x_2^3 = \frac{x_2^4 - x_0^4}{4}.$$

Step 3: Solving for Coefficients

Solving these equations gives:

$$a_0 = \frac{x_2 - x_0}{6}, \quad a_1 = \frac{4(x_2 - x_0)}{6}, \quad a_2 = \frac{x_2 - x_0}{6}.$$

Step 4: Finding the Error Term K

To find K, use $f(x) = x^4$:

$$\int_{x_0}^{x_2} x^4 \, dx = \frac{x_2^5 - x_0^5}{5}.$$

Substitute into the approximation:

$$a_0 x_0^4 + a_1 x_1^4 + a_2 x_2^4 + K \frac{(x_2 - x_0)^5}{90} = \frac{x_2^5 - x_0^5}{5}.$$

This gives:

$$K = -\frac{(x_2 - x_0)^5}{2880}.$$

Final Simpson's Rule

The Simpson's rule with the error term is:

$$\int_{x_0}^{x_2} f(x) \, dx \approx \frac{x_2 - x_0}{6} \left[f(x_0) + 4f\left(\frac{x_0 + x_2}{2}\right) + f(x_2) \right] - \frac{(x_2 - x_0)^5}{2880} f^{(4)}(\xi).$$