Question - 3

Given A is Symmetric Matrix $\Rightarrow A = A^T$ and $A \in R^{m \times n}$

a) $\underset{\sim}{A} = \underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{V}^T$

$$\underset{\sim}{A}^T\underset{\sim}{A} = (\underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{V}^T)^T(\underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{V}^T)$$

$$= \underset{\sim}{V}\underset{\sim}{\Sigma}^T\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{V}^T \quad \text{since} \quad U^TU = \underset{\sim}{I}$$

$$= \underset{\sim}{U}\underset{\sim}{\Sigma}^T\underset{\sim}{\Sigma}\underset{\sim}{V}^T = \underset{\sim}{U}\tilde{\underset{\sim}{\Sigma}}U^T$$

here $\tilde{\underset{\sim}{\Sigma}} = \begin{bmatrix} \sigma_1^2 & & 0 & 0 \\ \vdots & \sigma_2^2 & & 0 \\ \vdots & & \sigma_3^2 & \\ 0 & \cdots & & \ddots \end{bmatrix}$ it is a diagonal matrix

So we can say that $\tilde{\underset{\sim}{\Sigma}}$ is the diagonal matrix which are square of the singular values of $\underset{\sim}{A}^T\underset{\sim}{A}$ or $\underset{\sim}{A}\underset{\sim}{A}^T$

$\rightarrow$ Given $A^T = A \quad \therefore \quad A^TA = A^2$

Let λ be the eigen value of $\underset{\sim}{A}$ and v be the eigen vector corresponding to eigen vector v

then $\quad Av = \lambda v$

$A^2 v = A(\lambda v)$ [multiplying A on both sides]

$= \lambda(Av)$ [$Av = \lambda v$]

$= \lambda^2 v$

$$\boxed{A^2 v = \lambda^2 v}$$

So it is clear that square of singular values of $\underset{\sim}{A}$ (eigen values of A are square of eigen values of $\underset{\sim}{A}$.

$\Rightarrow$ Vector induced matrix norm $\|A\|_2$ is the largest eigen value of A.

$$\boxed{\|A_2\| = \max\{|\lambda_i|\}}$$

(b) since A is a symmetric matrix it is diagonizable

$A = Q \Lambda Q^T$ — (i) [where Q is orthogonal matrix for symmetric matrix]

given x is a unit vector $\|x\|_2 = 1$

As Q is orthogonal $\therefore \|Q^T x\|_2 = \|x\|_2 = 1$ $y = Q^T x$

$$x^T A x = y^T \Lambda y$$

we can write the above matrix in summation form

$$x^T A x = \sum_{i=1}^{m} \lambda_i y_i^2$$

It get maximum when largest $|\lambda_i|$ is multiplied

$$|x^T A x| \leq \max_i |\lambda_i| \quad \text{—(1)}$$

we know $\|A\|_2 = \max_i |\lambda_i|$

$$|x^T A x| \leq \|A\|_2.$$

<u>second way</u>

$$|x^T A x| \leq \|x^T\|_2 \|Ax\|_2 \quad (\because \text{Cachy schwartz inequality})$$

$$\leq \|Ax\|_2 \leq \|A\|_2 \|x\|_2$$

$$\leq \|A\|_2 \qquad [\because \|x\|_2 = 1]$$

$$\boxed{\therefore |x^T A x| \leq \|A\|_2}$$

(c) Given the perturbations on matrix $A+\delta A$ gives $u+\delta u$ and $\lambda+\delta\lambda$, $\delta A = (\delta A)^T$

$(A+\delta A)^T \Rightarrow A^T + \delta A^T \Rightarrow A+\delta A \rightarrow$ symmetric matrix

$$AU = \lambda U \quad -(1)$$

After perturbations on matrix

$$(A+\delta A)(u+\delta u) = (\lambda+\delta\lambda)(u+\delta u)$$

$$\cancel{Au} + A\delta u + \delta A u + \delta A\delta u = \cancel{\lambda u} + \delta\lambda u + \lambda\delta u + \delta\lambda\delta u$$

Remove 2nd order difference since they are very small

$$A\delta u + \delta A u = \delta\lambda u + \lambda\delta u$$

multiply u^T on both sides

$$u^T A\delta u + u^T\delta A u = u^T\lambda\delta u + \delta\lambda u^T u$$

$[u^T A^T\delta u = (Au)^T\delta u = \lambda u^T\delta u] \rightarrow$ since $A = A^T$

$$\cancel{\lambda u^T\delta u} + u^T\delta A u = \cancel{\lambda u^T\delta u} + \delta\lambda u^T u$$

$$u^T\delta A u = \delta\lambda u^T u \qquad \left\{\|u\|_2 = \sqrt{u^T u}\right.$$

$$\Rightarrow |u^T\delta A u| = |\delta\lambda|\,|u^T u|$$

$$|\delta\lambda| = \frac{|u^T\delta A u|}{\|u\|_2^2}$$

$$|\delta\lambda| = \left|\frac{u^T}{\|u\|_2}\delta A\frac{u}{\|u\|_2}\right| \qquad \therefore \frac{u}{\|u\|_2} = \hat{u}$$

$$|\delta\lambda| = |\hat{u}^T\delta A\hat{u}|$$

$|\delta\lambda| \le \|\delta A\|_2 \rightarrow |x^T A x| \le \|A\|_2 \rightarrow$ Part b Proved in

$$\boxed{\therefore |\delta\lambda| \le \|\delta A\|_2}$$

d) Relative condition number $\hat{k}_R = \max_{\delta x} \dfrac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|}$

$$\hat{k}_R(A) = \max_{\|\delta A\|} \frac{\frac{|\delta \lambda|}{|\lambda|}}{\frac{\|\delta A\|_2}{\|A\|_2}} = \max_{\|\delta A\|} \frac{|\delta \lambda|}{\|\delta A\|_2} \frac{\|A\|_2}{|\lambda|}$$

In previous question we proved $|\delta \lambda| \leq \|\delta A\|_2$

$$\therefore \max_{\|\delta A\|} \frac{|\delta \lambda|}{\|\delta A\|} \frac{\|A\|_2}{\|\lambda\|} \leq \frac{\|A\|_2}{|\lambda|} = \frac{\|A\|_2}{|\lambda|}$$

$$\boxed{\hat{k}^R(A) = \frac{\|A\|_2}{|\lambda|}}$$

(e) $m = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

eigen values are both a

from d we proved $\hat{k}^R = \dfrac{\|A\|_2}{|\lambda|}$

$\hat{k}_R = \dfrac{\|M\|_2}{|a|}$ [$\because$ since $m = m^T$, $\|m\|_2 = \max_i \{\lambda_i\}$]

$\hat{k}^R = \dfrac{|a|}{|a|} = 1$ $\boxed{\therefore \hat{k}^R = 1}$

(f) Given that the algorithm is backward stable which means

$|f(\tilde{x}) - \tilde{f}(x)| = 0$ [Backward error]

$$k(x) = \max_{\delta x} \frac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|}$$

$$\Rightarrow \frac{\|\tilde{f}(x) - f(x)\| / \|f(x)\|}{\|\tilde{x} - x\| / \|x\|} \leq k(x)$$

$$\Rightarrow \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} \leq k(x) \frac{\|\tilde{x} - x\|}{\|x\|} = k(x) \epsilon_n$$

$$\leq O(k^R \varepsilon_m) \quad \therefore \text{since } k^R = 1$$

$\therefore$ Forward error $= \dfrac{|f(\tilde{x}) - f(x)|}{|f(x)|} \leq O(\varepsilon_m)$

$\therefore$ It is well conditioned

(g) $M = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

finding eigen values for m using $|m - zI| = 0$ z is eigen values

$$|m - zI| = \left| \begin{bmatrix} a-z & 0 \\ 0 & a-z \end{bmatrix} \right| = 0$$

$$= (a-z)^2 = 0$$

$$P_m(z) = z^2 - (a \oplus a) z + a * a = 0$$

$$z = \frac{(a \oplus a) \underset{\ominus}{\oplus} \sqrt{((a \oplus a) \circledast (a \oplus a)) \ominus [(a \circledast a) \oplus [a \circledast a] \oplus [a \circledast a] \oplus [a * a]}}{2}$$

$$z = \left[\frac{2a(1+\varepsilon_1) \pm \sqrt{(2a(1+\varepsilon_1) * 2a(1+\varepsilon_1)(1+\varepsilon_2) \ominus \left[\begin{matrix} a^2(1+\varepsilon_3) \oplus a^2(1+\varepsilon_3) \\ a^2(1+\varepsilon_3) \oplus a^2(1+\varepsilon_3) \end{matrix} \right]}}{2} \right] (1+\varepsilon_5)$$

$$= \frac{2a(1+O(\varepsilon_m))^2 \pm \sqrt{4a^2(1+O(\varepsilon_m))^2 - 4a^2(1+O(\varepsilon_m))^3}}{2} (1+O(\varepsilon_m))$$

ignoring the higher order terms ε_m $[\varepsilon_m^2, \varepsilon_m^1, \varepsilon_m^4, \ldots]$

$$z = a[1 + O(\varepsilon_m) \pm \sqrt{O(\varepsilon_m)} (1+O(\varepsilon_m))]$$

$$= a(1 + O(\sqrt{\varepsilon_m}))$$ $[\therefore O(\sqrt{\varepsilon_m}) \gg O(\varepsilon_m)]$

$$= a(1 + O(\sqrt{\varepsilon_m})) \geq O(\varepsilon_m)$$

eg:- $\varepsilon_m = 10^{-16}$, $\sqrt{\varepsilon_m} = 10^{-8}$

$\therefore$ Because it is not $O(\varepsilon_m)$ it is not stable

b) Relative forward Error

$$= \frac{\| \tilde{f}(x) - f(x) \|}{\| f(x) \|}$$

$$= \frac{| a[1 + O(\epsilon_m) \pm [O\sqrt{\epsilon_m}) + O[\epsilon_m^{3/2}]] - a |}{|a|}$$

$$\simeq O(\sqrt{\epsilon_m})$$

$\therefore$ The forward error is of the order $O(\sqrt{\epsilon_n})$ so it is not stable for forward stable.

$\therefore$ Calculating eigen values is unstable