DS 215: Assignment 2

Full Marks: 100

1. In class, we stated that "if an efficient estimator exists, the maximum likelihood method will produce it". Can you prove that this statement is, in general, true?

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2. Consider the vector MAP estimator or

$$\hat{\boldsymbol{\theta}} \;=\; \arg\max_{\boldsymbol{\theta}} p\left(\boldsymbol{\theta}|\boldsymbol{x}\right).$$

Show that this estimator minimizes the Bayes risk for the cost function

$$\mathcal{C}\left(\boldsymbol{\varepsilon}\right) = \begin{cases} 1 & \text{when } ||\boldsymbol{\varepsilon}|| > \delta \\ 0 & \text{when } ||\boldsymbol{\varepsilon}|| < \delta \end{cases}$$

where $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}} \in \mathrm{R}^p$, $\boldsymbol{\varepsilon} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$, $||\boldsymbol{\varepsilon}||^2 = \sum_{i=1}^{p} \varepsilon_i^2$, and $\delta \to 0$.

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3. Consider the signal model:

$$s[n] = \begin{cases} A & \text{when } 0 \leq n \leq M-1 \\ -A & \text{when } M \leq n \leq N-1 \end{cases}$$

Find the LSE of A and the minimum LS error. Assume that $x[n] = s[n] + w[n]$ for $n = 0, 1, \cdots N-1$ are observed. If now $w[n]$ is WGN with variance σ^2, find the PDF of the LSE.

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4. The data $x[n] = A + w[n]$ for $n = 0, 1, \cdots N-1$ are observed. The unknown parameter A is assumed to have a prior PDF:

$$f(A) = \begin{cases} \lambda \exp\left(-\lambda A\right) & \text{when } A \geq 0 \\ 0 & \text{when } A < 0 \end{cases}$$

Here $\lambda > 0$, $w[n]$ is WGN with variance σ^2 and is independent of A. Find the MAP estimator of A.

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5. In fitting a line through experimental data we assume the model

$$x[n] = A + Bn + w[n], \qquad -M \leq n \leq M$$

where $w[n]$ is WGN with variance σ^2. If we have some prior knowledge of the slope B and intercept A such as

$$\begin{bmatrix} A \\ B \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix} \right),$$

find the MMSE estimator of A and B as well as the minimum Bayesian MSE. Assume that A and B are independent of $w[n]$. Which parameter will benefit most from the prior knowledge?

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