# DS 215: Assignment 1

Full Marks: 100

**1**. Let $X_1, X_2, \cdots X_n$ be $n$ independent random variables each having density function:

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $S_n = X_1 + X_2 + \cdots X_n$, show that

$$\Pr\left[\left|\frac{S_n}{n}\right| \geq \epsilon\right] \leq \frac{1}{3n\epsilon^2}$$

**10**

**2**. Let $X_1, X_2, \cdots X_n$ be random variables with the same mean $\mu$ and with the following covariance function

$$\mathrm{COV}(X_i, X_j) \ = \sigma^2 \rho^{|i-j|},$$

where $|\rho| < 1$.

**a.** Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

**b.** Does the weak law of large numbers hold for the sample mean?

**8+7 = 15**

**3**. Let $X_1, X_2, \cdots X_n$ be iid random variables having density function

$$f(x) = \begin{cases} \frac{1}{2a} & \text{for } |x| \leq a \\ \\ 0 & \text{otherwise} \end{cases}$$

**a.** Find the characteristic function of

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}$$

**b.** Prove that as $n \to \infty$, the characteristic function approaches $e^{-\omega^2/2}$.

**10+6 = 16**

**4**. Let $U_0, U_1, \cdots$ be a sequence of iid zero-mean, unit-variance Gaussian random variables. A 'low-pass filter' takes the sequence $U_i$ and produces the output sequence $X_n = (U_n + U_{n-1})/2$, and a 'high-pass filter' produces the output sequence $Y_n = (U_n - U_{n-1})/2$.

**a.** Find the joint pdf of $X_n$ and $X_{n-1}$.

**b.** Find the joint pdf of $Y_n$ and $Y_{n+m},\ m > 1$.

**c.** Find the joint pdf of $X_n$ and $Y_m$.

**d.** Does the sequence $X_n$ converge in the mean square sense?

**e.** Does the sequence $X_n$ converge in distribution?

**6+6+6+6+4 = 28**

**5**. Let $M_n$ be the discrete-time process defined as the sequence of sample means of an iid sequence:

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

**a.** Find the mean, variance, and covariance of $M_n$.

**b.** Does $M_n$ have independent increments?

**c.** Does $M_n$ have stationary increments?

**6+5+5 = 16**

**6**. A discrete-time random process $X_n$ is defined as follows. A fair coin is tossed; if the out- come is heads then $X_n = 1$ for all $n$. Otherwise, $X_n = -1$ for all $n$.

**a.** Is $X_n$ a WSS random process?

**b.** Is $X_n$ a stationary random process?

**c.** Does the answers in parts **a** and **b** change if the coin is biased, i.e., probability of heads $p \neq \frac{1}{2}$?

**6+5+4 = 15**