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QUESTION-1

Find the roots of the given functions using Newton's method, the Secant method, and Modified Newton's method. For the Secant method, start with initial guesses (or interval) of $p_0 = 0$ and $p_1 = 1$. Iterate until you reach a relative tolerance of 10^{-6} between successive iterates.

Convergence Rate

$$e_{n+1} = ke_n^{\alpha}, \quad e_n = ke_{n-1}^{\alpha}, \quad e_{n+1} = \frac{e_n^{\alpha+1}}{e_{n-1}^{\alpha}}$$

$$\log e_{n+1} = (\alpha+1)\log e_n - (\alpha)\log e_{n-1}$$

$$\alpha = \frac{\log\left(\frac{e_{n+1}}{e_n}\right)}{\log\left(\frac{e_n}{e_{n-1}}\right)}$$

Method	Root	Iterations
Newton's Method	-0.5884017765009963	5
Secant Method	-0.5884017765009013	7
Modified Newton's Method	-0.5884017765009963	5

Table 1: Results for $f_1(x)$: Root and Iterations

Method	Convergence (Mean)	Convergence	Theoretical Convergence
Newton's Method	Mean: 2.4641	1.098	Quadratic (2)
Secant Method	Mean: 1.6915	1.509	Superlinear (1.618)
Modified Newton's Method	Mean: 2.2532	1.62	Quadratic (2)

Table 2: Convergence rates for $f_1(x)$: convergence totheoretical convergence.

For $f_1(x)$, Newton's and Modified Newton's Methods show high precision with a residual error of -1.110×10^{-16} and both achieve quadratic convergence in 5 iterations. The Secant Method, while less precise with a residual error of 1.981×10^{-13}, has a superlinear convergence rate and requires 7 iterations.

For $f_2(x)$, Modified Newton's Method is exceptionally accurate with an error of 1.233×10^{-32} and achieves quadratic convergence in just 5 iterations. Newton's Method is also precise with an error of 1.930×10^{-12} and it is linear convergence because f'(x)=0requires 18 iterations. The Secant Method, though less accurate with an error of 5.240×10^{-12}, has a superlinear convergence rate and requires 33 iterations.

In summary, for equations with multiple roots, the Modified Newton's Method converges very quickly and accurately.

Method	**Root**	**Iterations**
Newton's Method	-0.5884011102599648	18
Secant Method	-0.5884006785832132	33
Modified Newton's Method	-0.5884017765009963	5

Table 3: Results for $f_2(x)$: Root and Iterations

Method	**Convergence (Mean)**	Convergence	**Theoretical Convergence**
Newton's Method	Mean: 1.0964	0.999	Linear (1)
Secant Method	Mean: 1.1491	0.998	Superlinear (1.618)
Modified Newton's Method	Mean: 2.2532	1.628	Quadratic (2)

Table 4: Convergence rates for $f_2(x)$: convergence to theoretical convergence.

QUESTION-3

Newton's method for solving systems of nonlinear equations can be expressed as:

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - J^{-1}(\mathbf{x}^{(n)}) \cdot \mathbf{F}(\mathbf{x}^{(n)})$$

Where:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

is the vector of functions, and

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

is the Jacobian matrix of partial derivatives.

$$\begin{pmatrix} \theta_2^{(k+1)} \\ \theta_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} \theta_2^{(k)} \\ \theta_3^{(k)} \end{pmatrix} - J^{-1} \cdot \begin{pmatrix} f_1(\theta_2^{(k)}, \theta_3^{(k)}) \\ f_2(\theta_2^{(k)}, \theta_3^{(k)}) \end{pmatrix}$$

$$J^{-1} = \frac{1}{-48(\sin\theta_2\cos\theta_3 - \sin\theta_3\cos\theta_2)}\begin{bmatrix} 8\cos\theta_3 & -6\cos\theta_2 \\ 8\sin\theta_3 & -6\sin\theta_2 \end{bmatrix}$$

Now consider the initial guess $\theta_2 = 30°$ and $\theta_3 = 0°$. The angles should be consider in Radian for computation.

The solution for the angles is as follows:

- $\theta_2 = 32.0151803593°$
- $\theta_3 = 355.6290125950°$(-4.3710°)

Number of iterations required: 4

$\alpha = \frac{\log\left(\frac{e_{n+1}}{e_n}\right)}{\log\left(\frac{e_n}{e_{n-1}}\right)}$

Observered convergence Rate =1.99324=approx(2)

The observed convergence Rate is approximately equal to netwon method of convergence=2

Question - 2

To derive Cubic Convergence on fixed point iteration function $g(x)$ to solve $f(x)=0$, we have to start with the given form

$P_{n+1} = g(P_n)$

$P_n = g(P_n)$

$|P_{n+1} - P| = g(P_n) - g(P) \Rightarrow \varepsilon_{n+1} = g(P_n) - g(P)$

Expand with taylor series around P

$$g(P_n) = g(P) + g'(P)(P_n - P) + \frac{g''(P)(P_n-P)^2}{2!} + \frac{g'''(P)(P_n-P)^3}{3!} + \cdots$$

For cubic Convergence $g'(p), g''(p) = 0$

$$\boxed{\varepsilon_{n+1} = \frac{g'''(P)}{3!}(P_n - P)^3 \Rightarrow \varepsilon_{n+1} = \frac{g''(P)}{3!}\varepsilon_n^3}$$

$\alpha = 3$ and $\lambda = \frac{g'''(p)}{3!}$

$$\boxed{g(x) = x - \phi(x) f(x) - \psi(x) f^2(x)}$$

$$g'(x) = 1 - \phi'(x) f(x) - \phi(x) f'(x) - \psi'(x) f^2(x) - 2\psi(x) f(x) f'(x)$$

$$g'(P) = 1 - \phi(P) f'(P) = 0 \Rightarrow \phi(P) = \frac{1}{f'(P)} \longrightarrow (1) \quad [\text{since } f(p) = 0]$$

$\phi(x)$ should satisfy the above equation

$$\boxed{\therefore \phi(x) = \frac{1}{f'(x)} \Rightarrow \phi'(x) = \frac{-f''(x)}{f'(x)^2}}$$

$$g''(p) = 2\phi'(p) f'(p) + \phi(p) f''(p) + 2\psi(p) f'(p)^2 = 0 \quad [\text{since } f(x), f'(x) = 0]$$

$$\Rightarrow \frac{-2f''(p)}{f'(p)} + \frac{f''(p)}{f'(p)} + 2\psi(p)(f'(p))^2 = 0$$

$$\boxed{\psi(x) = \frac{f''(p)}{2[f'(p)]^3}}$$

$$\boxed{\therefore g(x) = x - \frac{f(x)}{f'(x)} - \frac{f^2(x) f''(x)}{2[f'(x)]^3}}$$

$$g'''(p) = \frac{3(f'''(p))^2 - f'(p) f''(p)}{(f'(p))^2}$$

$\therefore$ the asymptotic error constant

$$\boxed{\lambda = \frac{3[f''(p)]^2 - f'(p) f''(p)}{6[f'(p)]^2}}$$