(i) Given $x_1, x_2, x_3, \ldots, x_n$ are n independent random variables each with

$$f(x) = \begin{cases} 1/2 & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

as $x_1, x_2, x_3, \ldots, x_n$ have some density function and independent they are i.i.ds Given $S_n = x_1 + x_2 \cdots x_n$ we have to show that

$$P\left[\left|\frac{S_n}{n}\right| \ge \epsilon\right] \le 1/3n\epsilon^2$$

from chebshev's inequality

$$P(|x - E[x]| \ge k) \le \frac{var(x)}{k^2} \quad \text{— (1)}$$

Let take $x = \frac{S_n}{n}$, $k = \epsilon$ } — (2)

$$E[x] = E[S_n/n] = \frac{E[S_n]}{n} = \frac{E[x_1 + x_2 + \cdots + x_n]}{n}$$

$$= \frac{E[x_1] + E[x_2] + \cdots + E[x_n]}{n}$$

$$= \frac{nE[x_i]}{n} \quad [\because x_i \text{ is uniform random variable in the } [-1, 1])$$

$$= E[x_i] = \frac{1-1}{2} = 0 \quad \left[\because E[x_i] = \frac{a+b}{2}\right]$$

$$var[x] = var\left[\frac{S_n}{n}\right] = \frac{var[S_n]}{n^2} = \frac{var[x_1 + x_2 + \cdots + x_n]}{n^2}$$

$$= \frac{var[x_1] + var[x_2] + \cdots + var[n]}{n^2} \quad (\because x_i\text{'s are iid's}]$$

$$= \frac{n\, var[x_i]}{n^2} = \frac{var[x_i]}{n}$$

$$var[x_i] = \frac{(1-(-1))^2}{12} = \frac{4}{12} = \frac{1}{3} \quad \left[var[x] = \frac{(a-b)^2}{12}\right]$$

$$var[x] = \frac{1}{3n} \quad \text{— (4)}$$

From (2), (3), (4) in (1)

$$P\left[\left|\frac{S_n}{n} - 0\right| \ge \epsilon\right) \le \left(\frac{1}{3n}\right)/\epsilon^2$$

$$\boxed{\therefore P\left[\left|\frac{S_n}{n}\right| \ge \epsilon\right) \le \frac{1}{3n\epsilon^2}}$$

(2) Given $x_1, x_2, \ldots, x_n$ are random variables with mean 'μ' and co-varience function. $\text{Cov}(x_i, x_j) = \sigma^2 \rho^{|i-j|}$ where $|\rho| < 1$

(a) $S_n = x_1 + x_2 + \ldots x_n$

$$E[S_n] = E[x_1 + x_2 + \ldots + x_n] = E[x_1] + E[x_2] + \ldots + E[x_n]$$
$$= n\mu$$

$$\text{Var}[S_n] = \text{Var}[x_1 + x_2 + \ldots + x_n]$$
$$= \sum_{i=1}^{n} \text{Var}(x_i) + 2 \sum_{1 \le i \le j \le m} \text{Cov}(x_i, x_j) \quad \text{—} (7)$$

$$\text{Var}(x_i) = \text{Cov}(x_i, x_i) = \sigma^2 \rho^{|i-i|} = \sigma^2$$

then $\sum_{i=1}^{n} \text{Var}(x_i) = n\sigma^2$ — (1)

$$2\sum_{1 \le i \le j \le n} \text{Cov}(x_i, x_j) = 2\sigma^2 \sum_{1 \le i \le j \le n} \rho^{|i-j|}$$

$$= 2\sigma^2 \sum_{k=1}^{n-1} (n-k)\rho^k$$

$$= 2\sigma^2 \left[n\rho \left[\frac{1-\rho^{n-1}}{1-\rho} \right] - \frac{\rho(1-\rho^{n-1})}{(1-\rho)^2} - \frac{(n-1)\rho^n}{(1-\rho)} \right] \quad \text{—} (2)$$

put (1) and (2) in (7)

$$\text{Var}(S_n) = n\sigma^2 + 2\sigma^2 \left[\frac{n\rho(1-\rho^{n-1})}{1-\rho} - \frac{\rho(1-\rho^{n-1})}{(1-\rho)^2} + \frac{(n-1)\rho^n}{1-\rho} \right)$$

(b) Weak law of large numbers is not applicable as X_i are not iid

But $\lim_{n \to \infty} \text{Var}\left(\frac{S_n}{n}\right) = \lim_{n \to \infty} \frac{\sigma^2}{n^2} + 2\sigma^2 \left[\frac{n\rho(1-\rho^{n-1})}{1-\rho} - \frac{\rho(1-\rho^{n-1})}{(1-\rho)^2} + \frac{(n-1)\rho^n}{1-\rho} \right)$

$= 0$

using Chebyshev's inequality

$$P\left[\left| \frac{S_n}{n} - \mu \right| \ge \epsilon \right) \le \text{Var}\left(\frac{S_n}{n}\right) / \epsilon^2$$

$P\left[\left| \frac{S_n}{n} - \mu \right| \ge \epsilon \right] \le 0$ thence $\frac{S_n}{n} \to \mu$

The weak law of large numbers holds but the premise is not satisfied.

③ Given $x_1, x_2, \cdots, x_n$ be iid random variables having density function

$$f(x) = \begin{cases} 1/2a & \text{for} \quad -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

ⓐ

$$\phi_{S_n}(jw) = E\left(e^{-jwS_n}\right)$$

$$= E\left[e^{-jw\frac{(x_1+x_2+\cdots+x_n)}{\sqrt{n}}}\right]$$

$$= E\left[\prod_{i=1}^{m} e^{-jwx_i/\sqrt{n}}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{-jwx_i/\sqrt{n}}\right] \quad [\because \text{iid's}]$$

$$\phi_{S_n}(jw) = \left(E\left[e^{-jwx_i/\sqrt{n}}\right]\right)^n \longrightarrow (1)$$

$$E\left[e^{-jwx_i/\sqrt{n}}\right) = \int_{-a}^{a} \frac{1}{2a} e^{-jwx_i/\sqrt{n}}\, dx$$

$$= \frac{\sqrt{n}}{2ajw}\left[e^{jaw/\sqrt{n}} - e^{-jaw/\sqrt{n}}\right]$$

$$E\left[e^{-jwx_i/\sqrt{n}}\right] = \frac{\sqrt{n}}{aw}\sin\left(\frac{aw}{\sqrt{n}}\right) - (2)$$

eq(2) in eq(1)

$$\phi_{S_n}(jw) = \left(\frac{\sqrt{n}}{aw}\sin\left(\frac{aw}{\sqrt{n}}\right)\right]^n$$

ⓑ Let

$$Y = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{aw}\sin\left(\frac{aw}{\sqrt{n}}\right)\right)^n$$

$$\log Y = \lim_{n\to\infty} n\log\left(\frac{\sin(aw/\sqrt{n})}{aw/\sqrt{n}}\right)$$

$$\log Y = \lim_{n\to\infty} \frac{\log\left[\frac{\sin(aw/\sqrt{n})}{aw/\sqrt{n}}\right)}{1/n}$$

L'Hosptial rule

Let $a\omega/\sqrt{n} = h \quad \Rightarrow \quad n = a^2\omega^2/h^2$

if $n \to \infty$, $h \to 0$

$$= \lim_{h\to 0} \frac{a^2\omega^2 \ln\left(\frac{\sin(h)}{h}\right)}{h^2} = \frac{0}{0}$$

LH rule

$$= \lim_{h\to 0} \frac{a^2\omega^2 (h\cos h - \sin h)}{2h^2 \sin h} = \frac{0}{0}$$

LH rule

$$= \lim_{h\to 0} \frac{a^2\omega^2 (\cancel{\cos h} - h\sin h - \cancel{\cos h})}{2(2h\sin h + h^2\cos h)} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{a^2\omega^2 (-\sin h - h\cos h)}{4\sin h + 4h\cos h + 4h\cos h - 2h^2\sin h} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{a^2\omega^2 [-\cos h - \cos h - h\sin h]}{4\cos h + 4\cos h - 4h\sin h + 4\cos h - 4\sin h - 4h\sin h - 2h^2\cos h}$$

$$= \frac{-a^2\omega^2(2)}{12}$$

$$\log Y = \frac{-a^2\omega^2}{6}$$

$$Y = e^{-a^2\omega^2/6}$$

$$\boxed{\therefore \lim_{n\to\infty} \phi_{S_n}(j\omega) = e^{-a^2\omega^2/6}}$$

depending on appropriate value of a it approxies $e^{-\omega^2/2}$

(4) Given $v_0, v_1, v_2, \dots$ are iid's with mean = 0 and varience = 1 which follow gaussian random variable

(a) Joint PDF of X_n and X_{n-1} where $X_n = \frac{u_n + u_{n-1}}{2}$

$$x_{n-1} = \frac{u_{n-1} + u_{n-2}}{2}$$

$$E[x_n] = E\left[\frac{u_n + u_{n-1}}{2}\right] = \frac{1}{2}\left(E[u_n] + E(u_{n-1})\right) \quad [\because \text{iid's}]$$

$E[x_n] = 0$ similarly $E[x_{n-1}] = 0$

$$\text{var}[x_n] = \text{var}\left[\frac{u_n + u_{n-1}}{2}\right] = \frac{1}{4}\left[\text{var}[u_n] + \text{var}[u_{n-1}]\right)$$

$$= \frac{1+1}{4} = \frac{1}{2}$$

Similarly $\text{var}[x_{n-1}] = \frac{1}{2}$

$$\text{cov}(x_n, x_{n-1}) = E[x_n x_{n-1}] - E[x_n]E[x_{n-1}]$$

$$= E\left[\left(\frac{u_n + u_{n-1}}{2}\right)\left(\frac{u_{n-1} + u_{n-2}}{2}\right)\right] - 0 \times 0$$

$$= E\left[\frac{u_n u_{n-1} + u_n u_{n-2} + u_{n-1} + u_{n-1} u_{n-2}}{4}\right]$$

$$= \frac{1}{4}\left[E[u_n]E[u_{n-1}] + E[u_n]E[u_{n-2}] + E[u_{n-1}^2] + E[u_{n-1}]E[u_{n-2}]\right)$$

$$= \frac{1}{4}\left(0 + 0 + \underset{1}{\text{var}(u_{n-1})} + \underset{0}{E[u_{n-1}]} + 0\right]$$

$$\text{cov}(x_n, x_{n-1}) = \frac{1}{4}$$

mean vector $= \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ coverience matrix $= \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$

Since u_n, u_{n-1}, u_{n-2} are jointly Gaussian bcz they are iid's, so even x_n, x_{n-1} PDF is Gaussian

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}\right)$$

(b) Joint PDF of Y_n and Y_{n-1} where

$$Y_n = \frac{u_n - u_{n-1}}{2} \quad ; \quad y_{n+m} = \frac{U_{n+m} - U_{n+m-1}}{2}$$

$$E[Y_n] = E\left[\frac{u_n - u_{n-1}}{2}\right] = \frac{1}{2}\left[E(u_n) - E[u_{n-1}]\right] = 0$$

$[\because \text{iid's}]$

similarly $E[Y_{n+m}] = 0$

$$var(Y_n) = var\left(\frac{u_n - u_{n-1}}{2}\right) = \frac{1}{4}\left(var(u_n) + var(u_{n-1})\right)$$

$[\because \text{iid's}]$

$var(Y_n) = \frac{1}{2}$ similarly $var(Y_{n+m}) = \frac{1}{2}$

$$cov(Y_n, Y_{n+m}) = E[Y_n Y_{n+m}] - E[Y_n]E[Y_{n+m}]$$

$$= E\left[\left(\frac{u_n - u_{n-1}}{2}\right)\left(\frac{u_{n+m} - u_{n+m-1}}{2}\right)\right] - 0$$

$$= \frac{1}{4} E\left[u_n u_{n+m} - u_n u_{n+m-1} - u_{n+m} u_{m-1} + u_{n-1} u_{m+n-1}\right]$$

$$cov(Y_n, Y_{n+m}) = 0$$

since $cov(Y_n, Y_{n+n}) = 0$ Y_n, Y_{n+m} are independent

$$f_{y_n, y_{n+m}}(Y_1, Y_2) = \frac{1}{\pi} e^{-(Y_1 + Y_2)}$$

It is Gaussian with mean 0 and varience $\frac{1}{2}$

(c) Join PDF of X_n & Y_m

$$X_n = \frac{U_n + U_{n-1}}{2} \qquad Y_n = \frac{U_m - U_{m-1}}{2}$$

$E[X_n] = 0$ and $E[Y_m] = 0$

$var[X_n] = \frac{1}{2}$ and $var[Y_m] = \frac{1}{2}$

$$cov(X_n, Y_n) = E[X_n Y_m] - E[X_n]E[Y_m]$$

$$= E\left[\frac{(u_n + u_{n-1})}{2}\frac{(u_m - u_{m-1})}{2}\right] - 0$$

$$= \frac{1}{4}\left(E\left[u_n u_m - u_n u_{m-1} + u_{n-1} u_m - u_{n-1} u_{m-1}\right]\right)$$

Case 1 :- if $n = m+1$

$cov(X_n, Y_n) = \frac{1}{4}(0 - 0 + 1 - 0)$

$cov(X_n, Y_n) = 1/4$

covarience matrix $= \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$

Case 2 :- if $m = n+1$

$cov(X_n, Y_n) = \frac{1}{4}(0 - 1 + 0 - 0)$

$= -1/4$

covarience matrix $= \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}$

Case 3

All other cases

Independent pair $U_n U_m$, $U_n U_{m-1}$, $U_n U_{m-1}$, $U_{n-1} U_m$, $U_{n-1} U_{m-1}$

So $cov(X_n, Y_n) = 0$

then similarly to "b".

(d) if $X_n \xrightarrow{ms} X$ then even $X_{n+1} \xrightarrow{ms} X$

$\Rightarrow X_{n+1} - X_n \xrightarrow{ms} 0$

$\Rightarrow U_{n+1} - U_{n-1} \xrightarrow{\text{distribution}} 0$

$Var(U_{n+1} - U_{n-1}) \longrightarrow 0$

But $Var(U_{n+1} - U_{n-1}) = Var(U_{n+1}) + Var(U_{n-1})$

$= 2$

So it doesnot converge

(e) as each $X_i \sim N(0, 1/2)$ so it converges in distribution to $N(0, 1/2)$

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(5) Given $X_1, X_2, \dots X_n$ is sequence of sample means of iid sequence

$$M_n = \frac{X_1 + X_2 + \dots + X_n}{2}$$

let $E[X_i] = \mu$
and $var(X_i) = \sigma^2$

(a) Mean of $M_n = E[M_n]$

$$= E\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right]$$

$$= \frac{1}{n}\left[E[X_1] + E[X_2] + \dots + E[X_n]\right]$$

[$\because$ iid's]

$$= E[X_i] = \mu$$

$$var[M_n] = var\left[\frac{X_1 + X_2 + \dots X_n}{n}\right] = \frac{Var[X_i]}{n}$$

[$\because$ iid's]

$$= \frac{\sigma^2}{n}$$

$$cov(M_m, M_n) = cov\left(\frac{\sum_{i=1}^{m} X_i}{m}, \frac{\sum_{j=1}^{n} X_j}{n}\right)$$

$$= \frac{1}{mn}\left(\sum_{i=1}^{m} cov(X_i, \sum_{j=1}^{n} X_j)\right)$$

$$\frac{1}{mn}\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{ij}\right.$$

if $n \le m$

$$= \frac{\sigma^2}{mn}(n) = \frac{\sigma^2}{m}$$

if $m < n$

$$= \frac{\sigma^2}{mn} \times m = \frac{\sigma^2}{n}$$

(b) to show it is independent increment $m_{n+1} - m_n$ should be dependent on m_n

$$m_{n+1} - m_n = \frac{x_1 + x_2 \ldots\ldots + x_n}{n+1} - m_n$$

$$m_{n+1} - m_n = \frac{\langle m_n \rangle n + x_{n+1}}{n+1} - m_n$$

$$= \frac{n}{n+1} m_n - m_n + \frac{x_{n+1}}{n+1}$$

$$m_{n+1} - m_n = \frac{-m_n}{n+1} + \frac{x_{n+1}}{n+1} \quad \text{(+)}$$

It is dependent on previous increment so not independent increments

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(c) The distribution of $m_{n+1} - m_n$ should be same for all n's

from equation (+)

$$m_{n+1} - m_n = \frac{-m_n}{n+1} + \frac{x_{n+1}}{n+1}$$

$$\mathrm{var}(m_{n+1} - m_n) = \frac{\sigma^2}{(n+1)^2 n} + \frac{\sigma^2}{(n+1)^2}$$

So the varience of $m_{n+1} - m_n$ depends on n. So it is not stationary

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6) a) $E[x_n] = E[x_n / head] P(head) + E[x_n / tail] P(tail)$

$= 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$

$R_x(m, n) = E[x_m x_n]$

$= E[x_m x_n / Head] P(Head) + E[x_m x_N / tail] P[tail]$

$= \frac{1}{2} + \frac{1}{2} = 1$

The process is stationary.

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b) X_n's are determined by flipping a coin, and it takes same value based on heads or tails.

if heads :- then

$x_1, x_2, x_3 \dots x_n$ takes values $1, 1, \dots$

if shifted the joint PDF of X_i's are still same

⇒ similar for tails also

So it is stationary radom process

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c) No, the process is still stationary because even if the probability of head is p and 1-p for tails

the cases doesn't change same as 'b' part just the probability of getting the case changes.

— the END —