DS 298: Work Assignment - 2

Due March 5, 2025

The goal of this work is to analyze the errors E in the matrix-multiplication Algorithm-1 for various values of the dimension n and number of samples c, in each case of the three different classes of matrices described.

Let $M^{m\times p} \approx A^{m\times n}B^{n\times p}$ be the evaluation considered where $m = p = n$ here. Use values of n (x-axis in plots) as 100, 200, 400, 800, 1600. The relative error $E = \|M - AB\|_F/\|AB\|_F$ is to be traced (in the y-axis) for three different number of samples $c = \log_2 n, (\log_2 n)^2, 0.2n$ rounded to the nearest integer, in a single plot. Average over 10 runs of the algorithms for each data point in the plot. This plot is to be replicated for the three matrix classes I, II, III.

The trial matrices are to be generated using the singular value decomposition $U\Sigma V^T$. The singular values given by Σ_{kk} should be fixed as i) $e^{-\frac{(k-1)}{10}}$ for Class-I matrices, ii) $e^{-\frac{10(k-1)}{n}}$ for Class-II matrices, and iii) as $\frac{n-k+1}{n}$ for Class-III matrices.

Algorithm-1 for the matrix multiplication, and Algorithm 2 for generating the trial matrices, are briefly described in the next page. Note that the probabilities of sampling the k^{th} column and row in estimating the product AB is given by

$$p_k = \frac{\|A^{(k)}\|_2\|B_{(k)}\|_2}{\sum_n \|A^{(k)}\|_2\|B_{(k)}\|_2}.$$

For a comparison, also produce results in another set of three plots when the columns and rows are deterministically chosen using the top c values of p_k (with no random sampling involved).

Note : Submit the responses with the plots, and the codes, as separate files all zipped into a single folder identified by your name in full, and upload it on the MS-Teams channel for DS 298.

Algorithm 1 : Matrix multiplication $M \approx AB$

Inputs : c, p_k, $A^{(k)}$, $B_{(k)}$ for $k \in \{1, 2, \dots n\}$.
Outputs : E, $M \longleftarrow [\ ]$; an approximation of product AB, and its error.
While trials $t \leq c$ do {
Random sampling :
For a uniformly distributed random integer $k \in \{1, 2, \dots n\}$
If $max\{p_k\}U < p_k$
Accept k and $t = t + 1$; Sample indices k with a probability mass p_k using rejection sampling.
Sum rank-one matrices : $M \longleftarrow M + A^{(k)}B_{(k)}/(cp_k)$; evaluate outer products of the accepted column and row, and update the sum.
} break while loop
Evaluate error : $E \longleftarrow \|M - AB\|_F / \|AB\|_F$; evaluate relative error in the Frobenius norm.

Algorithm 2 : Generation of a trial matrix in a class

Inputs : $\Sigma \in \mathcal{R}^{n \times n}$; diagonal matrix of singular values.
Outputs : $A \in \mathcal{R}^{n \times n}$; trial matrix.
Initialize : $M_1 \longleftarrow rand[n, n]$ and $M_2 \longleftarrow rand[n, n]$; Initialize two random matrices.
Generate singular vectors : $Q_1 R_1 \longleftarrow M_1$ and $Q_2 R_2 \longleftarrow M_2$; QR factorization of the two matrices.
Build trial matrices : $A \longleftarrow Q_1 \Sigma Q_2^T$; generate test matrix of a given singular value distribution.