ASTR 4201/5201 (Fall 2016)
Homework #1
Due: Friday, September 2 in class

The problems labeled G1 is required for graduate students. Undergrads can solve it for up to 10 percentage points extra credit. Please remember to show your work.

1. Your ability to perceive the world three-dimensionally is partially due to parallax effects from the two different perspectives seen by your eyes. Assume that your eyes have a baseline of $2B = 7.5$ cm.

   (a) If you hold your thumb 0.5 m in front of your face, what is the parallax angle seen by your eyes?

   (b) Assume that the *GAIA* satellite will measure trigonometric parallaxes for 10^9 stars with a precision of 20 μas (microarcseconds). How far away would your thumb have to be from your face to have a parallax this small relative to your eyes? How does this compare to the Earth-Moon distance (semi-major axis=3.84×10^8 m)?

2. Polaris is famous for navigational purposes because it is located near the North Celestial Pole, so it is known as the North Star or Pole Star. *Hipparcos* measured a parallax of 7.5 mas (milliarcseconds) and the apparent V magnitude is 2.0 mag. For the purposes of this problem, assume that Polaris is a single star that has the same intrinsic color as the Sun (approximately true! note $(B-V)_\odot = 0.65$ mag) and thus a similar bolometric correction to the Sun. (If you need to look up solar properties, see Appendices A of C&O, which I attached to the syllabus).

   (a) What is the distance to Polaris?

   (b) Distance modulus?

   (c) Absolute V magnitude?

   (d) Absolute bolometric magnitude? And how many magnitudes is this brighter or fainter than the Sun's $M_{\rm bol}$?

   (e) What does this imply about the ratio of Polaris's luminosity to that of the Sun? And so what is Polaris's bolometric luminosity (in W)?

   (f) What is the radius of the star (in m), assuming a spherical star with $T_{\rm e}$=6000 K? And what is this value in units of the solar radius?

   (g) Assume that we knew that Polaris had intrinsically the same $(B-V)$ color as the Sun but that we measured $(B-V)$=1.0 mag. Given things we have discussed in class, what could be the cause of this difference? What would the true bolometric luminosity of Polaris be in that case (assuming standard Galactic parameters)?

3. It is hard for objects in the universe to be much cooler than the cosmic microwave background because of interactions with the ubiquitous CMB blackbody photons. The current CMB temperature is 2.7K.

   (a) What is the peak wavelength of the CMB radiation?

   (b) What region of the electromagnetic spectrum does this correspond to?

   Note that observations taken at longer wavelengths of astronomical objects that emit thermally are therefore always on the Rayleigh-Jeans tail.

4. Find an expression for the value of $\nu_{\rm max}/T$ at the maximum of $B_\nu(T)$. Note that this expression requires a numerical or graphical solution. Convert $\nu_{\rm max}$ into the equivalent wavelength, $\lambda_{\rm max}$. You should find that the peak of $B_\nu(T)$ is not at the same wavelength as stated for the peak of $B_\lambda(T)$ by Wien's displacement law. What are the two peak wavelengths (for $B_\nu(T)$ and $B_\lambda(T)$) in the case of the Sun?

5. In class, we derived the radiation transfer equation (assuming the sign convention of CO) in terms of the optical depth:

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

   At a depth $\tau_\lambda = \tau_{\lambda,0}$ along a ray, we have incident radiation $I_\lambda(\tau = \tau_{\lambda,0})$ that emerges with intensity $I_\lambda(\tau = 0)$ after passing through some material along the line of sight. Assume that S_λ is a constant along the line of sight. Solve the differential equation to obtain the radiative transfer equation for $I_\lambda(\tau = 0)$ in terms of $\tau_{\lambda,0}$. (I wrote down the solution in class but did not solve it.)

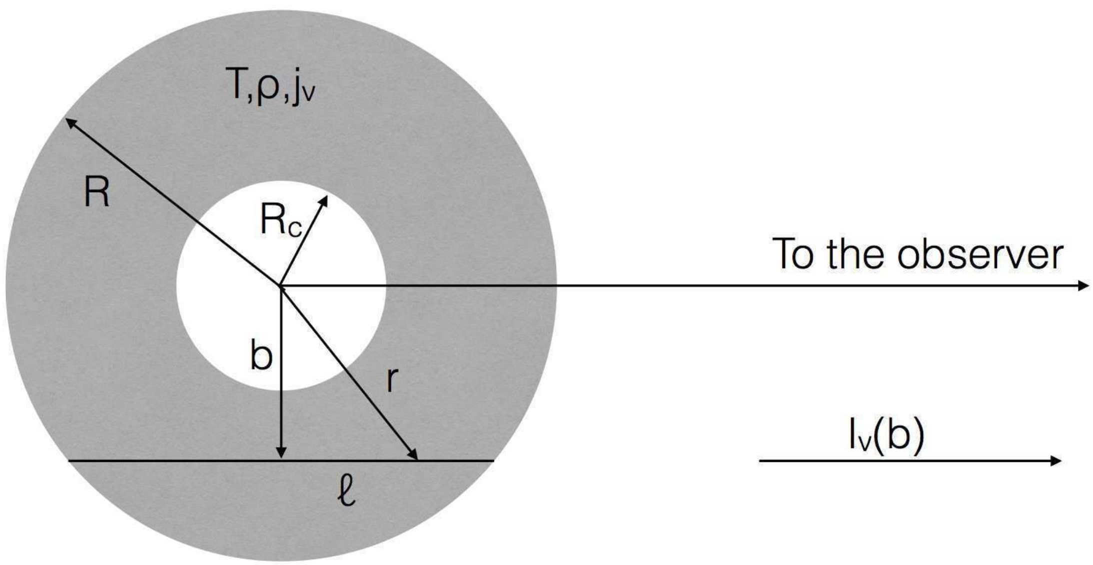
G1. Consider a uniform spherical cloud of radius R, temperature T, and mass density ρ emitting thermally and isotropically with an emission coefficient j_ν, as shown in the figure. Assume the same definition for j_ν as Carroll & Ostlie, i.e., $dI_\nu = j_\nu \rho ds$.

(a) Assume that the cloud is optically thin, homogeneous, and spatially resolved when seen from Earth. If b is the distance perpendicular to the line of sight (known as the impact parameter), what is $I_\nu(b)$?

(b) Assume that the cloud is optically thin, homogeneous, and spatially unresolved when seen from Earth. What is the flux density f_ν measured at Earth if the cloud is at distance d?

(c) What is the flux f_ν measured at Earth if there is a central cavity of radius R_c? Assume that $j_\nu = 0$ in the cavity.

(d) What are the answers to the three previous questions if the cloud is optically thick?



ASTR 4201/5201 (Fall 2016)
Homework #2
Due: Friday, September 9 in class

The problem labeled G1 is required for graduate students. Undergrads can solve it for up to 10 percentage points extra credit. Please remember to show your work.

1. For this question, assume all atoms and ions are described by the Bohr model.
   (a) Consider the ion O VIII. How much energy is needed to remove the final electron from the ground state? What about for Fe XXVI? In what region of the electromagnetic spectrum are these two transitions?
   (b) Radio astronomers can sometimes observe recombination lines from hydrogen in very high quantum levels ($n > 200$). One popular one is the n=167 $\rightarrow$ n=166 transition. What is the frequency of this transition? What is the radius (in the Bohr model) of a hydrogen atom in the n=167 state (note: it is *huge* for an atom)?
   (c) The existence of H atoms in this very highly excited state provides a constraint on the maximum possible density. Assuming a pure H composition at a certain number density (n_H), there is a mean interparticle spacing d. Given the radius you calculated in the previous question (r_n), at what number density n_H is $d = 2r_n$? At higher densities, the electrons cannot have such large orbital radii and remain bound to the nucleus (and the Bohr model breaks down).

2. Some of the brightest stars in the sky and their spectral classes are: Sirius (A1V), Arcturus (K2II), α Centauri (G2V), Vega (A0V), Rigel (B8Ia), Betelgeuse (M2Ib), Spica (B1V)

   Using only the information from their spectral classes:
   (a) Order these stars from hottest to coolest.
   (b) Indicate which of these stars will fall above the main sequence in an H-R diagram.
   (c) Which is these stars is most like the Sun?

3. Consider the following list of possible electronic transitions between configurations and terms. For each one, state whether it is permitted, forbidden, or semiforbidden by the electric dipole selection rules under the L-S coupling scheme. If it is not permitted, then specify which rules (potentially more than one!) are being violated.
   (a) $(2s^2)\ ^1S_0 \leftrightarrow (2s2p)\ ^3P_1^o$
   (b) $(2s^22p^4)\ ^3P_2 \leftrightarrow (2s^22p^4)\ ^1D_2$
   (c) $(1s^2)\ ^1S_0 \leftrightarrow (2s2p)\ ^1P_1^o$
   (d) $(3s)\ ^2S_{1/2} \leftrightarrow (3p)\ ^2P_{1/2}^o$
   (e) $(2s^22p^3)\ ^4S_{3/2}^o \leftrightarrow (2s^22p^3)\ ^2D_{3/2}^o$

4. In class we examined the excitation of hydrogen and found that in thermal equilibrium the first excited level has a substantial population relative to the ground state only at a fairly high temperature.

   (a) Consider an atom that emits a photon of wavelength 5000 Å when transitioning between its first excited state (with degeneracy g=5) and the ground state (with g=3). Assuming a dense gas with a sufficiently high collision rate to populate the levels in LTE, at what temperature will the population in the first excited level be equal to that in the ground state (ignoring ionization)?

   (b) What if the separation between energy levels is such that the photon is emitted at 50μm in the infrared?

G1. Here we will explore one of the limitations of the Saha formula by attempting to apply it to the conditions at the center of the Sun. For the purposes of this problem, we will be considering a pure hydrogen composition so that the only constituents we have to deal with are neutral hydrogen, ionized hydrogen, and free electrons. This is a *terrible* approximation for the center of the Sun, where He actually represents more than half the mass density, but it simplifies the problem to make a more general point about stellar interiors.

Start with the Saha equation in the following form (as stated in class):

$$\frac{n^{II}}{n^{I}} = \frac{2Z_{II}}{n_e Z_I} \left(\frac{2\pi m_e k T}{h^2} \right)^{3/2} e^{-\chi_i/kT}$$

   (a) There are three densities here, but they are all linked. Remove these densities by recasting this equation in terms of y, the fraction of hydrogen that is ionized, and n_H, the total number density of hydrogen (neutral or ionized).

   (b) Assume the following values for the conditions at the center of the Sun: the temperature is T_c=1.57×10^7 K and the density is ρ_c=1.53×10^5 kg m^{-3}. For now, also make the same standard assumptions about the partition functions that I described in class. Solve for y, the ionization fraction. It is probably somewhat lower than you were expecting.

   (c) One possible objection to the previous calculation concerns our approximation for the partition function for bound hydrogen, but that is not the main problem. Returning to the issue raised in Question 1c, above what mass density ρ is the mean interparticle spacing d less than twice the Bohr radius for the n=1 ground state? Explain the relevance of this calculation for the ionization state of hydrogen in the solar interior.

ASTR 4201/5201 (Fall 2016)
Homework #3
Due: Friday, September 16 in class

Please remember to show your work.

1. This problem concerns the amplitude of isotopic effects in atomic transition wavelengths. The strongest effect corresponds to the largest fractional difference in nuclear mass between two isotopes and thus is seen between normal hydrogen and deuterium. Deuterium is much less common than normal hydrogen (D/H $\approx 1.5\times10^{-5}$ in the local interstellar medium) but its primordial abundance is an important constraint on Big Bang nucleosynthesis. However, it is very hard to measure, in part because of destruction by fusion inside stars and in part because of the difficulty in separating it from regular hydrogen.

   For the purposes of this problem, you can use the measured mass of the deuteron (deuterium nucleus) of $3.3435837 \times 10^{-27}$ kg.

   (a) Apply the Bohr model of the H atom, but actually calculate the reduced mass and do not approximate it as the mass of the electron. Recalculate the wavelength of Hα emission (= Balmer α), but keep at least 5 significant figures of precision when inputting fundamental constants and calculating the wavelength.

   (b) Repeat the above calculation, but for the case of a deuterium nucleus. Note the fractional wavelength separation from the value above ($\Delta\lambda/\lambda$).

   (c) If the intrinsic line profile of the regular Hα line is too broad, the much weaker deuterium line cannot be separated or resolved. Given the wavelength offset calculated above, estimate the corresponding intrinsic velocity dispersion of an object or gas cloud that would blend the two lines and prevent you from seeing a separate deuterium line.

   (d) How hot would a gas cloud have to be so that its thermal Doppler width would blend the two lines?

   (e) The previous results make it seem theoretically promising to be able to detect the deuterium Balmer-α line separately from the hydrogen one. Why might it actually be very difficult in practice?

2. In class, I showed how to use detailed balance to relate the Einstein coefficients for a 2-level atom. I also discussed the importance of the stimulated emission term for getting the source function in LTE to be the blackbody function. For the purposes of this problem, consider a population of 2-level atoms, neglect collisions, and assume thermodynamic equilibrium so that the ambient radiation field is a blackbody, $B_\nu(T)$.

(a) Find the ratio of the rate of stimulated emission to the total rate of emission from the transitions from level 2 to level 1 (i.e., the fraction of the total emission due to stimulated emission). Show that this ratio is equal to $e^{-h\nu_0/kT}$, where $h\nu_0$ is the energy difference between the two states.

(b) For conditions in the solar photosphere (i.e., T=5777 K), calculate the relative importance of the stimulated emission term for both the Lyman-α transition and the 21 cm line of neutral hydrogen. You can treat each transition as occurring in a 2-level atom so that you can use the previous result. (The 21 cm line is due to the alignment/anti-alignment of the nuclear and electron spins in the H atom. It is very important for studying neutral interstellar gas, as we will cover later in the semester, but for now the important thing is its wavelength.)

3. Problem 9.27 in Carroll & Ostlie, which I repeat on the next page along with the relevant figure and table for the benefit of those who do not own C&O. You may use the fact that the electron pressure near the solar photosphere is 1 N m^{-2}.

4. Start with the solution to the radiative transfer equation in the form:

$$I_\lambda(0) = I_{\lambda,0}e^{-\tau_{\lambda,0}} + S_\lambda(1 - e^{-\tau_{\lambda,0}})$$

Note that this is separately true at each wavelength because $I_{\lambda,0}$, S_λ, and τ_λ are all functions of wavelength. In this problem, we are considering a wavelength-dependent incident beam of radiation $I_{\lambda,0} = I_0(\lambda/5000\text{ Å})^{-1}$ that is passing through a gas cloud of wavelength-dependent optical depth τ_λ and we want to know what specific intensity $I_\lambda(0)$ emerges from the other side. Let $I_0 = 10^4$ J/m^2/s/sr/Å and assume that τ has the functional form

$$\tau_\lambda = \tau_0\ (\lambda/5000\text{ Å})^3\ (1 - e^{-h\nu/kT})$$

and that the cloud is emitting thermally with $T = 10,000$ K.

Now make a log-log plot of the spectra for $\tau_0 = \{10^{-2},\ 10^{-1},\ 1,\ 10,\ 100,\text{ and } 300\}$ all on the same plot, with an abcissa range of 1000–10000 Å. That is, plot $\log(I_\lambda(0))$ versus $\log(\lambda)$, where λ has units of Å and $I_\lambda(0)$ has units of J/m^2/s/sr/Å. Overplot the Planck function $B_\lambda(T)$. ***Pay attention to the units of B_λ and I_0!***

Give a simple explanation for the behavior at low and high optical depths.

G1. There is no official grad student problem this week, but grad students should start downloading and setting up MESA and the MESA SDK on their computers by following the instructions from the website: `http://mesa.sourceforge.net/`

9.27 Pressure broadening (due to the presence of the electric fields of nearby ions) is unusually effective for the spectral lines of hydrogen. Using the general curve of growth for the Sun with these broad hydrogen absorption lines will result in an overestimate of the amount of hydrogen present. The following calculation nevertheless demonstrates just how abundant hydrogen is in the Sun.

The two solar absorption lines given in Table 9.4 belong to the Paschen series, produced when an electron makes an upward transition from the $n = 3$ orbital of the hydrogen atom.

(a) Using the general curve of growth for the Sun, Fig. 9.22, repeat the procedure of Example 9.5.5 to find N_a, the number of absorbing hydrogen atoms per unit area of the photosphere (those with electrons initially in the $n = 3$ orbital).

(b) Use the Boltzmann and Saha equations to calculate the total number of hydrogen atoms above each square meter of the Sun's photosphere.

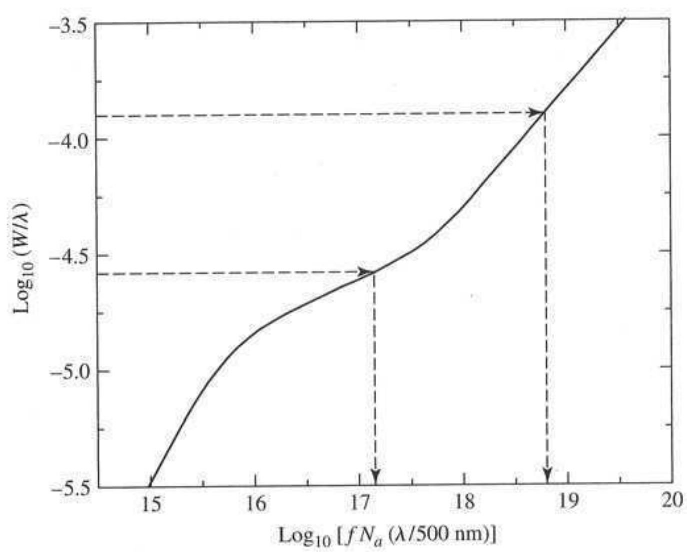


FIGURE 9.22 A general curve of growth for the Sun. The arrows refer to the data used in Example 9.5.5. (Figure adapted from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

TABLE 9.4 Data for Solar Hydrogen Lines for Problem 9.27. (Data from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

λ (nm)	W (nm)	f
1093.8 (Paγ)	0.22	0.0554
1004.9 (Paδ)	0.16	0.0269

ASTR 4201/5201 (Fall 2016)
Homework #4
Due: Friday, September 23 in class

Please remember to show your work.

1. In this problem, we will be making simple *estimates* of radiative diffusion in the Sun. You may approximate the Sun as a sphere of constant density.

   (a) What is the mean density of the Sun in kg m^{-3}? (As I've noted this in class, your value should be close to a very familiar terrestrial value— perhaps easier to see if expressed in g cm^{-3}).

   (b) Assuming a standard (homogeneous) solar composition (X=0.7, Y=0.28, Z=0.02), the average density from above, and that all material is completely ionized (so that Thomson scattering dominates the opacity), what is the mean free path of a photon in the interior of the Sun? Note how small this is relative to the radius of the Sun!

   (c) We will now make another simple estimate. Take the central temperature of the Sun (T_c=1.57×10^7 K) and make an assumption that the temperature declines linearly to its value at the surface (obviously wrong, but bear with me). Compute this average, linear value of $\Delta T/\Delta R$. Compare this to the answer of the previous question and explain how this relates to our assumptions about local thermal equilibrium in stellar interiors.

   (d) Assuming that a photon is undergoing a random walk with only the electron scattering opacity from above, how many steps does it take for a photon generated in the center of the Sun to escape?

   (e) Given the answer from the previous section, how much time does it take a photon generated in the center of the Sun to reach the surface?

   (f) We used only Thomson scattering opacity for this estimate. How would a more realistic treatment of the opacity affect the answers to the two previous questions?

2. We are going to consider free-free emission observed in the radio for a cloud of warm gas in an H II region.

   (a) Radio astronomers like to use a unit of specific intensity called the "brightness temperature". This is the temperature of a blackbody that would produce the same specific intensity (I_ν) that the telescope measured in an observation. Its utility rests on two facts: the specific intensity is constant along a ray in empty space, and that radio observations are always on the Rayleigh-Jeans tail of blackbody emission (as we saw on HW #1). Set I_ν equal to an expression for the Rayleigh-Jeans tail of a blackbody of temperature T_b. Invert this to get the brightness temperature in terms of I_ν. Note that T_b is always well defined, even if our source is not actually emitting blackbody radiation.

(b) Start with the solution for the radiative transfer equation in the form:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

Think of this in terms of the opposite sign convention from Caroll & Ostlie: an initial ray with intensity $I_\nu(0)$ is entering a cloud and exiting with $I_\nu(\tau_\nu)$ out the other side. You may assume that the cloud is radiating thermally under LTE at a constant temperature T, so that the source function is a blackbody. Also, we are concerned with observations at radio wavelengths, where the Rayleigh-Jeans limit is obtained. Rewrite the solution for the radiative transfer equation in terms of T_b and T in place of I_ν and S_ν, making sure to take the Rayleigh-Jeans limit.

(c) In class, I showed that the emission coefficient (defined by $dI_\nu = j_\nu \rho ds$) for free-free emission could be found from

$$4\pi\rho j_\nu = \frac{32\pi e^6}{3(4\pi\epsilon_0)^3 m_e c^3}\left(\frac{2\pi}{3km_e}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT}\overline{g_{\nu,ff}}$$

Now assume a pure ionized hydrogen composition. Using Kirchhoff's Law, find an expression for the absorption coefficient α_ν (defined by $dI_\nu/ds = -\alpha_\nu I_\nu$) and take the low-frequency limit of the exponential term. You should recover the power-law dependencies that I wrote down on the board in class, along with a constant coefficient.

(d) I stated in class that the Gaunt factor for free-free radiation ($\overline{g_{\nu,ff}}$) was of order unity in the optical. A more exact expression applicable in the radio is

$$\overline{g_{\nu,ff}} = \frac{\sqrt{3}}{2\pi}\left[\ln\left(\frac{8k^3T^3(4\pi\epsilon_0)^2}{\pi^2 Z^2 e^4 m_e \nu^2}\right) - 2.88608\right]$$

Evaluate this for a frequency of 1 GHz and $T = 10^4$ K (it should be >1.0).

(e) Ignoring other forms of opacity, the optical depth has the form $\tau_\nu = \int \alpha_\nu ds$. Also note that in class I defined the emission measure by $EM = \int n_e^2 ds$. Show that the optical depth can be written

$$\tau_\nu = C(EM)T^{-3/2}\nu^{-2}\overline{g_{\nu,ff}},$$

where C is the same constant coefficient you found above for α_ν.

(f) If $\tau_\nu \gg 1$ at low frequencies, show that $T_b \approx T$, assuming no background source and the results from part 2b. Conversely, show that $T_b \approx T\tau_\nu$ at optically thin high frequencies.

(g) Show that a spherical uniform gas cloud of radius R_c at a distance d has an observed flux of

$$F_\nu = \pi I_\nu (R_c/d)^2$$

where I_ν is the specific intensity emerging along a ray passing through the center of the cloud and to the observer, and thus that

$$F_\nu = \frac{2\pi k}{c^2}\left(\frac{R_c}{d}\right)^2 \nu^2 T_b.$$

(h) Using the results from questions 2e and 2f, take the low and high frequency limits of the second expression in 2g.

(i) Assume that the Orion nebula has $R_c = 0.6$ pc and is at a distance of d=500 pc. The low-frequency spectrum (F_ν) is proportional to ν^2, but above a frequency of $\sim$1 GHz, it flattens out to a roughly constant value of 4×10^{-24} W m^{-2} Hz^{-1}. Estimate the density and temperature of the Orion nebula using this information and the limits from the previous question.

3. Carroll & Ostlie problem 9.22, which I reproduce below:

9.22 Consider a horizontal plane-parallel slab of gas of thickness L that is maintained at a constant temperature T. Assume that the gas has optical depth $\tau_{\lambda,0}$, with $\tau_\lambda = 0$ at the top surface of the slab. Assume further that incident radiation of intensity $I_{\lambda,0}$ enters the bottom of the slab from outside. Use the general solution of the transfer equation (9.54) to show that when looking at the slab from above, you see blackbody radiation if $\tau_{\lambda,0} \gg 1$. If $\tau_{\lambda,0} \ll 1$, show that you see absorption lines superimposed on the spectrum of the incident radiation if $I_{\lambda,0} > S_\lambda$ and emission lines superimposed on the spectrum of the incident radiation if $I_{\lambda,0} < S_\lambda$. (These latter two cases correspond to the spectral lines formed in the Sun's photosphere and chromosphere, respectively; see Section 11.2.) You may assume that the source function, S_λ, does not vary with position inside the gas. You may also assume thermodynamic equilibrium when $\tau_{\lambda,0} \gg 1$.

Eqn. 9.54 is:

$$I_\lambda(0) = I_{\lambda,0}e^{-\tau_{\lambda,0}} - \int_{\tau_{\lambda,0}}^{0} S_\lambda e^{-\tau_\lambda} d\tau_\lambda$$

G1. This week, we are just going to make sure that you can run MESA. You should have downloaded and compiled it last week, so follow the instructions at

`http://mesa.sourceforge.net/starting.html`

to create the tutorial working directory, i.e.,

`cp -r $MESA_DIR/star/work tutorial`

and run the default model (`./rn`). If you are successful, two windows should show up on your screen, one H-R diagram (L vs. T_{eff}) and one central T-ρ diagram while the model runs.

Then follow the instructions at

`http://mesa.sourceforge.net/pgstar.html`

under the section "Output to files" to generate hardcopy plots. Simply create a subdirectory in your working directory with a name like `'png'` to output your files. Then add a few lines to your `inlist_pgstar` file to make some plots. Here is a minimal set of output options:

```
file_device = 'png'

file_extension = 'png'

file_digits = 5

file_white_on_black_flag = .false.  !  better for printing

TRho_Profile_file_dir = 'png'

TRho_Profile_file_flag = .true.

TRho_Profile_file_prefix = 'trho_profile_'
```

There are also postscript output options. For credit on this problem, please print out the final T-ρ profile to show that you have been able to make the tutorial model run and generate output files.

ASTR 4201/5201 (Fall 2016)
Homework #5
Due: Friday, September 30 in class

Please remember to show your work.

1. Convert the Maxwell-Boltzmann velocity distribution for the number density of particles with velocities between v and v+dv (n_v dv) into the similar expression for the kinetic energy distribution (n_E dE). We used this result to derive the rate equation for nuclear reactions.

2. Now we will consider a few aspects of nuclear fusion reactions.

   (a) Consider the reaction ${}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{15}_{8}\mathrm{O} + \gamma$ (from the CNO cycle). What is the Coulomb energy between the nitrogen nucleus and hydrogen nucleus (proton) when they are separated by 1 fm?

   (b) What temperature would the protons have to have in order for the classical thermodynamic energy to exceed this barrier?

   (c) In class we found that the nuclear reaction rate for A+B was equal to

   $$r_{AB} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_A n_b}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/kT} dE$$

   where μ_m is the reduced mass ($m_A m_B/(m_A+m_B)$) and the product of exponentials in the integrand produces what is known as the Gamow peak. The function $S(E)$ is taken to be a function of energy that incorporates all of the nuclear physics involved in the reaction and that is slowly varying in the neighborhood of the Gamow peak. Differentiate the integrand to find the expression for the energy of the Gamow peak that I gave in class.

   (d) Using the expression for b given in class or your book, determine the energy of the Gamow peak for this nitrogen plus hydrogen reaction in the Sun ($T_c = 1.57 \times 10^7$ K). Note that the energy at which this function peaks is entirely due to numbers that depend on the charges, masses, and temperature of the particles involved and not due to any parameters related to internal nuclear structure (although the amplitude, and hence the overall rate, certainly has such a dependence through $S(E)$!).

3. I have listed a few nuclear reactions below. In each case, there is a mysterious particle X or missing numbers y. Use the conservation laws for nuclear reactions to determine the identity of any particle X or number y. (show your work, don't just write down the answer...)

   (a) ${}^{8}_{5}\mathrm{B} \rightarrow {}^{y}_{4}\mathrm{Be} + \nu_e + X$

   (b) $\gamma + {}^{28}_{14}\mathrm{Si} \rightarrow {}^{24}_{y}\mathrm{Mg} + X$

   (c) ${}^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \rightarrow {}^{y}_{16}\mathrm{S} + n$

   (d) ${}^{21}_{10}\mathrm{Ne} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{24}_{12}\mathrm{Mg} + X$

   (e) ${}^{27}_{14}\mathrm{Si} \rightarrow {}^{y}_{13}\mathrm{Al} + e^{+} + X$

4. In this problem, we consider the energy generation rates for the *pp* chain and the CNO cycle. Use the power-law relationships for the energy generation rates expanded about $T = 1.5 \times 10^7$ K to find the ratio of energy generation in the center of the Sun by the two processes. You may assume that the order unity correction factors are in fact equal to 1.0 and may use the values appropriate for the standard solar model ($T_{c,\odot} = 1.5696 \times 10^7$ K, $\rho_{c,\odot} = 1.527 \times 10^5$ kg m^{-3}) as well as the derived central composition of the Sun (X=0.3397 and $X_{CNO} = 0.0141$). Note that the central composition of the Sun deviates from its initial value because it has been fusing hydrogen into helium for 4.6 billion years already. Does your result justify the statement that the *pp* chain is the dominant energy generation mechanism in the Sun?

5. I wrote down the power-law approximations for various burning stages in class. These were in the form: $\epsilon \propto \rho^{\lambda} T^{\nu}$. However, the burning rates are actually proportional to a product of power law and exponential terms. For example, Hansen, Kawaler, & Trimble give an expression for the energy generation rate of the CNO cycle as

$$\epsilon_{CNO} \approx \frac{4.4 \times 10^{21} \rho X Z}{T_9^{2/3}} e^{-15.228/T_9^{1/3}} \text{ J kg}^{-1} \text{ s}^{-1}$$

   where T_9 is the temperature divided by 10^9 K. Use this to find an expression for the power-law index of the temperature dependence, ν. You can do this by evaluating $\left|\frac{\mathrm{d}\ln\epsilon}{\mathrm{d}\ln T}\right|$ for both expressions and solve for ν. Evaluate your expression at $T = 1.5 \times 10^7$ K and you should recover the exponent in the expression I gave in class.

G1. This would be a good time to start familiarizing yourself with MESA. MESA has been described in the literature in three journal articles. Start reading the first one:

`http://adsabs.harvard.edu/abs/2011ApJS..192....3P`

In case you are interested, the other two described subsequent improvements to the code to handle more complex situations:

`http://adsabs.harvard.edu/abs/2013ApJS..208....4P`

`http://adsabs.harvard.edu/abs/2015ApJS..220...15P`

ASTR 4201/5201 (Fall 2016)
Homework #6
Due: Friday, October 7 in class

The questions labeled G1 is required for graduate students. Please remember to show your work. **Note:** You will need the handout from Wednesday's (Sep. 28) lecture (also available on blackboard).

1. The data in the table below come from two locations in the interior of a single main-sequence stellar model:

r	$M(r)$	$L(r)$	$T(r)$	$\rho(r)$	$\overline{\kappa}$
$0.242R_\odot$	$0.199M_\odot$	$340L_\odot$	2.52×10^7 K	1.88×10^4 kg m^{-3}	0.044 m^2 kg^{-1}
$0.670R_\odot$	$2.487M_\odot$	$528L_\odot$	1.47×10^7 K	6.91×10^3 kg m^{-3}	0.059 m^2 kg^{-1}

   (a) Is energy transport at these two locations in the star convective or radiative? You may assume that radiation pressure is negligible so that the gas behaves like a monatomic ideal gas with a mean molecular weight ($\overline{\mu}_m$) of 0.7 amu.

   (b) Explain your results in terms of the four situations where convection occurs that I listed in class and our discussion about the internal structures of main-sequence stars.

2. In class I used Sirius (the brightest star in the night sky) as my example of an astrometric binary. This was first noticed in 1844 by Bessel, the astronomer who also made the first parallax measurement of a star. He had trouble understanding his observations of Sirius because it had too much of a "wobble" for a simple parallax. Eventually it became clear that the system was a binary with a period of 50 yr and a total semimajor axis (of the reduced mass) of $7.61''$. You may assume that this is in the plane of the sky (so ignore inclination) and that Sirius has a parallax measured by *Hipparcos* of $0.379''$. We can now resolve the system into two components, Sirius A and the much fainter Sirius B, making this a visual binary. Assume that the ratio of the distances from the center of mass is $a_A/a_B = 0.466$.

   (a) Find the masses of both members of the system.

   (b) Sirius A has an absolute bolometric magnitude of 1.36 and that of Sirius B is 8.79. Determine their luminosities in terms of the Solar luminosity (and note the huge ratio between the two!).

   (c) Astronomers were shocked to learn that Sirius B is very blue, with $T_e \approx 25{,}200$ K. Estimate its radius and compare to those of the Sun and Earth.

   (d) Use this to compute the average density of Sirius B and compare to that of the Earth.

3. Solve the Lane-Emden equation for the case of n=0 by applying the appropriate boundary conditions to find $D_0(\xi)$. Our use of the dimensionless variable ξ and function D_0 obscures the physical significance of this solution. Find an expression for the actual density ρ in terms of the physical radius r. Does this seem like it describes a real star?

4. On Friday, we talked about using homology relationships to scale stellar structure from a known model of a star. Section 5 of the handout shows how to use the four equations of stellar structure combined with an assumption that the four variables (r, ρ, L, T) have power-law dependencies on the stellar mass M to arrive at a set of four coupled linear equations for the four exponents (e.g., $T \propto M^{\alpha_T}$). This is a linear algebra exercise that can either be solved by repeatedly eliminating a variable, or as an exercise in matrix multiplication (eqn. 32).

   (a) The handout leaves the energy generation equation in terms of two power-law dependencies, α and β. Insert the correct values for the CNO cycle (using the power-law approximation in the neighborhood of T=1.5×10^7 K) and solve the system of equations to get the four power-law exponents for (r, ρ, L, T) as functions of mass.

   (b) The trick of using linear algebra to solve the system of equations could make the coefficients resulting from this analysis appear to be almost random numbers. However, they do contain all of the input physics relating how various parameters scale with each other. To demonstrate this, use the virial theorem to find a scaling relationship to relate the average temperature T to the mass and radius. (Find the proportionality, don't worry about numerical coefficients because they don't matter for this problem.) Now write this relationship as an expression for α_T in terms of any other α parameters. Do your derived coefficients for the CNO cycle analysis obey this relationship?

   (c) As the handout states, the temperature in this approximation is some average internal T. If you want to compare to observations of stars, we only see the surface temperature, $T_{\text{eff}} \propto (L/R^2)^{1/4}$. Using the power-law dependencies from the previous section, find an expression for the dependence of luminosity on T_{eff}. Hint: using the expressions from 4a, eliminate the mass dependence to find a power-law relationship for the relationship between radius and luminosity. Insert this into the T_{eff} relationship to find T_{eff} as a function of L and convert that to $L \propto T_{\text{eff}}^{\alpha_{L,T}}$ and find $\alpha_{L,T}$.

   (d) Use Figure 10.13 from Carroll & Ostlie (repeated below) and the fact that the slope of a line on a log-log plot tells you the power-law dependence between the two quantities to estimate the power-law relationship between L and T_{eff} in the plotted stellar models (over the range of stellar masses where CNO burning is appropriate) and compare to your previous result.

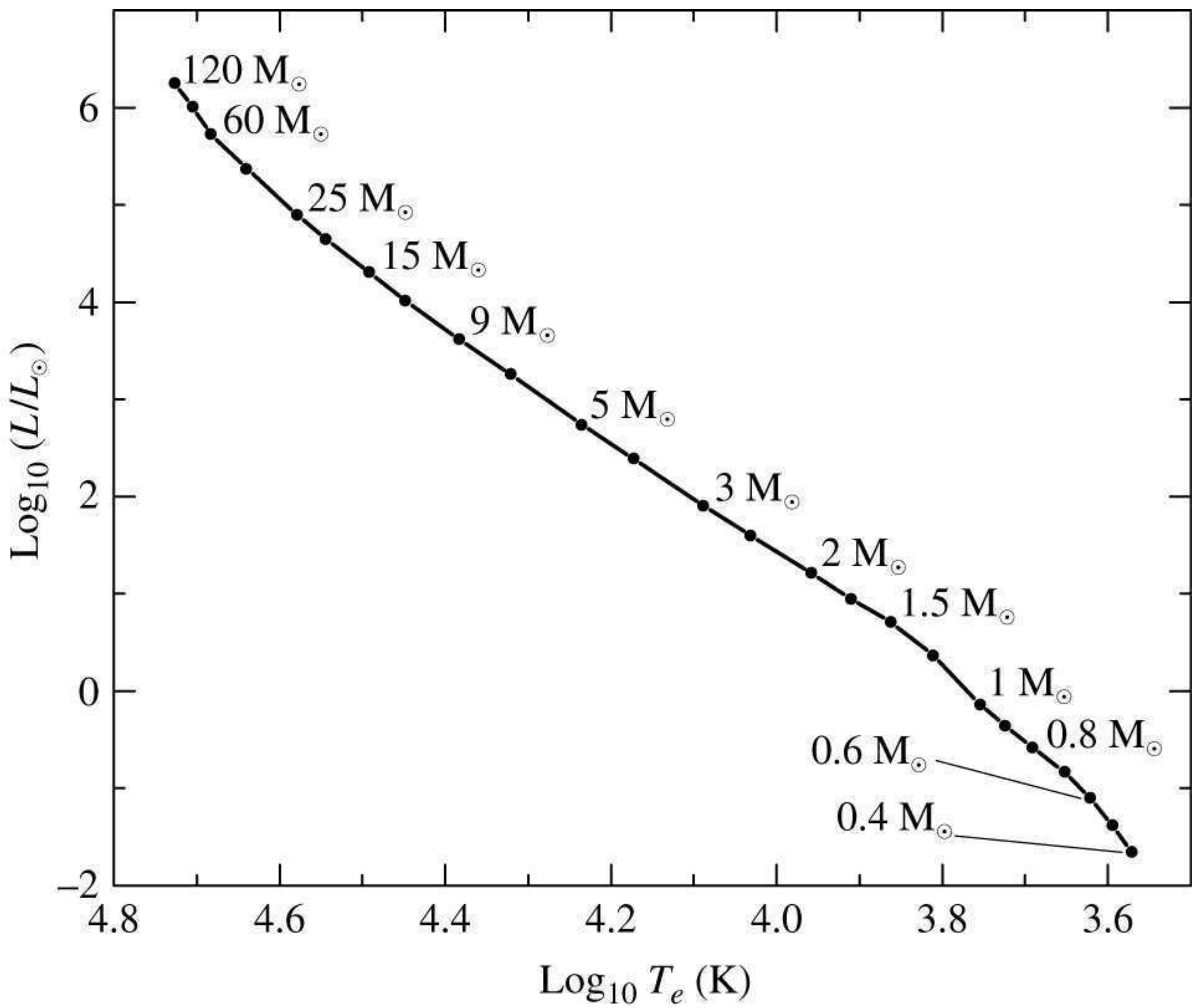


Figure 1: Figure 10.13 from Carroll & Ostlie. A set of models for the Zero Age Main Sequence (ZAMS).

G1. We will now combine the results from the previous problem and revisit problem 4 from Homework 5. In problem 4a, we derived some power-law scaling relationships for the average properties of stars relative to a fiducial model, assuming that the CNO cycle is dominant. You may assume some representative values for the conditions at the center of the present-day Sun: $T_{c,\odot} = 1.5696 \times 10^7$ K, $\rho_{c,\odot} = 1.527 \times 10^5$ kg m^{-3}, X=0.3397, and $X_{CNO} = 0.0141$.

(a) Repeat the analysis of problem 4a and solve for the power-law dependencies, but this time assume that the *pp* chain dominates the energy generation, as is appropriate for the Sun.

(b) Now assume that the relevant central properties (temperatures, densities, etc.) scale in the same way with mass as the average properties. Substitute the scaling relationships from problem G1(a) into the energy generation rates to get their scalings with stellar mass relative to the Sun. Above what stellar mass will the temperature be sufficiently high that the energy generation of the CNO process will dominate over the *pp* chain? The value you get will be slightly lower than the actual value I quoted in class due to the crude nature of the approximation. Also note that the steep temperature dependencies mean that you will need to retain at least three of the significant figures quoted above for $T_{c,\odot}$. This isn't totally fair, but how does the answer change if you subsititute in the initial composition of the Sun (i.e., X=0.7)?

ASTR 4201/5201 (Fall 2016)
Homework #7
Due: Friday, October 21 in class

Please remember to show your work.

1. I have compiled some properties of solar neutrinos in the table below (sources: Prialnik 2010; Haxton et al. 2013). The fluxes at Earth are those implied after correction for neutrino oscillations (i.e., what we would receive in the absence of oscillations) and therefore are a mix of measurement and theory.

Source	Flux ($\mathrm{m^{-2}\ s^{-1}}$)	Energy Range (MeV)	Average Energy (MeV)
$p + p \rightarrow\ {}^2_1\mathrm{D} + e^+ + \nu$	6.05×10^{14}	≤ 0.42	0.263
${}^7_1\mathrm{Be} + \mathrm{e}^- \rightarrow\ {}^7_3\mathrm{Li} + \nu$	4.82×10^{13}	0.86 (90%), 0.38 (10%)	0.80
${}^8_5\mathrm{B} \rightarrow\ {}^8_4\mathrm{Be} + e^+ + \nu$	5.0×10^{10}	≤ 15	7.2

   (a) What is the total neutrino luminosity of the Sun as a fraction of the (electromagnetic) solar luminosity?

   (b) Use this information along with the location of each of the neutrino emitting reactions in relationship to the *pp* chain to estimate the branching ratios. That is, what fraction of the ^{4}He made in the Sun is produced by each of the three branches of the *pp* chain? Note: be careful about the number and type of neutrinos produced for each ${}^4_2$He nucleus by each branch. The numbers you get should be fairly close to those shown in Figure 1 of Handout 4 (the "News & Views" by Haxton), and somewhat different than those implied by Fig. 10.8 of Carroll & Ostlie.

   (c) The branching ratios of the *pp* chain are a sensitive test of the central temperature of the Sun. I have reproduced below a plot from a paper by Peter Parker et al. (no not that one) that shows the fraction of ^{4}He produced by each of the three *pp* branches. Use the numbers you calculated in the previous step to estimate the relevant temperature in the solar core. The number you get should be somewhat smaller than the usual number we quote for the center of the Sun. This is because the energy generation region of the Sun extends over some range in radii and actually peaks in a region of lower temperature.

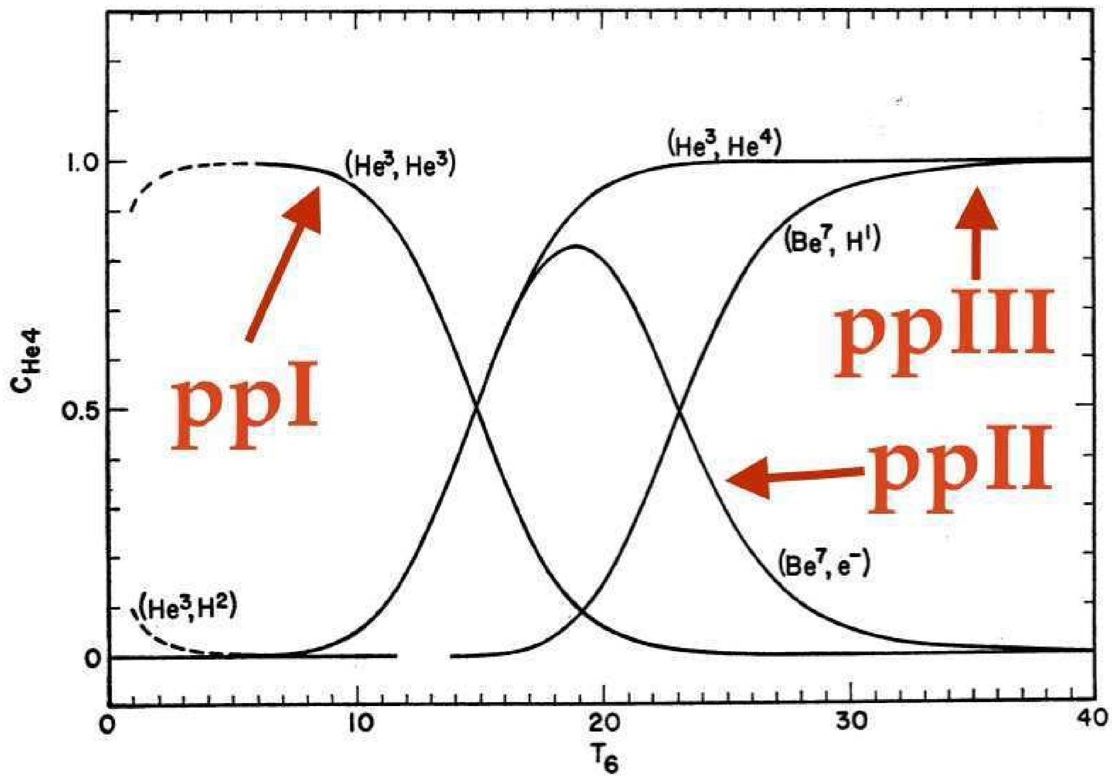


Figure 1: Fraction of ^{4}He produced by the different *pp* chains as a function of temperature, for an assumed density and composition. The curves are labeled by the unique reactions. Figure modified from Figure 8 of Parker, Bahcall, & Fowler 1964.

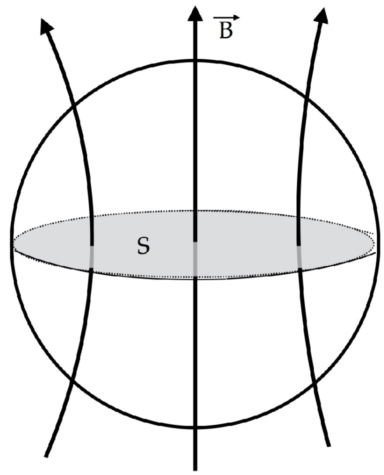
2. I stated in class that the frequency of CO J=1→0 emission was 115 GHz. A more precise value is 115.271 GHz.

   (a) Use this value to estimate the mean equilibrium separation (r_e) of the two atomic nuclei.

   (b) Now estimate the frequency of the fundamental rotational transition (J=1→0) for the isotopologue $C^{18}O$.

   (c) If you require a minimum thermal energy of collisions in a dense cloud to populate the upper state of the rotational energy levels of a molecule, then for the CO J=1 level, the necessary $T_{\rm min}$ is only ∼6 K. If the temperature in a molecular cloud is 20 K (on the warm side), which rotational levels of CO would be likely to have significant populations?

3. Now we will consider two of the barriers to collapse for a protostar that I listed in class. The first consideration is the magnetic field. Imagine a spherical cloud of radius r threaded by a large-scale magnetic field B (ignoring the turbulent fluctuations). The key parameter is the magnetic flux, $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$, where $\mathbf{S}$ is a surface going across the equator, so $B \approx \Phi/(\pi r^2)$. See the figure below:



   (a) A typical value for the magnetic field in a clump of a molecular cloud with density n=10^{10} m^{-3} is $B \approx$50 μG. What is the magnetic flux per unit mass (Φ/M) in a clump with a total mass of 1 $M_\odot$ (assume a constant density and a composition of only H_2)?

   (b) Young T Tauri stars have typical surface magnetic fields of ~2 kG. What is their magnetic flux per unit mass? Assume that the T Tauri star has $M \approx 1M_\odot$ and $r \approx 3R_\odot$. How does this number compare to the previous one?

4. Now let's consider angular momentum. Start with a molecular cloud core with typical parameters: $r \approx 0.1$ pc, $M = 1M_\odot$, and angular velocity $\Omega = 10^{-14}$ s^{-1}.

   (a) Compute the gravitational binding energy (ignore factors of order unity due to assumptions about the density distribution) as well as the rotational energy. Which one is bigger? You may assume that the moment of inertia is $I = 0.4Mr^2$.

   (b) Compute the angular momentum of the core.

   (c) In the T Tauri stage, $r \approx 3R_\odot$ and the typical rotation period is about a week. Let $I = 0.2Mr^2$ and estimate the rotational angular momentum.

   (d) The Sun today has a mean internal rotational period of about ~27 days. Let $I = 0.1Mr^2$ and estimate the rotational angular momentum.

   (e) Now compute the orbital angular momentum of Jupiter. The comparison of these last four numbers shows the magnitude of the required angular momentum transport.

ASTR 4201/5201 (Fall 2016)
Homework #8
Due: Friday, October 28 in class

The question labeled G1 is required for graduate students. Undergrads can get up to 15 percentage points extra credit for solving it. Please remember to show your work.

1. In this problem, we will explore the characteristic scales involved in the detection of exoplanets using the radial velocity technique.

   (a) Imagine that you are a distant observer looking at the Sun and measuring its radial velocity over time. Using Kepler's third law, estimate the amplitude of the radial velocity signal induced by Jupiter.

   (b) Now estimate the amplitude of this signal induced by the Earth.

   (c) There are two common ways of expressing the spectral resolution of a spectrograph. One is to quote $\Delta\lambda$, which is the full-width at half-maximum (in Å) of an unresolved absorption or emission line, but this is quite frequently a function of wavelength. The second is to quote the dimensionless resolution $R = \lambda/\Delta\lambda$. A typical high-resolution spectrograph used for radial velocity searches for planets has a resolution of $R \approx 50,000$. At $\lambda = 5000$ Å, what is the resolution ($\Delta\lambda$) of the spectrograph (in Å)? What is the velocity (Doppler) shift corresponding to a wavelength shift by an amount equal to this resolution?

   (d) The spectrum is dispersed on your CCD detector such that different CCD columns correspond to different wavelengths of light. It is typical to have $\sim$3 CCD pixels per resolution element $\Delta\lambda$. How many CCD pixels do the velocity shifts calculated in 1a) and 1b) correspond to? It should be a very small fraction of a pixel—which is why this is hard. Planets like Jupiter are feasible to detect with current technology, but planets like Earth are not quite (yet).

2. In class I stated several times that the luminosity on the main sequence scales with stellar mass as $L \propto M^{3.5}$. Given the age of the universe, most recently estimated as 13.8 Gyr by the *Planck* collaboration, and making an assumption that all stars fuse the same fraction of their mass into helium on the main sequence (approximately true for stars that are not fully convective!), what is the lowest possible mass for a star to have evolved off of the main sequence already? (We are also assuming no stars are affected by binary evolution...)

3. This problem considers several of the changes that are going to happen to the Sun after it evolves off the main sequence moves onto the red giant branch (RGB). When the Sun is on the RGB, it will have a dense, degenerate helium core surrounded by a low density envelope.

   (a) The gravitational potential energy of a sphere with constant density is given by $U_{\rm gr} = -\frac{3}{5}GM^2/R$. Calculate this value for the Sun right now, assuming a constant density. Also calculate it for when the Sun expands as a red giant to $R \approx 100R_\odot$, again assuming constant density (and ignore mass loss for now). What explains this difference in gravitational potential energies? Did the system's energy change or is one of our assumptions fatally flawed in this comparison? Explain.

   (b) The radius of a fully degenerate star is given by the expression

   $$R = 2.6 \times 10^7 \mu_e^{-5/3} \left(M/M_\odot\right)^{-1/3} \text{ m}$$

   where μ_e is the dimensionless mean molecular weight per electron (e.g., μ_e=1 for pure hydrogen). Assume that the evolved Sun now has a pure helium core with a mass of 0.25 $M_\odot$. What is the radius of the core? What is its density? In Homework 4, you calculated the average density of the Sun to be 1.41×10^3 kg m^{-3}, while the central density in the present-day Sun is 1.54×10^5 kg m^{-3}. Compare the density to these two numbers.

   (c) When the Sun is in the same evolutionary state as for the previous two questions, what is the average density in the envelope? Compare to the average density of air at sea level on Earth ($\sim$1.2 kg m^{-3}).

4. Low-mass stars like the Sun obey the core mass-luminosity relationship:

   $$L = 2.3 \times 10^5 L_\odot \left(\frac{M_c}{M_\odot}\right)^6$$

   as they burn H in a shell and climb the RGB.

   (a) What is the energy released per unit mass when fusing hydrogen into helium?

   (b) The "ash" from the hydrogen fusion is newly-formed helium, which is added to the core. Using the previous answer, derive a relationship between the luminosity L and the rate by which the core mass grows (dM_c/dt).

   (c) Use the core mass-luminosity relationship to solve this equation and get an expression for the core mass as a function of time $M_c(t)$, starting from an initial core mass of $M_{c,0}$.

   (d) Assume that a star reaches the base of the RGB with a core equal to 15% of its total mass and that at the peak of the RGB the core mass equals 0.45 $M_\odot$ when the helium core flash occurs. How long does this ascent take for stars of initial masses 1 $M_\odot$ and 2 $M_\odot$?

   (e) Compare your results in the previous question with the main sequence lifetimes of these two stars. The MS lifetimes can be calculated from the elapsed times between points 1 and 3 in Table 13.1 in Carroll & Ostlie (reproduced below).

G1. I said in class that small dust particles are removed from the Solar System by something known as the Poynting-Robertson effect, which is a consequence of special relativity. The essence of the Poynting-Robertson effect is that a grain will emit thermal radiation isotropically in its own rest frame, but that the emission is boosted in the direction of motion due to relativity (alternatively, relativistic aberration means that the radiation from the central star that an orbiting dust grain sees is not purely radial). This preferential emission direction causes it to feel a net force backwards which causes it to fall into an orbit with a smaller radius as it loses angular momentum.

(a) Start by assuming that a spherical dust grain of mass m and radius a is orbiting a star in a circular Keplerian orbit at radius r. If this grain absorbs a fraction $\langle Q \rangle$ of the stellar radiation energy that falls on it and then immediately re-radiates this energy via thermal emission, what is the luminosity of the grain?

(b) The radiative flux from the grain can then be approximated as having a spherical component (that causes no loss of angular momentum) and a boosted component that is emitted in the direction of motion. This boosted component represents a fraction of $\beta = v/c$ of the total emitted luminosity, where v is the orbital velocity. Derive the following expression for the net change in orbital angular momentum J of the dust grain produced by this emission (note that the linear momentum carried by photons is $p = E/c$).

$$\left(\frac{dJ}{dt}\right)_{\rm PR} = -\beta \frac{L_\star \langle Q \rangle}{4rc} a^2$$

(you can assume that r is approximately constant for the absorption/reemission process).

(c) Now find an expression for the timescale for Poynting-Robertson drag to significantly change the orbital angular momentum ($\tau_{\rm PR} = J/(-dJ/dt)_{\rm PR}$).

(d) If the grain is significantly larger than the typical wavelength of light emitted by the central star, then $\langle Q \rangle \approx 1$. You can also assume that the mean density of a dust grain is $\sim 3\times10^3$ kg m^{-3} (the typical density of silicates). What are the timescales for inspiral for particles of sizes 10 μm and 1 cm from an orbit of radius 1 AU around a star with luminosity equal to $L_\odot$? (Although keep in mind that the actual luminosity of the Sun when it was forming was higher than the current $L_\odot$.)

TABLE 13.1 The elapsed times since reaching the zero-age main sequence to the indicated points in Fig. 13.1, measured in millions of years (Myr). (Data from Schaller et al., *Astron. Astrophys. Suppl.*, *96*, 269, 1992.)

Initial Mass	1	2	3	4	5
($M_\odot$)	6	7	8	9	10
25	0	6.33044	6.40774	6.41337	6.43767
	6.51783	7.04971	7.0591		
15	0	11.4099	11.5842	11.5986	11.6118
	11.6135	11.6991	12.7554		
12	0	15.7149	16.0176	16.0337	16.0555
	16.1150	16.4230	16.7120	17.5847	17.6749
9	0	25.9376	26.3886	26.4198	26.4580
	26.5019	27.6446	28.1330	28.9618	29.2294
7	0	42.4607	43.1880	43.2291	43.3388
	43.4304	45.3175	46.1810	47.9727	48.3916
5	0	92.9357	94.4591	94.5735	94.9218
	95.2108	99.3835	100.888	107.208	108.454
4	0	162.043	164.734	164.916	165.701
	166.362	172.38	185.435	192.198	194.284
3	0	346.240	352.503	352.792	355.018
	357.310	366.880	420.502	440.536	
2.5	0	574.337	584.916	586.165	589.786
	595.476	607.356	710.235	757.056	
2	0	1094.08	1115.94	1117.74	1129.12
	1148.10	1160.96	1379.94	1411.25	
1.5	0	2632.52	2690.39	2699.52	2756.73
	2910.76				
1.25	0	4703.20	4910.11	4933.83	5114.83
	5588.92				
1	0	7048.40	9844.57	11386.0	11635.8
	12269.8				
0.8	0	18828.9	25027.9		

Figure 1: Table 13.1 from Carroll & Ostlie

ASTR 4201/5201 (Fall 2016)
Homework #9
Due: Friday, November 4 in class

The questions labeled G1 is required for graduate students. Undergrads can get up to 10 percentage points extra credit for solving it. Please remember to show your work.

1. We first consider η Carinae, the LBV that I discussed in Monday's class. For the purposes of this problem, assume that the distance is known to be 2.3 kpc, and that the interstellar extinction along the line of sight is $A_V = 1.7$ mag.

   (a) In class I quoted some numbers for the Eddington limiting luminosity for a fully ionized gas, but I assumed that the composition was pure hydrogen. Let's make that more realistic by using the expression for the electron-scattering opacity from earlier in the class of $\kappa_{\rm es} = 0.020(1 + X)$ m^2 kg^{-1}, where X is the hydrogen abundance, which we can assume to be $X \approx 0.7$. Now what is the Eddington luminosity in units of $L_\odot$ as a function of mass in $M_\odot$?

   (b) During the Great Eruption of the 19th century, η Car lost enough mass to form the Homunculus nebula. Dust then formed in the nebula, which now enshrouds the central stars and re-radiates most of their luminosity at infrared wavelengths. The total IR flux from the Homunculus currently has been measured to be about 2.6×10^{-8} J m^{-2} s^{-1}. Assuming that this represents 90% of the total bolometric flux from the object, what is the total bolometric luminosity of η Car in units of $L_\odot$?

   (c) Given the results from part 1a, what is the minimum central mass required for this luminosity to not be in conflict with the Eddington limit? The primary star is believed to be around 120 $M_\odot$. How does this compare?

   (d) The Great Eruption lasted for about 20 years, during which time η Car had a visual brightness around $m_V \approx 0$ mag. Assume that during the eruption the object was not yet enshrouded so that only the interstellar extinction is relevant. What was the bolometric luminosity during this outburst in units of $L_\odot$? You may assume no bolometric correction relative to the Sun.

   (e) What was the total amount of radiated energy during this long outburst? How does it compare to a typical supernova explosion ($\sim 10^{42} - 10^{43}$ J)?

2. The buoyancy (Brunt-Väisälä) frequency is defined by

$$N = \sqrt{-Ag} = \sqrt{\left(\frac{1}{\gamma P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right) g}$$

   In the case that $A < 0$, oscillations can occur. Conversely, show that the situation with $A > 0$ is equivalent to our condition for adiabatic convection. You may assume that the ideal gas law holds and that μ is constant.

3. Use the observed color-magnitude diagram (CMD) for M3 presented below along with the sequence of open cluster CMDs to estimate the age and distance to M3. Ignore reddening and other differences between M3 and the open clusters in that plot.

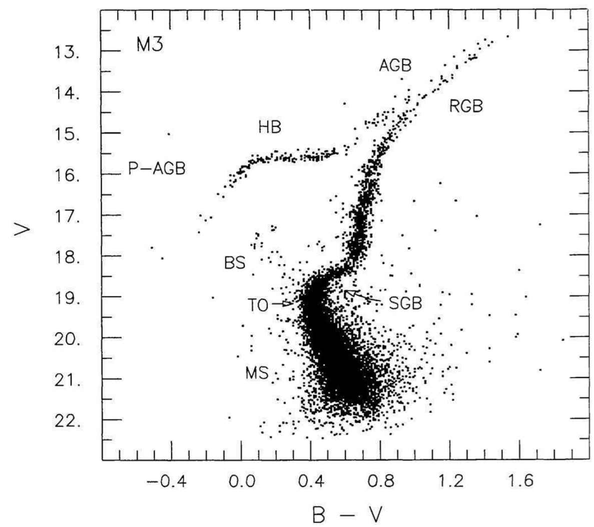


Figure 1: Color-magnitude diagram of M3 from Renzini & Fusi Pecci (1988).

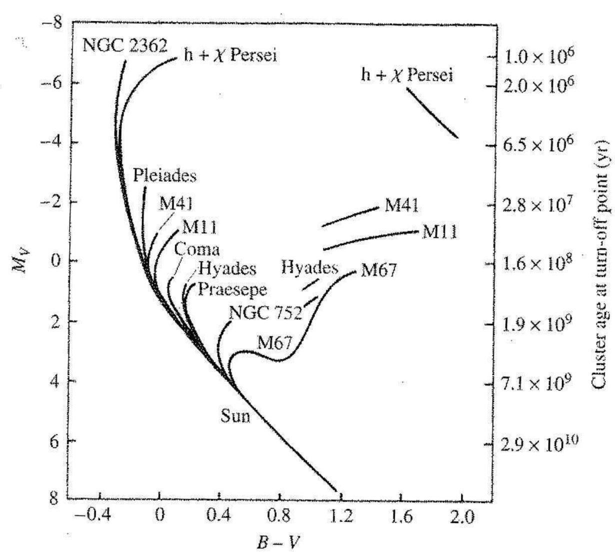


Figure 2: Compilation of open clusters from Sandage.

4. The most distant Cepheid variable stars are observed using the Hubble Space Telescope. Modern studies of distant Cepheids have also moved to the near-infrared both to avoid problems caused by dust and because the stars themselves exhibit less scatter. Figure 14.6(a) in Carroll & Ostlie shows the observed near-infrared (H band) period-luminosity relationship (in apparent magnitudes) for Cepheids in the Large Magellanic Cloud. Assume a distance of 50kpc for the LMC. If HST can measure Cepheids as faint as H$\approx$26.5 mag for a period of $\sim$15 d in a distant galaxy, what is the maximum distance out to which these Cepheids can be studied?

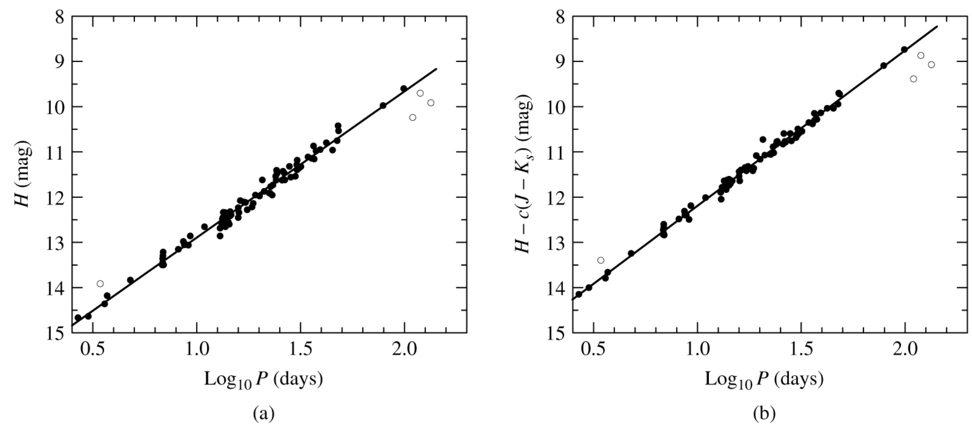


Figure 3: Figure 14.6 from Carroll & Ostlie. The left panel shows the observed apparent H-band magnitude versus period.

G1. For this problem, we will consider a single massive star cluster. Assume that the IMF in the cluster follows the Salpeter form, with a single power law such that $dN/dM = c(M/M_\odot)^{-2.35}$, where c is a normalization constant.

(a) Assume that the total mass of stars in the cluster is 10^4 $M_\odot$ and that stars form over a mass range of 0.1–100 $M_\odot$. What is the value of c?

(b) What is the total number of stars in this cluster?

(c) What are the mean and median masses of stars in this cluster?

(d) Assume that all of the stars in the cluster start on the ZAMS at the same time and that they obey the scaling relationship that $L \propto M^{3.5}$. What is the total luminosity of the cluster in $L_\odot$ at that time?

(e) What is the total number of stars with masses greater than $10M_\odot$? What fraction of the total luminosity comes from stars with initial masses above $10M_\odot$?

(f) Using arguments similar to those from Problem 2 of last week's homework (HW 8) and the same mass-luminosity relationship as in the previous question, what are the most massive stars left on the main sequence 1 Gyr after the formation of the cluster (in units of $M_\odot$)?

(g) What is the total luminosity of the stars remaining on the main sequence at this time? (i.e., ignore the luminosity from any stars that have evolved off the main sequence)

ASTR 4201/5201 (Fall 2016)
Homework #10
Due: **Monday**, November 14 in class

The question labeled G1 is required for graduate students. Undergrads can get up to 15 percentage points extra credit for solving it. Please remember to show your work.

1. In class, I showed how to derive the pressure-density relationship for degenerate electrons from the pressure integral and the Fermi momentum. Note that the Fermi momentum did not depend on the mass of the particle because it was derived from the uncertainty principle (it only depends on h, the number density, and the factor of 2 for spin degeneracy). Here, we consider the degenerate matter of a neutron star, where the composition and degeneracy pressure are both dominated by pure neutrons.

   (a) Derive an expression for the pressure as a function of the mass density (ρ) for pure neutrons in the non-relativistic limit.

   (b) Now consider the equation of hydrostatic equilibrium. Make the unrealistic assumption that the density is constant throughout the star and integrate it to obtain the following expression for the central pressure:

   $$P_c = \frac{2}{3}\pi G\rho^2 R^2$$

   You may assume "zero" boundary conditions so that the pressure at the surface of the star is $P(R) = 0$.

   (c) Combine the results from the previous two questions (again assuming a constant density sphere) to relate the mass and radius of a neutron star. What do you derive for the radius of a 1.4 $M_\odot$ neutron star? You will be off by a factor of $\sim$2.5, which is not terrible given that we assumed a constant density and ignored all nuclear physics.

2. The nearby pulsar known as Geminga has a period of $P = 0.237$ s with a derivative of $\dot{P} = 1.1 \times 10^{-14}$. Assuming that $\theta = 90°$, what is the strength of the magnetic field at the poles? You may assume that it has typical properties for a neutron star of $R = 10$ km and $M = 1.4M_\odot$, and that the moment of inertia equals $0.4MR^2$ (appropriate for a uniform density sphere).

3. We expect that the formation of a neutron star during core collapse will release 2.5×10^{46} J of gravitational binding energy that will mostly emerge as neutrinos.

   (a) If the average neutrino energy produced in the process is 4.2 MeV, what was the total neutrino fluence (number m^{-2}) expected to be received at Earth from SN 1987A? You may assume a distance of 50 kpc for the LMC.

   (b) If the fluence is split equally amongst 6 different types of neutrinos (3 flavors plus their antiparticles), how does the value from part (a) compare to the total electron antineutrino fluence inferred from Kamiokande and IMB of $F_{\bar{\nu}_e} = 1.3 \times 10^{14}\ \bar{\nu}_e\ \mathrm{m}^{-2}$ (with $\sim$50% uncertainty)?

   (c) The observed neutrino pulse from SN 1987A lasted $\sim$10 s. How does the average flux for all flavors of neutrinos (number m^{-2} s^{-1}) over this time interval compare to the solar neutrino flux (see HW #7)?

4. Supernova light curves are powered by the decay of radioactive elements. Near peak, this is usually the decay chain $^{56}\mathrm{Ni} \rightarrow {}^{56}\mathrm{Co} \rightarrow {}^{56}\mathrm{Fe}$. The radioactive decay constants (λ, defined by $N(t) = N_0 e^{-\lambda t}$, where N is the number of atoms) for the two decays are 1/(8.8 d) and 1/(111.3 d), respectively.

   (a) Show that the derivative of the logarithmic luminosity obeys the simple relationship $d \log_{10} L/dt = -0.434\lambda$.

   (b) Using the light curve of SN 1987A below (or the version shown as Figure 15.12 in Carroll & Ostlie), measure the slope of the observed light curve of SN 1987A between 300 and 700 days after explosion and compare it to your prediction from the previous question, assuming that the light curve is completely powered by the decay of ^{56}Co.

   (c) Each ^{56}Co decay releases a total of 3.73 MeV in energy in the form of γ-ray photons and positrons (plus some neutrinos that escape the ejecta). Assume that the ^{56}Ni decay timescale is sufficiently short that we can approximate the ^{56}Co production as all occurring at the time of the explosion. Use the luminosity of SN 1987A at $t = 500$ d to estimate the initial amount of ^{56}Ni produced in the explosion.

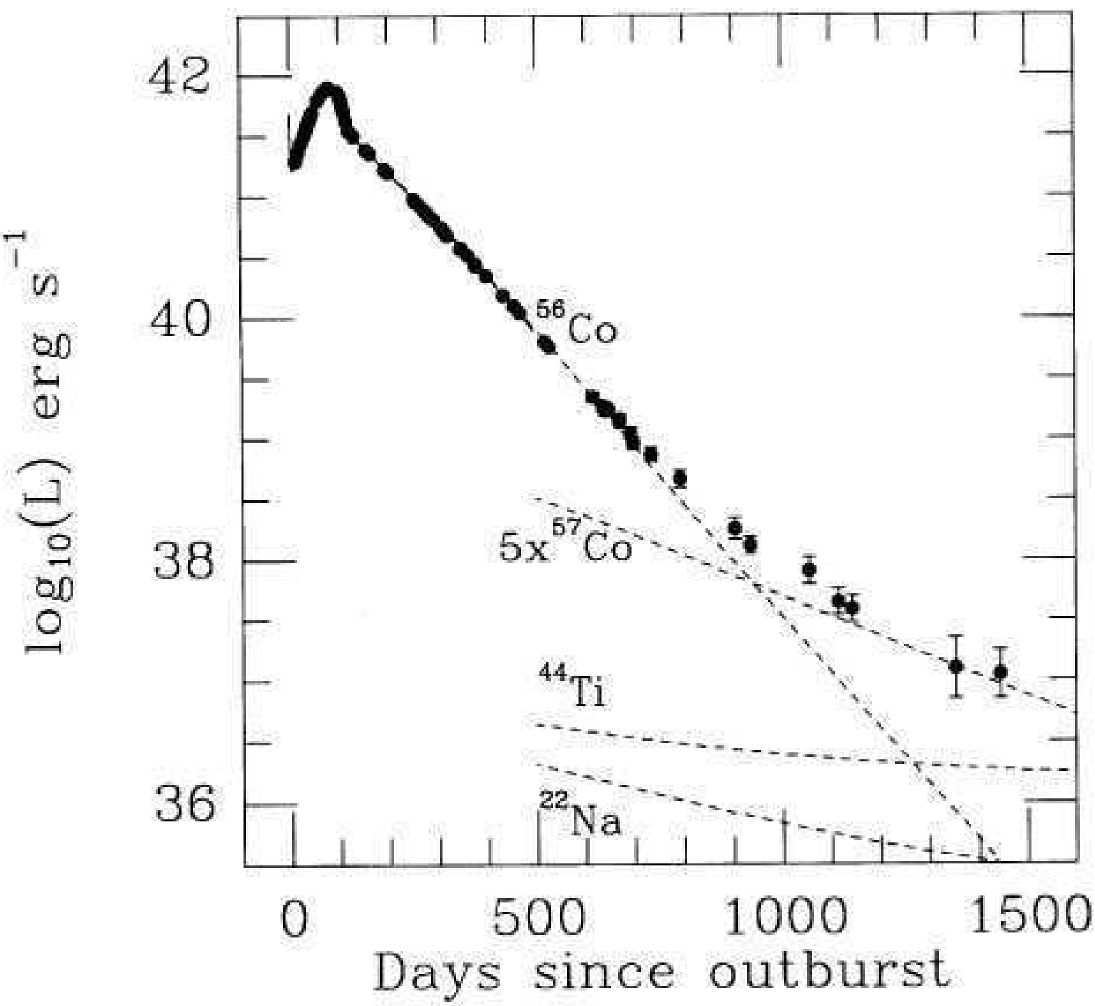


Figure 1: Bolometric light curve of SN 1987A from Suntzeff et al. 1992. Note the units (10^7 erg = 1 J).

G1. As a reminder, the solutions to the Lane-Emden equation obey the following properties:

$$R = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n} \xi_1$$

$$M = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \Theta_n$$

$$\text{where } \Theta_n = \left(-\xi^2 \frac{dD_n}{d\xi}\right|_{\xi_1} \quad \text{and } K \text{ is defined by } P = K\rho^{1+1/n}$$

For $n = 3/2$, $\xi_1 = 3.65375$ and $\Theta_{3/2} = 2.71406$.

(a) Find a general expression for the ratio of the central density (ρ_c) to average mass density of a spherical polytrope in terms of ξ_1 and Θ_n. Show that for $n = 3/2$, this ratio is 5.99.

(b) Consider a cold carbon-oxygen white dwarf in the non-relativistic regime. I showed in class that in this limit the radius scales like $R \propto M^{-1/3}$. However, as the central density increases, the electrons become increasingly relativistic and the actual radius shrinks below this expression, falling to zero at the Chandrasekhar mass.

i. If relativistic effects begin to cause deviations from the non-relativistic behavior when the Lorentz factor ($\gamma = 1/\sqrt{1-\beta^2}$) equals 1.1, what is the momentum of the electrons at this γ?

ii. At what mass density does this correspond to the Fermi momentum?

iii. At what white dwarf mass is ρ_c equal to this density? Carbon-oxygen white dwarfs above this mass will show deviations from the non-relativistic behavior.

ASTR 4201/5201 (Fall 2016)
Homework #11
Due: **Monday**, November 21 in class

Please remember to show your work.

1. In class, I gave a result for the temperature of a thin accretion disk that can be put in the form

$$T = T_{\rm disk} \left(\frac{R_{in}}{r}\right)^{3/4} (1 - \sqrt{R_{in}/r})^{1/4} \quad \text{where } T_{\rm disk} = \left(\frac{3GM\dot{M}}{8\pi\sigma R_{in}^3}\right)^{1/4}$$

and R_{in} is the inner radius of the disk, which can be taken to be the surface of the accreting compact object (if it isn't a black hole).

   (a) Show that the temperature reaches a maximum at $r = (49/36)R_{in}$ and that this maximum is $T = 0.488\ T_{\rm disk}$.

   (b) Assume that each annulus of the accretion disk emits as a blackbody and integrate from $r = R_{in}$ to $r = \infty$ to find the total emitted luminosity from the disk in terms of M, $\dot{M}$, and R_{in}.

2. Now consider the case of accretion through a disk onto a neutron star, with typical parameters of $R \approx 10$ km and $M \approx 1.4\ M_\odot$.

   (a) If the accretion luminosity from the disk is 10% of the Eddington luminosity, what is the mass accretion rate?

   (b) What does this imply about the peak temperature of the disk? In what region of the electromagnetic spectrum does this emission peak?

   (c) Assume that the accretion rate is the same fraction of the Eddington limit, but that the compact object is a 0.6 $M_\odot$ white dwarf with $R \approx 9 \times 10^6$ m. The peak temperature for this disk has its peak emission in what region of the electromagnetic spectrum?

3. This question concerns the phases of the interstellar medium (ISM) in the Galaxy. Use the information given in Table 1.3 from the handout I distributed in class on Monday for the relevant ISM properties. Assume that the temperature of the HIM is $T = 7.5 \times 10^5$ K.

   (a) Verify that hydrogen has the appropriate ionization level under the conditions listed for both the cold neutral medium and for H II regions. Even though it isn't really true, you can assume collisional and thermodynamic equilibrium.

   (b) Compute the gas pressures for material in the CNM, WNM, HIM, and dense H_2 phases. Do your numbers justify the statements in the righthand column about which phases are or are not in pressure equilibrium?

4. A pulsar is emitting pulses of synchrotron radiation that are linearly polarized at a constant (intrinsic) polarization angle. The light from the pulsar passes through the WNM (assume a constant average density of $\sim$0.6 cm^{-3}=6$\times$10^5 m^{-3}) on its way to us on Earth. The measured properties of the pulses are:

λ (cm)	Polarization Angle χ (degrees)	Arrival time
0.04	56.7	t_0
0.2	56.8	t_0 + 3.5$\times$10^{-5} s
1.0	59.5	t_0 + 9.2$\times$10^{-4} s
6.0	156	t_0 + 0.0332 s

   (a) Estimate the distance to the pulsar

   (b) and the mean magnetic field along the line of sight to us.

5. I discussed in class how the merger of two white dwarfs is a popular progenitor model for Type Ia supernovae. The system is formed after both stars evolve to produce white dwarfs and then their orbits decay due to the emission of gravitational waves. In this problem, we will calculate the necessary initial conditions for this process to occur in a reasonable timescale. The comparison between your numbers in the last two questions with both the solar radius and the radius of the Sun when it becomes a red giant demonstrates the importance of common envelope evolution to allowing WD mergers to occur within the age of the universe.

   (a) Show that the orbital kinetic energy of an equal-mass binary in circular orbits with separation a and individual masses M is

$$E_K = \frac{GM^2}{2a}$$

   and that the total orbital energy (kinetic + potential) is -1 times this amount (i.e., the virial result— but do not directly assume virial equilibrium, show it).

   (b) As the stars orbit each other, they emit gravitational waves. This loss of energy causes the orbit to shrink, which increases the rate at which gravitational waves are emitted. The total power lost to gravitational radiation by such a system is:

$$\dot{E}_{GW} = -\frac{2c^5}{5G}\left(\frac{2GM}{c^2 a}\right)^5$$

   Equate this expression to the time derivative of the expression for the total orbital energy from part (a) to find a differential equation for $a(t)$ and solve it.

   (c) What is the maximum initial separation that a WD binary can have if the components are to merge within 10 Gyr? Assume that the white dwarfs have $1M_\odot$ each, and that the merger occurs when $a = 0$. Express this answer in terms of the solar radius.

   (d) Some Type Ia SNe appear to explode as soon as 200 Myr after the formation of the system. What is the maximum initial separation for a binary to merge in this time, again expressing the answer in terms of the solar radius?

ASTR 4201/5201 (Fall 2016)
Homework #12
Due: Friday, December 2 in class

You will want to refer to the handout with all the nebular tables and figures that I gave out in class last Friday. Hint: Tables 3.6, 3.12, 14.2, and 15.1 and Figure 5.1 have various useful parameters. Please remember to show your work.

1. The black hole spin parameter a is defined as cJ/GM^2, where J is the angular momentum of the black hole, and $|a| < 1$.
   (a) What is the maximum angular momentum of a 5 $M_\odot$ black hole?
   (b) We know that neutron stars are also formed in the collapse of stellar cores, and that many are born with rapid spins. Assume that the Crab pulsar (spin $P =$ 33 ms) has a typical mass and radius for a neutron star and estimate its angular momentum. (For simplicity, you may assume that the Crab pulsar has a moment of inertia given by $I = 0.4MR^2$, appropriate for a constant density sphere.)
   (c) What would be the spin parameter for a black hole with this same mass and angular momentum?

2. It can be shown (e.g., Rybicki & Lightman 6.7b) that the synchrotron power ($-$dE/dt) emitted by a relativistic electron with Lorentz factor $\gamma \gg 1$ is equal to

$$P_{\rm em} = \left(\frac{4}{3}\right) c\sigma_{\rm T}\beta^2\gamma^2 {\rm U}_B$$

   where ${\rm U}_B$ is the energy density ($=B^2/2\mu_0$) in the magnetic field of strength B in Tesla, σ_T is the Thomson scattering cross-section, $\beta = v/c$, and γ is the Lorentz factor.
   (a) Set up a differential equation for the Lorentz factor as a function of time for an electron with energy $E = \gamma m_e c^2$. You may assume the relativistic limit so $\beta \approx 1$.
   (b) Solve it to find an expression for the lifetime of an electron with initial Lorentz factor γ_0 to radiate half of its initial energy.
   (c) Insert numbers into the coefficient to give a simple expression for the lifetime in terms of B and γ_0.
   (d) The Crab nebula has electrons emitting synchrotron radiation up to observed frequencies of more than 10^{21} Hz. The mean magnetic field in the nebula has been determined to be 1.24×10^{-8} T. Assuming that this emission is being emitted by electrons whose peak synchroton frequency ($\nu_{\rm sync} = \gamma^2 eB/(2\pi m_e)$, ignoring the factors of order unity) is near this value, what is the Lorentz factor of those electrons?
   (e) What is the lifetime of these electrons and how does it compare to the known age of the remnant (explosion 1054 A.D.)? What can you conclude about particle acceleration in the Crab Nebula?

3. I stated in class that the Innermost Stable Circular Orbit around a non-spinning black hole was located at 3 Schwarzschild radii ($3R_s$). In other words, orbits located closer than this are unstable and matter will fall into the black hole. Consider an observer located at rest at the ISCO around a Schwarzschild black hole. One hour of time experienced by this observer will correspond to how many hours of time experienced by a distant inertial observer? Is it more or less than seven years?

4. For this problem, consider an ionized H II region around a massive star embedded in a neutral medium. Assume that Case B recombination conditions apply, $n_e = 10^3$ cm^{-3}, and $T = 10^4$ K.

   (a) What is the typical radius of an H II region around an O5V star in such a medium?

   (b) Show that in steady state the nebula absorbs 2.2 ionizing photons from the central star per Hα photon that it emits. And what is the ratio for Hβ?

   (c) The nearby galaxy NGC 7714 has a spectrum dominated by H II region emission lines. It is at a distance of 39 Mpc and has a measured Hβ emission line flux of 1.9×10^{-12} erg cm^{-2} s^{-1}. Assuming that the galaxy's emission line flux can be approximated by a superposition of many different H II regions, how many ionizing photons are required to produce this Hβ flux?

   (d) If we assume that the average H II region is being powered by an O5V star, how many such stars does it take to be responsible for the Hβ emission that we see?

5. Now consider a nebula in the low-density limit that is emitting collisionally-excited [O III] lines.

   (a) What is the ratio of emission line fluxes (i.e., erg cm^{-2} s^{-1}) predicted by atomic physics for [O III] 5007Å/[O III] 4959Å?

   (b) If the line ratio emitted by the nebula for [O III] λ4363 : [O III] λ5007 is 1.2:270, what is the electron temperature in the nebula? (For this one, you do not need to do a detailed calculation but please use the appropriate graph in the handout.)

G1. Continuing on with the properties of [O III]:

   (c) Calculate the excitation coefficient q_{12} for ${}^3P_0 \rightarrow {}^1D_2$ for [O III] (i.e., collisional excitations from the ground state to the level that will emit λ5007 and λ4959). (Note: this is defined, along with other useful information, in the thermal equilibrium handout, but you will also need information from the tables and figures in the other handout.)

   (d) Estimate the critical density for the ${}^1D_2$ level (the upper level for the λ5007 transition), assuming the temperature calculated in part (b). (note: the answer is given in one of the tables in the packet for T=10^4 K, but I want you to actually calculate it...)