Homework 11:Differential Equations

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Nov 20,2015

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1 Question 1: Solution of a Differential Equation

In this question I solved the differential equation:

$$y'(x) = 2(y+1) \quad , -2 < x < 2 \quad , y(0) = 0 \tag{1}$$

whose exact solution is :

$$y(x) = e^{2x} - 1 \tag{2}$$

using five different numerical methods to find the solution,viz.:

– Euler's Method

– Improved Euler's Method

– Fourth Order Runge-Kutta Method

– Adaptive Runge-Kutta Method

– Picard's Method

And assessed their errors and robustness. **Note: in the plots $1b$ and $1d$ we may see that there is sudden transition of y value from high value to low value this is not due to the mistake but it is due to the way I printed values in my datafile. In the data file i have printed from 0 to $+2$ first, then 0 to -2. If i take from -2 to 2 the result will be same, nature of curve will remain same, error bar will remain same and there will be no sudden transion.**

1.1 part a: Euler method with different step sizes

In this part I used Euler' method to solve the differential equation. I used step-size $h = 0.05, 0.10, .015, 0.20$ and plotted the results with error bar.
From the graph we can say that when step size increases the error also increases. This means smaller value of step-size gives better result for the Euler's method.

The solution directory is :

```
location             : hw11/qn1/qn1a
source code          : hw11qn1a.f90 and euler.f90
plots                : hw11qn1a.eps
datafiles            : euler05.dat, euler10.dat, euler15.dat, euler20.dat, also eule
```

The figures are shown below:

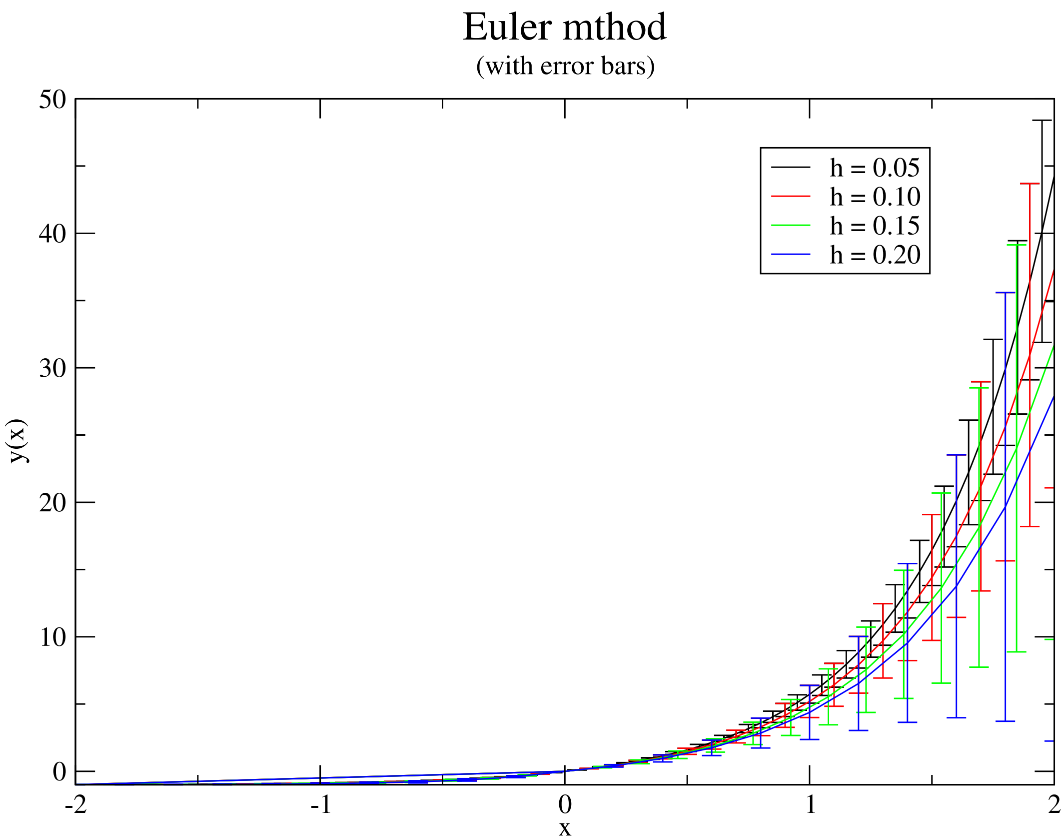

Figure 1: Euler method with different step-sizes

1.2 part b: Comparison of Euler, Modified Euler, and fourth order Runge Kutta methods

In this part I solved the given differential equation using three different methods, viz. Euler, Improved Euler and Fourth Order Runge-Kutta methods. From the datafile and plot, we can see that Runge-Kutta method is slightly better than Improved-Euler-Method, and Improved Euler method is much better than Euler method.

The solution directory is :

```
location              : hw11/qn1/qn1b
source code           : hw11qn1b.f90
plots                 : hw11qn1b.eps
datafiles             : hw11qn1b.dat
provided codes        : rk4.f90 and test1rk4.f90
```

The figures are shown below:

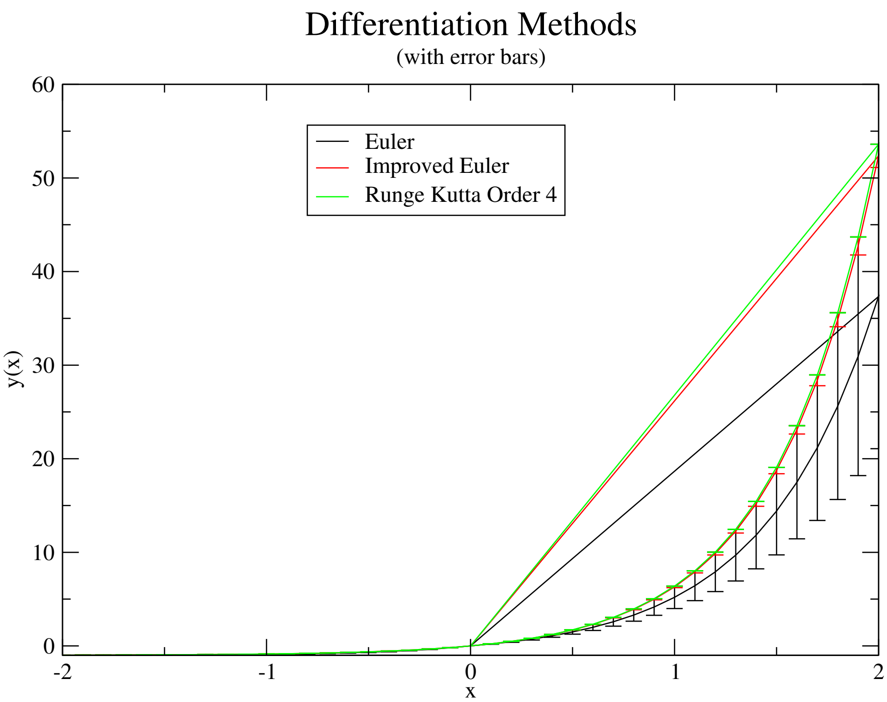

Figure 2: Euler, Modified Euler, and fourth order Runge Kutta methods

1.3 part c: Adaptive Runge-Kutta Method

In this part I used Adaptive Runge-Kutta Method ($difsis.f90$) to solve the given differential equation. I found that final value of h was found to be 0.875 in difsis, so I chose that value of step-size in Euler method (qn 1a) and compared the results. I found that adaptive Runge-Kutta method is much better than Euler's method. Also looking datafiles I found that Adaptive Runge-Kutta method is better than Fourth Order Runge Kutta and Improved-Euler method.

The solution directory is :

```
location             : hw11/qn1/qn1c
source code          : hw11qn1c.f90   and euler.f90 (inside qn1a)
plots                : hw11qn1c.eps
datafiles            : hw11qn1c.dat   and euler875.dat (inside qn1a)
provided subroutines : difsis.f90
```

The figures are shown below:

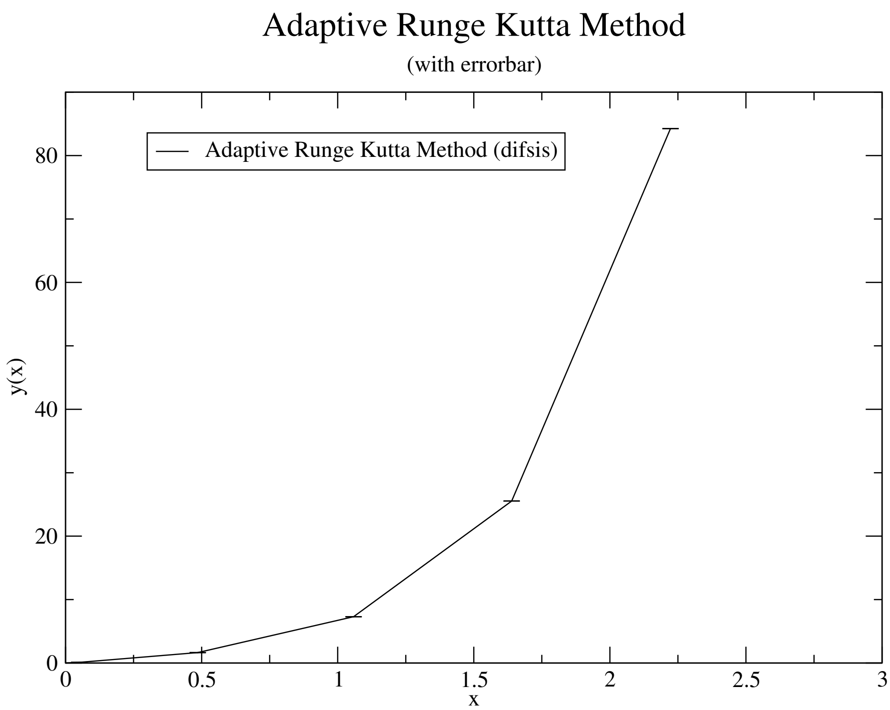

Figure 3: Adaptive Runge-Kutta Method

1.4 part d: Picard's Iteration Method

In this part I used Picard's method to solve the given differential equation. For iterations = 4, I found significant error. But for higher iterations There is less error and result is pretty accurate.

The solution directory is :

```
location              : hw11/qn1/qn1d
source code           : hw11qn1d.f90
plots                 : hw11qn1d.f90
datafiles             : picard4.dat, picard8.dat, picard12.dat, picard16.dat
```

The figures are shown below:

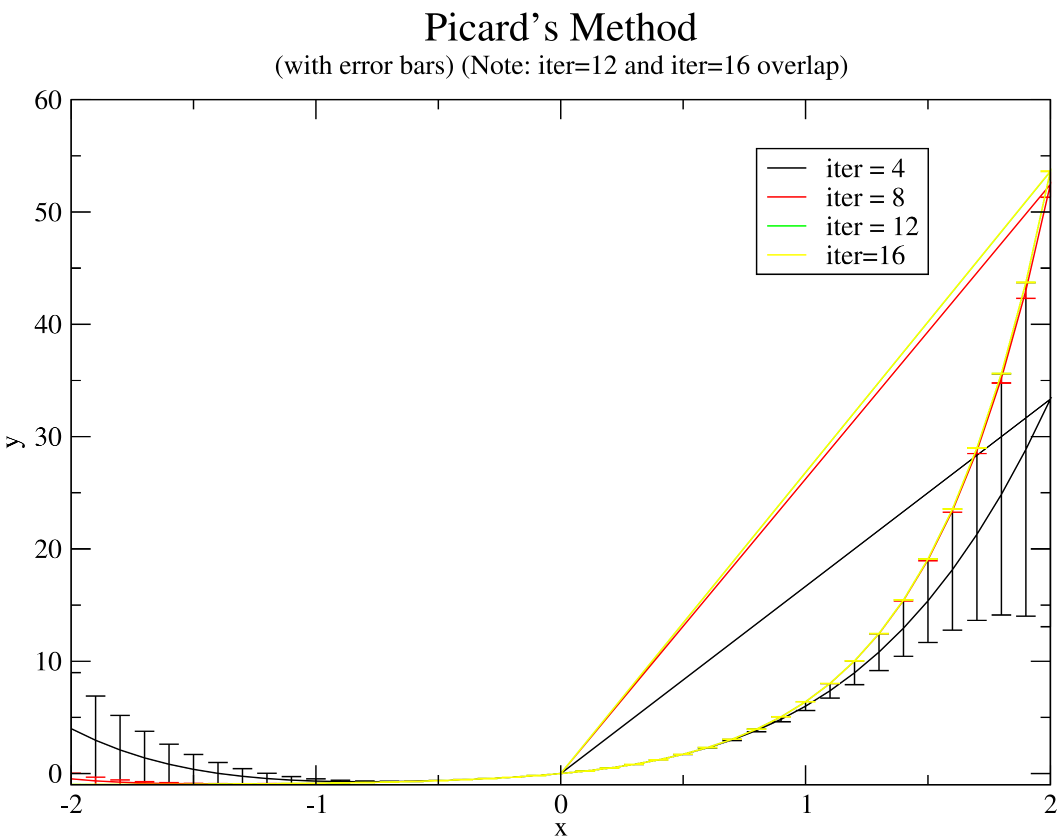

Figure 4: Picard's Method with different iterations

2 Question 2: The Ideal Harmonic Oscillator

In this part I solved the Newton's equation of motion for the ideal harmonic oscillator:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega_0^2 x \tag{3}$$

with frequency

$$\omega_0 = \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \tag{4}$$

and analytic solution

$$x(t) = Asin(\omega_0 t + \phi) \tag{5}$$

where, A is amplitude and ϕ is phase constant.
First I rewrite the second-order differential equation as two coupled first-order differential equations:

$$\frac{dx(t)}{dt} = v(t) \tag{6}$$

$$\frac{dv(t)}{dt} = -\omega_0^2 x(t) \tag{7}$$

Then I used Runge-Kutta method to solve this system of coupled first order differential equations.

The solution directory is :

```
location             : hw11/qn2
source code          : hw11qn2.f90
datafiles            : positiontime05.dat, positiontime10.dat
plots                : positiontime05.eps, positiontime10.eps
datafiles            : energyerror05.dat,  energyerror10.dat
datafiles            : energyerror05a.dat, energyerror10a.dat
plots                : energyerror.eps,    energytime.eps,stability.eps
provided subroutines : rk4.f90
hints                : Landau 2E, Chapter 15
```

2.1 part 2.1:

Here, I picked values of k and m such that the period T is a nice number to work with. I chose $T = 1$.

2.2 part 2.2:

I tried out step sizes starting at $h = 0.10$ and then took smaller sizes ($h = 0.05$). I solved for several periods. Here, the solution look smooth and have a period that never changes even after many oscillations.

The figures are shown below:

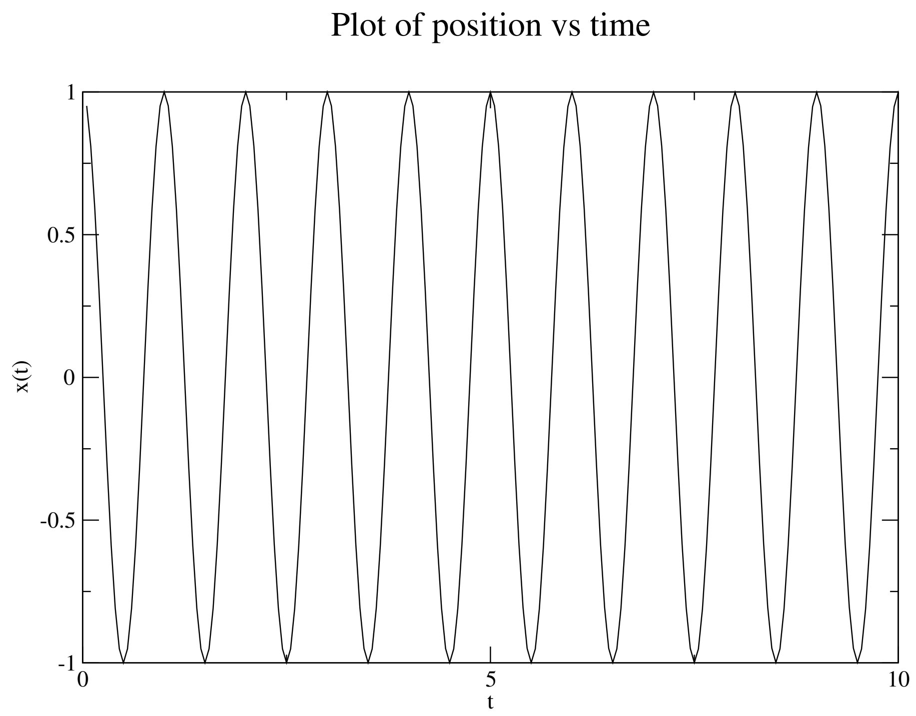

Figure 5: position vs time plot at $h = 0.05$

The figures are shown below:

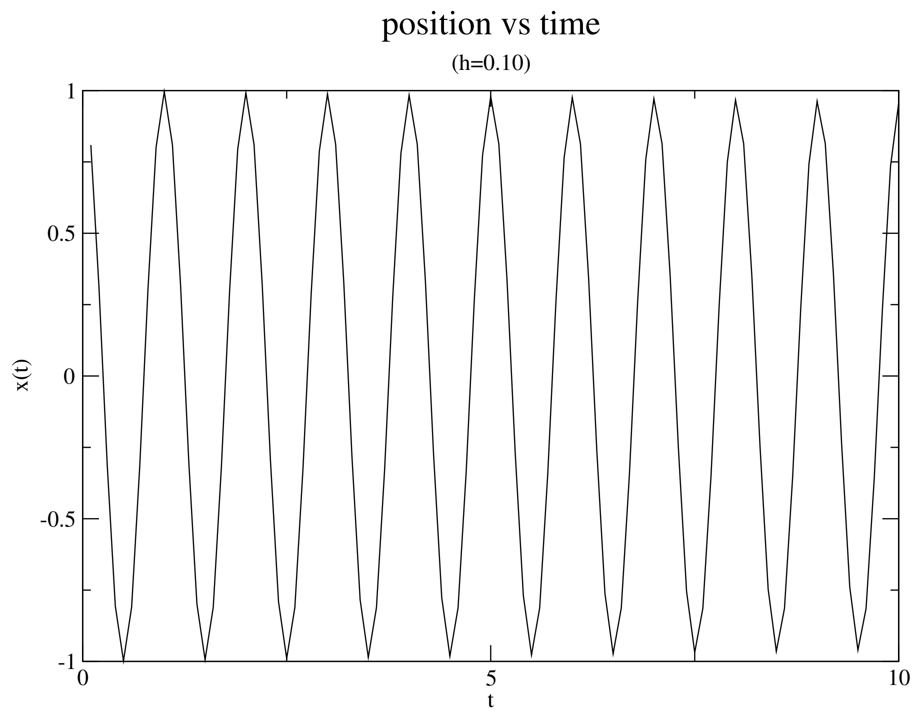

Figure 6: position vs time plot at $h = 0.10$

2.3 part 2.3:

Here, I plotted the computed solution together with the analytic solution. I also compared the computed solutions with analytic one. I computed a relative error at $t = 9.5T, 19.5T, and \quad 29.5T$ for different step sizes as in part 2.2.

	relative error at	
t	h=0.05	h=0.10
9.5	0.169	0.266
19.5	0.207	0.853
29.5	0.559	0.936

The figures are shown below:

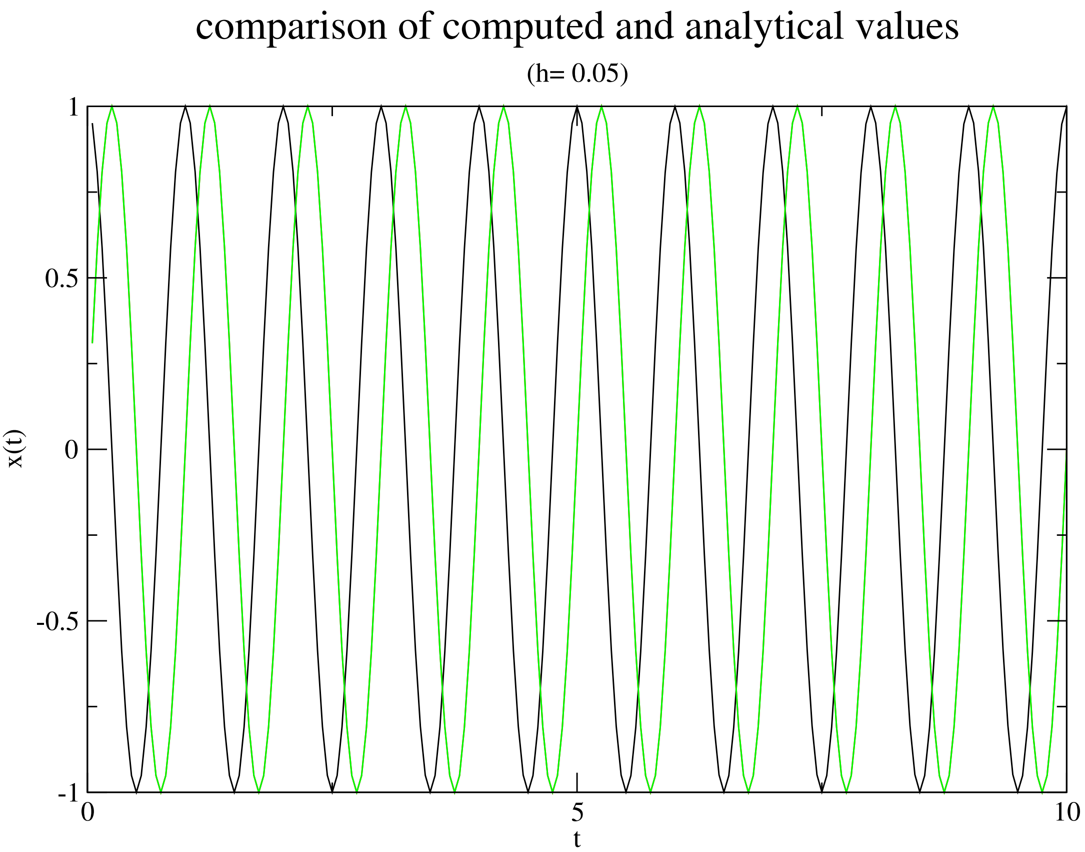

Figure 7: ideal harmonic oscillator when $h = 0.05$

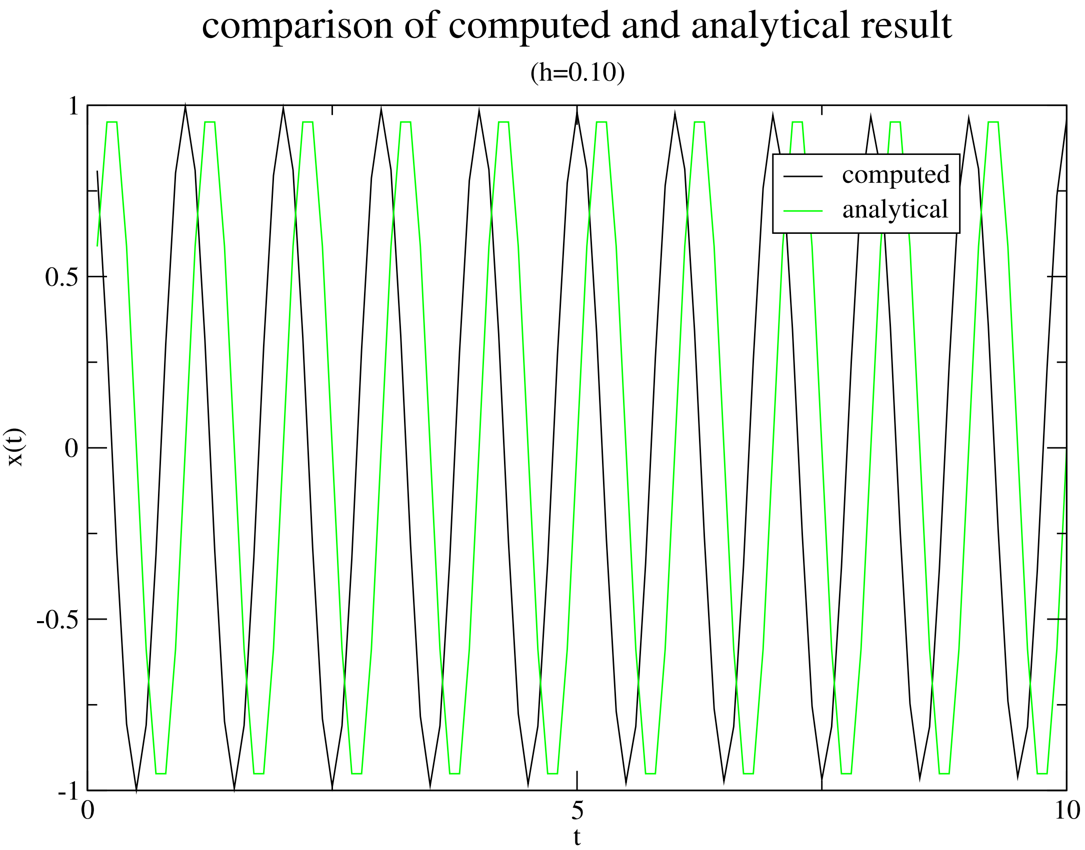

Figure 8: ideal harmonic oscillator when $h = 0.10$

2.4 part 2.4:

Here, we have not explicitly built energy conservation into the solution of the differential equation. Nonetheless, since no friction is included, the total energy must be a constant of motion. This is a demanding test of the accuracy of the solution. Here, when $h = 0.05$, I found constant value of energy to be $0.197389D + 02$.

2.5 part 2.5:

The total energy at any time t, given by

$$E = \frac{1}{2}kx(t)^2 + \frac{1}{2}mv(t)^2 \tag{8}$$

must be constant. I checked the numerically computed energy is constant at the different times given when $h = 0.5$ and $h = 0.10$ and plotted the relative error in percent as function of step size h. The figures are shown below:

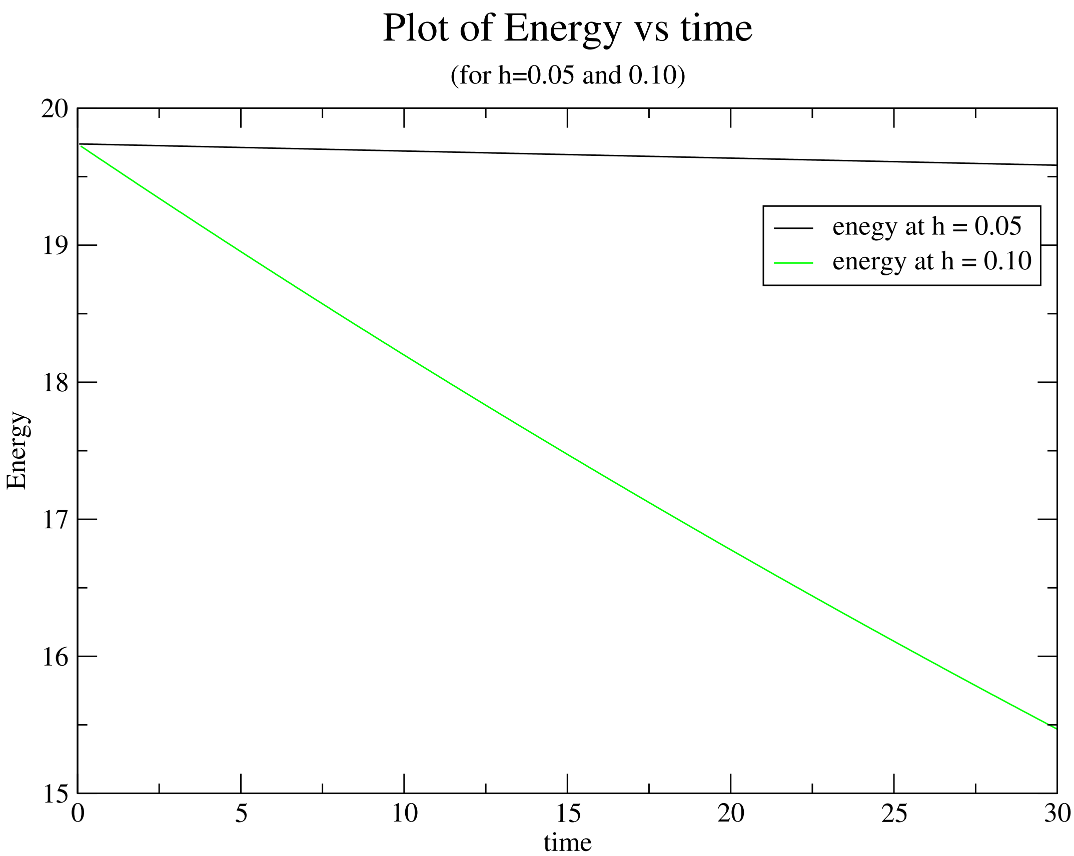

Figure 9: Plot of energy vs time

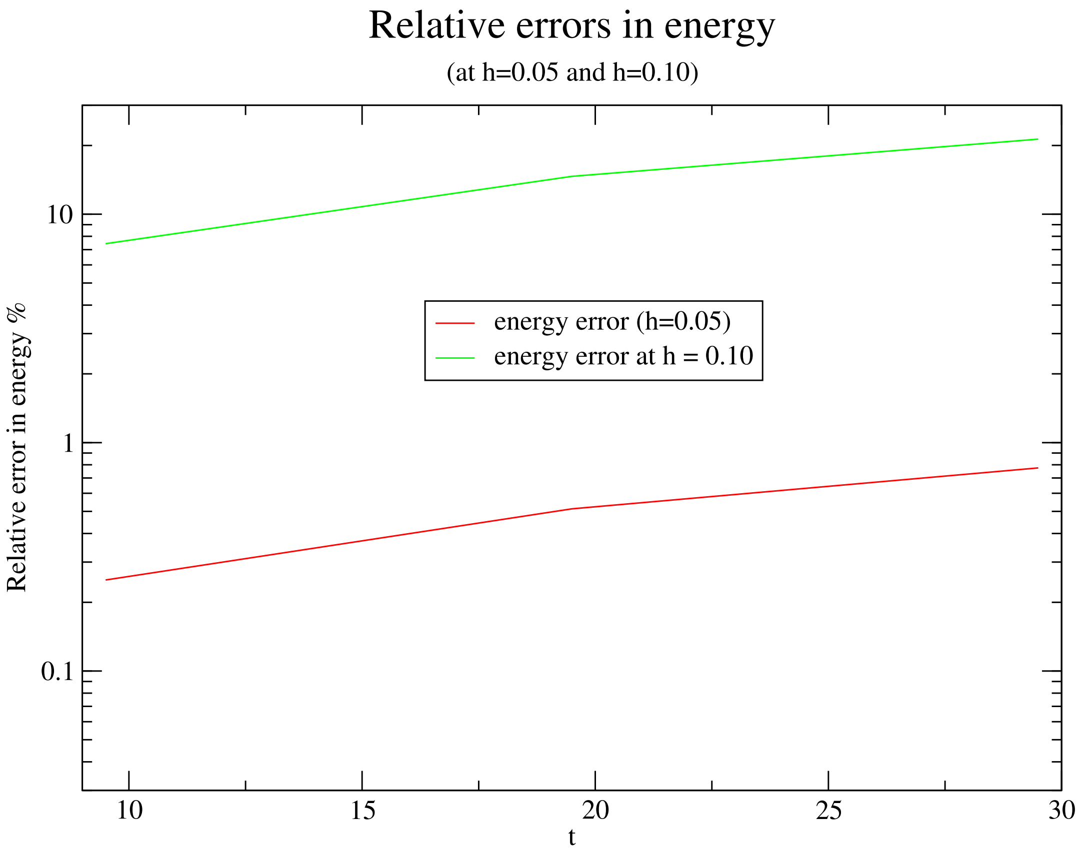

Figure 10: Plot of relative energy in percentage as a function of step-size

2.6 part 2.6:

The long-term *stability* of the solution is given by

$$log[\frac{|E(t) - E(t=0)|}{E(t=0)}]$$

I plotted this stability versus time curve. The figures are shown below:

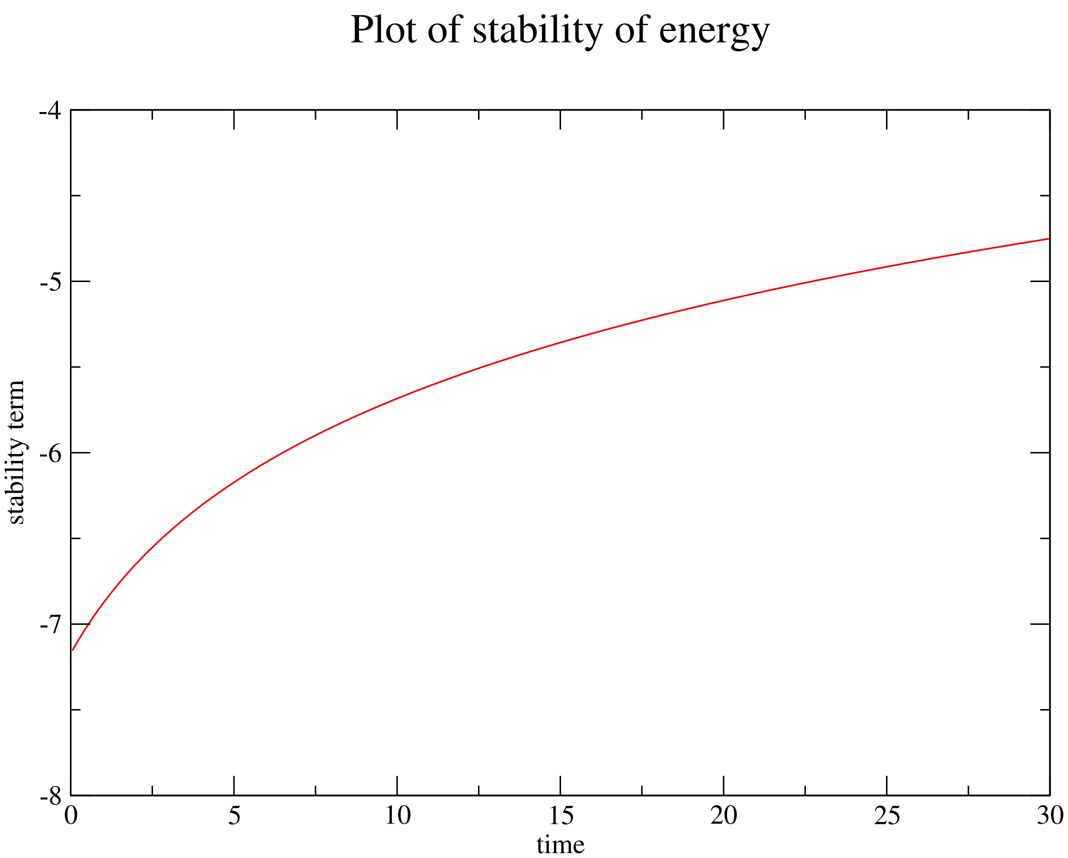

Figure 11: Plot of relative energy in percentage as a function of step-size

2.7 part 2.7:

Here, I added the viscous friction term to this model. I added a force

$$F_f = -bv \tag{9}$$

where, b is a parameter and v is the velocity.
I modified the code and investigated the qualitative changes of the solution that occur for increasing values of b:

- **Underdamped:** $b \leq 2m\omega_0$
- **Critically damped:** $b = 2m\omega_0$
- **Over damped:** $b \geq 2m\omega_0$

Here, In my code I have taken $\omega_0 = 2\pi \quad and \quad m = 1$.

I plotted the behavior of those three cases and demonstrated the behavior of the solution.

The figures are shown below:

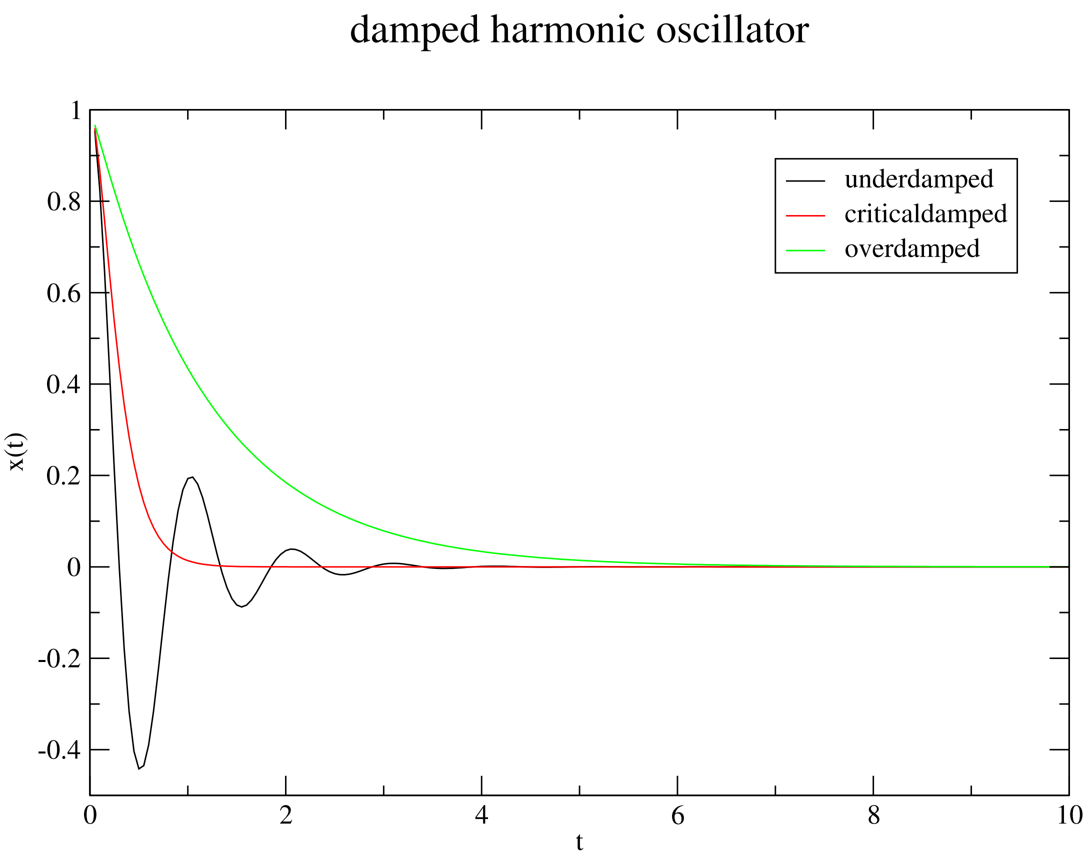

Figure 12: Plot of relative energy in percentage as a function of step-size