Appendix B. Cubic Hermitian Splines [illegible]

Consider a function $f(x)$ given at certain grid points x_i, where $i = 1, 2, 3, \ldots, n$.

Let x be positioned in the interval $[x_i, x_{i+1}]$.

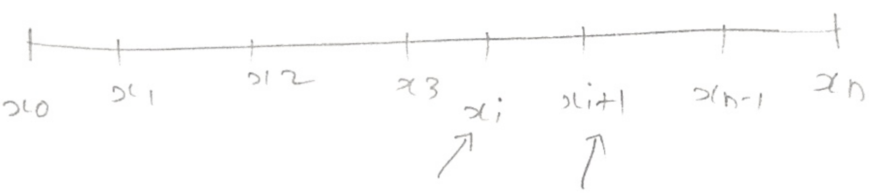
For the sake of simpler notation we call the end points

$$x_i \equiv x_1$$
$$x_{i+1} \equiv x_2$$

then one defines a unique cubic polynomial $f_i(x)$ by the following constraints:

$$\left.\begin{array}{ll} f_i(x_1) = f(x_1) & f_i'(x_1) = f'(x_1) \\ f_i(x_2) = f(x_2) & f_i'(x_2) = f'(x_2) \end{array}\right\} \quad B.1$$

Therefore these interpolating functions $f_i(x)$ and their derivatives $f_i'(x)$ are continuous at the grid points x_i.



$$x_i \equiv x_1 \qquad x_{i+1} \equiv x_2$$

$f(x)$ where $x \in [x_i, x_{i+1}]$ ie $x \in [x_1, x_2]$

To find the cubic splines we start with cubic polynomials.

$$\boxed{p(x) = a + b(x - x_1) + c(x - x_1)^2 + d(x - x_1)^3} \quad \text{--- (a)}$$

$$\boxed{p'(x) = b + 2c(x - x_1) + 3d(x - x_1)^2} \quad \text{--- (b)}$$

Now,

$$\begin{cases} p(x_1) = f(x_1) = a & \text{--- (a1)} \\ p(x_2) = f(x_2) = a + b(x_2 - x_1) + c(x_2 - x_1)^2 + d(x_2 - x_1)^3 & \text{--- (a2)} \end{cases}$$

$$\begin{cases} p'(x_1) = f'(x_1) = b & \text{--- (b1)} \\ f'(x_2) = p'(x_2) = b + 2c(x_2 - x_1) + 3d(x_2 - x_1)^2 & \text{--- (b2)} \end{cases}$$

Now, $f(x_2) - f(x_1) = b(x_2 - x_1) + c(x_2 - x_1)^2 + d(x_2 - x_1)^3$

$$\boxed{f(x_2) - f(x_1) = f'(x_1)\Delta + c\Delta^2 + d\Delta^3} \quad \text{--- (c)}$$

where, $\Delta \equiv x_2 - x_1 \equiv x_{i+1} - x_i$

again, $f'(x_2) - f'(x_1) = 2c\Delta + 3d\Delta^2$

from (c), $2\dfrac{(f(x_2) - f(x_1) - f'(x_1)\Delta)}{\Delta} = 2c\Delta + 2d\Delta^2$ (multiply by 2)

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subtracting, $f'(x_2) - f'(x_1) - \dfrac{2(f(x_2) - f(x_1) - f'(x_1)\Delta)}{\Delta} = d\Delta^2$

$$d = \frac{1}{\Delta^2}\left[f'(x_2) - f'(x_1) - \frac{2}{\Delta}\left(f(x_2) - f(x_1) \right) \right]$$

Again, to find c,

$$f'(x_2) - f'(x_1) - \frac{3}{\Delta}\left[f(x_2) - f(x_1) \right] = -3\, f'(x_1) - c(x_2 - x_1)$$

$$\Rightarrow \quad c = \frac{1}{x_2 - x_1}\left[-f'(x_2) - 2f'(x_1) + \frac{3}{x_2 - x_1}\left(f(x_2) - f(x_1) \right) \right]$$

$$\Rightarrow f_i(x) = f(x_1) + f'(x_1)(x - x_1) + \left[\frac{1}{x_2 - x_1}\left(-f'(x_2) - 2f'(x_1) \right) + \frac{3}{x_2 - x_1} \cdot \left(f(x_2) - f(x_1) \right) \right]$$

$$+ \frac{(x - x_1)^3}{(x_2 - x_1)}\left[f'(x_1) + f'(x_2) - \frac{2 f(x_2)}{x_2 - x_1} + \frac{2 f(x_1)}{(x_2 - x_1)} \right]$$

or,

$$f_i(x) = f(x_1)\left[1 - \frac{3(x - x_1)}{(x_2 - x_1)^2} + \frac{2(x - x_1)^3}{(x_2 - x_1)^3} \right]$$

$$+ f(x_2)\left[\frac{3(x - x_1)^2}{(x_2 - x_1)} - \frac{2(x - x_1)^3}{(x_2 - x_1)} \right]$$

$$+ f'(x_1)\left[(x - x_1) - \frac{2(x - x_1)^2}{(x_2 - x_1)} + \frac{(x - x_1)^3}{(x_2 - x_1)} \right]$$

$$+ f'(x_2)\left[-\frac{(x - x_1)^2}{(x_2 - x_1)} + \frac{(x - x_1)^3}{(x_2 - x_1)} \right]$$

comparing with (B.2)

$$f_i(x) = f(x_1)\,\phi_1(x) + f(x_2)\,\phi_2(x) + f'(x_1)\,\phi_3(x) + f'(x_2)\,\phi_4(x) \quad \text{——— (B.2)}$$

we get,

$$\phi_1(x) = 1 - \frac{3(x-x_1)^2}{(x_2-x_1)^2} + \frac{2(x-x_1)^3}{(x_2-x_1)^3}$$

$$\phi_2(x) = \frac{3(x-x_1)^2}{(x_2-x_1)} - \frac{2(x-x_1)^3}{(x_2-x_1)^3}$$

$$\phi_3(x) = (x-x_1) - \frac{2(x-x_1)^2}{(x_2-x_1)} + \frac{(x-x_1)^3}{(x_2-x_1)^2}$$

$$\phi_4(x) = -\frac{(x-x_1)^2}{(x_2-x_1)} + \frac{(x-x_1)^3}{(x_2-x_1)^2}$$

further simplifying $\phi_i(x)$

$$\phi_1(x) = 1 - \frac{3(x-x_1)^2}{(x_2-x_1)^2} + \frac{2(x-x_1)^3}{(x_2-x_1)^3}$$ multiply

$$= \frac{1}{(x_2-x_1)^3}\left[(x_2-x_1)^3 - 3(x-x_1)^2(x_2-x_1) + 2(x-x_1)^3\right]$$

$$\phi_1(x) = \frac{1}{(x_2-x_1)^3}\Big[(x_2^3 - 3x_1x_2 + 2x_1x_2^2) + (6x x_1 x_2 - 4x^2x_2 - 2x x_2^2) + (x^2x_2 - 3x^2x_1 + 2x^3) \Big]$$

$$= \frac{1}{(x_2-x_1)^3}\left[(x_2^2 - 2x x_2 + x^2) \cdot (x_2 - 3x_1 + 2x) \right]$$

$$\boxed{\phi_1(x) = \frac{(x_2-x)^2}{(x_2-x_1)^3}\left[(x_2-x_1) + 2(x-x_1) \right]}$$

again,

$$\phi_2(x) = \frac{3(x-x_1)^2}{(x_2-x_1)^2} - \frac{2(x-x_1)^3}{(x_2-x_1)^3} \quad \text{LCM}$$

$$= \frac{3(x-x_1)^2(x_2-x_1)^3 - 2(x-x_1)^3}{(x_2-x_1)^3}$$

$$= \frac{(x-x_1)^2}{(x_2-x_1)^3}\left[3(x_2-x_1) - 2(x-x_1) \right]$$

$$\boxed{\phi_2(x) = \frac{(x-x_1)^2}{(x_2-x_1)^3}\left[(x_2-x_1) + 2(x_2-x) \right]}$$

again, $\phi_3(x) = (x-x_1) - \dfrac{2(x-x_1)^2}{(x_2-x_1)} + \dfrac{(x-x_1)^3}{(x_2-x_1)^2}$

$$= \frac{(x-x_1)(x_2-x_1)^2 - 2(x-x_1)^2(x_2-x_1) + (x-x_1)^3}{(x_2-x_1)^2}$$

$$= \frac{x-x_1}{(x_2-x_1)^2}\left[(x_2-x_1)^2 - 2(x-x_1)(x_2-x_1) + (x-x_1)^2\right]$$

$$= \frac{x-x_1}{(x_2-x_1)^2}\left[x_2^2 - 2xx_2 + x^2\right]$$

$$\boxed{\phi_3(x) = \frac{(x-x_1)(x_2-x)^2}{(x_2-x_1)^2}}$$

again, $\phi_4(x) = -\dfrac{(x-x_1)^2}{x_2-x_1} + \dfrac{(x-x_1)^3}{(x_2-x_1)^2}$

$$= \frac{-(x-x_1)^2(x_2-x_1) + (x-x_1)^3}{(x_2-x_1)^2}$$

$$\boxed{\phi_4(x) = \frac{(x-x_1)^2}{(x_2-x_1)^2}(x-x_2)}$$

Now, we approximate the derivatives $f'(x_1)$ and $f'(x_2)$ with the help of quadratic polynomial which is uniquely defined by the function values at a grid point and its two neighbors.

we define parabola as,

$$q(x) = \alpha + \beta(x - x_i) + \gamma(x - x_i)^2$$

$$q'(x) = \beta + 2\gamma(x - x_i)$$

Then,

$$q(x_i) = \alpha = f(x_i)$$

$$q'(x_i) = \beta = f'(x_i)$$

$$q(x_{i+1}) = \alpha + \beta(x_{i+1} - x_i) + \gamma(x_{i+1} - x_i)^2$$

$$q'(x_{i+1}) = \beta + 2\gamma(x_{i+1} - x_i)$$

$$q(x_{i-1}) = \alpha + \beta(x_{i-1} - x_i) + \gamma(x_{i-1} - x_i)^2$$

$$q'(x_{i-1}) = \beta + 2\gamma(x_{i-1} - x_i)$$

Then,

$$\beta = (-1)\, f(x_{i-1}) \frac{x_{i+1} - x_i}{(x_i - x_{i-1})(x_{i+1} - x_{i-1})}$$

$$+ f(x_i) \frac{x_{i+1} - 2x_i + x_{i-1}}{(x_i - x_{i-1})(x_{i+1} - x_i)}$$

$$+ f(x_{i+1}) \frac{x_i - x_{i-1}}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}$$

$$\gamma = f(x_{i-1}) \frac{1}{(x_i - x_{i-1})(x_{i+1} - x_{i-1})}$$

$$+ f(x_i) \frac{(-1)}{(x_{i+1} - x_i)(x_i - x_{i-1})}$$

$$+ f(x_{i+1}) \frac{1}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}$$

Then,

$$p(x) = f(x_{i-1}) \cdot \phi_3(x) \underbrace{(-1) \frac{x_{i+1} - x_i}{(x_i - x_{i-1})} \frac{1}{(x_{i+1} - x_{i-1})}}_{S_{i-1}(x)}$$

continue :

$$+ f(x_i)\Big\{ \underbrace{\phi_1(x) + \phi_3(x)\frac{x_{i+1}-2x_i-x_{i-1}}{(x_i-x_{i-1})(x_{i+1}-x_i)} + \phi_4(x)\,(-1)\,\frac{x_{i+2}-x_{i+1}}{(x_{i+1}-x_i)(x_{i+2}-x_i)}}_{S_i(x)}$$

$$+ f(x_{i+1})\Big\{ \underbrace{\phi_2(x) + \phi_3(x)\frac{x_i-x_{i+1}}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})} + \phi_4(x)\frac{x_{i+2}-2x_{i+1}-x_i}{(x_{i+1}-x_i)(x_{i+2}-x_{i+1})}}_{S_{i+1}(x)}$$

$$+ f(x_{i+2})\cdot \underbrace{\phi_4(x)\frac{x_{i+1}-x_i}{(x_{i+2}-x_{i+1})(x_{i+2}-x_i)}}_{S_{i+2}(x)}$$

$\therefore$

$$\boxed{\begin{aligned} f(x) = p(x) = {} & f(x_{i-1})\,S_{i-1}(x) \\ & + f(x_i)\,S_i(x) \\ & + f(x_{i+1})\,S_{i+1}(x) \\ & + f(x_{i+2})\,S_{i+2}(x) \end{aligned}}$$

proved !