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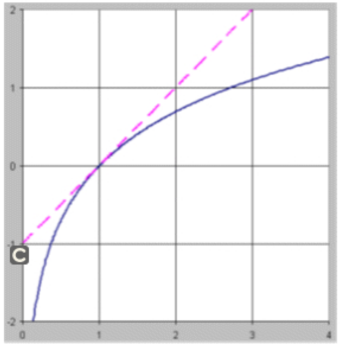
The logarithm transformation

Introduction to logarithms: Logarithms are one of the most important mathematical tools in the toolkit of statistical modeling, so you need to be very familiar with their properties and uses. A logarithm function is defined with respect to a "base", which is a positive number: **if b denotes the base number, then the base-b logarithm of X is, by definition, the number Y such that b^Y = X.** For example, the base-2 logarithm of 8 is equal to 3, because 2^3 = 8, and the base-10 logarithm of 100 is 2, because 10^2 = 100. There are three kinds of logarithms in standard use: the base-2 logarithm (predominantly used in computer science and music theory), the base-10 logarithm (predominantly used in engineering), and the *natural* logarithm (predominantly used in mathematics and physics *and in economics and business*). In the natural log function, the base number is the transcendental number "*e*" whose deciminal expansion is 2.718282…, so the natural log function and the exponential function (e^x) are inverses of each other. The only differences between these three logarithm functions are multiplicative scaling factors, so logically they are equivalent for purposes of modeling, but the choice of base is important for reasons of convenience and convention, according to the setting.

In standard mathematical notation, and in Excel and most other analytic software, the expression LN(X) is the natural log of X, and EXP(X) is the exponential function of X, so EXP(LN(X)) = X and LN(EXP(X)) = X. This means that *the EXP function can be used to convert natural-logged forecasts (and their respective lower and upper confidence limits) back into real units.* You cannot use the EXP function to directly unlog the *error statistics* of a model fitted to natural-logged data. You need to first convert the forecasts back into real units and then recalculate the errors and error statistics in real units, if it is important to have those numbers. However, the error statistics of a model fitted to natural-logged data can often be interpreted as approximate measures of *percentage* error, as explained below, and in situations where logging is appropriate in the first place, it is often of interest to measure and compare errors in percentage terms.

In general, the expression **LOG_b(.)** is used to denote the base-b logarithm function, and LN is used for the special case of the natural log while LOG is often used for the special case of the base-10 log. In particular, LOG means base-10 log in Excel. In Statgraphics, alas, the function that is called LOG is the natural log, while the base-10 logarithm function is LOG10. **In the remainder of this section (and elsewhere on the site), both LOG and LN will be used to refer to the *natural* log function, for compatibility with Statgraphics notation.** Also, the symbol "≈" means *approximately* equal, with the approximation being more accurate in relative terms for smaller absolute values, as shown in the table below.

Change in natural log ≈ percentage change: The natural logarithm and its base number *e* have some magical properties, which you may remember from calculus (and which you may have hoped you would never meet again). For example, the function e^X is its own derivative, and the derivative of LN(X) is 1/X. But for purposes of business analysis, its great advantage is that **small changes in the natural log of a variable are directly interpretable as percentage changes,** to a very close approximation. The reason for this is that the graph of Y = LN(X) passes through the point (1, 0) and has a slope of 1 there, so it is tangent to the straight line whose equation is Y = X-1 (the dashed line in the plot below):



This property of the natural log function implies that

LN(1+r) ≈ r

when r is much smaller than 1 in magnitude. Why is this important? Suppose X increases by a small percentage, such as 5%. This means that it changes from X to X(1+r), where r = 0.05. Now observe:

LN(X (1+r)) = LN(X) + LN(1+r) ≈ LN(X) + r

Thus, when X is increased by 5%, i.e., multiplied by a factor of 1.05, the natural log of X changes from LN(X) to LN(X) + 0.05, to a very close approximation. Increasing X by 5% is therefore (almost) equivalent to adding 0.05 to LN(X).

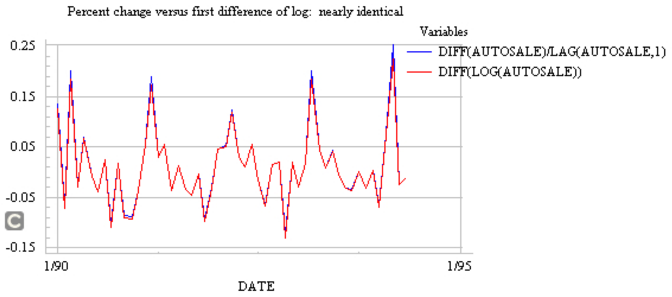
From now on I will refer to changes in natural logarithms as "diff-logs." (In Statgraphics, the diff-log transformation of X is literally DIFF(LOG(X)).) The following table shows the exact correspondence for percentages in the range from -50% to +100%:

% change in X	diff-log of X
-50%	-0.693
-40%	-0.511
-30%	-0.357

-20%	-0.223
-10%	-0.105
-5%	-0.051
-2%	-0.020
0%	0.000
2%	0.020
5%	0.049
10%	0.095
20%	0.182
30%	0.262
40%	0.336
50%	0.405
100%	0.693

As you can see, percentage changes and diff-logs are almost exactly the same within the range +/- 5%, and they remain very close up to +/- 20%. For large percentage changes they begin to diverge in an asymmetric way. Note that the diff-log that corresponds to a 50% decrease is -0.693 while the diff-log of a 100% increase is +0.693, exactly the opposite number. This reflects the fact that a 50% decrease followed by a 100% increase (or vice versa) takes you back to the same spot.

The percentage change in Y at period t is defined as $(Y_t-Y_{t-1})/Y_{t-1}$, which is only *approximately* equal to $LN(Y_t) - LN(Y_{t-1})$, but the approximation is almost exact if the percentage change is small, as shown in the table above. In Statgraphics terms, this means that DIFF(Y)/LAG(Y,1) is virtually identical to DIFF(LOG(Y)). If you don't believe me, here's a plot of the percent change in auto sales versus the first difference of its logarithm, zooming in on the last 5 years. The blue and red lines are virtually indistinguishable except at the highest and lowest points. (Again, LOG means LN in Statgraphics.)



If the situation is one in which the percentage changes are potentially large enough for this approximation to be inaccurate, it is better to use log units rather than percentage units, because this takes compounding into account in a systematic way, and it is symmetric in terms of sequences of gains and losses. A diff-log of -0.5 followed by a diff-log of +0.5 takes you back to your original position, whereas a 50% loss followed by a 50% gain (or vice versa) leaves you in a worse position.

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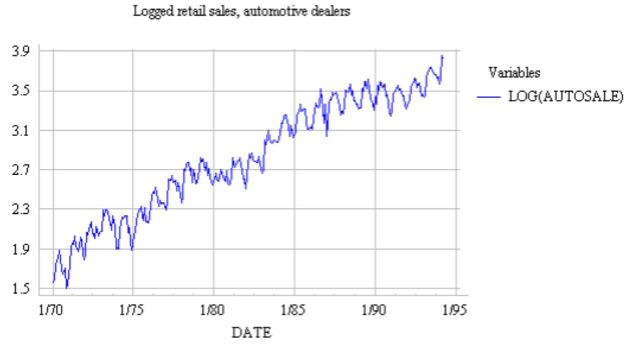
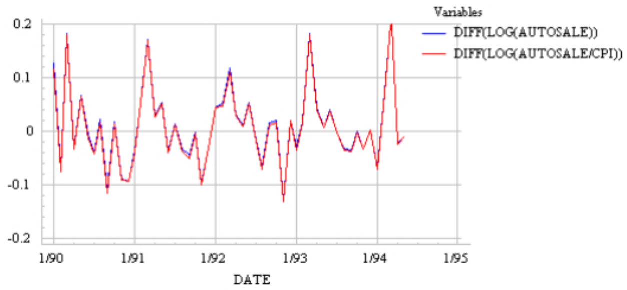
Linearization of exponential growth and inflation: **T**he logarithm of a product equals the sum of the logarithms, i.e., LOG(XY) = LOG(X) + LOG(Y), regardless of the logarithm base. Therefore, logging converts *multiplicative* relationships to *additive* relationships, and by the same token it converts *exponential* (compound growth) trends to *linear* trends. By taking logarithms of variables which are multiplicatively related and/or growing exponentially over time, we can often explain their behavior with linear models. For example, here is a graph of LOG(AUTOSALE). Notice that **the log transformation converts the *exponential* growth pattern to a *linear* growth pattern, and it simultaneously converts the *multiplicative* (proportional-variance) seasonal pattern to an *additive* (constant-variance) seasonal pattern.** (Compare this with the original graph of AUTOSALE.) These conversions make the transformed data much more suitable for fitting with linear/additive models.



Logging a series often has an effect very similar to deflating: it straightens out exponential growth patterns and reduces heteroscedasticity (i.e., stabilizes variance). Logging is therefore a **"poor man's deflator"** which does not require any external data (or any head-scratching about which price index to use). Logging is not *exactly* the same as deflating--it does not *eliminate* an upward trend in the data--but it can straighten the trend out so that it can be better fitted by a linear model. Deflation by itself will not straighten out an exponential growth curve if the growth is partly real and only partly due to inflation.

If you're going to log the data and then fit a model that implicitly or explicitly uses *differencing* (e.g., a random walk, exponential smoothing, or ARIMA model), then it is usually redundant to deflate by a price index, as long as the rate of inflation changes only slowly: the percentage change measured in nominal dollars will be nearly the same as the percentage change in constant dollars. In Statgraphics notation, this means that, DIFF(LOG(Y/CPI)) is nearly identical to DIFF(LOG(Y)): the only difference between the two is a very faint amount of noise due to fluctuations in the inflation rate. To demonstrate this point, here's a graph of the first difference of logged auto sales, with and without deflation:

Percent change in nominal and constant dollars: nearly identical



By logging *rather* than deflating, you avoid the need to incorporate an *explicit* forecast of future inflation into the model: you merely lump inflation together with any other sources of steady compound growth in the original data. Logging the data before fitting a random walk model yields a so-called geometric random walk--i.e., a random walk with geometric rather than linear growth. A geometric random walk is the default forecasting model that is commonly used for stock price data. (Return to top of page.)

Trend measured in natural-log units ≈ percentage growth: Because changes in the natural logarithm are (almost) equal to *percentage* changes in the original series, it follows that the slope of a trend line fitted to logged data is equal to the average *percentage* growth in the original series. For example, in the graph of LOG(AUTOSALE) shown above, if you "eyeball" a trend line you will see that the magnitude of logged auto sales increases by about 2.5 (from 1.5 to 4.0) over 25 years, which is an average increase of about 0.1 per year, i.e., 10% per year. It is much easier to estimate this trend from the logged graph than from the original unlogged one! The 10% figure obtained here is *nominal* growth, including inflation. If we had instead eyeballed a trend line on a plot of logged *deflated* sales, i.e., LOG(AUTOSALE/CPI), its slope would be the average *real* percentage growth.

Usually the trend is estimated more precisely by fitting a statistical model that explicitly includes a local or global trend parameter, such as a linear trend or random-walk-with-drift or linear exponential smoothing model. When a model of this kind is fitted in conjunction with a log transformation, its trend parameter can be interpreted as a percentage growth rate.

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Errors measured in natural-log units ≈ percentage errors: Another interesting property of the logarithm is that errors in predicting the logged series can be interpreted as approximate percentage errors in predicting the original series, albeit the percentages are relative to the forecast values, not the actual values. (Normally one interprets the "percentage error" to be the error expressed as a percentage of the actual value, not the forecast value, although the statistical properties of percentage errors are usually very similar regardless of whether the percentages are calculated relative to actual values or forecasts.)

Thus, if you use least-squares estimation to fit a linear forecasting model to *logged* data, you are implicitly minimizing mean squared *percentage* error, rather than mean squared error in the original units, which is probably a good thing if the log transformation was appropriate in the first place. And if you look at the error statistics in logged units, you can interpret them as percentages if they are not too large--say, if their standard deviation is 0.1 or less. Within this range, the standard deviation of the errors in predicting a logged series is approximately the standard deviation of the percentage errors in predicting the original series, and the mean absolute error (MAE) in predicting a logged series is approximately the mean absolute percentage error (MAPE) in predicting the original series. (I am using a benchmark of 0.1 here because at that point a 2 standard deviation variation, the critical value for a 95% confidence interval, would be 0.2, and the correspondence between diff-logs and percentages begins to fall off pretty rapidly beyond that as shown in the table above. If the error standard deviation in logged units is larger than 0.1, you ought to calculate confidence limits in logged units and then un-log their lower and upper values separately by using the EXP function.)

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Coefficients in log-log regressions ≈ proportional percentage changes: In many economic situations (particularly price-demand relationships), the marginal effect of one variable on the expected value of another is linear in terms of *percentage* changes rather than *absolute* changes. In such cases, applying a natural log or diff-log transformation to both dependent and independent variables may be appropriate. This issue will be discussed in more detail in the regression chapter of these notes. In particular, part 3 of the beer sales regression example illustrates an application of the log transformation in modeling the effect of price on demand, including how to use the EXP (exponential) function to "un-log" the forecasts and confidence limits to convert them back into the units of the original data.
