Multinomial Distribution

The formula for k outcomes is

$$p = \frac{n!}{(n_1!)(n_2!)\dots(n_k!)} p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

Note that the binomial distribution is a special case of the multinomial when k = 2.

Multinomial Distribution

Let a set of random variates $X_1, X_2, \dots, X_n$ have a probability function

$$P(X_1 = x_1, \dots, X_n = x_n) = \frac{N!}{\prod_{i=1}^{n} x_i!} \prod_{i=1}^{n} \theta_i^{x_i} \quad (1)$$

where x_i are nonnegative integers such that

$$\sum_{i=1}^{n} x_i = N, \quad (2)$$

and θ_i are constants with $\theta_i > 0$ and

$$\sum_{i=1}^{n} \theta_i = 1. \quad (3)$$

Then the joint distribution of $X_1, \dots, X_n$ is a multinomial distribution and $P(X_1 = x_1, \dots, X_n = x_n)$ is given by the corresponding coefficient of the multinomial series

$$(\theta_1 + \theta_2 + \dots + \theta_n)^N. \quad (4)$$

In the words, if $X_1, X_2, \dots, X_n$ are mutually exclusive events with $P(X_1 = x_1) = \theta_1, \dots, P(X_n = x_n) = \theta_n$. Then the probability that X_1 occurs x_1 times, ..., X_n occurs x_n times is given by

$$P_N(x_1, x_2, \dots, x_n) = \frac{N!}{x_1! \cdots x_n!} \theta_1^{x_1} \cdots \theta_n^{x_n}. \quad (5)$$

(Papoulis 1984, p. 75).

The mean and variance of X_i are

$$\mu_i = N\theta_i \quad (6)$$
$$\sigma_i^2 = N\theta_i(1 - \theta_i). \quad (7)$$

The covariance of X_i and X_j is

$$\sigma_{ij}^2 = -N\theta_i\theta_j. \quad (8)$$

Questions

Problem 1

Suppose a card is drawn randomly from an ordinary deck of playing cards, and then put back in the deck. This exercise is repeated five times. What is the probability of drawing 1 spade, 1 heart, 1 diamond, and 2 clubs?

Solution: To solve this problem, we apply the multinomial formula. We know the following:

- The experiment consists of 5 trials, so $n = 5$.
- The 5 trials produce 1 spade, 1 heart, 1 diamond, and 2 clubs; so $n_1 = 1$, $n_2 = 1$, $n_3 = 1$, and $n_4 = 2$.
- On any particular trial, the probability of drawing a spade, heart, diamond, or club is 0.25, 0.25, 0.25, and 0.25, respectively. Thus, $p_1 = 0.25$, $p_2 = 0.25$, $p_3 = 0.25$, and $p_4 = 0.25$.

We plug these inputs into the multinomial formula, as shown below:

$$P = [\ n! / (\ n_1! * n_2! * \ldots n_k!\ )\ ] * (\ p_1^{n_1} * p_2^{n_2} * \ldots * p_k^{n_k}\ )$$

$$P = [\ 5! / (\ 1! * 1! * 1! * 2!\ )\ ] * [\ (0.25)^1 * (0.25)^1 * (0.25)^1 * (0.25)^2\ ]$$

$$P = 0.05859$$

Thus, if we draw five cards with replacement from an ordinary deck of playing cards, the probability of drawing 1 spade, 1 heart, 1 diamond, and 2 clubs is 0.05859.

Problem 2

Suppose we have a bowl with 10 marbles - 2 red marbles, 3 green marbles, and 5 blue marbles. We randomly select 4 marbles from the bowl, with replacement. What is the probability of selecting 2 green marbles and 2 blue marbles?

Solution: To solve this problem, we apply the multinomial formula. We know the following:

- The experiment consists of 4 trials, so $n = 4$.
- The 4 trials produce 0 red marbles, 2 green marbles, and 2 blue marbles; so $n_{red} = 0$, $n_{green} = 2$, and $n_{blue} = 2$.
- On any particular trial, the probability of drawing a red, green, or blue marble is 0.2, 0.3, and 0.5, respectively. Thus, $p_{red} = 0.2$, $p_{green} = 0.3$, and $p_{blue} = 0.5$

We plug these inputs into the multinomial formula, as shown below:

$$P = [\ n! / (\ n_1! * n_2! * \ldots n_k!\ )\ ] * (\ p_1^{n_1} * p_2^{n_2} * \ldots * p_k^{n_k}\ )$$

$$P = [\ 4! / (\ 0! * 2! * 2!\ )\ ] * [\ (0.2)^0 * (0.3)^2 * (0.5)^2\ ]$$

$$P = 0.135$$

Thus, if we draw 4 marbles with replacement from the bowl, the probability of drawing 0 red marbles, 2 green marbles, and 2 blue marbles is 0.135.

The binomial distribution allows one to compute the probability of obtaining a given number of binary outcomes. For example, it can be used to compute the probability of getting 6 heads out of 10 coin flips. The flip of a coin is a binary outcome because it has only two possible outcomes: heads and tails. The multinomial distribution can be used to compute the probabilities in situations in which there are more than two possible outcomes. For example, suppose that two chess players had played numerous games and it was determined that the probability that Player A would win is 0.40, the probability that Player B would win is 0.35, and the probability that the game would end in a draw is 0.25. The multinomial distribution can be used to answer questions such as: "If these two chess players played 12 games, what is the probability that Player A would win 7 games, Player B would win 2 games, and the remaining 3 games would be drawn?" The following formula gives the probability of obtaining a specific set of outcomes when there are three possible outcomes for each event:

$$p = \frac{n!}{(n_1!)(n_2!)(n_3!)} p_1^{n_1} p_2^{n_2} p_3^{n_3}$$

where

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p is the probability,
n is the total number of events
n1 is the number of times Outcome 1 occurs,
n2 is the number of times Outcome 2 occurs,
n3 is the number of times Outcome 3 occurs,
p1 is the probability of Outcome 1
p2 is the probability of Outcome 2, and
p3 is the probability of Outcome 3.
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For the chess example,

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n = 12 (12 games are played),
n1 = 7 (number won by Player A),
n2 = 2 (number won by Player B),
n3 = 3 (the number drawn),
p1 = 0.40 (probability Player A wins)
p2 = 0.35(probability Player B wins)
p3 = 0.25(probability of a draw)
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$$p = \frac{12!}{(7!)(2!)(3!)} .40^7 .35^2 .25^3 = 0.0248$$