DRAFT VERSION MARCH 15, 2023
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1. MHD EQUATIONS WITH KRAMERS OPACITY

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0 \tag{2}$$

Dividing by ρ, we have

$$\frac{\partial \ln \rho}{\partial t} + \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln \rho = 0 \tag{3}$$

Momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{-\nabla P}{\rho} - \nabla \phi + \frac{1}{\rho} \nabla \cdot D \tag{4}$$

Here, $\mathbf{g} = -\nabla \phi$, and $D_{ik} = \mu(\partial_i u_k + \partial_k u_i - \frac{2}{3}\partial_l u_l \delta_{ik}) + \xi \partial_l u_l \nabla_{ik}$ is the viscous stress tensor. μ is the dynamic shear viscosity, and ξ is the bulk viscosity.
Transforming pressure to entropy and enthalpy definitions using

$$\nabla h = T \nabla s + \frac{\nabla P}{\rho} \text{ and } h = c_P T$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = T \nabla s - \nabla h - \nabla \phi + \frac{1}{\rho} \nabla \cdot D \tag{5}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla(h + \phi) + \frac{h}{c_P} \nabla s + \frac{1}{\rho} \nabla \cdot D \tag{6}$$

Entropy equation: From second law of thermodynamics, we have

$$\rho T ds = dq = \Phi - \nabla \cdot F \tag{7}$$

Here, dq is the heat transferred to the system, Φ is the viscous heating, and $\nabla \cdot F$ is the diffusive flux.
Sidenote: Second law of thermodynamics states that the entropy of a closed thermodynamic system either remains the same or increases over time.

Using, $h = c_P T$,

$$\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla)s = \frac{1}{\rho h} c_P \Phi - \frac{c_P}{\rho h} \nabla \cdot F \tag{8}$$

The generic form of F for diffusive fluxes is:

$$F = -K \nabla T \tag{9}$$

Here, in our case, with Kramers opacity, the radiative conductivity, K, is given by,

$$K(\rho, T) = \frac{16 \sigma_{SB} T^3}{3 \kappa \rho} \tag{10}$$

The opacity, κ, is given by,

$$\kappa = \kappa_0 \rho^a T^b \tag{11}$$

Here, a and b are free parameters. We then have,

$$K(\rho, T) = \frac{16\sigma_{SB} T^{3-b}}{3\kappa_0 \rho^{1+a}} \tag{12}$$

Using, 9 and $h = c_P T$ in 8, we have

$$\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla)s = \frac{1}{\rho h} c_P \Phi + \frac{1}{\rho h} \nabla \cdot (K \nabla h) \tag{13}$$

Let's focus on the diffusive term and see if we can simplify it,

$$\frac{1}{h} \nabla \cdot (K \nabla h) = \frac{1}{h} [(\boldsymbol{\nabla} K) \cdot \boldsymbol{\nabla} h + K \nabla^2 h] \tag{14}$$

$$= \boldsymbol{\nabla} K \cdot \boldsymbol{\nabla} \ln h + K \frac{\nabla^2 h}{h} \tag{15}$$

$$= \boldsymbol{\nabla} K \cdot \boldsymbol{\nabla} \ln h + K \nabla \cdot \left(\frac{\boldsymbol{\nabla} h}{h} \right) + K \left(\frac{\boldsymbol{\nabla} h}{h} \right)^2 \tag{16}$$

In $\ln h$ formalism, we have

$$\frac{1}{h} \nabla \cdot (K \nabla h) = \boldsymbol{\nabla} K \cdot \boldsymbol{\nabla} \ln h + K \nabla^2 \ln h + K (\boldsymbol{\nabla} \ln h)^2 \tag{17}$$

Finally, the entropy equation 13 can be written as,

$$\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})s = \frac{1}{\rho h} c_P \Phi + \frac{1}{\rho} (\boldsymbol{\nabla} K \cdot \boldsymbol{\nabla} \ln h + K \nabla^2 \ln h + K (\boldsymbol{\nabla} \ln h)^2) \tag{18}$$

Further simplifying the above equation,

$$\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})s = \frac{1}{\rho h} c_P \Phi + \frac{K}{\rho} (\boldsymbol{\nabla} \ln K \cdot \boldsymbol{\nabla} \ln h + \nabla^2 \ln h + (\boldsymbol{\nabla} \ln h)^2) \tag{19}$$

Rewriting the final set of equations here,

$$\frac{\partial \ln \rho}{\partial t} + \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln \rho = 0 \tag{20}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}(h + \phi) + \frac{h}{c_P} \nabla s + \frac{1}{\rho} \nabla \cdot D \tag{21}$$

$$\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = \frac{1}{\rho h} c_P \Phi + \frac{K}{\rho} (\boldsymbol{\nabla} \ln K \cdot \boldsymbol{\nabla} \ln h + \nabla^2 \ln h + (\boldsymbol{\nabla} \ln h)^2) \tag{22}$$

Here,

$s =$ entropy
$h =$ enthalpy
$c_P =$ specific heat at constant pressure
$\Phi =$ viscous heating
$K =$ radiative conductivity
$\phi =$ gravitational potential
$D =$ viscous stress tensor

Let's non-dimensionalize these equations,

$$\bar{\rho} = \frac{\rho}{\rho_0},\ \bar{\mathbf{u}} = \frac{\mathbf{u}}{\mathbf{u_0}},\ \bar{x} = \frac{x}{L},\ \bar{t} = \frac{t}{T},\ \bar{h} = \frac{h}{h_0},\ \bar{s} = \frac{s}{s_0} \tag{23}$$

Here, the variables with the bar sign are non-dimensional. Subscripts 0 represent characteristic values of the quantities for the given system, L is the typical length scale, and T is the typical timescale.

Let's focus on equation 20. Note that while substituting the variables from equation 23, I'd be dropping the bar signs for convenience and clarity in writing.

$$\frac{1}{T}\frac{\partial \ln\rho}{\partial t} + \frac{u_0}{L}\boldsymbol{\nabla}\cdot\mathbf{u} + \frac{u_0}{L}\mathbf{u}\cdot\boldsymbol{\nabla}\ln\rho = 0 \tag{24}$$

For the characteristic velocity, time and length scales, we have the following relation, $L = u_0 T$. Also an important point to note is that while non-dimensionalizing $\partial\ln\rho$, we don't need to have ρ's hanging around. The reason being derivative of $\ln\rho$ is of the form $\partial\rho/\rho$. Using all this information, we have, in non-dimensional form,

$$\partial_t \ln\rho + \nabla\cdot\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\ln\rho = 0 \tag{25}$$

Now, for equation 21, we have

$$\frac{u_0}{T}\frac{\partial\mathbf{u}}{\partial t} + \frac{c_0^2}{L}(\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{L}\nabla(h_0 h + \phi_0\phi) + \frac{h_0}{c_P L}s_0 h\boldsymbol{\nabla}s + \frac{1}{\rho_0\rho}\frac{1}{L}\frac{\mu u_0}{L}\nabla\cdot D \tag{26}$$

$$\frac{u_0^2}{L}\Big(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) = -\frac{h_0}{L}\nabla\Big(h + \frac{\phi_0}{h_o}\phi\Big) + \frac{h_0}{L}\frac{s_0}{c_P}h\boldsymbol{\nabla}s + \frac{\nu u_0}{L^2}\frac{1}{\rho}\nabla\cdot D \tag{27}$$

Here, we have used $\nu = \mu/\rho_0$, where ν is the kinematic viscosity, and μ is the dynamic shear viscosity.

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{h_0}{u_0^2}\nabla\Big(h + \frac{\phi_0}{h_o}\phi\Big) + \frac{h_0}{u_0^2}\frac{s_0}{c_P}h\boldsymbol{\nabla}s + \frac{1}{Lu_0/\nu}\frac{1}{\rho}\nabla\cdot D \tag{28}$$

Finally, the above equation simplifies to,

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\mathrm{Ma}^2}\nabla\Big(h + \frac{\phi_0}{h_o}\phi\Big) + \frac{1}{\mathrm{Ma}^2}h\boldsymbol{\nabla}s + \frac{1}{\Re}\frac{1}{\rho}\nabla\cdot D \tag{29}$$

Here, $\mathrm{Ma}^2 = \dfrac{u_0^2}{h_0}$, $\Re = \dfrac{Lu_0}{\nu}$ and $s \equiv c_P$. Let's focus to non-dimensionalize the entropy equation 30 now,

$$\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla s = \frac{1}{\rho h}c_P\Phi + \frac{K}{\rho}(\boldsymbol{\nabla}\ln K\cdot\boldsymbol{\nabla}\ln h + \nabla^2\ln h + (\boldsymbol{\nabla}\ln h)^2) \tag{30}$$

$$\frac{s_0}{T}\frac{\partial s}{\partial t} + \frac{u_0 s_0}{L}(\mathbf{u}\cdot\nabla)s = \frac{\mu}{\rho_0 h_0}\frac{u_0^2}{L^2}\frac{1}{\rho h}c_P\Phi + \frac{K}{\rho_0 L^2}\frac{1}{\rho}(\boldsymbol{\nabla}\ln K\cdot\boldsymbol{\nabla}\ln h + \nabla^2\ln h + (\boldsymbol{\nabla}\ln h)^2) \tag{31}$$

Here, the viscous heating has the dimensions, $[\Phi] = \mu u_0^2/L^2$. Refer to the defintion of the viscous stress tensor, D.

$$\frac{u_0 s_0}{L}\Big(\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla s\Big) = \frac{u_0^2}{\rho_0 h_0 L^2}\frac{\mu}{\rho h}c_P\Phi + \frac{K}{\rho_0 L^2}\frac{1}{\rho}(\boldsymbol{\nabla}\ln K\cdot\boldsymbol{\nabla}\ln h + \nabla^2\ln h + (\boldsymbol{\nabla}\ln h)^2) \tag{32}$$

$$\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla s = \Big(\frac{u_0^2}{h_0}\Big)\Big(\frac{c_P}{s_0}\Big)\Big(\frac{\mu}{\rho_0 u_0 L}\Big)\frac{1}{\rho h}c_P\Phi + \Big(\frac{K}{L\rho_0 u_0 s_0}\Big)\frac{1}{\rho}(\boldsymbol{\nabla}\ln K\cdot\boldsymbol{\nabla}\ln h + \nabla^2\ln h + (\boldsymbol{\nabla}\ln h)^2) \tag{33}$$

In the last term on the right-hand side, we multiply and divide by μc_P,

$$\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla s = \Big(\frac{u_0^2}{h_0}\Big)\Big(\frac{c_P}{s_0}\Big)\Big(\frac{\mu/\rho_0}{u_0 L}\Big)\frac{1}{\rho h}\Phi + \Big(\frac{K}{\mu c_P}\Big)\Big(\frac{c_P}{s_0}\Big)\Big(\frac{\mu/\rho_0}{Lu_0}\Big)\frac{1}{\rho}(\boldsymbol{\nabla}\ln K\cdot\boldsymbol{\nabla}\ln h + \nabla^2\ln h + (\boldsymbol{\nabla}\ln h)^2) \tag{34}$$

Here, we have, $\mathrm{Ma}^2 = \dfrac{u_0^2}{h_0}$, $\Re = \dfrac{Lu_0}{\nu}$, $s \equiv c_P$, and $Pr = \dfrac{\mu c_P}{K}$. The above equation can then be rewritten as,

$$\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla s = \frac{\mathrm{Ma}^2}{\Re}\frac{1}{\rho h}\Phi + \frac{1}{\Re\mathrm{Pr}}\frac{1}{\rho}(\boldsymbol{\nabla}\ln K\cdot\boldsymbol{\nabla}\ln h + \nabla^2\ln h + (\boldsymbol{\nabla}\ln h)^2) \tag{35}$$

The final set of non-dimensional equations are

$$\partial_t \ln \rho + \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \ln \rho = 0 \tag{36}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\mathrm{Ma}^2} \nabla \Big(h + \frac{\phi_0}{h_o} \phi \Big) + \frac{1}{\mathrm{Ma}^2} h \boldsymbol{\nabla} s + \frac{1}{\Re} \frac{1}{\rho} \nabla \cdot D \tag{37}$$

$$\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = \frac{\mathrm{Ma}^2}{\Re} \frac{1}{\rho h} \Phi + \frac{1}{\Re \mathrm{Pr}} \frac{1}{\rho} (\boldsymbol{\nabla} \ln K \cdot \boldsymbol{\nabla} \ln h + \nabla^2 \ln h + (\boldsymbol{\nabla} \ln h)^2) \tag{38}$$