Double Pendulum: Theory

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June 17, 2017

1 Introduction

As it turns out, the double pendulum is pretty complex. For now, I am modeling a double pendulum consisting of two identical sticks, one attached to a fixed anchor and the other attached to the other end of the anchored stick. Some research divulged that Lagrangian mechanics would yield an easier way to understand the motion of this system, so after some research and an MIT online lecture, here is my attempt at using Lagrangian mechanics to describe the behavior of this system:
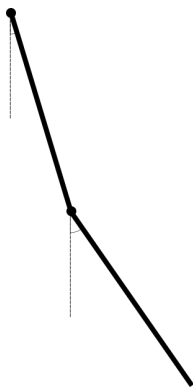
2 Describing the System

Figure 1: A crude image of this double pendulum

Each of the sticks of this pendulum has length ℓ and mass m. θ_1 represents the rotation of the stick attached to the anchor with respect to the vertical, and θ_2 represents the rotation of the other stick, also relative to the vertical. In order to write the Lagrangian for this system, we need to find a way to express its kinetic and potential energies. We will be using θ_1 and θ_2 as the two coordinates that describe this system, so we will work to express everything in terms of θ's.

The two sticks are the only elements of the system that account for its kinetic energy. Each stick has a component of linear and rotational kinetic energy, $\frac{1}{2}mv^2$ and $\frac{1}{2}I\dot{\theta}^2$. The rotational energy is with respect to the center of mass of each stick.

In order to calculate the speed of the center of mass of each stick, we must do some math. The cartesian coordinates of the centers of mass of the two sticks are as follows:

$$x_1 = \frac{\ell}{2}\sin(\theta_1)$$

$$y_1 = -\frac{\ell}{2}\cos(\theta_1)$$

$$x_2 = \ell\sin(\theta_1) + \frac{\ell}{2}\sin(\theta_2) = \ell(\sin(\theta_1) + \frac{1}{2}\sin(\theta_2))$$

$$y_2 = -\ell\cos(\theta_1) - \frac{\ell}{2}\cos(\theta_2) = -\ell(\cos(\theta_1) + \frac{1}{2}\cos(\theta_2))$$

In order to find the linear velocity , we can use the good old Pythagorean theorem to figure out that $v = \sqrt{\dot{x}^2 + \dot{y}^2}$, or, even more conveniently, that $v^2 = \dot{x}^2 + \dot{y}^2$ for each stick. Let's do that:

$$v_1^2 = \dot{x_1}^2 + \dot{y_1}^2 = (\dot{\theta}_1\frac{\ell}{2}\cos(\theta_1))^2 + (\dot{\theta}_1\frac{\ell}{2}\sin(\theta_1))^2 = \dot{\theta}_1^2\frac{\ell^2}{4}\cdot(\cos^2(\theta_1) + \sin^2(\theta_1)) = \dot{\theta}_1^2\frac{\ell^2}{4}$$

$$v_2^2 = \dot{x_2}^2 + \dot{y_2}^2 = \ell^2(\dot{\theta}_1^2\cos^2(\theta_1) + \frac{\dot{\theta}_2^2}{4}\cos^2(\theta_2) + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1)\cos(\theta_2)$$

$$+\ell^2(\dot{\theta}_1^2\sin^2(\theta_1) + \frac{\dot{\theta}_2^2}{4}\sin^2(\theta_2) + \dot{\theta}_1\dot{\theta}_2\sin(\theta_1)\sin(\theta_2))$$

$$= \ell^2(\dot{\theta}_1^2 + \frac{\dot{\theta}_2^2}{4} + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$v_1^2 + v_2^2 = \ell^2(\frac{5}{4}\dot{\theta}_1^2 + \frac{1}{4}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$K_{linear} = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{1}{2}m(v_1^2 + v_2^2) = \frac{1}{2}m\ell^2(\frac{5}{4}\dot{\theta}_1^2 + \frac{1}{4}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$= \frac{1}{8}m\ell^2(5\dot{\theta}_1^2 + \dot{\theta}_2^2 + 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

Because out selected independent coordinates describing the system are θ_1 and θ_2, describing the rotational kinetic energy of the system is much more simple. For each stick, $K_{rot} = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}I(\dot{\theta}_1^2 + \dot{\theta}_2^2)$. Each stick's moment of inertia about its center of mass is $\frac{1}{12}m\ell^2$, so $K_{rot} = \frac{1}{24}m\ell^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$.

We can now find the total kinetic energy of the system:

$$K = K_{linear} + K_{rot} = \frac{1}{8}m\ell^2(5\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + \frac{1}{24}m\ell^2(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})$$

$$= \frac{1}{24}m\ell^2(15\dot{\theta}_1^{\,2} + 3\dot{\theta}_2^{\,2} + 12\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})$$

$$= \frac{1}{24}m\ell^2(16\dot{\theta}_1^{\,2} + 4\dot{\theta}_2^{\,2} + 12\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$= \frac{1}{6}m\ell^2(4\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

In order to find the Lagrangian of the system, we must also describe the potential energy of the system at any given time. Gravitational potential energy accounts for the potential energy in this system, so let us describe the gravitational potential energy of the system with respect to the bottom position, where $\theta_1 = \theta_2 = 0$:

$$U_g = \sum mg\Delta h = mg(\Delta h_1 + \Delta h_2) = mg(y_1 + y_2)$$

$$= mg(-\frac{\ell}{2}\cos(\theta_1) - \ell(\cos(\theta_1) + \frac{1}{2}\cos(\theta_2)))$$

$$= mg\ell(-\frac{3}{2}\cos(\theta_1) - \frac{1}{2}\cos(\theta_2)) = -\frac{1}{2}mg\ell(3\cos(\theta_1) + \cos(\theta_2))$$

Now that we have calculated the kinetic and potential energies of the system, we can calculate the Lagrangian, $\mathcal{L} = T - V$, where T is the total kinetic energy and V is the total potential energy:

$$\mathcal{L} = T - V = \frac{1}{6}m\ell^2(4\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + \frac{1}{2}mg\ell(3\cos(\theta_1) + \cos(\theta_2))$$

With the Lagrangian, we can set up the Euler-Lagrange differential equation for each coordinate:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_i} - \frac{\partial\mathcal{L}}{\partial\theta_i} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = \frac{\partial}{\partial\dot{\theta}_1}(\frac{1}{6}m\ell^2(4\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + \frac{1}{2}mg\ell(3\cos(\theta_1) + \cos(\theta_2)))$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = \frac{1}{6}m\ell^2(8\dot{\theta}_1 + 3\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = \frac{d}{dt}(\frac{1}{6}m\ell^2(8\dot{\theta}_1 + 3\dot{\theta}_2\cos(\theta_1 - \theta_2))) = \frac{1}{6}m\ell^2(8\ddot{\theta}_1 + 3\ddot{\theta}_2\cos(\theta_1 - \theta_2) - 3\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2))$$

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = \frac{\partial}{\partial\theta_1}(\frac{1}{6}m\ell^2(4\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + \frac{1}{2}mg\ell(3\cos(\theta_1) + \cos(\theta_2)))$$

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -\frac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - \frac{3}{2}mg\ell\sin(\theta_1)$$

$$\frac{1}{6}m\ell^2(8\ddot{\theta}_1 + 3\ddot{\theta}_2\cos(\theta_1 - \theta_2) - 3\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)) + \frac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + \frac{3}{2}mg\ell\sin(\theta_1) = 0$$

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$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = \frac{\partial}{\partial \dot{\theta}_2}(\frac{1}{6}m\ell^2(4\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + \frac{1}{2}mg\ell(3\cos(\theta_1) + \cos(\theta_2)))$$

$$= \frac{1}{6}m\ell^2(2\dot{\theta}_2 + 3\dot{\theta}_1\cos(\theta_1 - \theta_2))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{6}m\ell^2(2\dot{\theta}_2 + 3\dot{\theta}_1\cos(\theta_1 - \theta_2))) = \frac{1}{6}m\ell^2(2\ddot{\theta}_2 + 3\ddot{\theta}_1\cos(\theta_1 - \theta_2) - 3\dot{\theta}_1\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2))$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{\partial}{\partial \theta_2}(\frac{1}{6}m\ell^2(4\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + \frac{1}{2}mg\ell(3\cos(\theta_1) + \cos(\theta_2)))$$

$$= \frac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - \frac{1}{2}mg\ell\sin(\theta_2)$$

$$\frac{1}{6}m\ell^2(2\ddot{\theta}_2 + 3\ddot{\theta}_1\cos(\theta_1 - \theta_2) - 3\dot{\theta}_1\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)) - \frac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + \frac{1}{2}mg\ell\sin(\theta_2) = 0$$

We can now solve these two differential equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$, which together will be used in our simulation.

$$\ell(8\ddot{\theta}_1 + 3\ddot{\theta}_2\cos(\theta_1 - \theta_2) - 3\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)) + 3\ell\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + 9g\sin(\theta_1) = 0$$

$$\ell(2\ddot{\theta}_2 + 3\ddot{\theta}_1\cos(\theta_1 - \theta_2) - 3\dot{\theta}_1\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)) - 3\ell\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + 3g\sin(\theta_2) = 0$$

$$\text{Let } S_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{3\ell\dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) - 3\ell\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - 9g\sin(\theta_1)}{\ell}$$

$$\text{Let } S_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{3\ell\dot{\theta}_1\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) + 3\ell\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - 3g\sin(\theta_2)}{\ell}$$

$$8\ddot{\theta}_1 + 3\ddot{\theta}_2\cos(\theta_1 - \theta_2) = S_1$$

$$2\ddot{\theta}_2 + 3\ddot{\theta}_1\cos(\theta_1 - \theta_2) = S_2$$

$$\ddot{\theta}_1 = \frac{S_1 - 3\ddot{\theta}_2}{8}$$

$$2\ddot{\theta}_2 + \frac{3}{8}S_1\cos(\theta_1 - \theta_2) - \frac{9}{8}\ddot{\theta}_2\cos(\theta_1 - \theta_2) = S_2$$

$$\ddot{\theta}_2 = \frac{8S_2 - 3S_1\cos(\theta_1 - \theta_2)}{16 - 8\cos(\theta_1 - \theta_2)}$$

$$\ddot{\theta}_2 = \frac{S_2 - 3\ddot{\theta}_1\cos(\theta_1 - \theta_2)}{2}$$

$$\ddot{\theta}_1 = \frac{2S_1 - 3S_2 - 9\ddot{\theta}_1\cos(\theta_1 - \theta_2)}{16}$$

$$\ddot{\theta}_1 + \frac{9}{16}\ddot{\theta}_1\cos(\theta_1 - \theta_2) = \frac{2S_1 - 3S_2}{16}$$

$$\ddot{\theta}_1 = \frac{2S_1 - 3S_2}{16 + 9\cos(\theta_1 - \theta_2)}$$

We now have a pair of second order differential equations that describes the motion of the system. Using the Runge-Kutta method (aka RK4), we can approximate solutions for θ_1 and θ_2 over some time interval. First, let us redefine this system as a single vector function:

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \ \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, \text{ and } \ddot{\theta} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$\ddot{\theta}(\theta,\dot{\theta}) = \begin{bmatrix} \frac{2S_1 - 3S_2}{16 + 9\cos(\theta_1 - \theta_2)} \\ \frac{8S_2 - 3S_1\cos(\theta_1 - \theta_2)}{16 - 8\cos(\theta_1 - \theta_2)} \end{bmatrix}$$

Now we will set up the RK4 method for this system.

$$\text{Let } z = \dot{\theta}(\theta,\dot{\theta}) = \frac{\mathrm{d}\theta}{\mathrm{d}t}, \text{ so } \frac{\mathrm{d}z}{\mathrm{d}t} = \ddot{\theta}$$

$$z = \dot{\theta}$$

$$\dot{z} = \ddot{\theta}(\theta,\dot{\theta})$$

$$\theta_{new} = \theta + \frac{h}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

$$\dot{\theta}_{new} = \dot{\theta} + \frac{h}{6}(l_0 + 2l_1 + 2l_2 + l_3)$$

Let our step be h, and k and l are as follows:

$$k_0 = hz = h\dot{\theta}_0$$

$$l_0 = h\dot{z} = h\ddot{\theta}_0$$

$$k_1 = hz(\theta_0 + \frac{k_0}{2}, \dot{\theta}_0 + \frac{l_0}{2}) = h \cdot (\dot{\theta}_0 + \frac{l_0}{2})$$

$$l_1 = h\dot{z}(\theta_0 + \frac{k_0}{2}, \dot{\theta}_0 + \frac{l_0}{2}) = h \cdot \ddot{\theta}(\theta_0 + \frac{k_0}{2}, \dot{\theta}_0 + \frac{l_0}{2})$$

$$k_2 = hz(\theta_0 + \frac{k_1}{2}, \dot{\theta}_0 + \frac{l_1}{2}) = h \cdot (\dot{\theta}_0 + \frac{l_1}{2})$$

$$l_2 = h\dot{z}(\theta_0 + \frac{k_1}{2}, \dot{\theta}_0 + \frac{l_1}{2}) = h \cdot \ddot{\theta}(\theta_0 + \frac{k_1}{2}, \dot{\theta}_0 + \frac{l_1}{2})$$

$$k_3 = hz(\theta_0 + k_2, \dot{\theta}_0 + l_2) = h \cdot (\dot{\theta}_0 + l_2)$$

$$l_3 = h\dot{z}(\theta_0 + k_2, \dot{\theta}_0 + l_2) = h \cdot \ddot{\theta}(\theta_0 + k_2, \dot{\theta}_0 + l_2)$$

Implementing this in code will get us our simulation. Hopefully.