Estimation

- We are going to be interested of solving e.g. the following estimation problems:
  - 2D homograpy. Given a point set $\mathbf{x}_i$ in $\mathcal{P}^2$ and corresponding points $\mathbf{x}'_i$ in $\mathcal{P}^2$, find the homography h such that $h(\mathbf{x}_i) = \mathbf{x}'_i$.
  - Camera projection. Given a point set $\mathbf{X}_i$ in $\mathcal{P}^3$ and corresponding points $\mathbf{x}_i$ in $\mathcal{P}^2$, find the mapping $\mathcal{P}^3 \rightarrow \mathcal{P}^2$.
  - The fundamental matrix. Given a point set $\mathbf{x}_i$ in one image and corresponding points $\mathbf{x}'_i$ in a second image, find the fundamental matrix $\mathtt{F}$ between the images. The fundamental matrix is a singular 3×3 matrix $\mathtt{F}$ the satisfies $\mathbf{x}'^{\top}_i \mathtt{F} \mathbf{x}_i = 0$ for all i.

Basic questions

- What is required for an exact, unique solutiion, i.e. how may corresponding points $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ are needed?
  A homography $\mathtt{H}$ has 8 degrees of freedom. Each point pair gives 2 independet equations $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$. Thus we need at least 4 points for an exact solution.
- How can we use more data to improve the solution? What is meant by "better"? We need to define a metric. What metrics are simple to calculate? Which are theoretically best?
- How do we handle low quality data, i.e. outliers?

Exact solution, the *Direct Linear Transform*

- Study the problem to determine a homography $\mathtt{H} : \mathcal{P}^2 \rightarrow \mathcal{P}^2$ from point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$.
- The transformation is given by $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$. Rewriting this gives $\mathbf{x}'_i \times \mathtt{H}\mathbf{x}_i = \mathbf{0}$, since $\mathbf{x}'_i$ and $\mathtt{H}\mathbf{x}_i$ are parallel vectors in $\mathcal{R}^3$.
- Let $\mathbf{h}^{j\top}$ be the jth row in $\mathtt{H}$ and $\mathbf{x}'_i = (x'_i, y'_i, w'_i)^{\top}$. Then we may write

$$\mathtt{H}\mathbf{x}_i = \begin{bmatrix} \mathbf{h}^{1\top}\mathbf{x}_i \\ \mathbf{h}^{2\top}\mathbf{x}_i \\ \mathbf{h}^{3\top}\mathbf{x}_i \end{bmatrix}$$

and

$$\mathbf{x}'_i \times \mathtt{H}\mathbf{x}_i = \begin{bmatrix} y'_i\mathbf{h}^{3\top}\mathbf{x}_i - w'_i\mathbf{h}^{2\top}\mathbf{x}_i \\ w'_i\mathbf{h}^{1\top}\mathbf{x}_i - x'_i\mathbf{h}^{3\top}\mathbf{x}_i \\ x'_i\mathbf{h}^{2\top}\mathbf{x}_i - y'_i\mathbf{h}^{1\top}\mathbf{x}_i \end{bmatrix}$$

- or

$$\begin{bmatrix} \mathbf{0}^{\top} & -w'_i\mathbf{x}_i^{\top} & y'_i\mathbf{x}_i^{\top} \\ w'_i\mathbf{x}_i^{\top} & \mathbf{0}^{\top} & -x'_i\mathbf{x}_i^{\top} \\ -y'_i\mathbf{x}_i^{\top} & x'_i\mathbf{x}_i^{\top} & \mathbf{0}^{\top} \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \mathbf{0}.$$

- This equation is on the form $\mathtt{A}'_i\mathbf{h} = \mathbf{0}$ where $\mathtt{A}'_i$ is a 3×9 matrix and $\mathbf{h}$ is a 9-vector with the row-wise elements of $\mathtt{H}$.

$$\mathbf{h} = \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix}, \ \mathtt{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}.$$

Exact solution, the *Direct Linear Transfo*

- The equation $\mathtt{A}'_i\mathbf{h} = \mathbf{0}$ is linear in $\mathbf{h}$.
- Each equation $\mathtt{A}'_i\mathbf{h} = \mathbf{0}$ has 2 linearly independent equations, i.e. one row can be removed. Removing the third row gives us

$$\begin{bmatrix} \mathbf{0}^{\top} & -w'_i\mathbf{x}_i^{\top} & y'_i\mathbf{x}_i^{\top} \\ w'_i\mathbf{x}_i^{\top} & \mathbf{0}^{\top} & -x'_i\mathbf{x}_i^{\top} \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \mathbf{0}$$

  or $\mathtt{A}_i\mathbf{h} = \mathbf{0}$, where $\mathtt{A}_i$ is a 2×9 matrix. If any of the points $\mathbf{x}'_i$ is an ideal point, i.e. $w'_i = 0$, another row has to be removed.
- The equation is valid for all homogenous representations $(x'_i, y'_i, w'_i)^{\top}$ of $\mathbf{x}'_i$, e.g. for $w'_i = 1$.
- Each point pair produces 2 equations in the elements of $\mathtt{H}$. With 4 point pairs, the matrix $\mathtt{A}'$ becomes 12×9 and the $\mathtt{A}$ matrix 8×9.
- Both matrices have rank 8, i.e. they have a one-dimensional null-space. The solution $\mathbf{h}$ can be determined from the null-space of $\mathtt{A}$.

LT — Over-determined solution (SVD)

- If we have more than 4 point pairs, the equation $\mathtt{A}\mathbf{h} = \mathbf{0}$ becomes over-determined.
- Without error in the points ("noise"), the rank of $\mathtt{A}$ will still be 8.
- With noise, the rank will be 9 and the only solution of $\mathtt{A}\mathbf{h} = \mathbf{0}$ is $\mathbf{h} = \mathbf{0}$, i.e. undefined.
- One solution of this problem is to add a contraint to $\mathbf{h}$, i.e. $\|\mathbf{h}\| = 1$. In that case the problem becomes

$$\begin{aligned} &\min_{\mathbf{h}} && \|\mathtt{A}\mathbf{h}\| \\ &\text{subject to} && \|\mathbf{h}\| = 1 \end{aligned}$$

or

$$\min_{\mathbf{h}} \frac{\|\mathtt{A}\mathbf{h}\|}{\|\mathbf{h}\|}.$$

DLT — Over-determined solution (SVI

- Study the *singular value decomposition (SVD)* of $\mathtt{A}$,

$$\mathtt{A} = \mathtt{UDV}^\top.$$

The matrix $\mathtt{D}$ is diagonal and contains the non-negative *singular values* of $\mathtt{A}$, sorted in descending order. The matrices $\mathtt{U}$ and $\mathtt{V}$ are orthogonal.

- The solution of the minimization problem

$$\min_{\mathbf{h}} \frac{\|\mathtt{A}\mathbf{h}\|}{\|\mathbf{h}\|}.$$

is the right singular vector $\mathbf{v}_n$ corresponding to the smallest singular value.

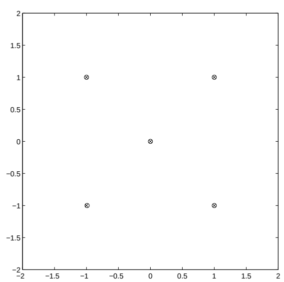
Example 1

For the points

$$\mathbf{x}_i = \begin{bmatrix} -1 & -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{x}'_i = \begin{bmatrix} -0.99 & -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



we get

$$\mathtt{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 0 & 0 & 0 & -0.99 & -0.99 & 0.99 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \text{ and } \mathbf{h} = \begin{bmatrix} 0.997 \\ -0.003 \\ 0.002 \\ -0.000 \\ 1.000 \\ -0.002 \\ -0.000 \\ -0.002 \\ 1 \end{bmatrix}$$

or

$$\mathtt{H} = \begin{bmatrix} 0.997 & -0.003 & 0.002 \\ -0.000 & 1.000 & -0.002 \\ -0.000 & -0.002 & 1 \end{bmatrix}.$$

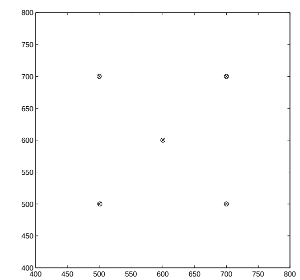
Example 2

For the points

$$\mathbf{x}_i = \begin{bmatrix} 500 & 500 & 600 & 700 & 700 \\ 500 & 700 & 600 & 500 & 700 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{x}'_i = \begin{bmatrix} 501 & 500 & 600 & 700 & 700 \\ 500 & 700 & 600 & 500 & 700 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



we get

$$\mathtt{A} = \begin{bmatrix} 0 & 0 & 0 & -500 & -500 & -1 & 250000 & 250000 & 500 \\ 500 & 500 & 1 & 0 & 0 & 0 & -250500 & -250500 & -501 \\ 0 & 0 & 0 & -500 & -700 & -1 & 350000 & 490000 & 700 \\ 500 & 700 & 1 & 0 & 0 & 0 & -250000 & -350000 & -500 \\ 0 & 0 & 0 & -600 & -600 & -1 & 360000 & 360000 & 600 \\ 600 & 600 & 1 & 0 & 0 & 0 & -360000 & -360000 & -600 \\ 0 & 0 & 0 & -700 & -500 & -1 & 350000 & 250000 & 500 \\ 700 & 500 & 1 & 0 & 0 & 0 & -490000 & -350000 & -700 \\ 0 & 0 & 0 & -700 & -700 & -1 & 490000 & 490000 & 700 \\ 700 & 700 & 1 & 0 & 0 & 0 & -490000 & -490000 & -700 \end{bmatrix} \text{ and } \mathbf{h} = \begin{bmatrix} 0.970 \\ -0.018 \\ 16.030 \\ -0.006 \\ 0.963 \\ 12.741 \\ -0.000 \\ -0.000 \\ 1.000 \end{bmatrix}$$

or

$$\mathtt{H} = \begin{bmatrix} 0.970 & -0.018 & 16.030 \\ -0.006 & 0.963 & 12.741 \\ -0.000 & -0.000 & 1.000 \end{bmatrix}.$$

Inhomogenous solution

- If we can fix of the elements in $\mathbf{h}$ we can remove that element and solve for the 8 remaining.
- If we e.g. assume that $\mathbf{h}_9 = \mathtt{H}_{33} = 1$ the point equations become

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w_i' & -y_i w_i' & -w_i w_i' & x_i y_i' & y_i y_i' \\ x_i w_i' & y_i w_i' & w_i w_i' & 0 & 0 & 0 & -x_i x_i' & -y_i x_i' \end{bmatrix} \tilde{\mathbf{h}} = \begin{bmatrix} -w_i y_i' \\ w_i x_i' \end{bmatrix},$$

where $\tilde{\mathbf{h}}$ contains the first 8 elements in $\mathbf{h}$.

- With 4 point pairs we get an equation $\mathtt{M}\tilde{\mathbf{h}} = \mathbf{b}$, where $\mathtt{M}$ is 8×8, that can be solved exactly.
- With more that 4 point pair we may solve

$$\min_{\tilde{\mathbf{h}}} ||\mathtt{M}\tilde{\mathbf{h}} - \mathbf{b}||$$

with a least squares method.

- Observe that this method works poorly if the correct solution has $\mathtt{H}_{33} = 0$.

Solutions from lines and points

- A line correspondence $\mathbf{l}_i \leftrightarrow \mathbf{l}_i'$ also gives 2 equations in the elements in $\mathtt{H}$ so a similar problem may be formulated from e.g. 4 line pairs or 2 point pairs and 2 line pairs.

Algebraic distance

- The DLT algorithm minimizes $||\epsilon|| = ||\mathtt{A}\mathbf{h}||$. Each point pair $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ contributes with an error vector ϵ_i that is called the *algebraic error vector* associated with the point pair $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ and the homography $\mathtt{H}$. The norm of ϵ_i is called the *algebraic distance* d_{alg} and is

$$d_{\text{alg}}(\mathbf{x}_i', \mathtt{H}\mathbf{x}_i)^2 = ||\epsilon_i||^2 = \left|\left| \begin{bmatrix} \mathbf{0}^\top & -w_i'\mathbf{x}_i^\top & y_i'\mathbf{x}_i^\top \\ w_i'\mathbf{x}_i^\top & \mathbf{0}^\top & -x_i'\mathbf{x}_i^\top \end{bmatrix} \mathbf{h} \right|\right|^2.$$

- In general, d_{alg} för två vektorer $\mathbf{x}_1$ och $\mathbf{x}_2$ is defined as

$$d_{\text{alg}}(\mathbf{x}_1, \mathbf{x}_2)^2 = a_1^2 + a_2^2 \text{ where } \mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T = \mathbf{x}_1 \times \mathbf{x}_2.$$

- Given a set of point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ the total error becomes

$$||\mathtt{A}\mathbf{h}||^2 = ||\epsilon||^2 = \sum_i ||\epsilon_i||^2 = \sum_i d_{\text{alg}}(\mathbf{x}_i', \mathtt{H}\mathbf{x}_i)^2.$$

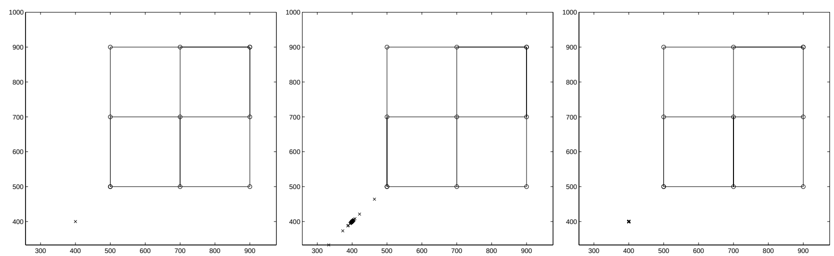
- The algebraic distance is easy to minimize, but is difficult to interpret geometrically. Furthermore it is transformation dependent and calculations based on the algebraic distance should be normalized.

Normalized DLT for 2D homographies

- Given $n \geq 4$ point pairs $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the homography $\mathtt{H}$ such that $\mathbf{x}_i' = \mathtt{H}\mathbf{x}_i$.
  - Determine the similarity transformation $\mathtt{T}$ such that the points $\{\tilde{\mathbf{x}}_i = \mathtt{T}\mathbf{x}_i\}$ have a center of gravity at the origin and a mean distance of $\sqrt{2}$ to the origin.
  - Determine the similarity transformation $\mathtt{T}'$ such that the points $\{\tilde{\mathbf{x}}_i' = \mathtt{T}'\mathbf{x}_i'\}$ have a center of gravity at the origin and a mean distance of $\sqrt{2}$ to the origin.
  - Determine the homography $\tilde{\mathtt{H}}$ for the point correspondences $\{\tilde{\mathbf{x}}_i \leftrightarrow \tilde{\mathbf{x}}_i'\}$.
  - Re-normalize such that $\mathtt{H} = \mathtt{T}'^{-1}\tilde{\mathtt{H}}\mathtt{T}$.

Pertubation sensitivity

- Assume we want to use a grid pattern $(x, y) \in [500, 900]$ as a reference coordinate system for the transformation between two images. Assume the images are approximately the same, i.e. $\mathtt{H} \approx \mathtt{I}$.
- How much will small measurement errors affect the estimation of another point $\mathbf{p} = (400, 400, 1)^\top$?
- Result of 100 monte carlo simulations where $\mathtt{H}$ was determined from point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$, where the points $\mathbf{x}'_i$ were perturbed with white noise of standard deviation σ=0.1 pixels.



Setup | Distribution of $\mathtt{H}\mathbf{p}$ for unnormalized estimation of $\mathtt{H}$ | Distribution of $\mathtt{H}\mathbf{p}$ for normalized estimation of $\mathtt{H}$

Geometrical distance

- We will now study a few error measures based on the geometric distance between measured and estimated point coordinates.
- Use the notation $\mathbf{x}$ for measured coordinate, $\hat{\mathbf{x}}$ for estimated coordinates, and $\bar{\mathbf{x}}$ for the true coordinate for a point. An estimated homographys is denoted $\hat{\mathtt{H}}$.

Errors in one image only

- If we have errors in one image only, an appropriate error measure is the Euclidian distance between the measured points $\mathbf{x}'_i$ and the transformed exact points $\mathtt{H}\bar{\mathbf{x}}_i$. This is called the *transfer* error and is denoted

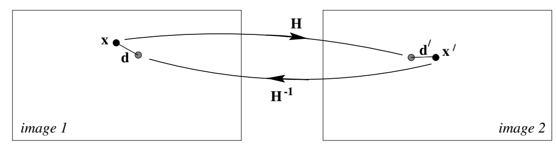
$$\sum_i d(\mathbf{x}'_i, \mathtt{H}\bar{\mathbf{x}}_i)^2,$$

where $d(\mathbf{x}, \mathbf{y})$ is the Euclidian distance between the cartesian points represented by $\mathbf{x}$ and $\mathbf{y}$.

Errors in both images

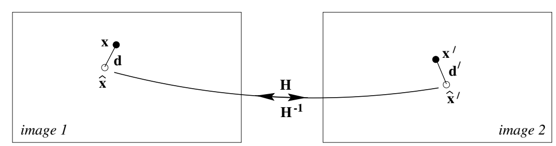
- If we have measurement errors in both images we need to take both errors into account.
- One solution is to sum the geometrical error form the forward transformation $\mathtt{H}$ and the backward transformation $\mathtt{H}^{-1}$. This is called the *symmetric transfer error*

$$\sum_i d(\mathbf{x}_i, \mathtt{H}^{-1}\mathbf{x}'_i)^2 + d(\mathbf{x}'_i, \mathtt{H}\mathbf{x}_i)^2.$$



- An alternate solution is to require a perfect matching and sum the errors in both images. This is called the *reprojection error*

$$\sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}'_i, \hat{\mathbf{x}}'_i)^2 \text{ subject to } \hat{\mathbf{x}}'_i = \mathtt{H}\hat{\mathbf{x}}_i,\ \forall i.$$



Statistical error

- If we assume the measurement error is Guassian distributed with variance σ^2, we may describe the measured coordinates as $x = \bar{x} + \delta x$, where the error δx is normally distributed with variance σ^2.
- Furthermore, if we assume the errors are *independent*, the *probability density function (pdf)* for a point measurement $\mathbf{x}$ given the true point $\bar{\mathbf{x}}$

$$\Pr(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right) e^{-d(\mathbf{x},\bar{\mathbf{x}})^2/(2\sigma^2)}.$$

- In the case of error in one image only we are interested in the probability for observing the correspondences $\{\bar{\mathbf{x}}_i \leftrightarrow \mathbf{x}'_i\}$. If the observations are independent the pdf becomes

$$\Pr(\{\mathbf{x}'_i\}|\mathtt{H}) = \Pi_i \left(\frac{1}{2\pi\sigma^2}\right) e^{-d(\mathbf{x}'_i,\mathtt{H}\bar{\mathbf{x}}_i)^2/(2\sigma^2)},$$

i.e. the probability that we will observe $\{\mathbf{x}'_i\}$ given that $\mathtt{H}$ is the true homography.

Maximum likelihood estimates

- If we take the logarithm we get the *log-likelihood function*

$$\log \Pr(\{\mathbf{x}'_i\}|\mathtt{H}) = -\frac{1}{2\sigma^2}\sum_i d(\mathbf{x}'_i,\mathtt{H}\bar{\mathbf{x}}_i)^2 + c,$$

where c is a constant.

- The *maximum likelihood estimate* (MLE) of the homography, $\hat{\mathtt{H}}$, maximizes the log-likelyhood function and minimizes

$$\sum_i d(\mathbf{x}'_i,\mathtt{H}\bar{\mathbf{x}}_i)^2,$$

i.e. the geometrical transfer error.

- For error in both images we get the pdf for the true correspondences $\{\bar{\mathbf{x}}_i \leftrightarrow \mathtt{H}\bar{\mathbf{x}}_i = \bar{\mathbf{x}}'_i\}$ as

$$\Pr(\{\mathbf{x}_i,\mathbf{x}'_i\}|\mathtt{H},\{\bar{\mathbf{x}}_i\}) = \Pi_i \left(\frac{1}{2\pi\sigma^2}\right) e^{-\left(d(\mathbf{x}_i,\bar{\mathbf{x}}_i)^2 + d(\mathbf{x}'_i,\mathtt{H}\bar{\mathbf{x}}_i)^2\right)/(2\sigma^2)},$$

whose MLE corresponds both of a homography $\hat{\mathtt{H}}$ and point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$ and minimizes

$$\sum_i d(\mathbf{x}_i,\hat{\mathbf{x}}_i)^2 + d(\mathbf{x}'_i,\hat{\mathbf{x}}'_i)^2$$

where $\hat{\mathbf{x}}'_i = \hat{\mathtt{H}}\hat{\mathbf{x}}_i$, i.e. the reprojection error.

Mahalanobis distance

- If we know the covariance matrix Σ for our observations we get the MLE by minimizing the *Mahalanobis distance*

$$\|\mathbf{X} - \bar{\mathbf{X}}\|^2_\Sigma = (\mathbf{X} - \bar{\mathbf{X}})^\top \Sigma^{-1} (\mathbf{X} - \bar{\mathbf{X}}).$$

- If the errors in both images are independent the corresponding error measure becomes

$$\|\mathbf{X} - \bar{\mathbf{X}}\|^2_\Sigma + \|\mathbf{X}' - \bar{\mathbf{X}}'\|^2_{\Sigma'},$$

where Σ and Σ' are the covariance matrices for measurements in the two images.

- A special case is if the measurements are independent but with different variance. Then the covariance matrix Σ becomes diagonal.

Iterative minimization

- To minimize a geometric distance an iterative method is often needed.
- If an inhomogenous formulation is possible a unconstrained algorithm may be used, e.g. Gauss-Newton. Otherwise a constained algorithm, e.g. SQP is the best choice.
- For the transfer error the vector of unknowns is $\mathbf{h}$ and the objective function becomes $\sum_i d(\mathbf{x}'_i,\mathtt{H}\bar{\mathbf{x}}_i)^2$, i.e. the residual function is

$$r(\mathbf{h}) = \begin{bmatrix} r_1(\mathbf{h}) \\ r_2(\mathbf{h}) \\ \vdots \\ r_n(\mathbf{h}) \end{bmatrix}, \text{ where } r_i(\mathbf{h}) = \begin{bmatrix} \frac{h_{11}\bar{x}_i + h_{12}\bar{y}_i + h_{13}\bar{w}_i}{h_{31}\bar{x}_i + h_{32}\bar{y}_i + h_{33}\bar{w}_i} - x'_i \\ \frac{h_{21}\bar{x}_i + h_{22}\bar{y}_i + h_{23}\bar{w}_i}{h_{31}\bar{x}_i + h_{32}\bar{y}_i + h_{33}\bar{w}_i} - y'_i \end{bmatrix}$$

- For a homogenous formulation a normalization constraint on $\mathbf{h}$ is necessary, e.g. $h_{11}^2 + \ldots + h_{33}^2 - 1 = \mathbf{h}^T\mathbf{h} - 1 = 0$.

Iterative minimization

- For the reprojection error we have to estimate $\hat{\mathbf{x}}'_i$ and $\hat{\mathbf{x}}_i$ in addition to $\mathbf{h}$.
- The components of the constaint $\hat{\mathbf{x}}'_i = \mathtt{H}\hat{\mathbf{x}}_i$ has to be normalized. For instance the implicit constraint $\hat{w}_i = 1$ may be used together with $\mathbf{h}^\top \mathbf{h} - 1 = 0$. Then the residual function becomes

$$r_i(\mathbf{h}) = \begin{bmatrix} \hat{x}_i - x_i \\ \hat{y}_i - y_i \\ \frac{\hat{x}'_i}{\hat{w}'_i} - x'_i \\ \frac{\hat{y}'_i}{\hat{w}'_i} - y'_i \end{bmatrix}$$

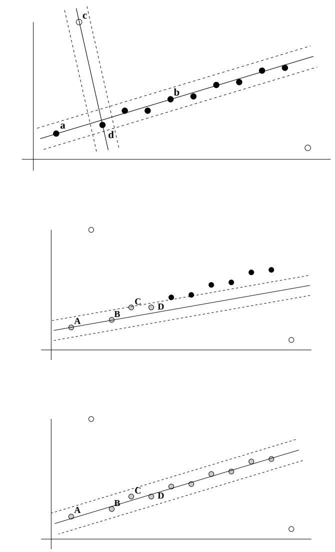
with constraints

$$\mathtt{H} \begin{bmatrix} \hat{x}_i \\ \hat{y}_i \\ 1 \end{bmatrix} - \begin{bmatrix} \hat{x}'_i \\ \hat{y}'_i \\ \hat{w}'_i \end{bmatrix} = 0,$$

$$\mathbf{h}^\top \mathbf{h} - 1 = 0.$$

Robust estimation

- How do we handle observations with large errors (outliers). One way is to use the *Random Sample Consensus* (RANSAC) algorithm.
- Given a model and a data set S containing outliers:
  - Pick randomly s data points from the set S and calculate the model from these points. For a line, pick 2 points.
  - Determine the *consensus set* S_i of s, i.e. the set of points being within t units from the model. The set S_i define the inliers in S.
  - If the number of inliers are larger than a threshold T, recalculate the model based on all points in S_i and terminate.
  - Otherwise repeat with a new random subset.
  - After N tries, choose the largest consensus set S_i, recalculate the model based on all points in S_i and terminate.



How to choose the distance limit t?

- If we assume the distance d from the model is normally distributed with standard deviation σ the limit t may be choosen as $t^2 = F_m^{-1}(\alpha)\sigma^2$, where F_m is the cumulative distiribution function for the χ^2 distribution with m degrees of freedom.
- Such a measurement satisfies $d^2 < t^2$ with probability α.
- A few examples:

Degrees of freedom	Model	t^2
1	line, fundamental matrix	$3.84\sigma^2$
2	homography, camera matrix	$5.99\sigma^2$
3	trifocal tensor	$7.81\sigma^2$

How many samples N?

- The number of samples N should be chosen such that the probability of having picked *at least* one sample without outliers is p.
- Assume w is the probability for an inlier, i.e. $\epsilon = 1 - w$ is the probability for an outlier.
- Then we need at least N samples of s points each, where $(1 - w^s)^N = 1 - p$ or

$$N = \frac{\log 1 - p}{\log(1 - (1 - \epsilon)^s)}.$$

How to choose an acceptable size of the consensus set T?

- A rule of thumb is to terminate if the size of the consensus set is equal to the number of expected inliers in the set, i.e. for n points

$$T = (1 - \epsilon)n.$$

Adaptive RANSAC

- It is possible to estimate T and N dynamically. Given a point set of n points:
  - $N = \infty$, $i = 0$.
  - Repeat while $i < N$
    - Pick a subset of s elements and count the number of inliers k.
    - Let $\epsilon = 1 - k/n$.
    - Let $N = \frac{\log 1-p}{\log(1-(1-\epsilon)^s)}$ for e.g. $p = 0.99$.
    - $i = i + 1$.
- This is called the *adaptive RANSAC algorithm*.

Example