EECS 442 – Computer vision
Conversation hour

- Review of linear algebra
- HW 0.4 discussion
- DLT algorithm

HW: Transformation and eigenvalues

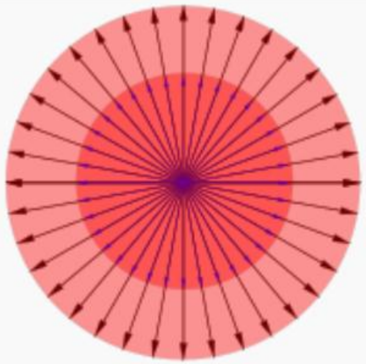
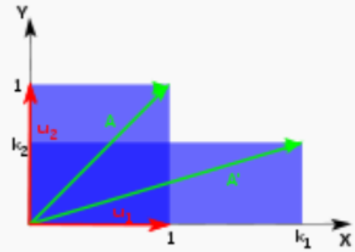
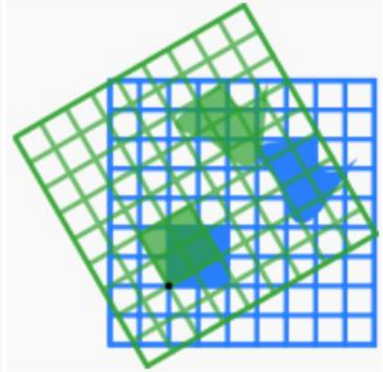
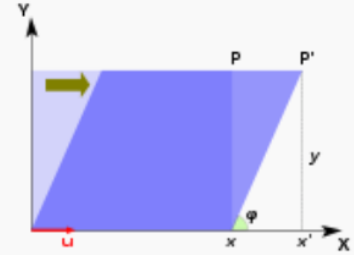
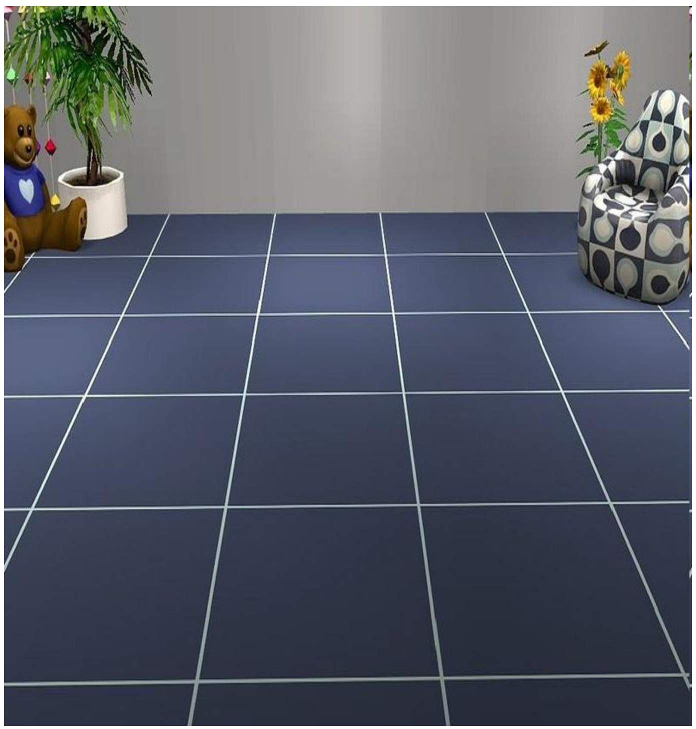
	scaling	unequal scaling	counterclockwise rotation by φ	horizontal shear
illustration				
matrix	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
characteristic equation	$\lambda^2 - 2\lambda k + k^2 = (\lambda - k)^2 = 0$	$(\lambda - k_1)(\lambda - k_2) = 0$	$\lambda^2 - 2\lambda \cos \varphi + 1 = 0$	$\lambda^2 - 2\lambda+1 = (1 - \lambda)^2 = 0$
eigenvalues λ_i	$\lambda_1=k$	$\lambda_1 = k_1, \lambda_2 = k_2$	$\lambda_{1,2} = \cos \varphi \pm i \sin \varphi = e^{\pm i\varphi}$	$\lambda_1=1$
algebraic and geometric multiplicities	$n_1 = 2, m_1 = 2$	$n_1 = m_1 = 1, n_2 = m_2 = 1$	$n_1 = m_1 = 1, n_2 = m_2 = 1$	$n_1 = 2, m_1 = 1$
eigenvectors	$\mathbf{u}_1 = (1,0), \mathbf{u}_2 = (0,1)$	$\mathbf{u}_1 = (1,0), \mathbf{u}_2 = (0,1)$	$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$	$\mathbf{u}_1 = (1,0)$

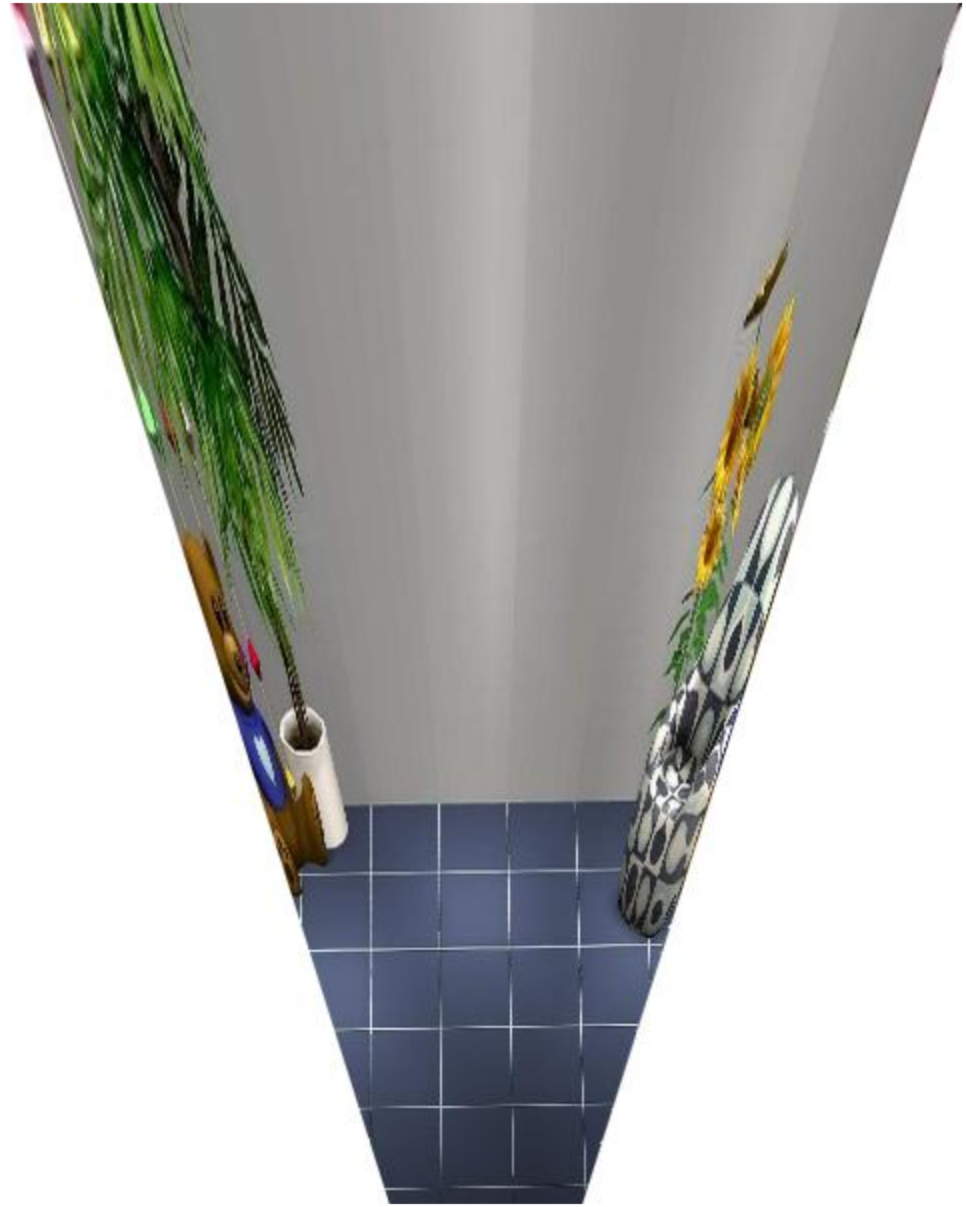
Illustration from wikipedia

Removing perspective distortion

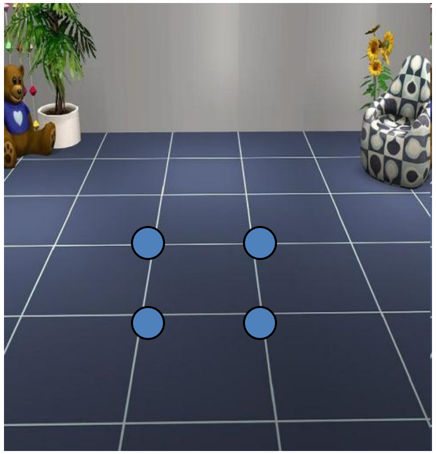
(rectification)

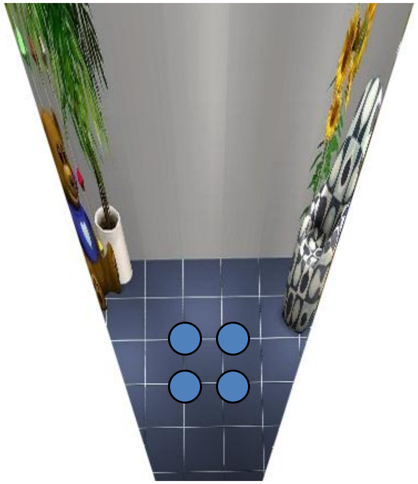
H_p





Computing H_p



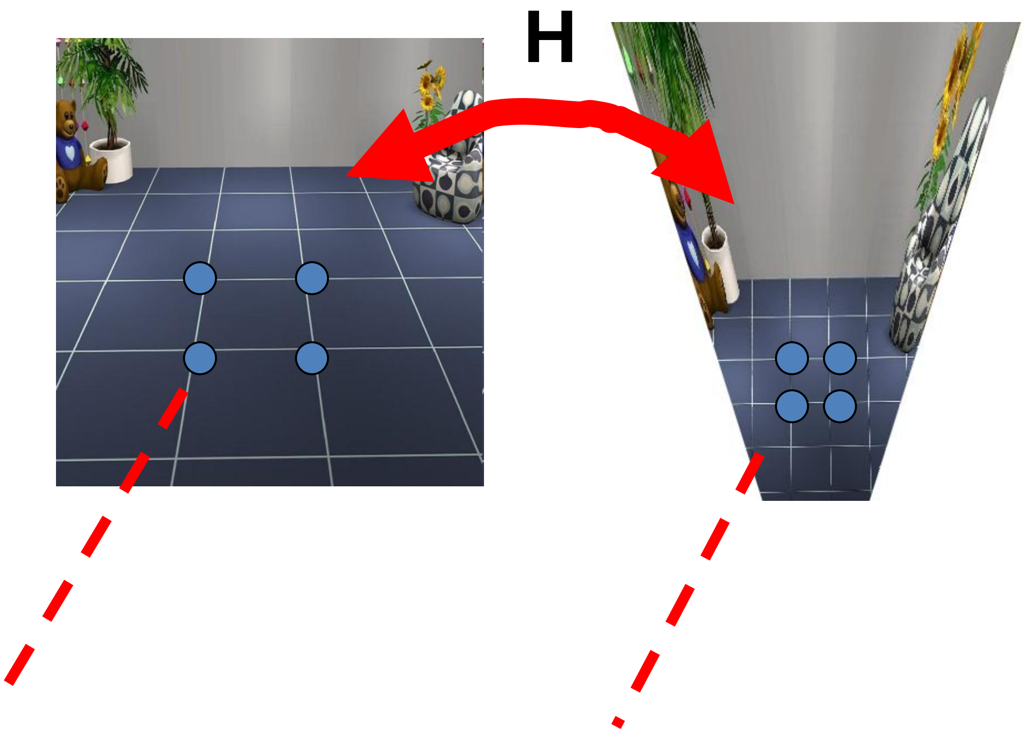


- 8 DOF
- how many points do I need to estimate H_p?

At least 4 points! (8 equations)

- There are several algorithms…

DLT algorithm (Direct Linear Transformation)



$\mathbf{x}_i$

$\mathbf{x}'_i$

$$\mathbf{x}'_i = \mathrm{H}\,\mathbf{x}_i$$

DLT algorithm (direct Linear Transformation)

$$x'_i \times H\, x_i = 0 \longrightarrow \underbrace{A_i}_{\text{Function of measurements}} \overbrace{\mathbf{h}}^{\text{Unknown [9x1]}} = 0$$

[2x9]

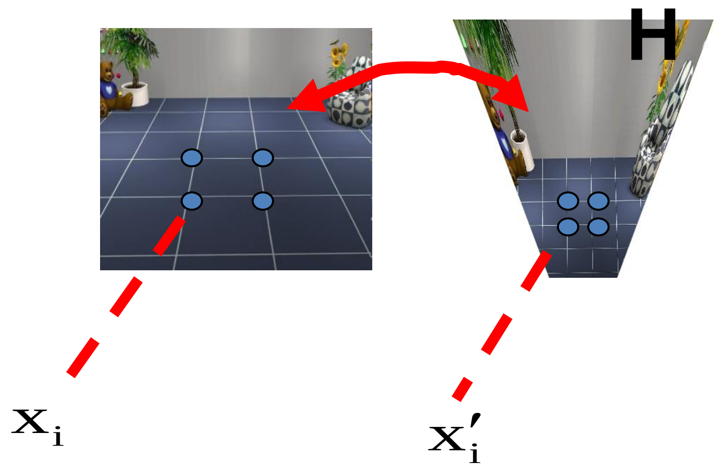
$$\underbrace{\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_9 \end{bmatrix}}_{\text{9x1}}$$

$$\mathtt{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

2 independent equations

DLT algorithm (direct Linear Transformation)

$$\overset{A_{2\text{x}9}}{\boxed{A_i}}\ \overset{h_{9\text{x}1}}{\boxed{h}} = 0$$



$$\begin{cases} A_1\ h = 0 \\ A_2\ h = 0 \\ \\ A_N\ h = 0 \end{cases} \rightarrow \quad A_{2N\times 9}\ h_{9\times 1} = 0$$

Over determined
Homogenous system

DLT algorithm (direct Linear Transformation)

How to solve $$A_{2N\times 9}\, h_{9\times 1} = 0$$?

Singular Value Decomposition (SVD)!

DLT algorithm (direct Linear Transformation)

How to solve $\mathrm{A}_{2\mathrm{N}\times 9}\ \mathrm{h}_{9\times 1} = 0$?

Singular Value Decomposition (SVD)!

$$\downarrow$$

$$U_{2n\times 9}\ \Sigma_{9\times 9} \boxed{V^{T}{}_{9\times 9}}$$

Last column of V gives h! $\rightarrow$ H!

Why? See pag 593 of AZ

DLT algorithm (direct Linear Transformation)

How to solve $$A_{2N\times 9}\, h_{9\times 1} = 0$$?

```
[U,D,V] = svd(A,0);
x = V(:,end);
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Clarification about SVM

$$P_{m\times n} = \boxed{U}_{m\times n} \; D_{n\times n} \; \boxed{V}^{T}_{n\times n}$$

Has n orthogonal columns

Orthogonal matrix

- This is one of the possible SVD decompositions
- This is typically used for efficiency
- The classic SVD is actually:

$$P_{m\times n} = \boxed{U}_{m\times m} \; D_{m\times n} \; \boxed{V}^{T}_{n\times n}$$

orthogonal

Orthogonal

Appendix:
Properties of SVD

Properties of the SVD

- Suppose we know the singular values of **A** and we know *r* are non zero

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \sigma_{r+1} = \dots = \sigma_p = 0$$

  - Rank(**A**) = *r*.
  - Null(**A**) = span$\{\mathbf{v}_{r+1},\dots,\mathbf{v}_n\}$
  - Range(**A**)=span$\{\mathbf{u}_1,\dots,\mathbf{u}_r\}$
- $||\boldsymbol{A}||_F^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_p^2$ $\qquad ||A||_2 = \sigma_1$
- *Numerical rank:* If k singular values of A are larger than a given number ε. Then the ε rank of A is k.
- Distance of a matrix of rank n from being a matrix of rank $k = \sigma_{k+1}$

Why is it useful?

- Square matrix may be singular due to round-off errors. Can compute a "regularized" solution
  - $$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=(\mathbf{U}\,\Sigma\,\mathbf{V}^{\mathbf{t}})^{-1}\mathbf{b} = \sum_{i=1}^{n}\frac{\mathbf{u}_i^t\mathbf{b}}{\sigma_i}\mathbf{v}_i$$
- If σ_i is small (vanishes) the solution "blows up"
- Given a tolerance ε we can determine a solution that is "closest" to the solution of the original equation, but that does not "blow up" $$\mathbf{x}_r = \sum_{i=1}^{k}\frac{\mathbf{u}_i^t\mathbf{b}}{\sigma_i}\mathbf{v}_i \qquad \sigma_k > \varepsilon,\ \ \sigma_{k+1} \le \varepsilon$$
- Least squares solution is the x that satisfies $\mathbf{A}^t\mathbf{A}\mathbf{x}=\mathbf{A}^t\mathbf{b}$
- can be effectively solved using SVD

Appendix:
DLT algorithm (direct Linear Transformation)
From:

Multiple View Geometry in Computer Vision,
by R. Hartley and A. Zisserman, Academic Press, 2002

4.1 The Direct Linear Transformation (DLT) algorithm

We begin with a simple linear algorithm for determining $\mathtt{H}$ given a set of four 2D to 2D point correspondences, $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$. The transformation is given by the equation $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$. Note that this is an equation involving homogeneous vectors; thus the 3-vectors $\mathbf{x}'_i$ and $\mathtt{H}\mathbf{x}_i$ are not equal, they have the same direction but may differ in magnitude by a non-zero scale factor. The equation may be expressed in terms of the vector cross product as $\mathbf{x}'_i \times \mathtt{H}\mathbf{x}_i = \mathbf{0}$. This form will enable a simple linear solution for $\mathtt{H}$ to be derived.

If the j-th row of the matrix $\mathtt{H}$ is denoted by $\mathbf{h}^{j\mathsf{T}}$, then we may write

$$\mathtt{H}\mathbf{x}_i = \begin{pmatrix} \mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \end{pmatrix}.$$

Writing $\mathbf{x}'_i = (x'_i, y'_i, w'_i)^\mathsf{T}$, the cross product may then be given explicitly as

$$\mathbf{x}'_i \times \mathtt{H}\mathbf{x}_i = \begin{pmatrix} y'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i - w'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ w'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i - x'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \\ x'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i - y'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \end{pmatrix}.$$

Since $\mathbf{h}^{j\mathsf{T}}\mathbf{x}_i = \mathbf{x}_i^\mathsf{T}\mathbf{h}^j$ for $j = 1, \ldots, 3$, this gives a set of three equations in the entries of $\mathtt{H}$, which may be written in the form

$$\begin{bmatrix} \mathbf{0}^\mathsf{T} & -w'_i\mathbf{x}_i^\mathsf{T} & y'_i\mathbf{x}_i^\mathsf{T} \\ w'_i\mathbf{x}_i^\mathsf{T} & \mathbf{0}^\mathsf{T} & -x'_i\mathbf{x}_i^\mathsf{T} \\ -y'_i\mathbf{x}_i^\mathsf{T} & x'_i\mathbf{x}_i^\mathsf{T} & \mathbf{0}^\mathsf{T} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = \mathbf{0}. \tag{4.1}$$

These equations have the form $\mathtt{A}_i\mathbf{h} = \mathbf{0}$, where $\mathtt{A}_i$ is a 3×9 matrix, and $\mathbf{h}$ is a 9-vector made up of the entries of the matrix $\mathtt{H}$,

$$\mathbf{h} = \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix}, \qquad \mathtt{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \tag{4.2}$$

with h_i the $i-$th elen[illegible]ions are in order here.

(i) The equation $\mathtt{A}_i\mathbf{h} = \mathbf{0}$ is an equation *linear* in the unknown $\mathbf{h}$. The matrix elements of $\mathtt{A}_i$ are quadratic in the known coordinates of the points.

(ii) Although there are three equations in (4.1), only two of them are linearly independent (since the third row is obtained, up to scale, from the sum of x'_i times the first row and y'_i times the second). Thus each point correspondence gives two equations in the entries of $\mathtt{H}$. It is usual to omit the third equation in solving for $\mathtt{H}$ ([Sutherland-63]). Then (for future reference) the set of equations becomes

$$\begin{bmatrix} \mathbf{0}^\mathsf{T} & -w'_i\mathbf{x}_i^\mathsf{T} & y'_i\mathbf{x}_i^\mathsf{T} \\ w'_i\mathbf{x}_i^\mathsf{T} & \mathbf{0}^\mathsf{T} & -x'_i\mathbf{x}_i^\mathsf{T} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = \mathbf{0}. \tag{4.3}$$

This will be written

$$\mathtt{A}_i\mathbf{h} = \mathbf{0}$$

where $\mathtt{A}_i$ is now the 2×9 matrix of (4.3).

(iii) The equations hold for any homogeneous coordinate representation $(x'_i, y'_i, w'_i)^\mathsf{T}$ of the point $\mathbf{x}'_i$. One may choose $w'_i = 1$, which means that (x'_i, y'_i) are the coordinates measured in the image. Other choices are possible, however, as will be seen later.

Solving for $\mathtt{H}$

Each point correspondence gives rise to two independent equations in the entries of $\mathtt{H}$. Given a set of four such point correspondences, we obtain a set of equations $\mathtt{A}\mathbf{h} = \mathbf{0}$, where $\mathtt{A}$ is the matrix of equation coefficients built from the matrix rows $\mathtt{A}_i$ contributed from each correspondence, and $\mathbf{h}$ is the vector of unknown entries of $\mathtt{H}$. We seek a non-zero solution $\mathbf{h}$, since the obvious solution $\mathbf{h} = \mathbf{0}$ is of no interest to us. If (4.1) is used then $\mathtt{A}$ has dimension 12×9, and if (4.3) the dimension is 8×9. In either case $\mathtt{A}$ has rank 8, and thus has a 1-dimensional null-space which provides a solution for $\mathbf{h}$. Such a solution $\mathbf{h}$ can only be determined up to a non-zero scale factor. However, $\mathtt{H}$ is in general only determined up to scale, so the solution $\mathbf{h}$ gives the required $\mathtt{H}$. A scale may be arbitrarily chosen for $\mathbf{h}$ by a requirement on its norm such as $\|\mathbf{h}\| = 1$.

4.1.2 Inhomogeneous solution

An alternative to solving for $\mathbf{h}$ directly as a homogeneous vector is to turn the set of equations (4.3) into a inhomogeneous set of linear equations by imposing a condition $h_j = 1$ for some entry of the vector $\mathbf{h}$. Imposing the condition $h_j = 1$ is justified by the observation that the solution is determined only up to scale, and this scale can be chosen such that $h_j = 1$. For example, if the last element of $\mathbf{h}$, which corresponds to H_{33}, is chosen as unity then the resulting equations derived from (4.3) are

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w'_i & -y_i w'_i & -w_i w'_i & x_i y'_i & y_i y'_i \\ x_i w'_i & y_i w'_i & w_i w'_i & 0 & 0 & 0 & -x_i x'_i & -y_i x'_i \end{bmatrix} \tilde{\mathbf{h}} = \begin{pmatrix} -w_i y'_i \\ w_i x'_i \end{pmatrix}$$

where $\tilde{\mathbf{h}}$ is an 8-vector consisting of the first 8 components of $\mathbf{h}$. Concatenating the equations from four correspondences then generates a matrix equation of the form