

## 27<sup>th</sup> International Mathematical Olympiad

Warsaw, Poland  
 Day I  
 July 9, 1986

- Let  $d$  be any positive integer not equal to 2, 5, or 13. Show that one can find distinct  $a, b$  in the set  $\{2, 5, 13, d\}$  such that  $ab - 1$  is not a perfect square.
- A triangle  $A_1A_2A_3$  and a point  $P_0$  are given in the plane. Define  $A_n = A_{n-3}$  for  $n \geq 4$ . Construct  $P_n$  from  $P_{n-1}$  by rotation through angle  $120^\circ$  clockwise about  $A_n$ . Prove that if  $P_{1986} = P_0$ , then triangle  $A_1A_2A_3$  is equilateral.
- To each vertex of a regular pentagon an integer is assigned such that the sum of all five numbers is positive. If three consecutive vertices are assigned  $x, y, z$  respectively and  $y < 0$ , then replace them by  $x+y, -y, z+y$ . Determine whether this procedure necessarily ends after finitely many steps.

## 27<sup>th</sup> International Mathematical Olympiad

Warsaw, Poland  
 Day II  
 July 10, 1986

- Let  $A, B$  be adjacent vertices of a regular  $n$ -gon ( $n \geq 5$ ). A triangle  $XYZ$  congruent to  $OAB$  moves such that  $Y, Z$  trace the boundary of the polygon while  $X$  remains inside. Find the locus of  $X$ .
- Find all functions  $f$  on non-negative real numbers satisfying:
  - $f(xf(y)) = f(x + y)$
  - $f(2) = 0$
  - $f(x) \neq 0$  for  $0 \leq x < 2$
- Given a finite set of lattice points in the plane, is it always possible to color them red and white so that for every line parallel to axes, the difference in counts on the line is at most 1?

## 28<sup>th</sup> International Mathematical Olympiad

Havana, Cuba  
 Day I  
 July 10, 1987

1. Let  $p_n(k)$  be number of permutations of  $\{1, \dots, n\}$  with exactly  $k$  fixed points.

Prove:

$$\sum_{k=0}^n k p_n(k) = n!$$

2. In an acute triangle  $ABC$ , let internal bisector of  $\angle A$  meet  $BC$  at  $L$  and circumcircle again at  $N$ . From  $L$ , drop perpendiculars to  $AB, AC$ . Prove quadrilateral  $AKNM$  and triangle  $ABC$  have equal areas.
3. Let  $x_1, \dots, x_n$  satisfy  $x_1^2 + \dots + x_n^2 = 1$ . Prove there exist integers  $a_1, \dots, a_n$ , not all zero, with

$$|a_i| \leq k - 1$$

and

$$|a_1x_1 + \dots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

## 28<sup>th</sup> International Mathematical Olympiad

Havana, Cuba

Day II

July 11, 1987

4. Prove there is no function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(f(n)) = n + 1987$ .
5. Let  $n \geq 3$ . Prove there exists  $n$  points in plane such that all pairwise distances are irrational and every triangle formed has rational area.
6. Let  $n \geq 2$ . Prove  $n^2 + k + n$  is prime for all integers  $k$  with  $0 \leq k \leq \sqrt{n/3}$  implies  $k^2 + k + n$  is prime for  $0 \leq k \leq n - 2$ .

## 29<sup>th</sup> International Mathematical Olympiad

Canberra, Australia

Day I

1. Consider two concentric circles of radii  $R > r$ . Let  $P$  be on smaller circle and  $B$  on larger. If perpendicular to  $BP$  at  $P$  meets smaller circle again at  $A$ , find:
- (i)  $BO^2 + CA^2 + AB^2$
  - (ii) Locus of midpoint of  $BC$
2. Let  $A_1, \dots, A_{2n+1}$  be subsets of a set  $B$  satisfying certain intersection properties. For which  $n$  can each element of  $B$  be assigned 0 or 1 so that each  $A_i$  has exactly  $n$  zeros?

3. Function  $f$  satisfies:

$$f(1) = 1, \quad f(3) = 3$$

$$f(2n) = f(n)$$

$$f(4n+1) = 2f(2n+1) - f(n)$$

$$f(4n+3) = 3f(2n+1) - 2f(n)$$

Determine number of  $n \leq 1988$  with  $f(n) = n$ .

## 29<sup>th</sup> International Mathematical Olympiad

Canberra, Australia

Day II

4. Show solution set of

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is union of disjoint intervals of total length 1988.

5. In right triangle  $ABC$ , prove  $S \geq 2T$  where  $S, T$  are areas of certain triangles formed by incenters.

6. Let  $a, b$  be positive integers such that  $ab + 1$  divides  $a^2 + b^2$ . Show

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square.