CS450: Numerical analysis of the Korteweg–De Vries equation

Yashraj Bhosale

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1 Introduction

The Korteweg–De Vries (KdV) equation [1] is a 1D mathematical model of non-linear waves, commonly applied to model shallow-water waves with weakly non-linear restoring forces, long internal waves in a density-stratified ocean, ion acoustic waves in a plasma and acoustic waves on a crystal lattice.

The KdV equation is a nonlinear, dispersive partial differential equation, where a function u of two dimensionless real variables, space coordinates x and time t is given by:

$$\partial_t u + \delta^2 \partial_x^3 u + u \partial_x u = 0 \tag{1}$$

with ∂x and ∂t denoting partial derivatives with respect to x and t, and δ represents the level of dispersion in the system.

In this study, I will present numerical solutions and analysis for this equation, utilising concepts presented in CS450. Beginning with discretisation description, I will present the validation of the numerical implementation against an analytical solution of a travelling soliton wave, followed by demonstration of spatial and temporal convergence. This is followed by showcase of the effects of the dispersion parameter δ on the solution accuracy, as well as comparison against the numerical scheme proposed by Zabusky and Kruskal [2]. Finally, I present solution to a different initial condition to demonstrate the robustness of the numerical implementation.

2 Discretisation

2.1 Spatial discretisation

In the present study, we discretise the dispersion term using a 2^{nd} order central difference scheme

$$\partial_x^3 u = \frac{u_{i+2} - 2u_{i+1} + 2u_{i+1} - u_{i-2}}{2\Delta x^3} + \mathcal{O}\left(\Delta x^2\right) \tag{2}$$

where Δx is the grid spacing and the subscript i denotes the grid spatial index. For discretising the advection term $u\partial_x u$, we use a 3^{rd} order conservative ENO scheme, where:

$$u\partial_x u = \partial_x(u^2/2) = \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} + \mathcal{O}\left(\Delta x^3\right) \tag{3}$$

where $f_{i+1/2}$ corresponds to the face flux, defined as

$$f_{i+1/2} = (u_{i+1/2} > 0)\left(\frac{-f_{i-1}}{6} + \frac{5f_i}{6} + \frac{f_{i+1}}{3}\right) + (u_{i+1/2} < 0)\left(\frac{-f_{i+2}}{6} + \frac{5f_{i+1}}{6} + \frac{f_i}{3}\right) \tag{4}$$

with the nodal flux defined as $f_i = u_i^2/2$ and the face velocity defined as $u_{i+1/2} = (u_{i+1} + u_i)/2$, respectively.

2.2 Temporal discretisation

For time-steppers we consider 2 candidates for temporal discretisation, with both the advection and dispersion terms treated explicitly. The first one is Euler Forward (EF):

$$u^{j+1} = u^j + \Delta t \; g(u^j) \tag{5}$$

where u^j is the solution at timestep j, Δt is the timestep, and $g(u^j)$ corresponds to the discrete advection and dispersion fluxes evaluated at timestep j. The second one is second order Runge-Kutta or RK2 scheme:

$$\begin{aligned} u^{j+1/2} &= u^j + \frac{\Delta t}{2}\, g(u^j) \\ u^{j+1} &= u^j + \Delta t\; g(u^{j+1/2}) \end{aligned} \tag{6}$$

Given the explicit treatment of fluxes, the dispersion term acts as the more stringent one in limiting timestep, with the stable timestep given by $\Delta t \leq \kappa \Delta x^3/\delta^2$, where κ is a prefactor, which is set as 0.04 throughout most of the study.

3 Results

We first begin by validating our implementation against an analytical solution, followed by convergence studies.

3.1 Validation and convergence

For validation, we note that that solitary waves known as "solitons" are the analytical solutions to the KdV equation. With an initial soliton defined as:

$$u_0 = u_\infty sech^2 \left(\frac{x - x_0}{\delta(u_\infty/12)^{-1/2}} \right) \tag{7}$$

where u_∞ and x_0 refer to the initial amplitude and center of the soliton, respectively. Throughout the study we consider a periodic domain of size [0, 1] (with grid spacing $\Delta x = 1/N$, where N is number of grid points), and set $u_\infty = 1$ and $x_0 = 0.3$. With this initial soliton, the soliton then travels with a speed of $c = u_\infty/3$ and after time t the analytical solution is given by a soliton of the same amplitude and center $x_0 + ct$. For validation we set $\delta = 0.02$ and time $t = 0.5/c$.

Figure 1 presents the evolution of the solution for Euler Forward (details in caption), where a clear blowup of the solution due to instability is seen. This can be attributed to the fact that the equation above has purely imaginary eigenvalues, and that Euler forward stability region does not include the imaginary axis.

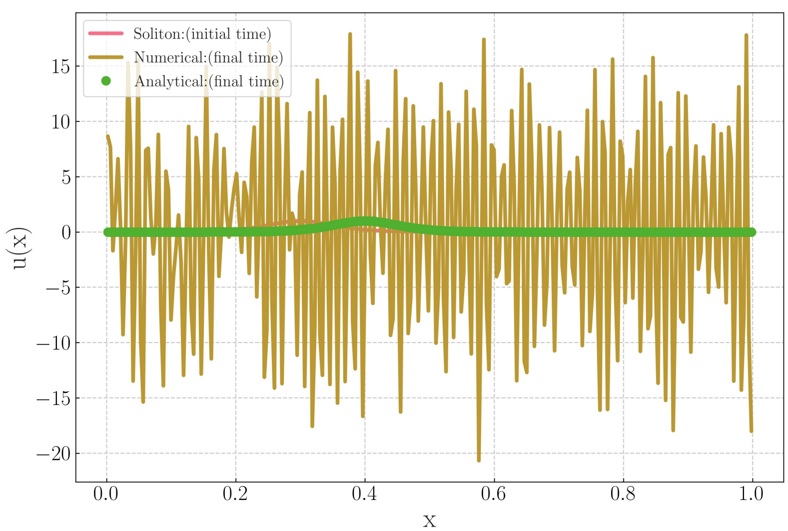

Figure 1: Solution for Euler forward ($\kappa = 0.04$) and N = 256 for initial soliton impulse with $u_\infty = 1$, $x_0 = 0.3$ and $\delta = 0.02$.

Figure 2 presents the evolution of the soliton for RK2 (details in caption). The solution is observed to be stable (owing the fact that RK2 stability region cuts the imaginary axis), as well as shows decent agreement with the analytical result.

To analyse the error convergence with grid spacing, we perform spatial convergence study by fixing $\kappa = 0.04$ and varying the grid size from 32 to 256, and plot the 2-norm and infinity-norm errors with grid spacing $1/N$. As seen in fig. 3, 2nd order convergence is observed, consistent with the spatial discretisation schemes used.

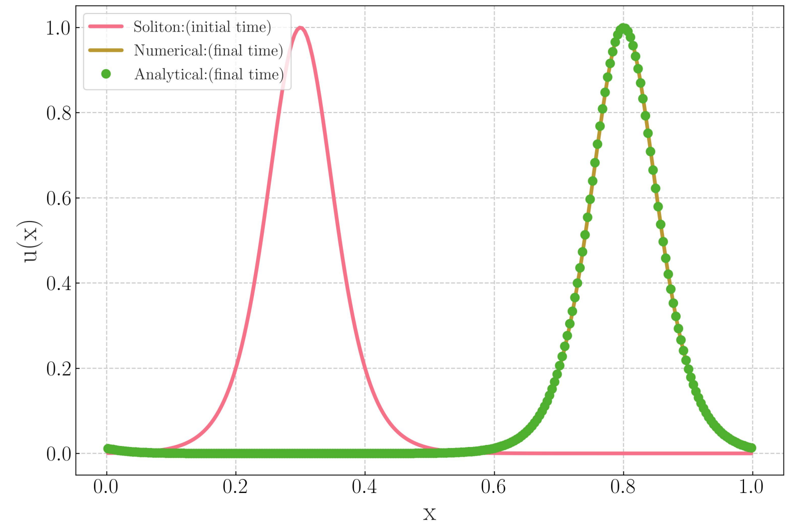

Figure 2: Solution for RK2 ($\kappa = 0.04$) and N = 256 for initial soliton impulse with $u_\infty = 1$, $x_0 = 0.3$ and $\delta = 0.02$.

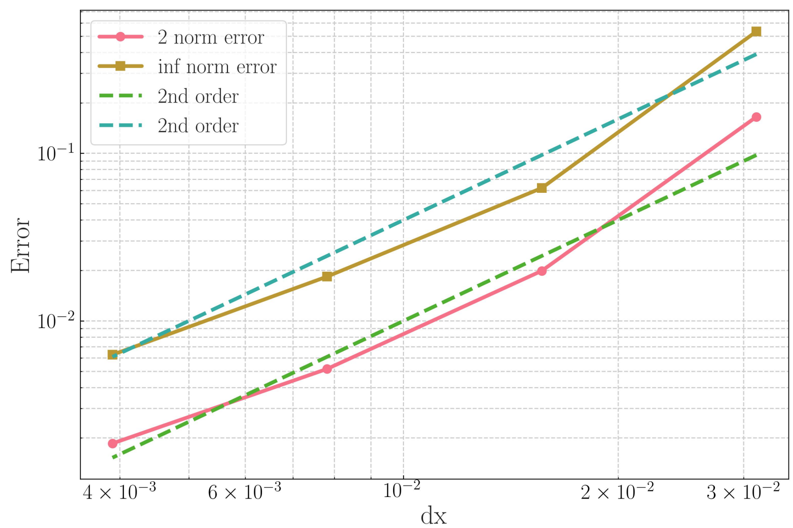

Figure 3: Spatial convergence for RK2 ($\kappa = 0.04$) and for initial soliton impulse with $u_\infty = 1$, $x_0 = 0.3$ and $\delta = 0.02$.

To analyse the error convergence with timestep size, we perform temporal convergence study by fixing N = 256 and varying the timestep prefactor κ from 0.32 to 0.04, and plot the 2-norm and infinity-norm errors with timestep Δt. As seen in fig. 4, 2nd order convergence is observed, consistent with the temporal discretisation scheme used.

3.2 Effect of dispersion parameter δ

We next investigate the effect of dispersion parameter δ on the numerical solution. As seen from the soliton equation and shown in fig. 5 (solid lines), for fixed u_∞, decreasing δ corresponds to decreasing dispersion and a sharper wave. Fixing N = 256 and $\kappa = 0.04$, fig. 5 shows the effect of varying δ on the numerical solution. With decreasing δ i.e. sharper waves, the numerical solution starts lagging behind and also shows amplitude decay. This is expected from the ENO3 scheme used for advection, which leads to artificial diffusion (amplitude decay) and dispersion (lag) for sharper wavemodes. This leads to increase in the error as δ decreases, as seen from fig. 6. To improve accuracy, resolving the sharper wavemodes is

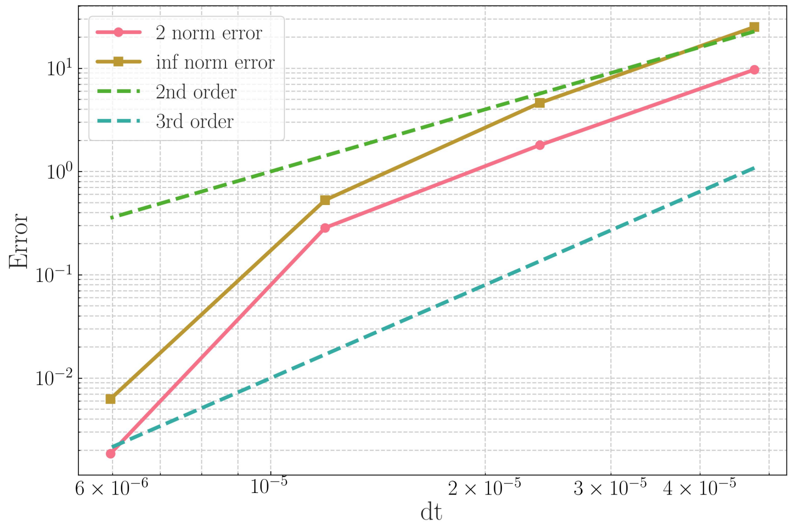

Figure 4: Temporal convergence for RK2 for N = 256, and for initial soliton impulse with $u_\infty = 1$, $x_0 = 0.3$ and $\delta = 0.02$.

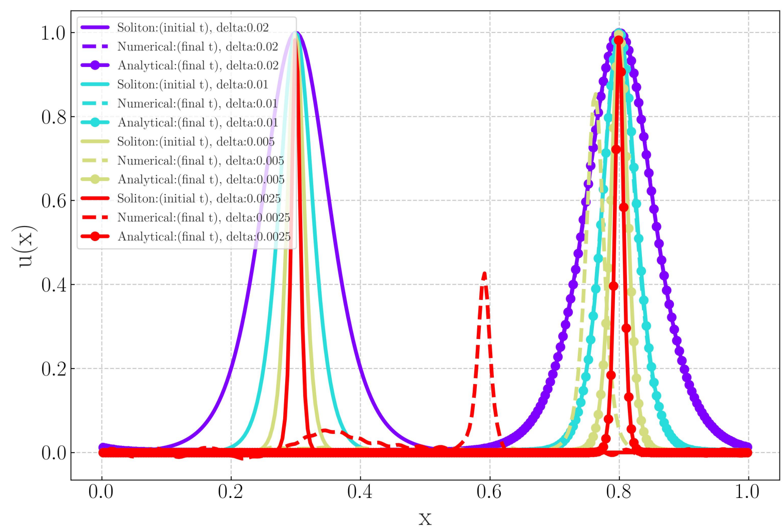

Figure 5: Effect of dispersion δ for RK2 ($\kappa = 0.04$) for N = 256, and for initial soliton impulse with $u_\infty = 1$ and $x_0 = 0.3$.

essential, which implies the need to increase the grid resolution i.e. N.

3.3 Comparison against Zabusky and Kruskal scheme

We next present a comparison against a numerical scheme for KdV equation previously proposed by Zabusky and Kruskal [2]. The spatio-temporal leap-frogging scheme can be described as follows:

$$
\begin{aligned}
g(u_i^j) &= -\delta^2 \frac{u_{i+2}^j - 2u_{i+1}^j + 2u_{i+1}^j - u_{i-2}^j}{2\Delta x^3} - \frac{(u_{i+1}^j + u_i^j + u_{i-1}^j)(u_{i+1}^j - u_{i-1}^j)}{6\Delta x} \\
u_i^{j+1} &= u_i^{j-1} + 2\Delta t \; g(u_i^j)
\end{aligned} \tag{8}
$$

To analyse the error convergence with grid spacing, we perform spatial convergence study by fixing $\kappa = 0.04$ and

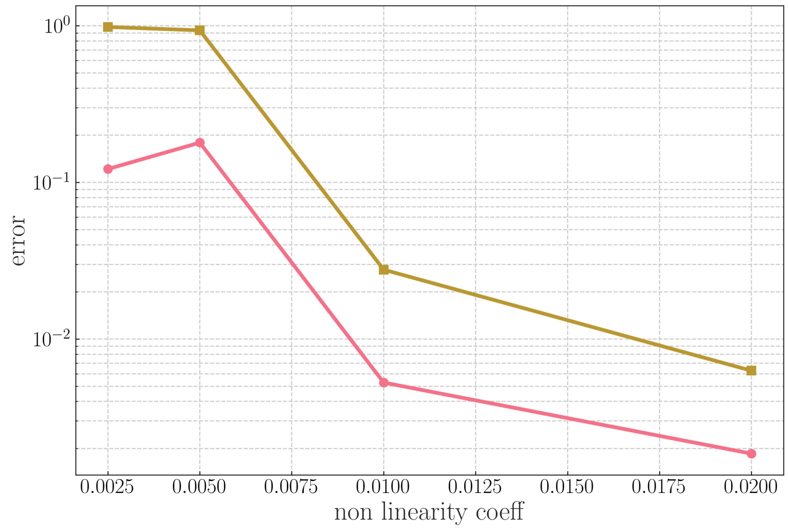

Figure 6: Error variation as a function of dispersion δ for RK2 ($\kappa = 0.04$) for N = 256, and for initial soliton impulse with $u_\infty = 1$ and $x_0 = 0.3$.

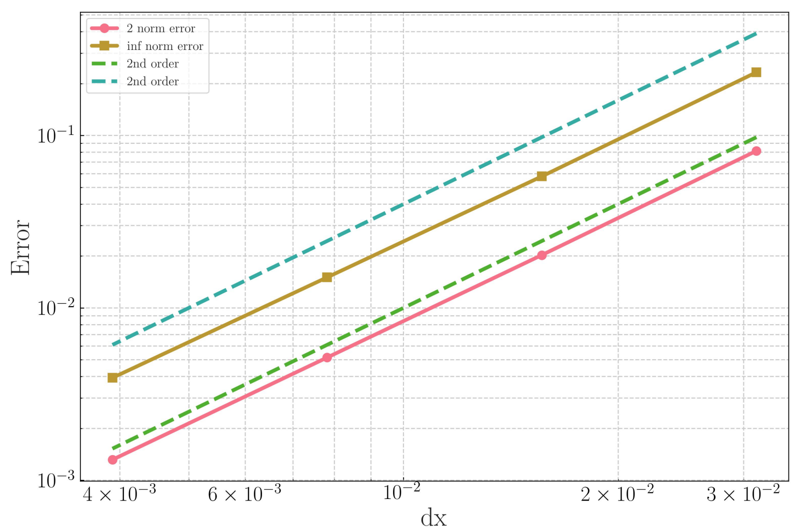

Figure 7: Spatial convergence for Zabusky and Kruskal scheme ($\kappa = 0.04$) and for initial soliton impulse with $u_\infty = 1$, $x_0 = 0.3$ and $\delta = 0.02$.

varying the grid size from 32 to 256, and plot the 2-norm and infinity-norm errors with grid spacing $1/N$. As seen in fig. 7, 2nd order convergence is observed, consistent with the spatial discretisation schemes used.

We next investigate the effect of dispersion parameter δ on the numerical solution. Fixing N = 256 and $\kappa = 0.04$, fig. 8 shows the effect of varying δ on the numerical solution. With decreasing δ i.e. sharper waves, the numerical solution starts lagging behind, although the error is significantly lower than the ENO3-RK2 scheme used in the previous subsections. Also unlike the previous scheme, the amplitude decay observed in this case is negligible. Thus for a given spatial resolution and the dispersion δ variation considered, the Zabusky and Kruskal scheme has a better performance compared to the RK2-ENO3 scheme used above.

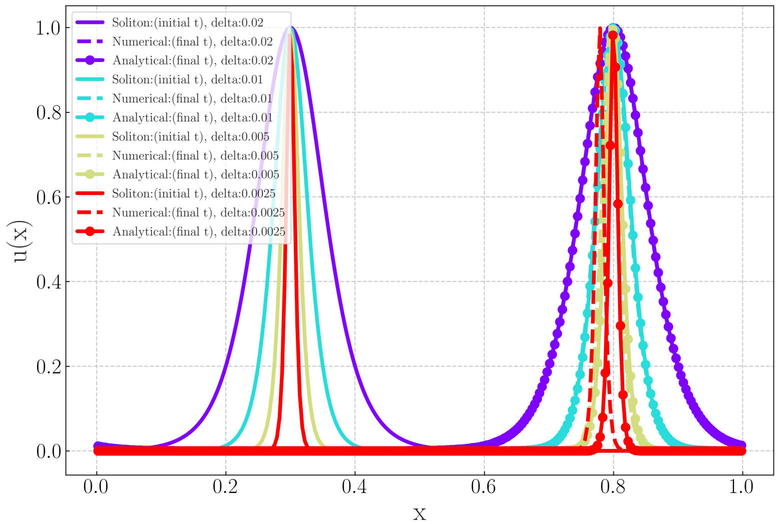

Figure 8: Effect of dispersion δ for Zabusky and Kruskal scheme ($\kappa = 0.04$) for N = 256, and for initial soliton impulse with $u_\infty = 1$ and $x_0 = 0.3$.

3.4 Variation in initial condition

Following [2], we next consider a different initial condition, a cosine wave $u_0 = \cos \pi x$. For N = 256, $\kappa = 0.04$, $\delta = 0.022$ and the Zabusky and Kruskal scheme, the solution evolution till final time $t = 1$ is presented in fig. 9. As seen in fig. 9, the solution evolves into a train of soliton waves, consistent with the observations of [2], thus demonstrating the robustness of the numerical scheme to variation in initial conditions. A temporal evolution of the cosine wave into soliton train can be found here.

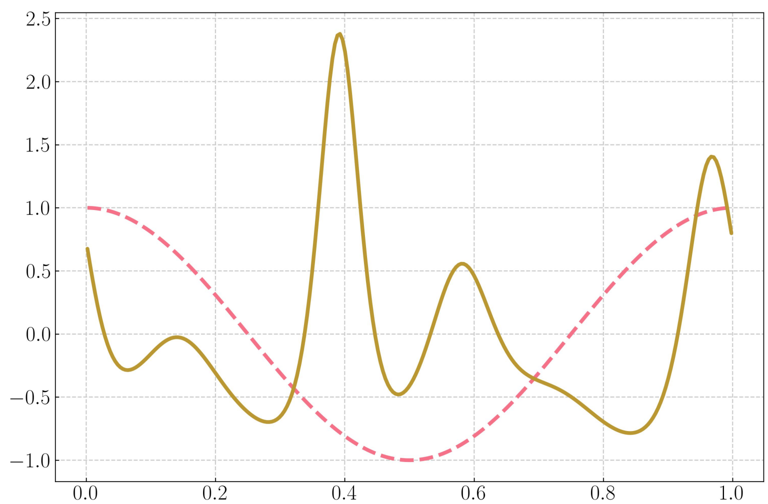

Figure 9: Solution for initial cosine wave $u_0 = \cos \pi x$ with Zabusky and Kruskal scheme ($\kappa = 0.04$) and N = 256, for $\delta = 0.022$.

4 Conclusion

In this study, numerical solutions for the KdV equation via different spatio-temporal discretisation schemes has been investigated, followed by error convergence analysis and error sensitivity analysis for model parameters and initial conditions.

Detailed codebase used for simulating the results in this report can be found here.

References

[1] Diederik Johannes Korteweg and Gustav De Vries. Xli. on the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 39(240):422–443, 1895.

[2] Norman J Zabusky and Martin D Kruskal. Interaction of" solitons" in a collisionless plasma and the recurrence of initial states. *Physical review letters*, 15(6):240, 1965.