1 Properties and Operations

1.1 Special Matrices

$$O_{mn} = \begin{bmatrix} 0_{11} & \cdots & 0_{1n} \\ \vdots & \ddots & \vdots \\ 0_{m1} & \cdots & 0_{mn} \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

1.2 List of Basic Property Theorems

1.2.1 Properties of Matrix Addition and Scalar Multiplication

$$A + B = B + A \tag{1}$$
$$A + (B + C) = (A + B) + C \tag{2}$$
$$(cd)A = c(dA) \tag{3}$$
$$1A = A \tag{4}$$
$$c(A + B) = cA + cB \tag{5}$$
$$(c + d)A = cA + dA \tag{6}$$

1.2.2 Properties of Zero Matrices

$$A + O_{mn} = A \tag{7}$$
$$A + (-A) = O_{mn} \tag{8}$$
$$\text{If } cA = O_{mn} \text{ then } c = 0 \text{ or } A = O_{mn} \tag{9}$$

1.2.3 Properties of Matrix Multiplication

$$A(BC) = (AB)C \tag{10}$$
$$A(B + C) = AB + AC \tag{11}$$
$$(A + B)C = AC + BC \tag{12}$$
$$c(AB) = (cA)B = A(cB) \tag{13}$$

1.2.4 Properties of Identity Matrix

$$AI_n = A \tag{14}$$
$$I_m A = A \tag{15}$$

1.3 Transposition

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
$$A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

1.3.1 Properties of Transposed Matrix

$$(A^T)^T = A \tag{16}$$
$$(A+B)^T = A^T + B^T \tag{17}$$
$$(cA)^T = c(A^T) \tag{18}$$
$$(AB)^T = B^T A^T \tag{19}$$

1.4 Inverse Matrices

1.4.1 Properties of Inverse Matrices

$$(A^{-1})^{-1} = A \tag{20}$$
$$(A^k)^{-1} = A^{-1}A^{-1} \cdots A^{-1} = (A^{-1})^k \tag{21}$$
$$(cA)^{-1} = \frac{1}{c}A^{-1} \tag{22}$$
$$(A^T)^{-1} = (A^{-1})^T \tag{23}$$
$$(AB)^{-1} = B^{-1}A^{-1} \tag{24}$$
$$AC = BC \rightarrow A = B \text{ if C is inversible} \tag{25}$$
$$CA = CB \rightarrow A = B \text{ if C is inversible} \tag{26}$$

Systems of Equations $Ax = y$ have a unique solution $x = A^{-1}b$.

Example:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3$$

$$A^{-1}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

1.4.2 Using Gauss-Jordan Elimination

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & a\prime & b\prime & c\prime \\ 0 & 1 & 0 & d\prime & e\prime & f\prime \\ 0 & 0 & 1 & g\prime & h\prime & i\prime \end{bmatrix}$$

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a\prime & b\prime & c\prime \\ d\prime & e\prime & f\prime \\ g\prime & h\prime & i\prime \end{bmatrix}$$

1.4.3 2x2 Quick Solution

First is a square matrix which can be inverted using a simple rearrangement and multiplication by the inverse of the **Determinant**.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2 Elementary Matrices

Are one single operation away from the I matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Every elementary matrix E is invertible and E^{-1} is also an elementary matrix.

2.1 Matrices as Products of Elementary Matrices

2.2 LU Factorization

$A = LU$
Matrix A is equal to the product of a Lower corner and Upper corner factor.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\0 & -4 & 2\end{bmatrix} \qquad R_3 + (-2)R_1 \to R_3 \qquad E_1 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-2 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\0 & 0 & 14\end{bmatrix} \qquad R_3 + (4)R_2 \to R_3 \qquad E_2 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 4 & 0\end{bmatrix}$$

The final matrix on the left is L.
$L = E_n \cdots E_2 E_1 A$
$U = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$

3 Determinants

3.1 Minors

Minors are the determinant of entries in a matrix that do not share a column or row with the current position.

$$\begin{bmatrix}a_{11} & \cancel{a_{12}} & a_{13}\\\cancel{a_{21}} & a_{22} & \cancel{a_{23}}\\a_{31} & \cancel{a_{32}} & a_{33}\end{bmatrix} \to M_{22} = \begin{vmatrix}a_{11} & a_{13}\\a_{31} & a_{33}\end{vmatrix}$$
$$\begin{bmatrix}a_{11} & \cancel{a_{12}} & \cancel{a_{13}}\\\cancel{a_{21}} & a_{22} & a_{23}\\\cancel{a_{31}} & a_{32} & a_{33}\end{bmatrix} \to M_{11} = \begin{vmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{vmatrix}$$

3.2 Cofactors

$C_{ij} = (-1)^{i+j} M_{ij}$
This follows the following sign pattern:

$$\begin{bmatrix}+ & - & + & - & + & - & \cdots\\- & + & - & + & - & + & \cdots\\+ & - & + & - & + & - & \cdots\\- & + & - & + & - & + & \cdots\\+ & - & + & - & + & - & \cdots\\\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

3.3 Determinants of Larger Matrices

The generalized Equation:

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

This can be applied to any single row or column. *Choose wisely.*

3.3.1 Alternate Method

$$det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Subtract these products

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

Add these products

$\mid A \mid = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

3.4 Adjoint Matrices

Adjoint matrices are the transpose of a matrix of Cofactors:

$$adj(A) = \begin{bmatrix} C_{11} & C_{21} & C_{n1} \\ C_{12} & C_{22} & C_{n2} \\ C_{1n} & C_{2n} & C_{nn} \end{bmatrix}$$

3.5 Properties of Determinants

$$|A||B| = |AB| \tag{27}$$

$$|cA| = c^n|A| \tag{28}$$

$$|A| = |A^T| \tag{29}$$

$$|A^{-1}| = \frac{1}{|A|} \tag{30}$$

$$A^{-1} = \frac{1}{|A|}adj(A) \tag{31}$$

$$\tag{32}$$

3.6 Cramer's Rule

For a system of equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{|A_1|}{|A|}$$
$$x_2 = \frac{|A_2|}{|A|}$$
$$\dots$$
$$A_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$
$$A_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

3.7 Applications of Determinants

3.7.1 Area of a Triangle

For a triangle with points (x_1, y_1),(x_2, y_2), and (x_3, y_3)

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

This can likewise be used to solve for the volume of a tetrahedron in 3-dimensional space.

$$\text{Area} = \pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

3.7.2 Colinearity

If

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Then the 3 points are on the same line. Likewise solving the following for x and y will give the equation of the line that passes through the other two points.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

This can also be used to test for Coplanar points in 3D space using the same expansion to 4x4 as seen above.

4 Vector Space

4.1 Properties of Vector Addition and Scalar Multiplication in a Plane

Let v, u, and w be vectors and c and d be scalars.

$$\begin{align} v + u &= u + v \tag{33} \\ (u + v) + w &= u + (v + w) \tag{34} \\ v + u &= (v_1 + u_1), (v_2 + u_2) \tag{35} \\ u + 0 &= u \tag{36} \\ u + (-u) &= 0 \tag{37} \\ c(u + v) &= cu + cv \tag{38} \\ (c + d)u &= cu + du \tag{39} \\ c(du) &= (cd)u \tag{40} \\ & \tag{41} \end{align}$$

Summary of important Vector Spaces

$$\begin{aligned} R &= \text{set of all real numbers} \\ R^2 &= \text{set of all ordered pairs} \\ R^3 &= \text{set of all ordered triples} \\ R^n &= \text{set of all n-tuples} \\ C(-\inf, \inf) &= \text{set of all continuous functions defined on the real number line} \\ C[a, b] &= \text{set of all continuous functions defined on a closed interval } [a, b], \\ &\quad \text{where } a \neq b \\ P &= \text{set of all polynomials} \\ P_n &= \text{set of all polynomials of degree} \leq n \text{(together with the zero polynomial)} \\ M_{m,n} &= \text{set of all } m \times n \text{matrices} \\ M_{n,n} &= \text{set of all } n \times n \text{square matrices} \end{aligned}$$

4.2 Subspaces

The Test for Subspace

1. If Subspace is non-empty. (Includes the 0 vector)
2. If u and v are in W, then $u + v$ is in W. (Closed by Addition)
3. If u is in W and c is any scalar, then cu is in W (Closed by Multiplication)

4.3 Linear Combinations of Vectors

Vector v in vector space V is a linear combination of vectors if it can be written in the following form:
$v = c_1u_1 + c_2u_2 + \ldots + c_ku_k$
Solvable by matrix method:

$$u_1 = (u_{11}, u_{12}, u_{13}) \begin{bmatrix} u_{11} & u_{21} & u_{31} & v_1 \\ u_{12} & u_{22} & u_{32} & v_2 \\ u_{13} & u_{23} & u_{33} & v_3 \end{bmatrix}$$

** Note that the matrix is built transposed from the way in which most matrices have been built in this course.
Once built, solve for row-echelon form. If the bottom row is 0s then the remainder has infinitely many solutions with a form of $c_it + \ldots$ from the rest of the matrix solution.

4.4 Spanning Sets

A set spans R^2 if its coefficient matrix has a non-zero determinant.
Same is true of R^3.

4.5 Linear Dependence and Independence

For a set $S = v_1, v_2, \ldots, v_n$
where there exists a non-trivial solution to:
$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$
(The trivial solution is for $c_{1 \cdots n} = 0$)
Is Linearly Dependant.
If only the trivial solution exists it is linearly-independent.
Testing for Dependence

If the coefficient matrix can reduced to identity matrix:

$$\begin{bmatrix} u_{11} & u_{21} & u_{31} & v_1 \\ u_{12} & u_{22} & u_{32} & v_2 \\ u_{13} & u_{23} & u_{33} & v_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

There will only be the trivial solution.
If the Gaussian reduction yields a all zero row there are infinitely many solutions, therefor a non-trivial solution, and the set is linearly-dependent.

4.6 Basis for Vector Space

The standard basis for $R^n$is $M_{n,n}$ in the identity matrix form. The standard basis for $P_n$is $S = 1, x, x^2, \ldots, x^n$.

Testing for Alternative Bases
And alternative basis will be Linearly-Independent and will be a spanning set.

5 Inner Product Spaces

5.1 Length and Dot Product

Vector Length is the result of a Pythagorean-like series.
$\|V_n\| = \sqrt{v_1^2 + v_2^2 + v_3^2 \cdots v_n^2}$

Inner Product:
$\langle X, Y \rangle = x_1y_1 + x_2y_2 + \cdots x_ny_n$

Unit Vectors have the same direction as the source vector but a length of 1.
$u = \frac{v}{\|v\|}$
Shown by:

$$\begin{aligned} \|u\| &= \left\| \frac{v}{\|v\|} \right\| \\ &= \frac{1}{\|v\|} \|v\| \\ &= 1 \end{aligned}$$

*Note that:

$$\|cv\| = |c| \, \|v\|$$

The distance between two vectors in R^n space is: $d(u,v) = \|u-v\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 \cdots (u_n-v_n)^2}$

5.2 Properties of Dot Products

$$\begin{aligned} u \cdot v &= u_1v_1 + u_2v_2 + \ldots + u_nv_n \\ u \cdot v &= v \cdot u \\ u \cdot (v+w) &= u \cdot v + u \cdot w \\ c(u \cdot v) &= (cu) \cdot v = u \cdot (cv) \\ v \cdot v &= \|v\|^2 \\ v \cdot v \geq 0, \text{ and } v \cdot v &= 0 \text{ if and only if } v = 0 \end{aligned}$$

5.3 Cauchy-Schwarz Inequality

If u and v are vectors in R^n, then
$|u \cdot v| \leq \|u\| \|v\|$
where $|u \cdot v|$ denotes the *absolute value* of $u \cdot v$.

5.4 Angle Between Vectors

The Cauchy-Schwarz inequality must be verified in order to use the following method to find the angle between two vectors.
$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$, $0 < \theta < \pi$

*Remember $(cos\theta, sin\theta)$

Orthogonal Unit Vectors:
In R^2, $v = (v_1, v_2)$, the Orthogonal vector is: $(v_2, -v_1)$
Therefor the Orthogonal Unit vectors are: $\left\langle \frac{v_1}{\|v\|}, -\frac{v_2}{\|v\|} \right\rangle$ and $\left\langle -\frac{v_1}{\|v\|}, \frac{v_2}{\|v\|} \right\rangle$

5.5 Inner Products

Axioms of Inner Products:

$$\begin{aligned} \langle u, v \rangle &= \langle v, u \rangle \\ \langle u, v+w \rangle &= \langle u, v \rangle + \langle u, w \rangle \\ c\langle u, v \rangle &= \langle cu, v \rangle \\ \langle v, v \rangle \geq 0, \text{ and } \langle v, v \rangle &= 0 \text{ if and only if } v = 0 \end{aligned}$$

5.6 Definitions in Inner Product Space

$$\textbf{Length of } u\text{: } \|u\| = \sqrt{\langle u, u\rangle}$$
$$\textbf{Distance} \text{ between } u \text{ and } v\text{: } d(u,v) = \|u - v\| = \sqrt{\langle u - v, u - v\rangle}$$
$$\textbf{Angle} \text{ between } u \text{ and } v\text{: } \cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}$$
$$u \text{ and } v \text{ are orthogonal when } \langle u, v\rangle = 0$$
$$\text{Cauchy-Schwarz Inequality: } |\langle u, v\rangle| \leq \|u\|\,\|v\|$$
$$\text{Triangle inequality: } \|u + v\| \leq \|u\| + \|v\|$$

Pythagorean Theorem: u and v are orthogonal if and only if

$$\|u + v\|^2 = \|u\|^2\,\|v\|^2$$

5.6.1 Orthogonal Projection

Let u and v be vectors in an inner product space V, such that $v \neq 0$. Then the orthogonal projection of u onto v is
$\text{proj}_v u = \frac{\langle u,v\rangle}{\langle v,v\rangle} v$

5.7 Gram-Schmitz Orthonormalization Process

1. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for product space V.

2. Let $B' = \{w_1, w_2, \ldots, w_n\}$, where
   $w_1 = v_1$
   $w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$
   $w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$
   $w_n = v_n - \frac{v_n \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_n \cdot w_2}{w_2 \cdot w_2} w_2 - \cdots - \frac{v_n \cdot w_n}{w_n \cdot w_n} w_n$
   B' is the *orthoganal basis* for V.

3. Let $u_i = \frac{w_i}{\|w_i\|}$ and
   $B" = \{u_1, u_2, \ldots, u_n\}$
   $B"$ is the *orthonormal basis* for V.

6 Eigenvectors and Eigenvalues

6.1 Definitions

$Ax = \lambda x$
Where A is an $n \times n$ matrix,
λ is the Eigenvalue, and
x is the non-zero Eigenvector.

$$|\lambda I - A| = 0$$
$$(\lambda I - A)x = 0$$

The **characteristic equation** is $|\lambda I - A = 0|$ solving this from polynomial form will give Eigenvalues.

6.1.1 Examples

Eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The characteristic polynomial is:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 - a\lambda - d\lambda + (-b)(-c) = (\lambda + i)(\lambda + j) = 0$$

The result is that Matrix A has the Eigenvalues $\lambda_1 = -i$ and $\lambda_2 = -j$.
Eigenvectors:

$$(\lambda_1)I - A = \begin{bmatrix} \lambda_1 - a & -b \\ -c & \lambda_1 - d \end{bmatrix}$$

Reduce this matrix into Row-Echelon form, or as near as possible.

$$\begin{bmatrix} \lambda_1 - a & -b \\ -c & \lambda_1 - d \end{bmatrix} \rightarrow \begin{bmatrix} i & j \\ 0 & 0 \end{bmatrix}$$

Each row is set equal to zero and a variable will be assigned.
In this case $x_2 = t$

$$x_1 - 4x_2 = 0x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \neq 0$$

This must be done for each Eigenvalue.

7 Linear Programming

7.1 Systems of Linear Equations

Linear Equation between two points:
$(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1 = 0$