CRITERION FOR ANDERSON LOCALIZATION IN ACTIVE RANDOM MEDIA

by

BENJAMIN HENRY PAYNE

A DISSERTATION

Presented to the Faculty of the Graduate School of the

MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

PHYSICS DEPARTMENT

2012

Dr. Alexey Yamilov, Advisor
Dr. Hui Cao
Dr. Paul Parris
Dr. John Story
Dr. Thomas Vojta

Anderson Localization occurs when wave interference effects are dominant. We find a criterion for non-conservative random media that specifies when the particle-based diffusion model becomes useless.

Abstract as breakdown of title "Criterion for Anderson Localization in Active Random Media"

- study = numerical, analytical, theory
- Anderson Localization = diffusion with interference effects for passive systems
- active = not passive: gain or absorption
- random media = randomly placed delta function "scatterers"

I would like to thank my parents, without whom none of this would have been possible. Thanks to the computers, without which none of the computations would have been feasible.

Thanks to NSF and VA for funding my education.

checklist:

- MST: A Yamilov, P Parris, J Story, T Vojta, Peacher
- UW: Bob Wilson , Mark Eriksson , Cavendish McKay , Richard A. Brualdi , Thad Walker , Jay K Martin , Paul Milewski , Michael Winokur , Fabian Waleffe
- high school , Dr. Scott Eckel , Beth Torrison , Richard Ihle , Mr Murphy , Cynthia Chin

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

Table Page

NOMENCLATURE

Symbol	Description

1. ONE DIMENSION

1.1. MOTIVATION: T/$\mathcal{E}$ AS A PARAMETER FOR LOCALIZATION WITH GAIN

Localization criteria developed for passive systems may not be applicable for random media with gain. Transmission coefficients and the magnitude of their fluctuations diverge close to lasing threshold. In realistic systems, saturation prevents the divergence but introduces the dependence on saturation parameter. This may not carry the desired information about wave transport.

The Thouless criteria may not account for gain: "The ensemble-averaged spectral correlation function might be dominated by the rare lasing configurations, thus the spectral correlation width could be equal to the lasing line width that depends on the properties of the gain material." [From NSF proposal.] [Need more background from NSF paper.] (Ioffe-Regel will not work in systems with gain?)

We confirm that transmission (T) and energy ($\mathcal{E}$) both diverge as critical gain is approached in diffusive systems. Either transmission [**?**] or energy alone is not useful at critical gain, but perhaps a ratio of the two (T/$\mathcal{E}$) will be non-divergent.

1.2. DIFFUSION: SLAB GEOMETRY

In starting to look for criteria for localization in systems with gain we first looked at diffusive systems. The reason for starting out in the diffusive model is because the diffusion equation can be solved analytically for the slab geometry, even when gain is added [**?**]. We need to see how light is transmitted through media in general.

We know there will not be any strong localization in systems described by the diffusion equation since the model does not keep track of phase. Localization is based on interaction of waves and this interference is dependent on the phase. Any model that does not account for phase or some derivative of phase will not be able to fully account for strong localization.

Gain cannot always be modelled by negative absorption. Close to lasing threshold the phase (i.e description of fields not intensities), fluctuations and the system

Figure 1.1. Transmission and reflection are plotted for increasing gain and absorption coefficients. In the left plot dotted green line is transmission and solid line blue is reflection. In the left plot the upper dotted green and upper solid blue lines are transmission and reflection with increasing amounts of gain, both diverging at the critical gain value. The lower dotted green and lower solid blue lines are absorption. In the right plot we see transmission/energy and reflection/energy. Now when gain is added neither T/$\mathcal{E}$ (upper dotted green line) nor R/$\mathcal{E}$ (lower solid blue line) diverge at critical gain, and they both reach the same value. T/$\mathcal{E}$ (lower dotted green line) and R/$\mathcal{E}$ (upper solid blue line) in the absorption regime do nothing unexpected.

dynamics become important. Far from lasing threshold, "negative absorption" is a reasonable way of introducing gain.

Starting from the intensity distribution from Ref. [**?**] we can find the transmission and reflection from Eq. 1

$$J_{\pm} = \frac{c}{4} I \mp \frac{D}{2} \frac{dI}{dz} \tag{1}$$

where J_{-} is the reflection flux, J_{+} is the transmission flux, $I(z)$ is the intensity, and z is the position within the slab. Substituting the expression for $I(z)$ found in Ref. [**?**], we find

$$J_{-}(0) = \frac{c}{4} \frac{2 z_o q}{D} \qquad \frac{\sinh(\alpha(L - z_p)) + \alpha z_o \cosh(\alpha(L - z_p))}{(1 + \alpha^2 z_o^2)\sinh(\alpha L) + 2\alpha z_o \cosh(\alpha L)} \tag{2}$$

and

$$J_{+}(L) = \frac{c}{4} \frac{2 z_o q}{D} \qquad \frac{\sinh(\alpha z_p) + \alpha z_o \cosh(\alpha z_p)}{(1 + \alpha^2 z_o^2)\sinh(\alpha L) + 2\alpha z_o \cosh(\alpha L)} \tag{3}$$

where $J_{incident} = q$. We can also find the energy by integrating intensity over the entire system:

$$\mathcal{E} = \frac{q}{4\pi D \alpha^2} \left(1 - \frac{\sinh(\alpha z_p) + \sinh(\alpha(L - z_p)) + \alpha z_o(\cosh(\alpha z_p) + \cosh(\alpha(L - z_p))}{\sinh(\alpha L(1 + \alpha^2 z_o^2)) + \qquad \cosh(\alpha L 2\alpha z_o)} \right) \tag{4}$$

We plot transmission, Eq. 3,and reflection, Eq. 2, with and without normalization by energy $\mathcal{E}$, Eq. 4, as functions of gain and absorption in Fig. 1.1.

Figure 1.2. Intensity distribution for the diffusive system. Top red dotted line is the sample with gain, middle solid blue line is passive sample, lower dotted green line is sample in the absorption regime. We can see that the amount of gain or absorption in the sample affects the field in the sample. This is verified in non-diffusive models.

As critical gain is reached both transmission and reflection diverge asymptotically in Fig. 1.1 a according to

$$q\frac{z_o + z_p}{\pi}\frac{\alpha_c^2}{\alpha_c - \alpha} \tag{5}$$

Plotting the ratios of reflection to energy ($\frac{J_-}{\mathcal{E}}$) and transmission to energy ($\frac{J_+}{\mathcal{E}}$) in Fig. 1.1 we see two things: T/$\mathcal{E}$ does not diverge at critical gain, and the ratios of R/$\mathcal{E}$ and T/$\mathcal{E}$ match at the critical gain. This tells us that T/$\mathcal{E}$ may be a good criteria for detecting localization (whereas both T and R are divergant at critical gain). Note that although R/$\mathcal{E}$ doesn't diverge, we don't use it because transmission may be easier to measure.

Another method for reaching a lasing state is to have a certain amount of gain for a material and then adjust the length of the sample until a critical length is reached such that the leakage (this is an open system) is compensated by the cumulative effect of the gain in the material. We see similar behavior for the reflection and transmission (divergence of both transmission and reflection in systems near the critical length) versus R/$\mathcal{E}$ and T/$\mathcal{E}$ ratios (both converge to same value at critical gain, both are non-divergent) in this alternate analytical diffusion model.

The last thing we investigate in the diffusive system is how the distribution of the intensity in the sample changes for active systems and systems with absorption. When gain is added there is more energy stored in the sample, as compared to the passive and absorption regimes. See Fig. 1.2. We also see that the intensity decreases as it passes through the sample. If light were shined on the other side of the sample we would see the decay in intensity flip (decreasing from a maxium at x=L to near zero at x=0).

1.3. LOCALIZATION: STACKS OF DIELECTRIC LAYERS MODEL

1.3.1. Description of setup.

Figure 1.3. Layer setup for alternating pairs of dielectric. Solid blue layer A has a constant width, while the width of B has uniformly random variance of width between 1.1 and .9. Light is incident on the left side of the sample with amplitude 1. Some light is reflected to the left and some is transmitted to the right.

We consider a passive system having layers of alternating ($\epsilon = 1$ and 1.2) dielectric material, resulting in alternating refractive index ($n = \sqrt{\epsilon}$). This pair of layers is repeated to create 1000 pairs. One last $\epsilon = 1$ layer is added to the end to create a symmetric stack of 2001 layers. Then the total sample has length L. Randomness is introduced by varying the width of each $\epsilon = 1.2$ layer. The variance of the random layers is such that localization length is between L/2 and L/5. Gain or absorption can be arbitrarily added by changing the value of the dielectric to be complex or negative, respectively, (but not both). Fig. 1.3 schematically shows the setup. These parameters put the model in the localization regime ($a \ll \lambda \ll L$). [What is a?] Where λ=wavelength, L is system length. The frequency range is chosen so that single parameter scaling is satisfied. [citation?]

Light is shone on one side of the material (from left to right) and the propagation of the light through the sample is calculated using Maxwell's equations and the transfer matrix method. The transfer matrix method is covered in detail around page 44 in a text book by Mello and Kumar[**?**]. See also Appendix A. As a check, the magnitude of transmission and reflection add to one is verified.

Using the computational model the cumulative energy in the system and the transmission are found for a given frequency. Then we scan many frequencies. Then in order to get generalized behavior we alter the random widths to produce another sample. Repeat to find ensemble behavior.

We chose not to take any averages; rather we look at a single realization of disorder and attempt to understand it. Once the behavior is understood for a single instance we can develop a general phenomenological explaination.

This computational configuration has been realized experimentally with one dimensional experiments using microwaves by Genack [**?**], Luna-Acosta [**?**] and John Scales [**?**]. Fig. 1.4 is the electric field plotted on a log scale for many frequencies. The electric field is related to the energy distribution in the sample. [Explain importance, compare to Genack Nov 07 paper]

Figure 1.4. The log of electric field in a random sample for a range of frequencies versus position in the sample. Without localization pure exponential decay is expected. This can be seen between $\omega = 1.45$ and 1.453 as a straight line descending from the incident side at x/L=0 down to a minimum at x/L=1. Everywhere else in the sample localization effects are observed. At $\omega = 1.4425$ a peak higher than the incident value can be seen. This would correspond to Fig. 1.6a. At $\omega = 1.4475$ there is an intial linear decline, then a peak, then further linear decline (exponential decay), similar to Fig. 1.6b. A 2-D version of this plot can be seen in [**?**].

Figure 1.5. Transmission as a function of frequency (omega) is compared to the total energy in the sample. This single represenative realization demonstrates that $T/\mathcal{E}$ in the passive system will not be smooth. Transmission and energy sometimes share peaks for certain frequencies, but some peaks in transmission have no decernable peak in energy (even on a log plot with no noise).

1.3.2. Results for passive random layers model.

From the computational model we see peaks in transmission do not always have the corresponding peaks in energy (Fig. 1.5). We will call the frequencies for which transmission peaks occur resonant frequencies. From this plot of transmission and energy versus frequency we can tell that the ratio $T/\mathcal{E}$ is not going to be flat since there are spikes in transmission that do not have the counterpart in energy.

Another quantity we can calculate in our computational model (in addition to the electric field and its derivative) is the amplitudes of the left and right traveling waves: A and B. Electric field in each layer is

$$\begin{aligned} \mathcal{E}(x) &= A\exp(iknx) + B\exp(-iknx) \\ A &= \frac{1}{2}(E - i\frac{1}{k}\frac{dxi}{dx}) \\ B &= \frac{1}{2}(E + i\frac{1}{k}\frac{dE}{dx}) \end{aligned} \tag{6}$$

For resonant frequencies $\|A\| \simeq \|B\|$, which implies that it is almost like a standing wave with very little energy leakage, which is expected.

If the light is incident on the same sample but from the other side (ie right-to-left orientation) we see a different plot for transmission and energy versus frequency. Again, some peaks in transmission have no corresponding peak in energy while others do.

Figure 1.6. In the left plot of the electric field (which is proportional to the energy stored in the sample) in the random media at resonant frequencies on a log scale there is a maximum near $\frac{L}{4}$. This means there was exponential growth to the center of localization, followed by exponential decay. The ratio of the amplitude at x=0 to x=L is approximately equal to the transmission. In the right plot we see a peak in electric field (energy) at $\frac{3L}{4}$. For light to reach the center of localization the light first exponentially decays, then grows, and falls again. The red dotted lines are the estimated slopes, which correspond to $\exp(\pm x/\xi)$.

[Need a T, $\mathcal{E}$ v. freq plot showing RL is different]

Fluctuating T/$\mathcal{E}$ is not of a significant concern, since we are in the passive system but the fact that it depends on illumination (from left of from right) is. It means that if we use this as a criterion, we will get a different value depending on how we perform the measurement. We will pay attention to this fact when we introduce the gain.

We investigate what is going on at those peak transmission frequencies. We would like to know why some peaks in transmission have corresponding peaks in energy while others do not. We also notice there are no cases where a peak in energy occurs and not transmission.

First, we confirm that for off-resonant frequencies we see exponential decay of the electric field through the sample. This is expected for random media in localization regime. When the exponential decay is plotted on a log scale we get a straight line.

Now we pick one resonant frequency and plot the electric field in the system. This is equivalent to looking at the energy inside the system.

Based on our analysis of many peaks in transmission, we conclude: (i) At the frequencies where peaks in transmission and energy occur together there is a center of localization at $0 < x < \frac{L}{2}$. (ii) Where there is no corresponding peak in energy for a peak in transmission the center of localization occurs $\frac{L}{2} < x < L$. We will use examples where the center of localization happens at $\frac{1}{4}$L, $\frac{1}{2}$L, and $\frac{3}{4}$L as canonical centers of localization and we will treat them as specific examples. Other positions can be interpolated.

We would like to know why the energy distributions change when the position of the center of localization changes. The linear (when viewed on a semilogy scale) slopes in these plots correspond to the localization length. This behavior has been

considered by Azbel, but only for $0 < x < \frac{1}{2}$L. [**?**]

We calculate the energy stored in the system for different positions of a single center of localization and plot Eq. 7 in Fig. 1.8, which explains different ratios of the energy peak to the peak transmission.

$$\mathcal{E}(x) = \begin{cases} \xi(-1 + 2\exp(\frac{x_1}{\xi}) - \exp(\frac{2x_1 - L}{\xi}) & if \quad x < \frac{1}{2}L \\ \xi(1 + 2\exp(\frac{2L - 3x_2}{\xi}) - 3\exp(\frac{-2x_2 - L}{\xi}) & if \quad x > \frac{1}{2}L \end{cases} \tag{7}$$

1.3.3. Passive non-random 1D models.

Normally when an electromagnetic wave is incident on a material that does not allow transmission the wave exponentially decays in the material.

Any wave that does make it through the material can be said to have tunneled through the material. Tunneling phenomenon is present in quantum mechanics and band gap materials. We analyze two idealized passive analytical models to confirm the behavior observed in the previous section.

The first system consists of a potential barrier and the energy of incident wave is smaller then height of the barrier. To introduce what would be a center of localization in a random system we add a small well in the barrier (making two close potential barriers). The second model is a periodic structure with a single defect acting as the analog of the center of localization. The advantage of these analytical models is that we can manually position what would be the center of localization by specifying where the well is and where the defect is, respectively. This is in comparison to the random model, where we can not specify where the center of localization should be.

On a linear scale the behavior of the wave magnitude in the double barrier potential model is not obvious, but when the same wave is plotted on a log scale (Fig. 1.7) it is easy to see the comparison to the random model. Straight lines of increasing and decreasing wave amplitude for a and c are the same as the canonical wave forms in Fig. 1.8 a and b.

In the periodic layers with single defect the plots are the similar to the potential barrier model results, but with a more pronounced peak. For this analytical model we manually position the defect (instead of the potential well).

Figure 1.7. A wave incident from the left (solid blue line) on a potential barrier with potential well (dotted green) on semilogy scale looks very similar to the electric field (aka energy) in the random system (Fig. 1.8). We hypothosize that tunneling may be the reason for the exponential decay and growth. Also seen in [**?**], but this is cleaner.

These three models (random dielectric layers, potential barrier, and periodic layers) are closely related since all three are wave based.

Based on the passive analytical models we gain confidence that the light is tunneling through the material. If there is a center of localization present in the random layers this is akin to the light tunneling to the potential well between the barriers and then tunneling out.

Also based on the analytical models we determine that the position of the center of localization/potential well/single defect directly affects how much energy is stored in the system.

This variation of stored energy based on the position of the center of localization explains why there are peaks in energy but not always corresponding peaks in energy. It also explains why peaks in energy imply peaks in transmission: the energy stored in the sample means there is a center of localization in the sample, which allows for increased transmission. Now when we see a transmission and energy plot versus frequency we can determine the location of the center of localization by inspection. If there is a peak in transmission and energy, then $0 < x < \frac{1}{2}L$. If there is a peak in transmission but not energy then the center of localization is between $\frac{1}{2}L < x < L$.

Presence of a center of localization for some frequency can be detected through the deviations from a simple exponential decay in the field distribution inside the sample.

1.3.4. Theory of passive 1-D systems.

We derive the T/$\mathcal{E}$ ratio. The transmission T as a function of frequency ω and center of localization position x_o is a lorentz distribution of the form

$$T = T(\omega, x_o) = \frac{(\exp(-\frac{L}{\xi}))^2}{(\omega - \omega_o)^2 + (\exp(-\frac{L}{\xi})\frac{1}{2}\exp(\frac{|L-2x_o|}{\xi}))^2} \tag{8}$$

Figure 1.8. The above plots are from the periodic layers with single defect model. The left half are from the defect occuring at $\frac{1}{4}$L and the right set of plots are from the defect at $\frac{3}{4}$L, all without gain. The top plot on both sides is transmission versus frequency centered at the resonant frequency. For both $\frac{1}{4}$L and $\frac{3}{4}$L the resonant frequency means a spike in transmission. However, the lower plot on both side (Energy versus frequency) shows that the amount of energy is dependent on the defect position (which corresponds to the center of localization in a random model). Looking at the behavior of the energy inside the sample for resonant and off-resonant frequencies, the behavior of the wave changes as close the resonant frequency. When off resonance there is pure exponential decay, which translates to a straight decreasing slope on the semilogy scale. On resonance there is a peak in energy for both defect positions. However, the total energy in the sample is much less for $\frac{3}{4}$L (remember that these are semilogy plots). This demonstrates why $T/\mathcal{E}$ is not constant in passive samples: the ratio is affected by the position of the defect (center of localization for random layers).

Figure 1.9. Energy distribution as a function of position of center of localization x_0 on resonant frequencies without gain (dotted red) and with 99% of critical gain (upper solid blue). See Eq. 9. These plots show why the peaks in transmission do not always have a corresponding peak in energy, as seen in Fig. 1.8. and Fig. 1.5. Even when gain is added the asymmetry does not go away.

Where we have appoximated cosh() as $\frac{1}{2}\exp()$.

From work done in Appendix B,

$$\mathcal{E}(x,\omega) = \frac{\xi}{2}(T(\exp\left(\frac{2(L-x_o)}{\xi}\right) - \exp\left(\frac{2(L-2x_o+x_1)}{\xi}\right) + \exp\left(\frac{2(L-x_o))}{\xi}\right) - 1) + 4(1 - \exp\left(-\frac{2x}{\xi}\right. \tag{9}$$

As a check on Energy we can look at the resonant frequencies as the center of localization position x_o varies. We recover Fig. 1.9.

Now we will look only at resonant frequencies ($\omega = \omega_0$) with sufficient gain that x_1 goes to zero (then y_2 has a range from zero to x_1. This includes passive systems where the defect is in the first half of the sample and if the defect is in the second half, then the gain is sufficient that there is no initial exponential decay regardless of

Figure 1.10. Transmission/Energy as a function of position of center of localization x_0 on resonant frequencies without gain. Eq. 9 divided by Eq. 13. This is why T/$\mathcal{E}$ will not work for a criteria; the ratio depends on where the center of localization occurs. One can not average over all possible positions.

Figure 1.11. T/$\mathcal{E}$ versus frequency for passive systems with the correction from Eq. 13. Frequency is zero on resonance. This plot show that if T/$\mathcal{E}$ is attempted to be used as a criteria for finding localization for an ensemble of samples, the peak and background obsure each other. The problem is the same for $\mathcal{E}$/T.

defect position. Visually, the plot has a single clusp.

$$\frac{\mathcal{E}}{T} = \frac{\xi}{2}(2\exp\left(\frac{2(L-x_0)}{\xi}\right) - \exp\left(\frac{2(L-2x_0)}{\xi}\right) - 1)) \tag{10}$$

If we assume x_0 is not near the edges of the sample $x_0 > 0$ and $x_0 < L$ with $L > \xi$ then we can drop the last two terms, leaving

$$\frac{\mathcal{E}}{T} = \xi\exp\left(\frac{2(L-x_0)}{\xi}\right) \tag{11}$$

Inverting that to get T/$\mathcal{E}$,

$$\frac{T}{\mathcal{E}} = \frac{1}{\xi}\exp\left(-\frac{2(L-x_o)}{\xi}\right) \tag{12}$$

Note that the log of either results in a non-exponential form which could be integrated. Neither T/$\mathcal{E}$ nor $\mathcal{E}$/T is particularly useful. When one averages $\mathcal{E}$/T the background dominates, whereas for T/$\mathcal{E}$ the resonant frequency behavior dominates. See Fig. 1.11.

When gain is added, the left boundary condition ($y_1 = 4$ at $x = 0$) is no longer valid. The entire field distribution grows in height, including the end points. This would be equivalent to letting x_1 (the turning point) become negative, which is non-physical. In order to prevent this we change the limits of integration to be max(x_1,0) instead of x_1 in Eq. A.14.

Figure 1.12. The log of the field in the sample for three canonical positions for the periodic 1-D structure with single defect. In this analytical model the single anomalous layer is at $\frac{1}{4}$L, $\frac{1}{2}$L, and $\frac{3}{4}$L, respectively and the amount of absorption or gain is varied. The lowest dotted blue straight line is full absorption. On a log scale the exponential scale appears as a more obvious straight negative slope line. As absorption is decreased and the passive system is approached a turning point appears. For the $\frac{1}{4}$L and $\frac{1}{2}$L systems this turning point disappears at the passive system incident side. Then gain is added to the model (solid red line). For the $\frac{1}{4}$L and $\frac{1}{2}$L defects the peak simply increases, as expected in active media. For the $\frac{3}{4}$L system the turning point moves toward the incident side. Thus in active media the wave form changes for defects occuring in the second half of the sample. Eventually the gain is sufficient that the $\frac{3}{4}$L position is symmetric with the $\frac{1}{2}$L position. Now in all three canonical positions with gain the field in the sample grows exponentially to a peak before falling.

Another problem arises with T from Eq. 8 in that the "worst case" (minimum for T) is when we are so far off resonance that only exponential decay occurs. As the above equations are given T goes to zero, which is incorrect. Physically, T asymptotically reaches the exponential decay value. Thus in order to make this correction, we shift the Eq. 8 up by the exponential decay background.

$$T = T(\omega, x_0) = \frac{(\exp(-\frac{L}{\xi}))^2}{(\omega-\omega_0)^2 + (\exp(-\frac{L}{\xi})\frac{1}{2}\exp(\frac{|L-2x_0|}{\xi}) - g)^2} + \exp\left(-\frac{L}{\xi}\right)^2 \quad (13)$$

1.3.5. Adding gain.

Now we understand why the ratio of transmission to energy (T/$\mathcal{E}$) is not smooth in passive systems: the ratio depends on the position of the center of localization. When gain is added to the random system we hope that T/$\mathcal{E}$ becomes smoother because we do not expect such a dramatic dependence on the position of the localization center.

In order to reach the critical gain we investigate the periodic system with single defect. We did not test the addition of gain in the double potential single well model.

When gain is added to the three canonical cases we see a rise in the peaks and the peak position remains the same. The tail for the $\frac{3}{4}$L does not change, only the peak increases. (Fig. 1.12).

Note that near the critical gain, the transmission and the field distribution inside the sample becomes very sensitive to the amount of gain added to the system.

At the critical gain the defect at $\frac{1}{4}$L and the defect at $\frac{3}{4}$L cases are symmetric with respect to each other about the y axis. The defect at $\frac{1}{2}$L is symmetric about x=$\frac{1}{2}$L and the defect at $\frac{1}{4}$L would be symmetric with the defect at $\frac{3}{4}$L if it were flipped around. [Foreshadowing, we will get back to this.]

At the critical gain the ratio of transmission to energy is smoother than the passive system, but there are some issues for when the center of localization is between $\frac{1}{2}$L and L. We have found that T/$\mathcal{E}$ is a "good" parameter for 1-D systems with gain, in that it is not divergent (unlike T and $\mathcal{E}$ are separately). It can be measurable. However, it is not perfect as it depends on sample orientation.

1.3.6. Closed and open systems theory for passive 1-D models.

The idea of flipping the sample orientation warrants attention. If there is a center of localization at $\frac{1}{4}$L and the sample is flipped (sample goes from L to 0 instead of the standard 0 to L) then we know there will be a center of localization at $\frac{3}{4}$L. The waveform for these two localization centers is the same in the region of localization.

An equivalent to flipping the sample orientation is to change the orientation of the light. If one keeps the sample fixed and shines light on the x=L side of the sample then we get the same result as fixing the light and flipping the sample orientation. The math is slightly different for the two cases.

We will call the case where light is incident on the left side "LR" as light moves left to right. Conversely we will call the light being incident on the right side "RL". Also, we will call the resulting LR field distribution $\psi_1(x)$ and the RL wave $\psi_2(x)$. The flipping we will term "reciprocity."

A reality check for flipping the sample (or flipping the light source) is that the transmission should be the same for the two cases since the light is traveling through the same material. Note that flipping both the sample and the light gets one back to the original setup.

Thouless criteria says that a small change in boundary conditions should not affect energy spectrum. However, we find that flipping the beam (which is a small change in boundary conditions) does cause an enormous change *in the waveform in the sample*. This affects how much energy is in the system, ie the waveform for

Figure 1.13. For the two orientations, left-to-right and right-to-left, two wave functions are assigned: ψ_1 and ψ_2. The wronskian for is the same for both since the transmission is the same for both orientations.

localization at $\frac{1}{4}$L becomes the waveform for localization at $\frac{3}{4}$L. There is a lot less energy in the system for the $\frac{3}{4}$L waveform as compared to $\frac{1}{4}$L. A small change in the boundary conditions leads to a significant change in how much energy is stored in the sample.

How can reciprocity help to identify localization? Experimentally one can shine light on either side of a given sample, so reciprocity is straight forward to execute. What follows is the mathematical analysis of using reciprocity.

We start by showing that the two wave forms are functionally independent by finding the Wronskain for the left sides of the two orientations. It can be shown that for our Maxwell equation the Wronskian is position independent.

$$\begin{vmatrix} 1\exp(ikx) + r\exp(-ikx) & t'\exp(-ikx) \\ ik\exp(ikx) - ikr\exp(-ikx) & ikt'\exp(ikx) \end{vmatrix} = -2ikt' \tag{14}$$

Similarly, the Wronskain for the right sides is:

$$\begin{vmatrix} t\exp(ikx) & 1\exp(-ikx) + r'\exp(ikx) \\ ikt\exp(ikx) & -ik\exp(-ikx) + ikr'\exp(-ikx) \end{vmatrix} = -2ikt \tag{15}$$

Note that the two Wronskians are non-zero and are the same since t = t'. Now that we have to independent functions we can use them as a basis to create a new linear combination.

An analog is the one dimensional potential well with an eigen state in the well.In a closed system we can find eigen states and eigen frequencies. For the passive random model this would be equivalent to adding mirrors to the edges of the sample so that no light escapes to the environment.

For the open system (both for potentials and in the case we are working with, layers of dielectric) there will be two solutions, a left and right traveling wave. For

closed systems there is one solution (composed of two waves - standing wave).

Closed system solution can be expressed in terms of $\psi_1(x), \psi_2(x)$:

$$\psi(x) = c_1\psi_1(x) + c_2\psi_2(x) \tag{16}$$

where ψ_1 and ψ_2 are the wave functions associated with the LR and RL orientations in the random system. To find the eigenstates apply the boundary conditions: the function ψ(x) is zero at the edges of the closed system.

$$\begin{aligned} \psi(x=0) &= c_1\psi_1(0) + c_2\psi_2(0) = 0 \\ \psi(x=L) &= c_1\psi_1(L) + c_2\psi_2(L) = 0 \end{aligned} \tag{17}$$

This can only be valid when the determinant is zero

$$det \begin{bmatrix} \psi_1(0) & \psi_2(0) \\ \psi_1'(L) & \psi_2'(L) \end{bmatrix} = 0 \tag{18}$$

This ψ_1 and ψ_2 can be interpreted as either the electric field at the edges or related to the reflection and transmission coefficients:

$$\begin{aligned} E(0) &= \psi_1(0) = 1 + r \\ kE'(0) &= \psi_1'(0) = ki(1 - r) \end{aligned} \tag{19}$$

or equivalently t t' - (1+r) (1+r') = 0.

Now we can go back to the random system and find the determinant given ψ_1(x) and ψ_2(x). When the determinant is zero that implies the frequency is an eigen frequency (in a closed system) and the wave function is an eigenstate. Since the layers actually make an open system, we will call these "resonant frequencies." Scanning the range of frequencies yields points on the T and $\mathcal{E}$ versus frequency plot that correspond (perfectly) with the peaks in energy in transmission.

Note that when the transmission and energy versus frequency is plotted then the transmission for the LR and RL cases is the same (t=t'), whereas the energy is not the same for the LR and RL orientation. The energy peaks at different frequencies, resulting in two lines for energy, whereas there is one for transmission. The dots are

Figure 1.14. Transmission, energy, and determinant=0 for LR and RL orientations versus frequency. This is similar to Fig. 1.5, except now we can predict where the peaks occur.

Figure 1.15. A resonant frequency showing drop-off at x=L. Even though the center of localization may not be obvious for this resonant frequency, the fact that the electric field approaches zero on both sides of the sample means the field is approaching a closed system resonance.

where the determinant is zero.

We can test that the states at these resonant frequencies are orthogonal. See Deych's paper[**?**].

$$\frac{\int_0^L \psi_1 \psi_2^* \epsilon(x) dx}{\sqrt{\int_0^L \psi_1 \psi_1^* \epsilon(x) dx} \sqrt{\int_0^L \psi_2 \psi_2^* \epsilon(x) dx}} = 0 \tag{20}$$

Note that when the center of localization is at $\frac{1}{4}$L for resonant frequencies the electric field at x=0 is close to zero. (See Fig. 1.15.) Reason: the waves outside the material are cancelling out due to interference. This results in a standing wave. r=-1 (from 1+r=0).

This is a reliable and accurate method for find lasing threshold in computational models. To find the critical gain first find a resonant frequency by calculating where the determinant is zero. Then for each resonant frequency plot the transmission versus gain. The transmission will diverge at critical gain. The amount of gain added becomes very sensitive near the critical gain, but one can find arbitrarily small bounds on what the critical gain is by zooming in on the peak.

An alternative method for finding the critical gain is to look at the transmission versus frequency plot and measure the full width at half maximum (FWHM), γ.

$$g_{critical} = \frac{\gamma}{2} / \frac{2\pi}{\lambda} \tag{21}$$

Experimentally the light has to be shone from both left-to-right and right-to-left, but the result is to find what would be eigen frequencies of the closed system. We will still call these the resonant frequencies since we are in the open system.

Figure 1.16. A and B coefficients for the two orientations (left to right and right to left). The lower two lines (dotted green and dotted black) are the backward (B) and forward (A) propagation coeffients, respectively of the electric field for the left-to-right orientation (both start out near unity on the incident side). The forward propagation A is non-zero on the transmission side of the sample and dotted green B falls to zero. The upper red and blue lines are hard to distinguish through most of the sample because they overlap. The red and blue lines are A and B for the right-to-left orientation since both start out near unity on the right side and only near the transmission side does it become apparent that the solid red is forward propagating A whereas dotted blue falls to zero and is the B coefficient. There is very little difference between A and B for each orientation which means there is little energy difference in the two wave directions. Normally for non-resonant frequencies A and B are very different.

As a reality check, the wave coefficients A and B should be equal throughout the sample at resonant frequencies since the left and right-traveling waves are equal (see Fig. 1.16. A=B also implies that there is no oscillation in energy at the resonant frequency, thus there is no energy leakage out of the sample.

If A=B then the two complex waves should sum to a real wave:

$$A\exp(ikx) + A\exp(-ikx) = A\cos(kx) \tag{22}$$

Claim: the standing wave has constant phase throughout the sample [We should go back and study this.]

1.3.7. Overview of finding criteria for localization.

In forming a criteria for localization (in both passive systems and systems with gain) it is important to remember what distinguishes localization from diffusion. The diffusion equation does not keep track of phase, thus it can not account for localization, which is a result of wave interactions with phase. Regardless of whether one is in the absorption, passive, or active system, a criteria for localization needs to be able to account for phase. Constant phase means there is a center of localization occurring. Since phase is not (?) an experimentally measurable quantity then the criteria is going to need to be able to be dependent on or directly related to phase.

1.3.8. Add gain to random 1-D model.

In the passive system ψ_1 and ψ_2 form an orthonormal basis for ψ(x) where ψ_1 and ψ_2 refer to the two orientations for shining light on the sample, left-to-right and

right-to-left, respectively.

When gain is added to the system, we have a new $\tilde{\psi}$ which depends on the gain:

$$\tilde{\psi}(x,g) = \alpha(g)\psi_1(x) + \beta(g)\psi_2(x) \tag{23}$$

Where we can find the complex coefficients α and β from

$$\begin{aligned}
\beta_1 &= \frac{\int_0^L \psi_1 \tilde{\psi_1^*}\epsilon(x)dx}{\sqrt{\int_0^L \psi_1\psi_1^*\epsilon(x)dx}\sqrt{\int_0^L \tilde{\psi_1}\tilde{\psi_1^*}\epsilon(x)dx}} \\
\alpha_1 &= \frac{\int_0^L \psi_2 \tilde{\psi_1}\epsilon(x)dx}{\sqrt{\int_0^L \psi_2\psi_2^*\epsilon(x)dx}\sqrt{\int_0^L \tilde{\psi_1}\tilde{\psi_1^*}\epsilon(x)dx}}
\end{aligned} \tag{24}$$

The coefficients depend on the orientation of the light, so there is another set of coefficients for the other orientation. Similarly for β_2 and α_2,

$$\begin{aligned}
\beta_2 &= \frac{\int_0^L \psi_1 \tilde{\psi_2^*}\epsilon(x)dx}{\sqrt{\int_0^L \psi_1\psi_1^*\epsilon(x)dx}\sqrt{\int_0^L \tilde{\psi_2}\tilde{\psi_2^*}\epsilon(x)dx}} \\
\alpha_2 &= \frac{\int_0^L \psi_2 \tilde{\psi_2}\epsilon(x)dx}{\sqrt{\int_0^L \psi_2\psi_2^*\epsilon(x)dx}\sqrt{\int_0^L \tilde{\psi_2}\tilde{\psi_2^*}\epsilon(x)dx}}
\end{aligned} \tag{25}$$

A reality check on α and β is that

$$\|\alpha\|^2 + \|\beta\|^2 = 1 \tag{26}$$

These two sets of coefficients can be plotted with respect to gain. We scan gain (by adding complex refractive index) from zero (passive systems) up to the critical gain.

What we get from these two plots in Fig. 1.17 is that the two wave functions from the passive system are added up to get the wave form with gain, $\tilde{\psi}$. The coefficients α and β tell how much of each passive waveform to add. In the case of a wave function with center of localization at $\frac{1}{4}$L there is hardly any contribution from the wave function with center of localization at $\frac{3}{4}$L, so α and β remain at their original 0

Figure 1.17. The left most plot is the electric field for a resonant frequency (blue shows a center of localization at $\frac{1}{4}$L and is the LR orientation while red shows a center of localization at $\frac{3}{4}$L and is the RL orientation. The other two plots are the alpha and beta coefficients scanning gain from passive to critical gain at a resonant frequency for the two orientations. The center is LR and the right is RL. The crossover takes place very near critical gain. This explains why the amount of gain added is more sensitive near critical gain.

and 1 values. For the other canonical case α starts with hardly any contribution to $\tilde{\psi}$ but as gain is added α dominates.

1.4. SUMMARY

We were interested in finding a criteria for localization in media with gain. We suspected the ratio of transmission and energy may be a good parameter.

The analytical diffusion model was the first thing we investigated. Even though localization could not be present the diffusion equation demonstrated that T/$\mathcal{E}$ was not divergent at the lasing threshold (even though transmission and energy both diverged).

Next we created a computational model in Fortran for random layers of dielectric. In this system there is no diffusion and localization does occur. We used the transfer matrix method to propagate light using Maxwell's equations and recorded transmission, reflection, and electric field in the sample, (total energy in the sample).

Starting in the passive model we see that T/$\mathcal{E}$ is neither smooth nor constant due to peaks in transmission that lack a corresponding peak in energy. We then determined why this was occurring. The electric field in the sample at that specific frequency lead us to show that localization is wave "tunneling" though the medium. We explain this by looking at two analytical models, the double barrier potential with varying well position and the periodic structure with a single defect. Now we know why there are peaks in transmission but not always energy. This explains why T/$\mathcal{E}$ is not smooth.

Adding gain to the system makes T/$\mathcal{E}$ smoother and it is a good parameter (in that it does not diverge) but T/$\mathcal{E}$ is not constant due to the center of localization varying in the sample for different frequencies.

Flipping the sample and keeping the light that shines on the sample in the original orientation moves the center of localization to the opposite side. This is equivalent to keeping the sample fixed and flipping the light source orientation (now the light shines on the x=L side).

This change in boundary conditions (where the light is shined from) leads to insight on why the waveform of the electric field changes. Reciprocity in the passive system allows us to construct two waveforms that constitute a basis, similar to the closed system and looking for eigenstates. Adding gain to the system we use ψ_1 and ψ_2 as the basis. We use coefficients α and β to build the resultant waveform.

1.5. MEDIUM CREATION

On first inspection, creating a media filled with random scatterers seems a trivial task. This is true for low density, as one simply picks random positions and ensures that no two scatterers are too close to each other. However, localization occurs in high-density scatterers. There exists a maximum number of spheres that fits in a given volume, and as this upper limit is reach it becomes harder to find the last available volumes to put the scattering sphere in. For randomly positioned scatterers the upper limit is not well defined (realization specific), as the space between scatterers is not constant.

The initial algorithm implemented in code placed a scatterer in the medium by choosing x,y,z from a uniformly random distribution. If the new position conflicts with a previously choosen scatterer (distance between scatterers is too small) then another choice of x,y,z is made. This process is repeated until a valid scatterer position is found. The number of unsuccessful choices grows with the number of total scatterers for a given volume as ?

To reduce this problem alternative algorithms were developed.

Instead of attempting to generate a random media, start with a lattice of scatterers and then let the scatterers take a random walk. However, this results on average that each scatterer is roughly in the same place it started, a lattice.

Another option was to use a lattice and then allow the scattering strength α to be random. This is computationally simple but introduces ?? band gap effects.

Breaking the medium into smaller containers

Stacking scatterers in the x direction. Δx is not uniform, so the distribution for choosing x,y,z was found.

1.6. PROPAGATION ALTERNATIVES

For 1D and quasi1D the incident light is a plane wave. Using self-embedding technique the eigen values numerical error in the transfer matrix was limited. However, for certain parameters with quasi-1D the error prevented full investigation.

An alternative to plane wave is ray tracing, which introduces a different set of complications. Instead of numerical inaccuracy, which can not be overcome, the difficulty is computational time. Individual photons are propagated through the medium, and a large number of photons is needed. There is error introduced in a discrete incident wave. Even with a large number of incident photon paths, there exists the possibility of a path not being taken, since not all y-z choices are used. Normally that would be acceptable, but localization demands that all paths be taken, less the one not taken results in the closed path. By ignoring any path, we disregard a small number of localization occurrences. A finite number of incident photons ignores an infinite number of path possibilities. The plane wave, continuous in y-z, takes all paths but error can build up in the numerical accuracy of the transfer matrix eigenvalues.

2. QUASI-1D

In quasi-1D, the gain can be put in either the scatterers, the empty medium, or both. If one puts gain in the empty medium, then lasing can occur that is not due to localization (lasing due to boundary conditions of the waveguide). Since we are only interested in lasing due to strong localization, then the gain must be mathematically set in the scatterers.

Note: our scattering potentials are based on δ functions, unlike the potentials developed in [**?**] (in Appendix B).

Assume a rectangular metal waveguide with two dimensions, oriented such that the longer dimension is along the z-axis (length), and the shorter direction is the y-axis (width). The rectangular aspect ratio is such that there is no significant propagation along the y-axis (i.e. not two dimensional), but wide enough so that a scatterer does not cause only reflection (which would be one dimensional). Thus we define a quasi-1D waveguide between 1D and 2D. The motivation is to simplify a two-dimensional system using quantization of transverse momentum. This quasi-1D work scales to higher dimensions, unlike true one-dimensional models, which do not have closed channels. Two physical examples of quasi-1D systems are a mesoscopic metal wire and a disordered optical waveguide.

This model permits straightforward generalization to 3D waveguides. However, one would also need to consider the effect of polarization.

our quasi-1D model has the following adjustable parameters: N_c, L, α_s, gain, W, M, ω, sign of α_s. These set the density and N_o. For output, we get directly g, $\mathcal{E}$, $\mathcal{E}^2$, $J_{pw}(z)$, $W_{pw}(z)$, $J_{uf}(z)$, $W_{uf}(z)$, $A(z)$, $B(z)$, $\hat{T}$, $\hat{R}$, correlation of energy(z). From these output, we can get $D(z)$, ℓ_{tmfp}, $E(y,z)$.

2.1. ASSUMPTIONS AND ABSTRACTIONS

For a given model, assumptions must be enumerated.

Specifically, a feature of the quasi-1D waveguide is the delta-function scatterers in free space. This simplifies the math by removing scattering physics, leaving transport physics.

choice of δ-function scatterers has significant consequences:

- easier to model mathematically (no spherical scattering math), while retaining the desired phenomenon: scattering from one channel to another
- inelastic scattering: no energy loss due to scattering when passive, and phase remains coherent [scatterers only affect amplitude].
- non-physical
- no saturation limit for number of closed channels

No noise is assumed (no spontaneous emission). This means that the transmission with gain will be slightly different compared to real output. Jon's model (from Yale) has noise (1D only).

The gain mechanism is purely mathematical: no atomic level modeling is included. This is part of being mesoscopic regime: independent of atomic conditions.

No input beam properties are assumed, other than the capability to be selectively incident on a specific channel.

By chosing planar quasi-1D geometry (and thus scalar waves), we implicitly assume that polarization will not significantly alter transport phenomena. Although planar geometry may be experimentally realizable, it is not as popular as 3D quasi-1D

2.1.1. review of channels.

In a waveguide with two dimensions, the wave number $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$ is broken into orthogonal components as $k^2 = k_\parallel^2 + k_\perp^2$. Solving for $k_\parallel$,

$$k_\parallel = \sqrt{k^2 - k_\perp^2} \tag{27}$$

where $k_\perp$ is defined by the mode number of the sine function perpendicular to the direction of propagation, and is dependent on the waveguide width. The two components are thus

$$\begin{aligned} k_{\perp_n} &= \frac{n\pi}{W} \\ k_{\parallel_n} &= \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{n\pi}{W}\right)^2} \end{aligned} \tag{28}$$

Note that for sufficient channel index, $k_{\|}$ becomes imaginary, denoted $i\kappa_{\|}$. These channels, with an imaginary wave number, are the evanescent channels. The intensity decays exponentially because the exponent is real ($e^{i(i\kappa x)}$).

2.2. INTRODUCTION

In experimental studies of quasi-1D systems with randomly positioned scatterers there are open (propagating) channels and closed (evanescent) channels. Dorokhov-Mello-Pereyra-Kumar (DMPK) theory for quasi-1D geometry assumes (1) all open channels are the same, and (2) closed channels can be disregarded.

Since the closed channels decay exponentially one normally neglects these contributions in numerical simulations. If the detector is far way from the media then the closed channels won't contribute.

This is desirable, as the math is easier (which translates computationally to faster), and it is justified analytically by calculations done for single scatterers [**?**] [**?**] and by rationalizing that if the density of scatterers is low enough, then there is sufficient separation between scatterers that closed channels do not contribute to interaction. However, since we are interested in Anderson localization, we will be in the high density, low scattering strength regime. In this regime it would appear that we can not ignore the effect of closed channels. Then the ability to numerically model systems becomes questionable since real systems have an infinite number of closed channels.

A numerical simulation is described that models an arbitrary number of point scatterers in a quasi-1D waveguide where each has the same scattering strength. We can vary the number of open and closed channels, scattering strength, system dimensions, and scatterer density. The significance of closed modes can be studied through numerical simulation to determine how a finite number of these modes affect the wave propagation through the random medium. In a real system there are an infinite number of closed channels. It is shown that closed channels can be ignored, which is useful as accounting for an infinite number is difficult or impossible. Using this computational model we verify theoretical concepts. Self-embedding technique [**?**] extends the limits of numerical simulation when closed channels are added, but this does not allow for an arbitrary number of closed channels as numerical inaccuracy

still emerges.

The scattering matrices for one scatterer and two scatterers are also derived. We show that folding the closed channels is both possible, and in numerical simulation, necessary.

This work applies to higher dimensions in that it scales, unlike one-dimensional models which do not have closed channels. Physical examples of quasi-1D systems are mesoscopic metal wire and disordered optical waveguide.

Conductance g is observed to be uniform, even if contribution of the channels are non-uniform [need more here].

$$g = \sum_{a} \sum_{b} T_{a,b} \tag{29}$$

Scattering length as a function of three parameters, density, scattering strength, and number of closed, or evanescent, channels is determined. (?)

Quantization of transverse momentum allows for the simplification of this two-dimensional system to a one-dimensional system referred to as quasi-1D. This model permits straightforward generalization to 3D waveguides, in which one would need to consider the effect of polarization, such as long-range mesoscopic correlations (?) of fields with different polarization.

2.2.1. Numerical Simulation Model Description.

Evolution of electric field through the system is described with a set of transfer matrices in terms of open and closed channels of the waveguide. This ensures the continuity of both the electric field and its derivative throughout the waveguide. Parameters such as the length and width of the waveguide, the number of open and closed channels (real and evanescent modes), the number of scatterers, and the scattering strength of scatterers determine how the light will propagate through the waveguide. The wave vector k for both open and closed channels is calculated, n

being the mode number, W the width, c speed of light in a vacuum.

$$\begin{aligned} k_{\perp n} &= n\pi/W \\ k_{\parallel n} &= \sqrt{(w/c)^2 - k_{\perp n}^2} \end{aligned} \tag{30}$$

The wavelength is normalized to one for all simulations. Each channel, open and closed, has both a perpendicular and parallel wave vector components. From Equ. 30 if $K_{\parallel n}$ is real the channel is open, $K_{\parallel n}$ imaginary corresponds to closed. The total number of channels (N_{max}) is studied systematically to determine the effects of closed channels and includes the open channels.

Wave propagation is described as follows. The values for the electric field and its derivative are stored for each mode at each layer, scatterer or free space, in matrices. The electric field and its derivative for free space propagation of open channels are given by:

$$\begin{aligned} E_n(x+\Delta x) &= E_n(x)cos(K_{\parallel n}\Delta x) + K_{\parallel n}^{-1}E_n^{'}(x)sin(K_{\parallel n}\Delta x) \\ K_{\parallel n}^{-1}E_n^{'}(x+\Delta x) &= -E_n(x)sin(K_{\parallel n}\Delta x) + K_{\parallel n}^{-1}E_n^{'}(x)cos(K_{\parallel n}\Delta x) \end{aligned} \tag{31}$$

For the closed channels, the field equations are the similar to Equ. 31 except K is replaced by the imaginary κ.

The electric field after each scattering event must be equal to the initial field, where the first derivative is given by:

$$\begin{aligned} E_n(x+\Delta x) &= E_n(x) \\ E_n^{'}(x+\Delta x) &= E_n^{'}(x) - \alpha(\frac{\omega}{c})^2 \sum_{m=1}^{n_{max}} A_{nm}E_m(x) \end{aligned} \tag{32}$$

and

$$A_{nm} = \frac{2}{W}sin(K_{\perp n}y)sin(K_{\perp m}y) \tag{33}$$

Figure 2.1. Propagation or Free Space Matrices (left), and Scattering Matrices (right). "x" denotes non-zero terms, which are not necessarily the same. Each matrix is separated into four $N_{max}xN_{max}$ quadrants (long dashed portions), each containing information from open channels (upper left short dashed portions) as well as closed (lower right short dashed portions).

$$K_{\|n} = i\kappa_{\|n} \tag{34}$$

Alpha describes the scattering strength of the scatterers. Thus wave propagation can be described by $2N_{max}x2N_{max}$ matrices of two types: propagation or free space matrices, and scattering matrices. See Fig. 2.1

These matrices are combined to form a total matrix. This matrix contains the propagation of the CW electric field information through the waveguide. The total matrix is used to calculate the reflection and transmission coefficients for each mode.

2.2.2. Self-Embedding.

Due to the numerical instability which builds up in longer disordered waveguides, the computation of the simple product of individual matrices is not sufficient, as the eignvalues become divergent. The self-embedding method is introduced slow the growth of computational error inherent in numerical matrix multiplication. This method consistently corrects any deviation caused by rounding on every step of the multiplication and gives increased accuracy to the final result.

Start with the vector describing N open and N_c closed channels for both the electric field and the field derivative, total length $N + N_c + N + N_c$, on the incident side of the medium: $\vec{v}(0)$. There is a similar vector on the otherside of the quasi-1D waveguide, $\vec{v}(L)$, of the same dimension. For $\vec{v}(0)$, assume a an input beam in one channel, n_0. (Later on the input channel will be iterated over all open channels. The input could be to closed channels, but that is not investigated in this letter.)

Given $\vec{v}(0)$ and $\vec{v}(L)$ a pair of matrices is sought that satisfies

$$\hat{G}\vec{v}(0) + \hat{H}\vec{v}(L) = \vec{v}_{BC} \tag{35}$$

Figure 2.2. Boundary conditions (?). H on left, G on right.

Figure 2.3. conductance versus number of closed channels N_c for varying scattering strength and density. (Which is which?) Dotted blue is, solid green is, dot-dashed red is, and dashed black is .

such that $\vec{v}_{BC}$ has no dependence on variables, transmission, or reflection. Any $\hat{G}$ and $\hat{H}$ that satisfies these conditions can be used. We use Fig. 2.2 which results in $\vec{v}_{BC}$ having zeros in all elements except for the input channel, where there is a "2".

Now that $\hat{G}$ and $\hat{H}$ are set, S is

$$\hat{S} = (\hat{H} + \hat{G})^{-1} \tag{36}$$

Self-embedding uses inverses, which add to computational time.

Transmission and reflection vectors are related to the S matrix (part of the total matrix(?)) that is determined via the self-embedding procedure:

Each time the simulation self-embeds multiple matrix inversions must be calculated (?). These calculations take longer than a simple matrix multiplication, but the method prevents the buildup of numerical error (describe this more). More self-embedding loops means a longer simulation run time.

Even with self-embedding technique, the number of closed channels is computationally limited. The numerical limits depend on scattering density, system length (number of transfer matrices).

2.3. NUMERICAL SIMULATION RESULTS

This table shows the preliminary results of the scattering lengths obtained using this method. The scattering lengths describe the rate at which the energy is removed from the input channel. It is a function of two parameters, the scattering density and the strength of the individual scatterer. The results displayed on the following page show, as expected, the scattering lengths decrease as the scattering strength increases. This is also the trend for density.

Figure 2.4. Matrices describing scattering from an input channel to an output channel for 10 open channels; arbitrary units? Left: for few scattering events in the medium it is likely that light will not scattering into a different channel. Even at low scattering probability the channels are not uniform (higher concentration of (?) in lower channel numbers). As the scattering event probability is increased (higher scatterer density or longer system length), inter-channel scattering is observed. At high scattering density (?)

The figures above illustrate the transition which occurs in the transmission matrix (T_{ab}) as the length of the sample increases for a fixed density and scattering strength. A is the input channel number (Horizontal axis) and B is the output channel number (Vertical axis). One can see initially that the shortest sample (left) only the diagonal components of not equal to zero. This corresponds to the wave emerging from the same channel as it was launched into. One can see from the right panel, for a very long system, the light emerges in all output channels.

The model suggests that saturation (a finite number of closed channels) is attainable given that the parameters fit a constraint:

$$(m/(\hbar^2\kappa_n))(\lambda/W) << 1 \tag{37}$$

2.4. WHY CLOSED CHANNELS CAN BE RENORMALIZED: QUASI-1D THEORETICAL MODELS

In order to model quasi-1D system with random defects one must know how to deal with a single defect. It is then assumed that additional defects can be extrapolated. The math for a single defect in a waveguide can be dealt with using transfer matrices or Greene's functions

A matrix that describes open and closed channels is of size $(N + N_c)x(N + N_c)$ where N is the number of open channels, N_c is the number of closed channels. We derive in appendix A how to reduce matrices of rank N_c to rank N, called folding, as the closed channels are incorperated into the open channels. There is a conservation of information, in that the information stored in the original $(N + N_c)x(N + N_c)$ matrix is still contained within the new "folded" NxN matrix, albeit messier elements. Although messier, by induction (using the developed recursion relation) it has been

demonstrated that an NxN matrix can account for N open channels and an infinite number of closed channels N_c. The math done is a correction to [**?**] and is a more explicit reworking of [**?**].

With a single scatterer model scattering strength can be modeled, but the separation between two scatterers is not present. Thus if there are many scatterers the effect of density on scattering length can not be accounted for by a single scatterer. In order to model interaction of scatters by both the open and closed channels, one needs to derive the math for at minimum two scatterers, as is done in

2.5. CONCLUSION

Using a numerical simulation of a quasi-1D waveguide with densely packed randomly placed scatterers the effect of closed channels on was investigated. Using transfer matrices, open and closed channels are varied. The number of matrices multiplied is limited by numerical accuracy, and is extended using self-embedding technique.

However, even with a perfect numerical simulation the effect of closed channels can not be accounted for, as (?) does not reach an asymptote.

Thus analytically folding the closed channels is necessary. The effect on numerical simulations is acceptable by renormalizing conductance.

Verified that the numerical model, analytical single scatterer, and analytical two scatterer results match.

Increasing N_c is effectively the same as increasing system Length. Thus we can disregard N_c (?) Or maybe this is a way to account for it?

The folding (in the appendix) shows that one can disregard N_c by using a different NxN matrix

3. 3D WAVEGUIDE

I'll call this system quasi-1D cubic, to avoid confusion

3.1. APPLICTIONS

4. CONCLUSIONS

Here are my conclusions

APPENDIX A

transmission derivation

$$T = T(\omega, x_o) = \frac{(\exp(-\frac{L}{\xi}))^2}{(\omega - \omega_o)^2 + (\exp(-\frac{L}{\xi})\frac{1}{2}\exp(\frac{|L-2x_o|}{\xi}))^2} \tag{A.1}$$

Where we have appoximated cosh() as $\frac{1}{2}\exp()$.

From the behavior of transmission in media with defects and centers of localizaiton, we see two cases when on a resonant frequency: either the defect is in the first half of the sample, (**??**, plot 1).

$$\mathcal{E}(x) = \begin{cases} A\exp\left(\frac{2(x-x_o)}{\xi}\right) & 0 < x < x_o \\ B\exp\left(-\frac{2(x-x_o)}{\xi}\right) & x_o < x < L \end{cases} \tag{A.2}$$

or it is in the second half of the sample, (**??**, plot 2).

$$\mathcal{E}(x) = \begin{cases} A\exp\left(-\frac{2x}{\xi}\right) & 0 < x < L - 2x_o \\ B\exp\left(\frac{2(x-x_o)}{\xi}\right) & L - 2x_o < x < x_o \\ C\exp\left(-\frac{2(x-x_o)}{\xi}\right) & x_o < x < L \end{cases} \tag{A.3}$$

With either case, we see three distinct regions when the frequency is not on resonance but prior to the pure exponential decay regimes (**??**, plot 3):

$$\mathcal{E}(x) = \begin{cases} y_1 = A\exp\left(-\frac{2x}{\xi}\right) & 0 < x < x_1 \\ y_2 = B\exp\left(\frac{2(x-x_o)}{\xi}\right) & x_1 < x < x_o \\ y_3 = C\exp\left(-\frac{2(x-x_o)}{\xi}\right) & x_o < x < L \end{cases} \tag{A.4}$$

Where x_1 is the turning point.

From the off-resonance Eq. A.4, we apply boundry conditions to determine the coefficients in passive systems. We take gain into account and make corrections later.

At $x = 0$ then $y_1 = 4$, giving $A = 4$. Thus

$$\boxed{y_1 = 4\exp\left(-\frac{2x}{\xi}\right)} \qquad 0 < x < L - 2x_o \tag{A.5}$$

At $x = L$, $y_3 = T$. Solve for C,

$$C = T \exp\left(\frac{2(L - x_o)}{\xi}\right) \tag{A.6}$$

plug back in,

$$y_3 = T \exp\left(\frac{2(L - x_o)}{\xi}\right) \exp(-\frac{2(x - x_o)}{\xi}\right) \qquad x_o < x < L \tag{A.7}$$

simplify to get

$$\boxed{y_3 = T \exp\left(-\frac{2(x - L)}{\xi}\right)} \qquad x_o < x < L \tag{A.8}$$

at x_o, $y_2 = y_3$

$$B \exp\left(\frac{2(x_o - x_o)}{\xi}\right) = B = T \exp\left(-\frac{2(x_o - L)}{\xi}\right) \tag{A.9}$$

plug in to y_2

$$y_2 = T \exp\left(-\frac{2(x_o - L)}{\xi}\right) \exp\left(\frac{2(x - x_o)}{\xi}\right) \qquad L - 2x_o < x < x_o \tag{A.10}$$

simplify to get

$$\boxed{y_2 = T \exp\left(\frac{2(x + L - 2x_o)}{\xi}\right)} \qquad L - 2x_o < x < x_o \tag{A.11}$$

The turning point x_1 varies as a function of frequency. Boundry conditions on x_1 are that it should remain less than x_o and that it be non-negative. To solve for x_1, see where $y_1 = y_2$

$$4 \exp\left(-\frac{2x_1}{\xi}\right) = T \exp\left(\frac{2(x_1 + L - 2x_o)}{\xi}\right) \tag{A.12}$$

$$\boxed{x_1(\omega) = -\frac{\xi}{4} \log(\frac{1}{4}T) - \frac{1}{2}L + x_o} \tag{A.13}$$

Knowing y_1, y_2, and y_3 we can integrate to find the energy $\mathcal{E}$:

$$\mathcal{E}(x,\omega) = \int_0^{x_1} 4\exp\left(-\frac{2x}{\xi}\right)dx + \int_{x_1}^{x_o} T\exp\left(\frac{2(x+L-2x_o)}{\xi}\right)dx + \int_{x_o}^{L} T\exp\left(-\frac{2(x-L)}{\xi}\right) \tag{A.14}$$

solve, reduce to get the energy as a function of x, frequency, and transmission:

$$\mathcal{E}(x,\omega) = \frac{\xi}{2}(T(\exp\left(\frac{2(L-x_o)}{\xi}\right) - \exp\left(\frac{2(L-2x_o+x_1)}{\xi}\right) + \exp\left(\frac{2(L-x_o))}{\xi}\right) - 1) + 4(1 - \exp\left(-\frac{2x}{\xi} \tag{A.15}$$

BIBLIOGRAPHY

Ben Payne 710 Republic Drive Rolla, MO 65401 608.308.2413 ben.is.located@gmail.com

Education

Missouri Science and Technology (formerly University of Missouri-Rolla) Masters Physics, Dec 2008

University of Wisconin-Madison, 2001 - Dec 2006 double major: Applied Math, [Mechanical] Engineering, and Physics; Physics

Community College of the Air Force [Maxwell AFB] Associates degree in Aircraft maintenence,

graduated east HS, 2001

Conferences

Classification of Regimes of Wave in Non-Conservative Random Media Ben Payne and Alexey G. Yamilov Presented by A Yamilov, 20091011 http://www.opticsinfobase.org/abs 2009-FMC3 In search of a criterion of Anderson localization applicable in random media with absorption or gain, we explore the parameter space of the system and identify different regimes in wave transport. Frontiers in Optics, 2009 San Jose, CA 20091012

Effect of Evanescent Modes on Conductance Distribution in Disordered Waveguides Ben Payne, Tom Mahler, and Alexey G. Yamilov Presented by B Payne, 20091011 http://www.opticsinfobase.org/abstract.cfm?URI=FiO-2009-FMC6 We demonstrate that proper account of evanescent modes in wave propagation results in a reduction of transport mean free path whilst preserving the universality of the conductance distribution and its single parameter scaling. Frontiers in Optics, 2009 San Jose, CA 20091012

“On Criterion for Light Localization in Random Amplifying Media” Ben Payne, Alexey Yamilov, Jon Andersen, Hui Cao Frontiers in Optics, 2008 Rochester, NY 20081023 [12 minute presentation, 3 minutes for questions] http://www.opticsinfobase.org/abstract.c 2008-FThJ3

LAB EXPERIENCE Mark Eriksson, Quantum computing [28 Aug 2006 - Dec 2006] made custom electronic equipment

Work

Micom, LLC Network Administrator; Nov 2007 - Aug 2008 managed six buisness networks Server 2003, Windows XP support printer setup, network trouble shooting

scripting virus/adaware removal data recovery XP SP3 DVD virtualbox use

Air National Guard, July 2001 - July 2008 Title: crew chief deployed Nov 04-Jan 05 Qatar deployed June 06 - Aug 06 Balad, Iraq Rank: Staff Sargent (E-5) Skill: 5 level Security clearance: secret

Registrar's Office Network Support May 2002 - Dec 2007 Title: Network Administrator (Domain Admin privilege level on a 300 user network)

projects: desktop support, server support data server (manager) Setup of test domain wrote virus response protocol batch files, automation (au3) scripts, perl, matlab Unattended XP SP2 slipstreamed CD some linux experience (desktop) training of new employees setup of kiosk-type boxes Cat5e wiring, making of wires VMWare use inventory upgrading

Pleasant company

Big Mikes 199X -?

ACCOMPLISHMENTS

black belt in Karate, 199X eagle scout, XX 200X private pilot, Sept 2005

PROFESSIONAL ORGANIZATIONS

United States Parachute Association National Eagle Scout Association Optical Society of America

VOLUNTEER:

Missouri Science and Technology Skydiving Club Secretary 2008 - current Webmaster 2007-2008 President 2007-2008