Numerical Simulation of Wave Propagation through Quasi-1D Random Scattering Medium

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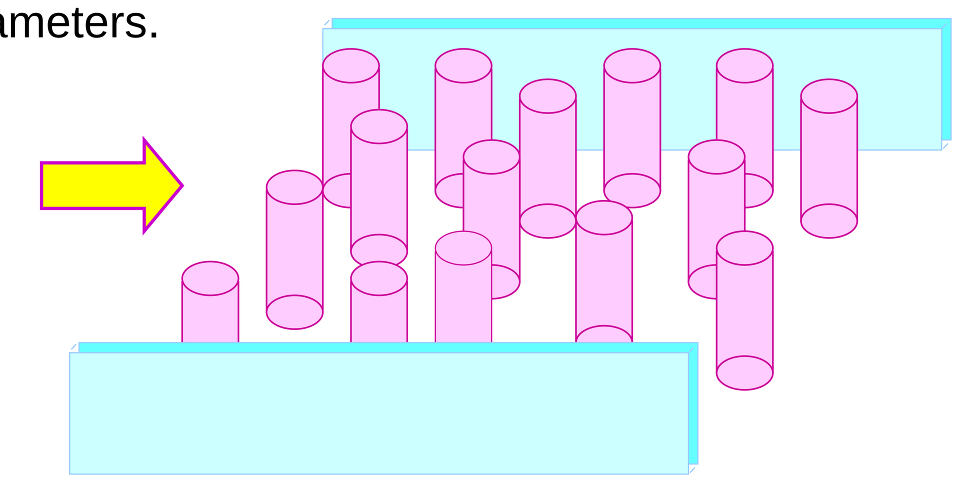
Abstract

Mesoscopic metal wire and disordered optical waveguide are two examples of quasi-1D random scattering media. The goal of this project is to develop a flexible numerical code for examining wave transport through the random media. Evolution of electric field from the beginning to the end of the system is described with a set of transfer matrices in terms of open and closed channels of the waveguide. To achieve numerical stability we also implement self-embedding procedure. The simulation will run under various regimes of wave transport: from diffusion to Anderson localization. Consistent account of closed channels in our approach will allow to determine the importance of evanescent fields on the statistics of mesoscopic transport.

Studied System

We consider wave propagation in two-dimensional disordered waveguide. Quantization of transverse momentum makes system quasi-1D. Transition from diffusive transport to Anderson localization can be achieved by varying system parameters.



One can directly compute transmission coefficients:

$$\langle T_{ab}\rangle = \langle t_{ab} t^*_{ab}\rangle \qquad \langle T_a\rangle = \left\langle \sum_a T_{ab}\right\rangle$$

Angular transmission — Total transmission

$$\langle G\rangle \frac{h}{e^2} = g = \langle T\rangle = \left\langle \sum_{ab} T_{ab}\right\rangle$$

Dimensionless conductance

Motivation

In the above formalism only open channels are usually considered. With advances in near-field optical spectroscopy, statistics of evanescent field came into focus. We devise an efficient numerical method that accounts for presence of closed channels that describe evanescent field.

This algorithm permits straightforward generalization to 3D waveguides where one can consider polarization related effects, such as long-range mesoscopic correlations of fields with different polarization.

Transfer-Matrix Formalism

Channels: The goal of this project is to simulate numerically the propagation of the incident wave as it is being scattered by the randomly positioned scatters. The simulation will keep track of and ensure continuity of the electric field and its first derivative after every disturbance the wave experiences. To find the electric field at and after each scatter, the value of the electric field and its first derivative at previous point was needed. Waveguide geometry leads to discretization of transverse momentum

$$K_{\perp n} = n\pi / W \qquad K_{\parallel n} = \sqrt{(\omega/c)^2 - K_{\perp n}^2}$$

where W is width of the system, n = 1, 2, …, n_{max}. The total number of channels (n_{max}) is determined based on the desired accuracy of the simulation. From the above equations, we can conclude which channels are open (where the electric fields propagate through space), and which channels are closed (where the electric fields exponentially decay through space – evanescent waves). If $K_{\parallel n}$ is real, the channel is open; if $K_{\parallel n}$ is imaginary, the channel is closed.

Construction of propagation matrices:

Free Space Open Channels:

$$E_n(x+\Delta x) = E_n(x)\cos(K_{\parallel n}\Delta x) + K_{\parallel n}^{-1}E_n'(x)\sin(K_{\parallel n}\Delta x)$$

$$K_{\parallel n}^{-1}E_n'(x+\Delta x) = -E_n(x)\sin(K_{\parallel n}\Delta x) + K_{\parallel n}^{-1}E_n'(x)\cos(K_{\parallel n}\Delta x)$$

Free Space Closed Channels:

$$E_n(x+\Delta x) = E_n(x)\cosh(\kappa_{\parallel n}\Delta x) + \kappa_{\parallel n}^{-1}E_n'(x)\sinh(\kappa_{\parallel n}\Delta x)$$

$$\kappa_{\parallel n}^{-1}E_n'(x+\Delta x) = E_n(x)\sinh(\kappa_{\parallel n}\Delta x) + \kappa_{\parallel n}^{-1}E_n'(x)\cos(\kappa_{\parallel n}\Delta x)$$

Scatters:

$$E_n(x+\Delta x) = E_n(x)$$

$$E_n'(x+\Delta x) = E_n'(x) - \alpha(\omega/c)^2 \sum_{m=1}^{n_{max}} A_{nm}E_m(x)$$

where $A_{nm} = \frac{2}{W}\sin(K_{\perp n}y)\sin(K_{\perp m}y),\quad K_{\parallel n} = i\kappa_{\parallel n}$, and α defines scattering strength of the scatters.

Thus wave propagation can be described by $2n_{max}$X$2n_{max}$ matrices of two types:

Propagation (Free Space) Matrices

$$\left[\begin{array}{cccc|cccc} x & & & & x & & & \\ & x & & & & x & & \\ & & x & & & & x & \\ & & & x & & & & x \\ \hline x & & & & x & & & \\ & x & & & & x & & \\ & & x & & & & x & \\ & & & x & & & & x \end{array}\right]$$

Scattering Matrices

$$\left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \hline x & x & x & x & 1 & & & \\ x & x & x & x & & 1 & & \\ x & x & x & x & & & 1 & \\ x & x & x & x & & & & 1 \end{array}\right]$$

where x's denote non-zero terms, that are not necessarily the same. Matrices are separated into four n_{max}Xn_{max} quadrants each containing information from open and closed channels.

Self-Embedding Ansatz

Due to the numerical instability which arises in the presence of closed channels, computation of simple product of individual matrices is not sufficient. The self-embedding method is introduced to correct any computation error within the simulation. This method consistently corrects any deviation due to rounding on every step of the calculation and insures the accuracy of the final result.

Boundary conditions are written in matrix form: $\hat{G}\vec{r} + \hat{H}\vec{t} = \vec{V}_{bc}$

$$\hat{H} = \left[\begin{array}{cc|cc} & & & \\ & & & \\ \hline i\hat{I} & & i\hat{I} & \\ & \hat{I} & & \hat{I} \end{array}\right] \qquad \hat{G} = \left[\begin{array}{cc|cc} \hat{I} & & -i\hat{I} & \\ & \hat{I} & & -\hat{I} \\ \hline & & & \\ & & & \end{array}\right]$$

Transmission and reflection vectors are related to S matrix that is determined via _self-embedding_:

$$Step-1 \qquad \hat{S}(0,0) = (\hat{G}+\hat{H})^{-1}$$

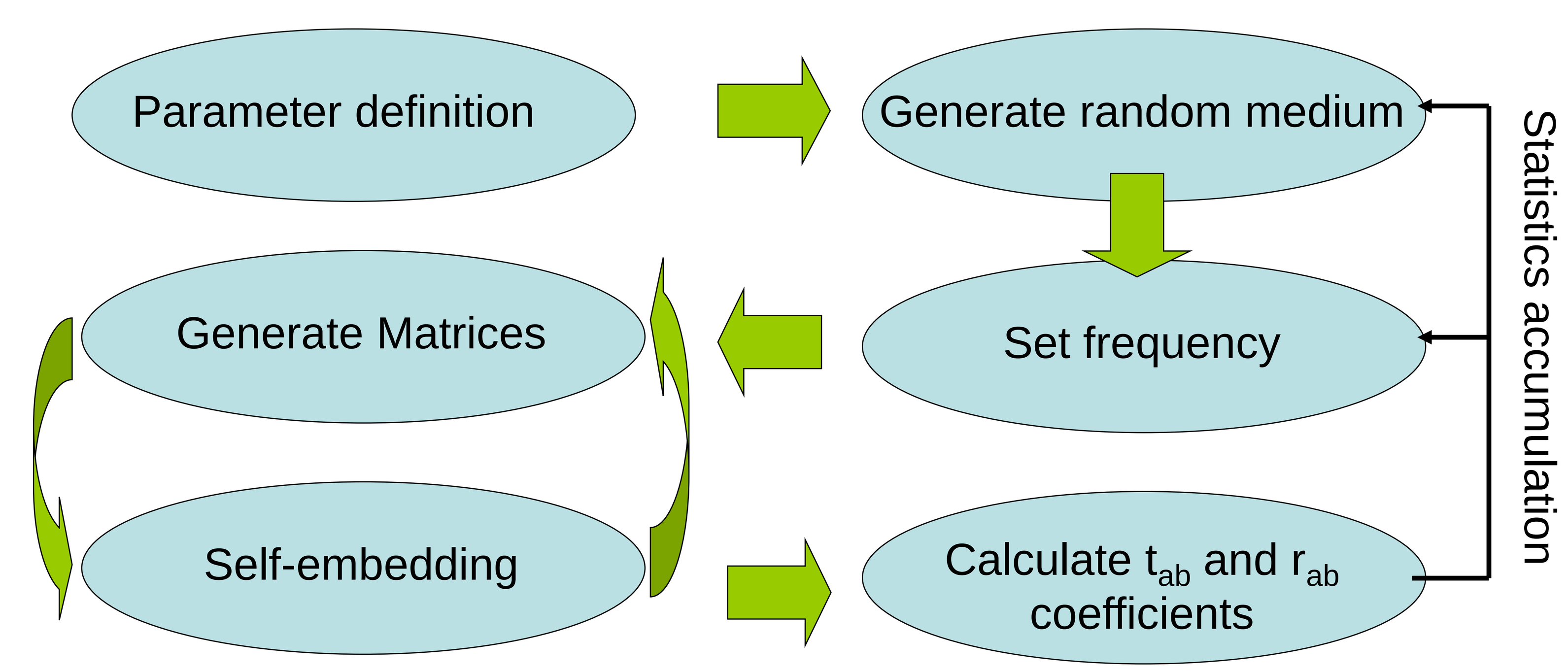
$$\hat{\Sigma}(J) = [\hat{I} - \hat{S}(J,J)\times\hat{H}\times(\hat{I}-\hat{M}(J))]^{-1}$$

$$Step-2 \qquad \hat{S}(J+1,J+1) = \hat{M}(J)\times\hat{\Sigma}(J)\times\hat{S}(J,J)$$

$$\hat{S}(0,J+1) = \hat{S}(0,J)\times[\hat{I}+\hat{H}\times(\hat{I}-\hat{M}(J))\times\hat{\Sigma}(J)\times\hat{S}(J,J)]$$

$$Step-3 \qquad \vec{t} = \hat{S}(N,N)v_{BC} \qquad \vec{r} = \hat{S}(0,N)v_{BC}$$

Algorithm and Implementation



Code testing

- Verify stability of algorithm: energy conservation should be satisfied.
- Determine the range of physical parameters.
- Accumulate large ensemble of transmission and reflection coefficients and plot the distributions of various transport coefficients. Compare to known results in limiting situations.