Answers to Homework 5

CIS 351

October 17, 2017

Part I.

For each pair of f's and g's in problems -1 through 6 below, state whether $f(n) \in \Theta(g(n))$, and if not, whether $f(n) \in O(g(n))$ or $f(n) \in \Omega(g(n))$ **and** *justify your answers with appropriate math!!!! Use the "Math Facts" on page 10 of the Big-O writeup*

1. $f(n) = n^3$ $\qquad g(n) = 10000000n^2$.

An answer for 1

$$\lim_{n\to\infty} \frac{n^3}{10000000 \cdot n^2} = \lim_{n\to\infty} \frac{n}{10000000} = \infty.$$

Therefore, $f(n) \notin \Theta(g(n))$ and $f(n) \notin O(g(n))$, but $\boxed{f(n) \in \Omega(g(n))}$.

2. $f(n) = (\log_2 n)^{100}$ $\qquad g(n) = log_2(n^{100})$.

An answer for 2

$$\begin{aligned} \lim_{n\to\infty} \frac{(\log_2 n)^{100}}{log_2(n^{100})} &= \lim_{n\to\infty} \frac{(\log_2 n)^{100}}{100 \cdot \log_2 n} && \text{by Math Facts (c)} \\ &= \lim_{n\to\infty} \frac{(\log_2 n)^{99}}{100} = \infty. \end{aligned}$$

Therefore, $f(n) \notin \Theta(g(n))$ and $f(n) \notin O(g(n))$, but $\boxed{f(n) \in \Omega(g(n))}$.

3. $f(n) = 49n + (\log_2 n)^4$ $\qquad g(n) = 5(\log_2 n)^{76}$.

An answer for 3

$$\lim_{n\to\infty} \frac{49n + (\log_2 n)^4}{5(\log_2 n)^{76}} = \lim_{n\to\infty} \left(\frac{49n}{5(\log_2 n)^{76}} + \frac{1}{5(\log_2 n)^{72}}\right) = \infty \quad \text{by Math Facts } (\ell.3).$$

Therefore, $f(n) \notin \Theta(g(n))$ and $f(n) \notin O(g(n))$, but $\boxed{f(n) \in \Omega(g(n))}$.

4. $f(n) = 6^{n/2}$ $g(n) = 6^n$.

An answer for 4

$$\lim_{n\to\infty} \frac{6^{n/2}}{6^n} = \lim_{n\to\infty} 6^{\frac{n}{2}-n} = \lim_{n\to\infty} \frac{1}{6^{n/2}} = 0.$$

Therefore, $f(n) \notin \Theta(g(n))$ and $f(n) \notin \Omega(g(n))$, but $\boxed{f(n) \in O(g(n))}$.

5. $f(n) = (\log n)^{100}$ Typo! $g(n) = 2^{\sqrt{n}}$.

An answer for 5

$$\lim_{n\to\infty} \frac{(\log n)^{100}}{2^{\sqrt{n}}} = \lim_{n\to\infty} \frac{(\log n)^{100}}{2^{n^{0.5}}} = 0 \quad \text{(By Math Fact } (\ell.2)).$$

Therefore, $f(n) \notin \Theta(g(n))$ and $f(n) \notin \Omega(g(n))$, but $\boxed{f(n) \in O(g(n))}$.

5. $f(n) = n^{100}$ Correct! $g(n) = 2^{\sqrt{n}}$.

An answer for 5

$$\lim_{n\to\infty} \frac{n^{100}}{2^{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^{100}}{2^{n^{0.5}}} = 0 \quad \text{(By Math Fact } (\ell.2)).$$

Therefore, $f(n) \notin \Theta(g(n))$ and $f(n) \notin \Omega(g(n))$, but $\boxed{f(n) \in O(g(n))}$.

6. $f(n) = 17^{\log_2 n}$ $g(n) = n^{\log_2 17}$.

An answer for 6

$$\begin{aligned}
\lim_{n\to\infty} \frac{17^{\log_2 n}}{n^{\log_2 17}} &= \lim_{n\to\infty} \frac{(2^{\log_2 17})^{\log_2 n}}{(2^{\log_2 n})^{\log_2 17}} && \text{(By Math Fact (g))} \\
&= \lim_{n\to\infty} \frac{2^{(\log_2 17)\cdot(\log_2 n)}}{2^{(\log_2 n)\cdot(\log_2 17)}} && \text{(By Math Fact (b))} \\
&= \lim_{n\to\infty} \frac{2^{(\log_2 17)\cdot(\log_2 n)}}{2^{(\log_2 17)\cdot(\log_2 n)}} && \text{(By algebra)} \\
&= 1.
\end{aligned}$$

Therefore, $\boxed{f(n) \in \Theta(g(n))}$.

7. Show that $f(n) \in \Theta(n^2)$ where $f(n) = (9 + 2\cos n) \cdot n^2 + 12n$.
Also explain why we *cannot* use the limit rule to do this problem.
Would we have the same problem with $f'(n) = (3 + \cos n) \cdot n + 12n^2$?

An answer for 7

Since $-1 \leq \cos n \leq 1$ for all n, we have: $7 \leq (9 + 2\cos n) \leq 11$ for each n. Hence, for all $n > 12$:

$$8 \leq \frac{(9 + 2\cos n) \cdot n^2 + 12n}{n^2} = (9 + 2\cos n) + \frac{12}{n} \leq 12$$

So by the definition of $\Theta(\cdot)$, $f(n) \in \Theta(n^2)$. We could not use the limit rule above because that limit does not exist. However,

$$\lim_{n\to\infty} \frac{(9 + 2\cos n) \cdot n + 12n^2}{n^2} = \lim_{n\to\infty} \frac{9 + 2\cos n}{n} + 12 = 12.$$

So we could use the limit rule to decide if $(9 + 2\cos n) \cdot n + 12n^2 \in \Theta(n^2)$.

Part II.

For each program fragment below, give a $\Theta(\,\cdot\,)$ expression (in parameter n) for the value of z after the program fragment is finished. *Justify your answers!!!!*

8.
```
int z = 1;
for (int i = 0 ; i <= n ; i++)
  z = z + 1;
```

An answer for 8

(final-value of i)-(initial-value of i) + 1 = $n - 0 + 1 = n + 1$.
Thus the loop goes through $n + 1$ iterations and so the final value of z is $n + 1 \in \Theta(n)$.

9.
```
int z = 1;
for (int k = 0 ; k <= n ; k++) {
  int j = n+1;
  while (j>0) {
    z = z + 1;
    j = j - 1;
  }
}
```

An answer for 9

Inner while-loop: Since the control variable, j, goes down instead of up, the number of iterations is:

$$\text{(initial-value of } j\text{)-(final-value of } j\text{) + 1} = (n+1) - 1 + 1 = n+1 \text{ times.}$$

Outer for-loop: The loop iterates $n - 0 + 1 = n + 1$ times and each time executes the inner loop, which iterates $n+1$-many times. Hence, there are $(n+1)^2$ many iterations of the `z=z+1` statement. So, the final value of $z = n^2 + 2n + 1 \in \Theta(n^2)$.

10.
```
int z = 1;
for (int k = 1 ; k <= n ; k++)
  for (int j = 0 ; j < k ; j++)
    z = z + 1;
```

An answer for 10

Inner loop: The loop iterates $(k-1) - 0 + 1 = k$ times.

Outer loop: In successive iterations the value of k goes from 1 to n in steps of 1 and the inner loop is executed k-many times. So the total number of executions of the `z=z+1` statement is

$$1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1) \quad \text{(By Math Facts (h)).}$$

Therefore, the final value of $z = \frac{n(n+1)}{2} + 1 \in \Theta(n^2)$.

11.
```
int z = 1;    // Hint: a log figures in the answer.
int m = 1;
while (m<n) {
  z = z + 1;
  m = 2*m;
}
```

An answer for 11

In the loop, m starts at 1 and doubles each time through the loop. Hence, at the end of the k-th iteration, the value of m is 2^k. Hence, the number of iterations cannot be more than the least value of k such that $2^k \geq n$, i.e., $k \geq \log_2 n$. Therefore, the number of iterations cannot be more than $(\log_2 n) + 1$ and cannot be less that $(\log_2 n) - 1$. Hence,

$$(\log_2 n) - 1 \leq \text{the final value of } z \leq (\log_2 n) + 1.$$

So $z \in \Theta(\log_2 n)$.