Advanced Methods Homework 4

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1 Linear models

1、 Then $\vec{y} \sim N_p(X\vec{\beta}, \Sigma)$ (actually ε should follow $N_n(\vec{0}, \Sigma)$, but the notation can vary)

Therefore the likelihood of samples is:

$$L = \frac{1}{\sqrt{2\pi}^n |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(\vec{y} - X\vec{\beta})' \Sigma^{-1} (\vec{y} - X\vec{\beta})}$$

So maximize L is just to minimize $(\vec{y} - X\vec{\beta})' \Sigma^{-1} (\vec{y} - X\vec{\beta})$

Suppose $\Sigma^{-1} = (\Sigma^{-\frac{1}{2}})^2$ where $\Sigma^{-\frac{1}{2}}$ is symmetrical

then we need to minimize $(\Sigma^{-\frac{1}{2}}\vec{y} - \Sigma^{-\frac{1}{2}}X\vec{\beta})'(\Sigma^{-\frac{1}{2}}\vec{y} - \Sigma^{-\frac{1}{2}}X\vec{\beta})$

which is just normal least squares.

So we get that the MLE of $\vec{\beta}$ is

$$\hat{\vec{\beta}} = (X'\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}}X)^{-1} X'\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}}\vec{y} = (X'\Sigma^{-1}X)^{-1} X'\Sigma^{-1}\vec{y}$$

Here we need that Σ is invertible, but if Σ is not invertible then $\vec{y}$ do not have likelihood, so you can't do MLE

(you need to first choose the maximum linear independent subgroup of $\vec{\varepsilon}$ and do MLE on the subgroup)

So suppose Σ invertible is reasonable.

2: The likelihood of $\vec{y}$ is:

$$L(\vec{\beta}, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}^{\,n}} e^{-\frac{1}{2\sigma^2}(\vec{y}-X\vec{\beta})'(\vec{y}-X\vec{\beta})}$$

The likelihood of $\breve{\vec{y}}$ is:

$$L_p(\breve{\vec{\beta}}, \breve{\sigma}^2) = \frac{1}{\sqrt{2\pi\breve{\sigma}^2}^{\,n}} e^{-\frac{1}{2\breve{\sigma}^2}(P\vec{y}-PX\breve{\vec{\beta}})'(P\vec{y}-PX\breve{\vec{\beta}})}$$

$$= \sqrt{2\pi\breve{\sigma}^2}^{\,-n} e^{-\frac{1}{2\breve{\sigma}^2}(\vec{y}-X\breve{\vec{\beta}})'(\vec{y}-X\breve{\vec{\beta}})}$$

So $L_p = L$, therefore the MLE are the same, suppose the MLE of original model is $\hat{\vec{\beta}}$ and $\hat{\sigma}^2$

then $\hat{\vec{\beta}} = \breve{\vec{\beta}}$ and $\hat{\sigma}^2 = \breve{\sigma}^2$

2 Multivariate normals

1: $\vec{a}' X \sim N_p(0, \vec{a}' I \vec{a}) = N(0, \vec{a}'\vec{a}) = N(0, \vec{a}'\vec{a})$

Since $\vec{a} \neq \vec{0}$ $\vec{a}'\vec{a} > 0$

therefore $\frac{\vec{a}' X}{\sqrt{\vec{a}'\vec{a}}} \sim N(0, \frac{\vec{a}'\vec{a}}{\sqrt{\vec{a}'\vec{a}}^{\,2}}) = N(0,1)$

2: If $AA' = I$: $AX \sim N_p(A\vec{0}, AIA') = N_p(\vec{0}, I)$

Since $AA' = I$, A is orthogonal which is transformation combined rotation and reflection. And since $N_p(0, I)$ is uniform in every direction so after rotation and reflection, it will remain unchanged.

3: $(\vec{y}-\vec{\mu})'\Sigma^{-1}(\vec{y}-\vec{\mu}) = \cancel{\Sigma^{-\frac{1}{2}}\vec{y}-\Sigma^{\frac{1}{2}}}$

$$\left(\Sigma^{-\frac{1}{2}}(\vec{y}-\vec{\mu})\right)'\left(\Sigma^{-\frac{1}{2}}(\vec{y}-\vec{\mu})\right)$$

And $\Sigma^{-\frac{1}{2}}(\vec{y}-\vec{\mu}) \sim N_p\left(\Sigma^{-\frac{1}{2}}(\vec{\mu}-\vec{\mu}),\ \Sigma^{-\frac{1}{2}}\Sigma(\Sigma^{-\frac{1}{2}})'\right)$

$$= N_p(0, I_p)$$

Therefore $\left(\Sigma^{-\frac{1}{2}}(\vec{y}-\vec{\mu})\right)'\left(\Sigma^{-\frac{1}{2}}(\vec{y}-\vec{\mu})\right) \sim \chi^2_p$

so $(\vec{y}-\vec{\mu})'\Sigma^{-1}(\vec{y}-\vec{\mu}) \sim \chi^2_p$

3 Inference and estimation in Linear models.

1: Denote $\vec{Y}_1 = (Y_{11}, Y_{12}, Y_{13}, \cdots, Y_{1J_1})^T$; $\vec{\mu}_1 = (\mu_1, \mu_1 \cdots \mu_1)^T$

$\vec{Y}_2 = (Y_{21}, Y_{22}, \cdots\cdots, Y_{2J_2})^T$; $\vec{\mu}_2 = (\mu_2, \mu_2 \cdots \mu_2)^T$

Then $\vec{Y}_1 \sim N_{J_1}(\vec{\mu}_1, \sigma^2 I_{J_1})$

$\vec{Y}_2 \sim N_{J_2}(\vec{\mu}_2, \sigma^2 I_{J_2})$

Then $\frac{1}{\sigma^2}\sum_{i=1}^{J_1}(Y_{1i}-\bar{\vec{Y}}_1)^2 \sim \chi^2_{J_1-1}$

$\frac{1}{\sigma^2}\sum_{i=1}^{J_2}(Y_{2i}-\bar{\vec{Y}}_2)^2 \sim \chi^2_{J_2-1}$

So $E S_1^2 = \frac{(J_1-1)\sigma^2}{J_1-1} = \sigma^2$

$E S_2^2 = \frac{(J_2-1)\sigma^2}{(J_2-1)} = \sigma^2$

So $E S_p^2 = \frac{1}{J_1+J_2-2}\left[(J_1-1)\sigma^2 + (J_2-1)\sigma^2\right] = \sigma^2$

So S_p^2 is unbiased.

Then we show the estimator of σ^2 in Linear model is just this estimator

Here $\vec{Y} = \begin{pmatrix} 1, 0 \\ 1, 0 \\ \vdots \\ 1, 0 \\ 0, 1 \\ 0, 1 \\ \vdots \\ 0, 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \vec{\varepsilon} = X \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \vec{\varepsilon}$

Then $S^2 = (\vec{Y} - \vec{X}\hat{\beta})'(\vec{Y} - \vec{X}\hat{\beta})/(n-2)$ where $n = J_1 + J_2$.

where $\hat{\beta} = (X'X)^{-1} X' \vec{y} = \begin{pmatrix} \sum_{i=1}^{J_1} Y_{1i} / J_1 \\ \sum_{i=1}^{J_2} Y_{2i} / J_2 \end{pmatrix}$

So $S^2 = \dfrac{(J_1 - 1) \dfrac{\sum (Y_{1i} - \overline{Y}_1)^2}{J_1 - 1} + (J_2 - 1) \dfrac{\sum (Y_{2i} - \overline{Y}_2)^2}{J_2 - 1}}{J_1 + J_2 - 2}$

$= \dfrac{1}{J_1 + J_2 - 2} \left[(J_1 - 1) S_1^2 + (J_2 - 1) S_2^2 \right] = S_p^2$

2:

(a) $X_{i+1} - X_i = (1, -1) \begin{pmatrix} X_{i+1} \\ X_i \end{pmatrix} \sim N\left[(1,-1)\begin{pmatrix} \mu \\ \mu \end{pmatrix}, (1,-1)\begin{pmatrix} \sigma^2, 0 \\ 0, \sigma^2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$

$\Rightarrow X_{i+1} - X_i \sim N(0, 2\sigma^2)$

$\Rightarrow EQ = \dfrac{1}{2(n-1)} \sum_{i=1}^{n-1} \cdot 2\sigma^2 = \sigma^2$

$\Rightarrow Q$ is unbiased estimator of σ^2

(b) Here $Q = \dfrac{1}{2(n-1)} (X_1, X_2 \cdots X_n) \begin{pmatrix} 1, -1, 0 \cdots & & & & \\ -1, 2, -1, 0 \cdots & & & & \\ 0, -1, 2, -1 & & & 0 & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \\ & 0 & & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$

$= \dfrac{1}{2(n-1)} \vec{X}' A \vec{X}$

So $\mathrm{var}(Q) = \dfrac{1}{4(n-1)^2} \left[2\,\mathrm{tr}\left[(A \cdot \sigma^2 I)^2 \right] + 4 (\mu, \mu \cdots \mu) A \sigma^2 I A \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \right]$

$= \dfrac{1}{4(n-1)^2} \left[2 \cdot (6n - 8) + 0 \right]$

$= \dfrac{3n-4}{(n-1)^2}$.

3:

(a) $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1, 1, 1, 1 \\ 1, -1, -1, 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}$

so $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 1,1,1,1 \\ 1,-1,-1,1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1,1,1,1 \\ 1,-1,-1,1 \end{pmatrix} \begin{pmatrix} 4&3&2&1 \\ 3&3&2&1 \\ 2&2&2&1 \\ 1&1&1&1 \end{pmatrix} \begin{pmatrix} 1,1 \\ 1,-1 \\ 1,-1 \\ 1,1 \end{pmatrix} \right]$

$= N_2\left[\begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 30, -2 \\ -2, 2 \end{pmatrix} \right] = N_2\left(\begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 30, -2 \\ -2, 2 \end{pmatrix} \right)$

(b) $X_1 | X_2 \sim N_1(\mu_{1|2}, \sigma_{1|2})$

where $\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2) = 8 - X_2$

$\sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}' = 28$

so $X_1 | X_2 \sim N_1(8 - X_2, 28)$

We can also compute directly:

$$f(X_1|X_2) = \frac{f(X_1, X_2)}{f(X_2)} \sim \frac{e^{-\frac{1}{2}\frac{(X_1-8)^2+2(X_1-8)X_2+15X_2^2}{28}}}{e^{-\frac{1}{2}\frac{X_2^2}{2}}}$$

$$\sim e^{-\frac{1}{2}\frac{(X_1-8)^2+2(X_1-8)X_2+X_2^2}{28}} = e^{-\frac{1}{2}\frac{[X_1-(8-X_2)]^2}{28}}$$

$\Rightarrow f(X_1|X_2) \sim N_1(8 - X_2, 28)$