Probability Theory Homework I

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1 (i) *Proof.* We have $a_1 > 0$ and since $a_n \to 0$, there exists N such that for every $n > N$, $a_n < a_1$. Therefore $\max_{1\leq n}\{a_n\}$ is $\max_{1\leq n\leq N}\{a_n\}$, and then the max exits since $N < \infty$ □

(ii) *Proof.* Suppose A is uncountable, then $A \setminus \{0\}$ is also uncountable. Since $A \setminus \{0\} = \cup_{n=1}^{\infty} A_n$ where $A_n = \{a \in A : a \geq 1/n\}$, then there must exits a N such that A_N is uncountable, and therefore $N + 1$ elements $\{a_1, \cdots, a_{N+1}\}$ in $A_N \subset A$ such that $\forall i : \ a_i \geq 1/N$. We'll have $\sum_{i=1}^{N+1} a_i \geq \frac{N+1}{N} > 1$. A contradiction, so A must be countable. □

2 (i) *Proof.* Suppose in (M, d_1): $x_n \to x$ and in (M, d_2): $x_n \to y$ and $x \neq y$, which infers $d_2(x, y) > 0$. Then we consider an array y_n, where $y_{2k+1} = x$ and $y_{2k} = x_k$. It's easy to see that y_n has limit x in (M, d_1), so according to the assumption in problem, y_n must tend to some $z \in M$. Therefore $d_2(y_{2k+1}, y_{2k}) \leq d_2(y_{2k+1}, z) + d_2(z, y_{2k})$ tends to 0. However when k is large enough:

$$d_2(y_{2k+1}, y_{2k}) \geq d_2(y_{2k+1}, y) - d_2(y, y_{2k}) = d_2(x, y) - d_2(y, x_k) > d_2(x, y)/2 > 0$$

A contradiction shows up, therefore $x = y$. □

(ii) *Proof.* For every sequence (x_n, y_n) in the set, if $(x_n, y_n) \to (x_0, y_0)$. Then since function $y - x^2$ is continuous with (x, y), we have $\lim(y_n - x_n^2) = (y_0 - x_0^2)$, and since $(x_n, y_n) \in \{(x, y) : y \geq x^2\}$, we have $\lim(y_n - x_n^2) \geq 0$ and therefore $y_0 \geq x_0^2$ and $(x_0, y_0) \in \{(x, y) : y \geq x^2\}$, which means the set is closed. □

(iii) (a) $(0, 1 + \frac{1}{n})$ are open but $\cap_{n=1}^{\infty}(0, 1 + \frac{1}{n}) = (0, 1]$ is not open.

(b) $\{(x, x^2) : x \in [0, 1]\} \cap \{(x, x) : x \in [0, 1]\} = \{(0, 0), (1, 1)\}$ is not connected.

3 (i) *Proof.* Suppose M is infinite, since M is compact, an infinite subarray of M must have an accumulation point, which means we can find an array $\{x_n\}$ of distinct elements in M such that $x_n \to x$ and $x \notin \{x_n\}$. Therefore $\{x_n\}$ is a subset of M, hence compact, but not closed. However, since M is a metric space, hence T_2 space, every compact set in M must be closed [1]. A contradiction, which means M is finite □

(ii) *Proof.* For every open cover $\{O_\alpha\}$ of $f(K)$, $\{f^{-1}(O_\alpha)\}$ will be an open cover of K. Since K is compact, there exists an finite subcover $\{f^{-1}(O_{\alpha_i})\}_1^N$. Then:

$$f(K) \subset f\left(\cup_1^N f^{-1}(O_{\alpha_i})\right) = \cup_1^N f(f^{-1}(O_{\alpha_i})) = \cup_1^N O_{\alpha_i}$$

So $\{O_{\alpha_i}\}_1^N$ is an finite open subcover of $f(K)$. Therefore, $f(K)$ is compact. □

4 (i) *Proof.* By using derivative, $f_n(x)$ is increase in $[0, \frac{1}{\sqrt{3}n}]$ and decrease in $[\frac{1}{\sqrt{3}n}, 1]$. So we have:

$$\sup_{x\in[0,1]} |f_n(x) - 0| = f_n(\frac{1}{\sqrt{3}n}) > \frac{1}{2} > 0$$

therefore f_n don't uniformly convergent to 0 in $[0, 1]$. But in $[\delta, 1]$, when n is large enough such that $\frac{1}{\sqrt{3}n} < \delta$, we have:

$$\sup_{x\in[\delta,1]} |f_n(x) - 0| = f_n(\delta) = \frac{\sqrt{n}\delta}{n^2\delta^2 + 1} \to 0$$

therefore, f_n uniformly convergent to 0 in every $[\delta, 1]$ where $\delta > 0$. □

(ii) $|(1+n^2)^{-2}\sin(nx)|, |n(1+n^2)^{-2}\cos(nx)|, |(-1)^n n^2(1+n^2)^{-2}\sin(nx)|$ are all less than $\frac{1}{1+n^2}$, where $\sum_1^\infty \frac{1}{1+n^2} < \infty$. Therefore (a), (b), (c) are all uniformly convergent to some function [3]. Since $(\frac{\sin(nx)}{(1+n^2)^2})' = n(1+n^2)^{-2}\cos(nx)$, which is continuous and (a) and (b) convergent uniformly, we have [6]:

$$\left(\sum_{n=1}^{\infty}(1+n^2)^{-2}\sin(nx)\right)' = \sum_{n=1}^{\infty} n(1+n^2)^{-2}\cos(nx)$$

And in the same way, we can prove that:

$$\left(\sum_{n=1}^{\infty} n(1+n^2)^{-2}\cos(nx)\right)' = \sum_{n=1}^{\infty}(-1)n^2(1+n^2)^{-2}\sin(nx)$$

So, $f'' = \sum_{n=1}^{\infty}(-1)n^2(1+n^2)^{-2}\sin(nx)$ exists and it's continuous since uniform convergence and the continuousness of $\sin(nx)$ [4].

5 (i) Since every nonempty interval in $\mathbb{R}$ contains rational and irrational number. Therefore every upper sum is 1 and every lower sum is 0, hence f is not Riemann integrable.

(ii) *Proof.* In this situation we have that f is uniformly continuous [2] in $[a, b]$, which means $\forall\epsilon > 0, \exists\delta > 0\ \forall|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.
For this (but arbitary) ϵ and δ, $\forall$ partition $0 = x_0 < x_1 < \cdots < x_N = 1$ with $\max_{0\le i\le N-1}(x_{i+1} - x_i) < \epsilon$. Denote U, L be the upper sum and the lower sum, then we have:

$$|U - L| \le \epsilon \cdot \sum_{i=0}^{N-1}(x_{i+1} - x_i) = \epsilon$$

which means $|U - L| \to 0$ when $\max_{0\le i\le N-1}(x_{i+1} - x_i) \to 0$. Therefore f is Riemann integrable. □

(iii) *Proof.* We have $0 \le f_n(x) < 1, \forall x \in [0, 1]$ and $\forall\delta > 0$, f_n uniformly convergent to 0, hence $\lim\int_\delta^1 f_n(x)dx = 0, \forall\delta > 0$ [5]. Therefore we have:

$$\int_0^1 f_n(x)dx = \int_0^\delta f_n(x)dx + \int_\delta^1 f_n(x)dx < \delta + \int_\delta^1 f_n(x)dx$$

then:

$$\overline{\lim_{n\to\infty}} \int_0^1 f_n(x)dx < \delta, \qquad \forall\delta > 0$$

which means $\overline{\lim}\int_0^1 f_n(x)dx = 0$, together with $f_n \ge 0$, we have $\lim\int_0^1 f_n(x)dx = 0$. □

Bonus:

Suppose the strategy is to switch the number with probability $p(x)$ when we get x, then calculate the probability that we end with a larger number if two numbers are x and $2x$. It's:

$$\mathbf{P}(\text{end with larger number}) = \frac{1}{2}p(x) + \frac{1}{2}(1 - p(2x)) = \frac{1}{2} + \frac{p(x) - p(2x)}{2}$$

So we only need $p(x) > p(2x)$, for example we can switch the number with probability $\frac{1}{1+x}$ when we get x, and then we will have more chance than 1/2 to get the biggest number for every x.

References

[1] **Theorem 26.3. "Every compact subspace of a Hausdorff space is closed"**. Munkres, James R. *Topology*. Prentice Hall, 2000.

[2] **Theorem 4.19. "Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X "**. Rudin, Walter. *Principles of mathematical analysis*. Vol. 3. New York: McGraw-hill, 1964.

[3] **Theorem 7.10. "$\cdots$ suppose $|f_n(x)| \leq M_n \cdots$ Then $\sum f_n$ converges uniformly $\cdots$ if $\sum M_n$ converges "**. Rudin, Walter. *Principles of mathematical analysis*. Vol. 3. New York: McGraw-hill, 1964.

[4] **Theorem 7.12. "If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E"**. Rudin, Walter. *Principles of mathematical analysis*. Vol. 3. New York: McGraw-hill, 1964.

[5] **Theorem 7.16. "$\cdots f_n \to f$ uniformly on $[a, b]$. Then $\cdots$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$ "**. Rudin, Walter. *Principles of mathematical analysis*. Vol. 3. New York: McGraw-hill, 1964.

[6] **Theorem 7.17. "Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$"**. Rudin, Walter. *Principles of mathematical analysis*. Vol. 3. New York: McGraw-hill, 1964.