Homework 2

1. Suppose that $f_X(x;p) = \frac{-(1-p)^x}{x \cdot logp}$ for $x = 1, 2, ...$ and for some parameter $p \in (0, 1)$. Find the mean and variance of X in terms of p.

2. Casella & Berger problems 3.1 and 3.2

3. Suppose $f_X(x) = \frac{1}{\sqrt{2\pi}} exp\left\{\frac{-(x-10)^2}{2}\right\}$. ($X$ is normally distributed with mean 10 and variance 1.)

   (a) What is the distribution of $Y = |X|$?

   (b) What is the distribution of $Z = X^4$?

4. A density function often used by engineers is the Rayleigh density, given by $f_X(X) = \frac{2x}{\theta} exp(-\frac{x^2}{\theta})$, $x > 0$. What is the distribution of $Y = X^2$?

Statistical Theory Homework 2

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1. First we show that $f_X(x;p)$ is a pmf. Since $\sum_{x=1}^{\infty}(1-t)^{x-1}$ is uniformly convergent to $\frac{1}{p}$ in every subset $[p,1]$. We have that:

$$\sum_{x=1}^{\infty}\frac{-(1-p)^x}{x} = \sum_x \int_1^p (1-t)^{x-1}dt = \int_1^p \sum_x (1-t)^{x-1}dt = \int_1^p \frac{1}{t}dt = \log p$$

Therefore $f_X(x;p)$ is a pmf. Then we have:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{\infty} x\frac{-(1-p)^x}{x\log p} = \frac{-(1-p)\frac{1}{1-1+p}}{\log p} = \frac{p-1}{p\log p} &(1)\\
\mathbb{E}[X^2] &= \sum_{x=1}^{\infty} x\frac{-(1-p)^x}{\log p} = \frac{1-p}{\log p}\frac{d}{dp}\sum_{x=1}^{\infty}(1-p)^x = \frac{p-1}{p^2\log p} &(2)\\
Var[X] &= \mathbb{E}[X^2]-\mathbb{E}^2[X] = \frac{(1-p)(p-1-\log p)}{p^2\log^2 p} &(3)
\end{aligned}$$

2. 3.1 $X = N_0 - 1 + Y$, where Y is uniform$(1, N_1 - N_0 + 1)$. Therefore $\mathbb{E}[X] = N_0 - 1 + \mathbb{E}[Y]$ and $Var[X] = Var[Y]$. For Y, we have:

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{N_1-N_0+1}\frac{i}{N_1-N_0+1} = \frac{N_1-N_0+2}{2} &(4)\\
\mathbb{E}[Y^2] &= \sum_{i=1}^{N_1-N_0+1}\frac{i^2}{N_1-N_0+1} = \frac{(N_1-N_0+2)(2N_1-2N_0+3)}{6} &(5)\\
Var[X] &= Var[Y] = \mathbb{E}[Y^2]-\mathbb{E}^2[Y] = \frac{(N_1-N_0)(N_1-N_0+2)}{12} &(6)\\
\mathbb{E}[X] &= \frac{N_1-N_0+2}{2}+N_0-1 = \frac{N_0+N_1}{2} &(7)
\end{aligned}$$

3.2 (a) If the lot is unacceptable, suppose there are M defective parts where $M \geq 6$. Then the probability that the manufacturer accepts an unacceptable lot is:

$$\mathbf{P}(\text{wrong}) = \frac{\binom{100-M}{K}}{\binom{100}{K}} \leq \frac{\binom{94}{K}}{\binom{100}{K}}$$

And the '=' situation in '$\leq$' above can be reached, so we need to let $\frac{\binom{94}{K}}{\binom{100}{K}} < 0.1$. Therefore K should be at least 32.

(b) If the lot is unacceptable, suppose there are M defective parts where $M \geq 6$. Then the probability that the manufacturer accepts an unacceptable lot is:

$$\mathbf{P}(\text{wrong}|M) = \frac{\binom{100-M}{K} + \binom{100-M}{K-1}\binom{M}{1}}{\binom{100}{K}}$$

Then we should choose the smallest K to make $\max_{100 \geq M \geq 6} \mathbf{P}(\text{wrong}|M) < 0.1$. The numerical solution shows that K need to be at least 51.

3. We give the pdf here for $|X|$ and X^4.

(a) First, consider the cdf $F(t) = \mathbf{P}(Y \leq t)$, if $t < 0$, then obviously $F(t) = 0$. Suppose $t \geq 0$, then $\mathbf{P}(Y \leq t) = \mathbf{P}(-t \leq X \leq t) = \int_{-t}^{t} f_X(x)dx$. And pdf $f_Y(y) = \frac{dF(y)}{dy}$, therefore we have:

$$f_Y(y) = (f_X(-y) + f_X(y))\mathbb{1}_{y\geq 0} = \frac{\mathbb{1}_{y\geq 0}}{\sqrt{2\pi}}\left(exp\{\frac{-(y+10)^2}{2}\} + exp\{\frac{-(y-10)^2}{2}\}\right)$$

(b) First, consider the cdf $F(z) = \mathbf{P}(Z \leq z)$, if $z < 0$, then obviously $F(z) = 0$. Suppose $z \geq 0$, then $\mathbf{P}(Y \leq z) = \mathbf{P}(-t^{\frac{1}{4}} \leq X \leq t^{\frac{1}{4}}) = \int_{-t^{\frac{1}{4}}}^{t^{\frac{1}{4}}} f_X(x)dx$. And pdf $f_Z(z) = \frac{dF(z)}{dz}$, therefore we have:

$$f_Z(z) = \frac{t^{-\frac{3}{4}}}{4}(f_X(-z^{\frac{1}{4}})+f_X(z^{\frac{1}{4}}))\mathbb{1}_{z\geq 0} = \frac{t^{-\frac{3}{4}}\mathbb{1}_{z\geq 0}}{4\sqrt{2\pi}}\left(exp\{\frac{-(z^{\frac{1}{4}}+10)^2}{2}\} + exp\{\frac{-(z^{\frac{1}{4}}-10)^2}{2}\}\right)$$

4. First, consider the cdf $F(y) = \mathbf{P}(Y \leq y)$, if $y < 0$, then obviously $F(y) = 0$. Suppose $y \geq 0$, then $\mathbf{P}(Y \leq y) = \mathbf{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \mathbf{P}(0 < X \leq \sqrt{y}) = \int_0^{\sqrt{y}} f_X(x)dx$. And pdf $f_Y(y) = \frac{dF(y)}{dy}$, therefore we have:

$$f_Y(y) = \frac{1}{2\sqrt{y}}f_X(\sqrt{y}))\mathbb{1}_{y\geq 0} = \frac{\mathbb{1}_{y\geq 0}}{\theta}exp(-\frac{y}{\theta})$$

which is a exponential distribution with mean θ.