BST 140.752
Problem Set 4
Due: December 14, 2017

1 Linear Mixed Model

1. Let $y_{ij} = \mu + u_i + \epsilon_{ij}$ for $i = 1, \ldots 4$, $j = 1, \ldots 3$, $u_i \sim N(0, \sigma_u^2)$ and $\epsilon_{ij} \sim N(0, \sigma^2)$.

   (a) Write the model in general linear mixed model format, i.e. $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}$. Specify each component of the model.

   (b) Calculate the Best Linear Unbiased Predictor (BLUP) for u_i

   (c) Find the best linear unbiased estimate (BLUE) for μ.

2. Let $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}$ for $u \sim N(0, \Sigma_u)$ and $\epsilon \sim N(0, \sigma^2 I)$.

   (a) Show that the BLUP is given by $E(\mathbf{u}|\mathbf{y})$.

   (b) Calculate the general form of the BLUP for $\mathbf{u}$, denoted $\hat{\mathbf{u}}$.

   (c) Find $\mathrm{var}(\hat{\mathbf{u}})$.

   (d) Find $\mathrm{var}(\hat{\mathbf{u}} - \mathbf{u})$.

3. Recall that using REML we obtained a set of $m + 1$ estimating equations for $\sigma_0^2, \ldots \sigma_m^2$ given by

$$\mathrm{tr}\Big((\mathbf{KVK}')^{-1}\mathbf{K}\mathbf{Z}_{(l)}\mathbf{Z}_{(l)}'\mathbf{K}'\Big) = \mathbf{y}'\mathbf{K}'(\mathbf{KVK}')^{-1}\mathbf{K}\mathbf{Z}_{(l)}\mathbf{Z}_{(l)}'\mathbf{K}'(\mathbf{KVK}')^{-1}\mathbf{K}\mathbf{y}$$

   where $\mathbf{K} = \mathbf{C}(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')$. Show that the expected value of the quadratic form on the right side of the equation is given by the left side.

2 Coding

1. Load the `Rail` data set in R from the package `nmle`. Use the function `help(Rail)` to familiarize yourself with its format.

   (a) Fit a fixed effect model of the form $y_{ij} = \mu + u_i + \epsilon_{ij}$ with $\epsilon_{ij} \sim (0, \sigma^2)$.

   (b) Fit a mixed model of the form $y_{ij} = \mu + u_i + \epsilon_{ij}$ with $u_i \sim (0, \sigma_u^2)$ and $\epsilon_{ij} \sim (0, \sigma^2)$.

   (c) Compute the BLUPs for the model above and compare them with the corresponding fixed effect estimate for the u_i.

2. Load the `Pixel` data set in R from the package `nmle`. Use the function `help(Pixel)` to familiarize yourself with its format.

(a) Visualize the data using some informative plots.

(b) Fit a linear mixed effect model where you have $y_{ijk} = \beta_0 + \beta_1 x_k + \beta_1 x_k^2 + u_i + u_{ij} + \epsilon_{ijk}$ where y_{ijk} represents pixel intensity, i is dog, j is side and k is day index and x_k is day. Note that sides of a dog are nested within the dog. Interpret the results.