Take home Final for Probability Theory, First Quarter, 2017

Due October 20 at 5PM.

1. In four sentences or less, explain the usefulness of measure theory, i.e., what can be done with measure theory that cannot be done without it. Be as specific as possible.

2. In four sentences or less, explain the usefulness of the general theory of integration. (It is enough to explain what can be done with it that cannot be done with only the Riemann integral.) Be as specific as possible.

3. In this problem, you will prove the important result that for two random variables X, Y on the measurable space (Ω, Σ): Y is $\sigma(X)$-measurable if and only if $Y = f(X)$ for some Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$. For this problem, you can assume the range of X equals $\mathbb{R}$, where the range of X is defined as $X(\Omega) = \{x \in \mathbb{R} : \text{ for some } \omega \in \Omega, X(\omega) = x\}$.

   (a) Show that if $Y = f(X)$ for some Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$, then Y is measurable with respect to $\sigma(X)$.

   (b) To show the converse, for all remaining parts of this problem, assume that Y is measurable with respect to $\sigma(X)$. Define the sets $A_{m,n} = \{\omega \in \Omega : m2^{-n} \leq Y(\omega) < (m+1)2^{-n}\}$ for integers $n > 0, m \in \mathbb{Z}$. Show that for any $n > 0$, the sets $\{A_{m,n}\}_{m \in \mathbb{Z}}$ partition Ω, i.e., these sets are disjoint and their union is Ω.

   (c) Show that for each $A_{m,n}$ there exists a Borel set $B_{m,n}$ such that $A_{m,n} = \{\omega \in \Omega : X(\omega) \in B_{m,n}\}$. Show that for any $n > 0$ the sets $\{B_{m,n}\}_{m \in \mathbb{Z}}$ partition $\mathbb{R}$.

   (d) Define $f_n(x) = m2^{-n}$, for $x \in B_{m,n}$. (We know for each $n > 0, x \in \mathbb{R}$, there is a unique m satisfying $x \in B_{m,n}$, by the previous part of this problem.) Show that for any $x \in \mathbb{R}$, we have $\lim_{n\to\infty} f_n(x)$ exists, and denote $f(x) = \lim_{n\to\infty} f_n(x)$. Show f is Borel-measurable.

   (e) Show that $Y = f(X)$.