Probability Theory III - Final

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1: Here we only consider the situation $1 > P(X<Y)$

if $P(X<Y)=1$ then $E[X|X<Y] = \frac{E[X1_{X<Y}]}{P(X<Y)} = E[X1_{X<Y}] = E[X]$

since $1_{X<Y} = 1$ a.s, therefore ~~the~~ we don't need to prove any thing

a) We prove here $E[X|X<Y] \le EX$.:

Proof: consider $Q = EX\, E1_{X<Y} - E[X1_{X<Y}]$, we prove $Q \ge 0$

if $Q \ge 0$, since $P(X<Y) > 0$, we directly have $E(X|X<Y) = \frac{E[X1_{X<Y}]}{E[1_{X<Y}]} \le EX$

now

$$Q = E\left[X E(1_{X<Y}) - X1_{X<Y}\right] = E\left\{X\left[E1_{X<Y} - 1_{X<Y}\right]\right\}$$

since X, Y are independent $E[\cdot] = E_X\{E_Y[\cdot]\}$ where $E_Y[\cdot]/E_X[\cdot]$ is doing expectation with regard to Y/X

$$\Rightarrow Q = E_X\left[X\left(E1_{X<Y} - E_Y 1_{X<Y}\right)\right] \cancel{= E_X[X\, P(X<Y)}$$

Denote $Q(t) = E1_{X<Y} - E_Y 1_{t<Y} = P(X<Y) - P(t<Y)$

then $q(t)$ is non-decreasing function and $\left.\begin{array}{l}\lim_{t\to+\infty} q(t) = P(X<Y) > 0 \\ \lim_{t\to-\infty} q(t) = P(X<Y) - 1 < 0\end{array}\right\}$

let $t_0 = \sup_t \{t : q(t) \le 0\}$ We have that $\left.\begin{array}{l} q(t) \le 0 \quad \forall t < t_0 \\ q(t) \ge 0 \quad \forall t > t_0\end{array}\right\}$ here $q(t), Q(t)$ are the same thing

Also $Q = E_X[X\, Q(X)] = E_X\left[X1_{X>t_0} Q(X)\right] + E_X\left[X1_{X<t_0} Q(X)\right] + E_X\left[X1_{X=t_0} Q(X)\right]$

~~$\ge E_X[t_0 1_{X>t_0} Q(X)]$~~

Consider $E_X[X 1_{X>t_0} Q(X)]$ when $X>t_0$ $Q(X)>0$

therefore $X 1_{X>t_0} Q(X) \geq t_0 1_{X>t_0} Q(X)$

$\Rightarrow E_X[X 1_{X>t_0} Q(X)] \geq E_X[t_0 1_{X>t_0} Q(X)]$

Similarly we will get

$$Q \geq E_X[t_0 1_{X>t_0} Q(X)] + E_X[t_0 1_{X<t_0} Q(X)] + E_X[t_0 1_{X=t_0} Q(X)]$$
$$= E_X[t_0 Q(X)] = t_0 E_X[Q(X)] = t_0 \{E 1_{X<Y} - E_X E_Y 1_{X<Y}\} = 0$$

Then we showed that $Q \geq 0$ which ends the proof.

b) We prove here $E[X|X<Y] \leq EX$

Proof: Now we have $P(X<Y) = P(Y<X)$

Since $P(X=Y)=0$, we have $P(X<Y)=\frac{1}{2}$

Also since $P(X=Y)=0$, $1_{X<Y} + 1_{X>Y} \overset{a.s}{=} 1$

$$\Rightarrow EX = E[X 1_{X<Y}] + E[X 1_{X>Y}] = E[X 1_{X<Y}] + E[Y 1_{Y>X}] \geq E[X 1_{X<Y}] + E[X 1_{Y>X}]$$
$$= 2E[X 1_{X<Y}] \Rightarrow E[X 1_{X<Y}] \leq \frac{1}{2} EX \Rightarrow E(X|X<Y) \leq EX$$

since $P(X<Y)=\frac{1}{2}$

c) We give a counter-example: let

$P(X=1, Y=1) = 0.54$, $P(X=2, Y=0) = 0.45$, $P(X=3, Y=4) = 0.01$

Then $E[X|X<Y] = 3$ $E[X] = 1.47 < 3 = E[X|X<Y]$

But X, Y are bounded $\Rightarrow E|X|^2 < +\infty$ $E|Y|^2 < +\infty$; $P(X<Y) = 0.01 > 0$

$$\mathrm{cov}(X,Y) = EXY - EXEY = 12 \times 0.01 + 0.54 - (0.54 + 2 \times 0.45 + 3 \times 0.01)(0.54 + 4 \times 0.01)$$
$$= 0.66 - 1.47 \times 0.58$$
$$= -0.1926 < 0 \Rightarrow \mathrm{corr}(X,Y) < 0.$$

So we have an counter-example

2:

1° a) $\to$ b) If a) holds $\forall A, B \in \mathcal{B}(\mathbb{R})$, let $1_A, 1_B$ be their charactorsbic function

Then $E[1_A(X) \cdot 1_B(Y) | G] = P(X \in A, Y \in B | G)$
$= P(X \in A | G) P(Y \in B | G)$
$= E[1_A(X) | G] E[1_B(Y) | G]$

Then the $\{A_i\}_{i=1}^n, \{B_j\}_{j=1}^m$ be borel sets and $\{a_i\}, \{b_j\}$ be any real number

We have $E[\sum_{i=1}^n a_i 1_{A_i}(X) \cdot \sum_{j=1}^m b_j 1_{B_j}(Y) | G] = \sum_{i,j} a_i b_j E[1_{A_i}(X) 1_{B_j}(Y) | G]$
$= \sum_{i,j} a_i b_j E[1_{A_i}(X) | G] E[1_{B_j}(Y) | G]$
$= E[\sum_{i=1}^n a_i 1_{A_i}(X) | G] \cdot E[\sum_{j=1}^m 1_{B_j}(Y) | G]$

Then b) hold for all simple function h, g.

Now first consider h, g be any positive bounded Borel function, then we can find $0 \le h_n \uparrow h$ $0 \le g_n \uparrow g$ where h_n, g_n are simple function.

Then $E[h_n(X) g_m(Y) | G] \uparrow E[h_n(X) g(Y) | G]$ when $m \uparrow +\infty$, fixed n since $h_n(X) g_m(Y)$ is montonic increase and positive and bounded

On the other side $E[h_n(X) g_m(Y) | G] = E(h_n(X) | G) E[g_m(Y) | G] \uparrow E[h_n(X) | G] E[g(Y) | G]$

$\Rightarrow E[h_n(X) g(Y) | G] = E[h_n(X) | G] E[g(Y) | G]$, let $n \uparrow +\infty$, in a same discussion we have $E[h(X) g(Y) | G] = E[h(X) | G] \cdot E[g(Y) | G]$

Finally ~~$E\,h(X)$~~ For any bounded borel h, g

$E[h(X) g(Y) | G] = E[h^+ g^+ | G] + E[h^- g^- | G] - E[h^+ g^- | G] - E[h^- g^+ | G]$
$= E[h^+ | G] E[g^+ | G] + \cdots = E[h | G] \cdot E[g | G]$

So b) holds.

2^0 b) $\Rightarrow$ c)

We need to prove $\forall A \in \sigma(\mathcal{G} \cup \sigma(Y))$

$$\int_A h(x)dp = \int_A E[h(x)|\mathcal{G}]\,dp \quad (1)$$

notice that $E[h(X)|\mathcal{G}]$ is $\mathcal{G}$ measurable therefore $\sigma[\mathcal{G} \cup \sigma(Y)]$ measurable therefore we only need to prove (1).

First we show that $\mathcal{A} = \{A : (1) \text{ holds}\}$ is an σ-algebra.

this is because ① $\int_\emptyset h(x)dp = 0 = \int_\emptyset E[h(x)|\mathcal{G}]dp$

$\int_\Omega h(x)dp = Eh(x) = E[E[h(x)|\mathcal{G}]] = \int_\Omega E[h(x)|\mathcal{G}]dp$ $\Big\} \Rightarrow \emptyset, \Omega \in \mathcal{A}$.

② $\int_{A^c} h(x)dp = E\,h(x) - \int_A h(x) = E h(x) - \int_A E(h(x)|\mathcal{G})$ if $A \in \mathcal{A}$

Therefore $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

③ $\forall A_i \in \mathcal{A}$. $A_i \cap A_j = \emptyset$

$$\int_{\cup A_i} h(x)dp = \cancel{\lim_n \int h(x) 1_{\cup_{i=1} A_i} dp} = \int h(x) 1_{\bigcup_{i=1}^{+\infty} A_i} dp$$

$$= \lim_n \int h(x) 1_{\bigcup_{i=1}^{n} A_i} dp$$

$$= \lim_n \sum \int_{A_i} h(x)\,dp = \lim_n \sum \int_{A_i} E[h(x)|\mathcal{G}]dp$$

$$= \lim_n \int E[h(x)|\mathcal{G}] 1_{\bigcup_{i=1}^{n} A_i} dp = \int_{\cup A_i} E[h(x)|\mathcal{G}]\,dp$$

$\Downarrow$

$\bigcup_{i=1}^{+\infty} A_i \in \mathcal{A}$

Since $\left|h(x) 1_{\bigcup_{i=1}^{n} A_i}\right| \le h(x)$ bounded

$\left|E[h(x)|\mathcal{G}] 1_{\bigcup_{i=1}^{n} A_i}\right| \le$ ~~the bounded~~ $\sup|h(x)|$ bounded.

so $E[h(x) 1_{\bigcup_{i=1}^{n} A_i}] \to E[h(x) 1_{\bigcup_{i=1}^{+\infty} A_i}]$

and $E[E[h|\mathcal{G}] 1_{\bigcup_{i=1}^{n} A_i}] \to E[E[h|\mathcal{G}] 1_{\bigcup_{i=1}^{+\infty} A_i}]$

Then we only need to prove $\mathcal{G} \subset \mathcal{A}$ and $\sigma(Y) \subset \mathcal{A}$. $\mathcal{G} \subset \mathcal{A}$ is given by definition of $E[h|\mathcal{G}]$

We only need to prove $\sigma(Y) \subset \mathcal{A}$: $\forall B \in \sigma(Y)$ $B = Y^{-1}(B_0)$ where B_0 is an borel set.

$$\int_B h(x)dp = E[h(x) 1_{B_0}(Y)] = E[E[h(x) 1_{B_0}(Y)|\mathcal{G}]] \quad \text{by b)}$$

$$= E\{E[h(x)|\mathcal{G}] \cdot E[1_{B_0}(Y)|\mathcal{G}]\}$$

and $\int_B E[h(x)|\mathcal{G}]dp = E[E[h(x)|\mathcal{G}] 1_{B_0}(Y)] = E[E[E[h(x)|\mathcal{G}] \cdot 1_{B_0}(Y)|\mathcal{G}]]$

$= E\{E(h(x)|\mathcal{G}) \cdot E[1_{B_0}(Y)|\mathcal{G}]\} = \int_B h(x)\,dp$

therefore c) holds.

3° c) ⇒ a): we have $\forall\, C \in \sigma[\sigma(Y) \cup \mathcal{G}]$ and bounded Borel h

$$\int_C h(X)\,dP = \int_C E[h(X)\mid \mathcal{G}]\,dP$$

Then $\forall$ Borel set A, B and $W \in \mathcal{G}$ let $h = 1_A(\cdot)$, $C = Y^{-1}(B) \cap W$.

We have $\int_{Y^{-1}(B)\cap W} 1_A(X)\,dP = \int_{Y^{-1}(B)\cap W} E(1_A(X)\mid \mathcal{G}]\,dP$

$$\Rightarrow \int_W 1_A(X)\,1_B(Y)\,dP = \int_W E[1_A(X)\mid \mathcal{G}]\cdot 1_B(Y)\,dP$$

Since $W \in \mathcal{G}$ and $E\big[E[1_A(X)\mid \mathcal{G}]\,1_B(Y) \mid \mathcal{G}\big] = E[1_A(X)\mid \mathcal{G}]\,E[1_B(Y)\mid \mathcal{G}]$

we have $\int_W E[1_A(X)\mid \mathcal{G}]\,E[1_B(Y)\mid \mathcal{G}]\,dP = \int_W E[1_A(X)\mid \mathcal{G}]\,1_B(Y)\,dP$

$$\Rightarrow \forall\, W \in \mathcal{G}: \int_W 1_A(X)\,1_B(Y)\,dP = \int_W E[1_A(X)\mid \mathcal{G}]\,E[1_B(Y)\mid \mathcal{G}]\,dP$$

$$\Rightarrow E\big[1_A(X)\,1_B(Y)\mid \mathcal{G}\big] = E[1_A(X)\mid \mathcal{G}]\;E[1_B(Y)\mid \mathcal{G}]$$ (the right hand side is obviously $\mathcal{G}$ measurable)

$$\Rightarrow P[X\in A, Y \in B \mid \mathcal{G}] = P[X \in A\mid \mathcal{G}]\,P(Y \in B\mid \mathcal{G})$$ $\forall\, A, B \in \mathcal{B}(\mathbb{R})$ therefore a) hold.

Combine 1°, 2°, 3°, the whole proof is finished.

~~Proof:~~

~~3: We prove b) in 2. $\forall$ bounded borel function h and g~~

$$\cancel{E[h(X)\cdot g(Y)\mid \mathcal{F}] = E\big[E[h(X)\cdot g(Y)\mid Z]\mid \mathcal{F}\big]}$$ ~~since $\mathcal{F} \subset \sigma(Z)$~~

~~since $X \perp Y$ given Z ⇒~~ $$\cancel{= E\big[E[h(X)\mid Z]\cdot E[g(Y)\mid Z]\mid \mathcal{F}\big]}$$

3. $\forall$ bounded borel function h and g

$$E[h(X)\cdot g(Y)\mid \mathcal{G}] = E\left[E[h(X)\cdot g(Y)\mid X, Z]\mid \mathcal{G}\right]$$ since $\mathcal{G} \subseteq \sigma(Z)$ so also $\mathcal{G} \subseteq \sigma(X, Z)$

$$= E\left[h(X)\cdot E[g(Y)\mid X, Z]\mid \mathcal{G}\right]$$

$$= E\left[h(X)\cdot E[g(Y)\mid Z]\mid \mathcal{G}\right]$$ since $X \perp Y$ given Z and use c) in 2.

In first ~~last~~ term we have proved that $E[g(Y)\mid Z]$ is a borel function of Z (because $E[g(Y)\mid Z]$ is $\sigma(Z)$ measurable) denote by $f(Z)$, we prove that f here can be a bounded function.

this is because $|f(Z)| = |E[g(Y)\mid Z]| \leq E[|g(Y)|\mid Z] \leq C < +\infty$

since $|x|$ is convex and $g(Y)$ is bounded

therefore use b) in 2. $X \perp Z$ give $\mathcal{G}$

we have. $E[h(X)\cdot E[g(Y)\mid Z]\mid \mathcal{G}] = E[h(X)\mid \mathcal{G}]\cdot E[E(g(Y)\mid Z)\mid \mathcal{G}]$

$$= E[h(X)\mid \mathcal{G}]\cdot E[g(Y)\mid \mathcal{G}]$$ since $\mathcal{G} \subseteq \sigma(Z)$

Therefore we get: $E[h(X)\cdot g(Y)\mid \mathcal{G}] = E[h(X)\mid \mathcal{G}]\,E[g(Y)\mid \mathcal{G}]$

$\forall$ bounded borel h, g.

by 2. we have that $X \perp Y$ given $\mathcal{G}$.