Advanced Method Homework 3

Bohao Tang

1: The quasi-likelihood is $Q(\mu;y) = \int_y^{\mu} \frac{y-t}{\sigma^2 V(t)} dt$.

$$= \int_y^{\mu} \frac{y-t}{\sigma^2 t^3} dt$$

$$= \frac{y}{2\sigma^2}\left[\frac{1}{y^2}-\frac{1}{\mu^2}\right] + \frac{1}{\sigma^2}\left[\frac{1}{\mu}-\frac{1}{y}\right].$$

$$= \frac{\mu^2 y - y^3 + 2y^2\mu - 2\mu^2 y}{2\sigma^2 y^2 \mu^2}$$

$$= \frac{2y^2\mu - \mu^2 y - y^3}{2\sigma^2 y^2 \mu^2}$$

2: Exercise 8.5:

a: $\text{Logit}(y_i) = x_i\beta + \sigma z \Rightarrow \log\frac{y_i}{1-y_i} = x_i\beta+\sigma z \Rightarrow y_i = \frac{\exp(x_i\beta+\sigma z)}{1+\exp(x_i\beta+\sigma z)}$

use taylor expansion, it similar to on $x_i\beta$

$$y_i \approx \frac{\exp(x_i\beta)}{1+\exp(x_i\beta)} + \left(\frac{e^t}{1+e^t}\right)'_{t=x_i\beta}\sigma z + \left(\frac{e^t}{1+e^t}\right)''_{t=x_i\beta}\frac{(\sigma z)^2}{2} + \cdots$$

$$\Rightarrow y_i = \frac{\exp(x_i\beta)}{1+\exp(x_i\beta)} + \frac{e^{x_i\beta}}{1+e^{x_i\beta}}\frac{1}{1+e^{x_i\beta}}\sigma z + \frac{e^{x_i\beta}(1-e^{x_i\beta})}{2(1+e^{x_i\beta})^3}\sigma^2 z^2$$

b: $$E y_i = \frac{\exp(x_i\beta)}{1+\exp(x_i\beta)} + 0 + \frac{e^{x_i\beta}(1-e^{x_i\beta})}{2(1+e^{x_i\beta})^3}\sigma^2 + \cdots$$

ignore σ^2 ~~term~~ and higher term since σ is small

$$\Rightarrow E y_i \approx e^{x_i\beta}/(1+e^{x_i\beta})$$

$$var(y_i) = \cancel{var\frac{e^{x_i\beta}}{1+e^{x_i\beta}}}\ var\left[\frac{e^{x_i\beta}}{1+e^{x_i\beta}}\frac{1}{1+e^{x_i\beta}}\sigma z + \cdots\right]$$

$$= [\mu_i(1-\mu_i)]^2\sigma^2 + \sigma^3(\cdots\cdots)$$ ignore σ^3 and higher term

$$\Rightarrow var(y_i) \approx [\mu_i(1-\mu_i)]^2\sigma^2$$

c. the QL equation is

$$\sum_i \frac{y_i-\mu_i}{V(\mu_i)} \cdot \frac{\partial \mu_i}{\partial \beta} = 0 \Leftrightarrow \sum \frac{\partial \mu_i}{\partial \beta} \cdot \frac{y_i-\mu_i}{\mu_i^2(1-\mu_i)^2} = 0 \qquad ①$$

and OLS for logit y_i is from the following equation

$$\sum_i (\operatorname{logit} y_i - \operatorname{logit} \mu_i) \frac{\partial \operatorname{logit} \mu_i}{\partial \beta} = 0$$

$$\Leftrightarrow \sum_i (\operatorname{logit} y_i - \operatorname{logit} \mu_i) \frac{d \operatorname{logit} \mu_i}{d\mu_i} \frac{\partial \mu_i}{\partial \beta} = 0$$

$$\Leftrightarrow \sum_i \frac{\operatorname{logit} y_i - \operatorname{logit} \mu_i}{\mu_i(1-\mu_i)} \frac{\partial \mu_i}{\partial \beta} = 0 \qquad ②$$

do taylor expansion to logit y_i

we have $\quad \operatorname{logit} y_i - \operatorname{logit} \mu_i = \frac{y_i-\mu_i}{\mu_i(1-\mu_i)} + o[(y_i-\mu_i)^2]$

There fore if $(y_i-\mu_i)^2$ is of small scale which means variance for y_i is not large

then solving equation ② is similar of solving $\sum_i \frac{y_i-\mu_i}{\mu_i^2(1-\mu_i)^2} \frac{\partial \mu_i}{\partial \beta} = 0$

which is exactly the QL equation.

d. then the proportion is given by $\frac{m_i}{n_i}$ where $m_i \sim$ binomial (n_i, p_i)

$\Rightarrow \mu_i = E \frac{m_i}{n_i} = p_i \qquad var\left(\frac{m_i}{n_i}\right) = \frac{1}{n_i^2} \cdot n_i p_i(1-p_i) = \frac{1}{n_i} p_i(1-p_i)$

$= \phi(1-\mu_i)\mu_i$ where $\phi = \frac{1}{n_i}$

If n_i is constant then ϕ is also constant $\Rightarrow V(\mu_i) = \phi\, \mu_i(1-\mu_i)$

If each leaf is cutting to a different number of tiny regions. or if the tiny regions are not independent and of equal probability of having blotch covered.

Then $V(\mu_i) = \phi\, \mu_i(1-\mu_i)$ is questionable.

3: Exercise 8.8:

$$U(\beta) = \sum_i \frac{y_i - \mu_i}{V(\mu_i)} \frac{\partial \mu_i}{\partial \beta} = \sum \frac{y_i - \beta}{\sigma^2} \cdot 1 = \frac{1}{\sigma^2} \sum (y_i - \beta)$$

$$\Rightarrow \hat{\beta} = \bar{y}$$

the model based estimate ~~fore~~ for $var(\hat{\beta})$ is from $var(\bar{y}) = \frac{\sigma^2}{n}$

then $\widehat{var(\hat{\beta})} = \frac{\hat{\sigma}^2}{n} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{(n-1)n}$.

the actual variance is $var(\hat{\beta}) = var(\bar{y}) = \frac{\sum var(y_i)}{n^2} = \frac{\sum \mu_i}{n^2} = \frac{\beta}{n}$

The robust estimator is given from

$$V \left[\sum_{i=1}^{n} \left(\frac{\partial \mu_i}{\partial \beta}\right) \frac{(y_i - \bar{y})^2}{[V(\mu_i)]^2} \left(\frac{\partial \mu_i}{\partial \beta}\right) \right] V$$

$$= \frac{\sigma^2}{n} \sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{\sigma^4} \frac{\sigma^2}{n} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n^2}$$

4: $X = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix}$ $\quad X^T X = \sum_{i=1}^{n} x_i x_i^T$ and $X_{(i)}^T X_{(i)} = \sum_{j=1}^{n} x_j x_j^T - x_i x_i^T$

$$= X^T X - x_i x_i^T$$

Then recall the sherman-morrison formula:

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u} \quad \text{where } u, v \text{ are vectors}.$$

Therefore

$$(1)\ h_{(i)} = x_i^T (X_{(i)}^T X_{(i)})^{-1} x_i = x_i^T [X^T X - x_i x_i^T]^{-1} x_i \quad \left(\text{let } \begin{matrix} u = -x_i \\ v = x_i \end{matrix} \Rightarrow\right)$$

$$= x_i^T (X^T X)^{-1} x_i + \frac{x_i^T (X^T X)^{-1} x_i x_i^T (X^T X)^{-1} x_i}{1 - x_i^T (X^T X)^{-1} x_i} = h_i + \frac{h_i^2}{1 - h_i} = \frac{h_i}{1 - h_i}$$

(2) $\hat{\mu}_{(i)} = \cancel{X_{(i)}^T} \hat{\beta}_{(i)} = x_i^T (X_{(i)}^T X_{(i)})^{-1} [X_{(i)}^T y_{(i)}]$

x_i^T

$$X_{(i)}^T y_{(i)} = \sum_{j=1}^{n} x_j y_j - x_i y_i = X \vec{y} - x_i y_i$$

$$\Rightarrow \hat{\mu}_{(i)} = x_i^T \left[(X^T X)^{-1} + \frac{(X^T X)^{-1} x_i x_i^T (X^T X)^{-1}}{1 - x_i^T (X^T X)^{-1} x_i} \right] (X \vec{y} - x_i y_i)$$

$$= \hat{y}_i - h_i y_i + \frac{h_i \hat{y}_i}{1 - h_i} - \frac{h_i^2 y_i}{1 - h_i}$$

$$\Rightarrow y_i - \hat{\mu}_{(i)} = \frac{y_i - \hat{y}_i}{1 - h_i} = \frac{y_i - \hat{\mu}_i}{1 - h_i}$$

Therefore

$$\frac{y_i - \hat{\mu}_{(i)}}{\sqrt{1 + h_{(i)}}} = \frac{y_i - \hat{\mu}_i}{1 - h_i} \cdot \frac{1}{\sqrt{1 + \frac{h_i}{1 - h_i}}}$$

$$= \frac{y_i - \hat{\mu}_i}{\sqrt{1 - h_i}}$$