Probability Theory II Homework 1

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1:

$$\int_{-\infty}^{+\infty} [F(x+5) - F(x)]\,dx$$

$$= \int_{-\infty}^{+\infty} \int_{x}^{x+5} f(t)\,dt\,dx$$

$$= \int_{-\infty}^{+\infty} dt \int_{t-5}^{t} f(t)\,dx$$ --- Since $f(t) \geq 0$ and by Tonelli theorem

$$= \int_{-\infty}^{+\infty} 5 f(t)\,dt$$

$$= 5$$ --- since $f(t)$ is a density

2: Proof:

$\forall A_{ij} \in \sigma(X_{i,j})$

$$P(\bigcap_{\substack{i=1 \\ j=1}}^{n} A_{ij}) = P(\bigcap_{i=1}^{n} [\bigcap_{j=1}^{n} A_{ij}])$$

$$= \prod_{i=1}^{n} P(\bigcap_{j=1}^{n} A_{ij})$$ --- since $\bigcap_{j=1}^{n} A_{ij} \in R_i$ ~~$\bigcap_{j=1} A_{ij} \in$~~ and R_i independent $\Rightarrow P(\bigcap_{i=1}^{n} \bigcap_{j=1}^{n} A_{ij}) = \prod_{i} P(\bigcap_{j} A_{ij})$

$$= \prod_{i=1}^{n} \prod_{j=1}^{n} P(A_{ij})$$ --- since. $A_{ij} \in G_j$ and G_j independent $\Rightarrow P(\bigcap_j A_{ij}) = \prod_j P(A_{ij})$

$$= \prod_{i,j}^{n} P(A_{ij})$$

$\Rightarrow X_{i,j}$ $(1 \leq i, j \leq n)$ are independent.

3:

$$E\left(\frac{X_1+\cdots+X_n}{n}\right)^2$$

$$= E\left[\frac{\sum_i X_i^2 + 2\sum_{i<j} X_i X_j}{n^2}\right]$$

$$\leq \frac{n\, r(0)}{n^2} + \frac{2}{n^2}\sum_{1\leq i<j\leq n} r(j-i)$$

$$= \frac{r(0)}{n} + \frac{2}{n^2}\left[r(n-1) + 2r(n-2) + 3r(n-3) + \cdots\cdots (n-1)r(1)\right]$$

Then $\forall \varepsilon>0$, $\exists K$, $n>k \Rightarrow |r(n)|<\varepsilon$.

and since $r(k)\to 0$, $|r(k)|$ is bounded $\Rightarrow \exists M<+\infty$ $|r(k)|\leq M$ $\forall k$.

Then for $n>\left[\frac{KM}{\varepsilon}\right]$ we have.

$$\frac{2}{n^2}\left[r(n-1)+2r(n-2)+\cdots+(n-1)r(1)\right]$$

$$\leq \frac{2}{n^2}\left[1+2+\cdots+(n-K+1)\right]\varepsilon + \frac{2}{n^2}(n-1)\cdot\underbrace{(M+M+\cdots+M)}_{K\text{ plus}}$$

$$\leq \frac{2}{n^2}\,\frac{n(n+1)}{2}\varepsilon + \frac{2}{n}KM \leq 4\varepsilon.$$

Therefore $\overline{\lim_n}\, E\left(\frac{\sum X_i}{n}\right)^2 \leq 0$ but $E\left|\frac{\sum X_i}{n}\right|^2 \geq 0$

$$\Rightarrow E\left(\left[\frac{\sum X_i}{n}\right]^2\right)\to 0 \Rightarrow \frac{\sum X_i}{n}\xrightarrow{L^2} 0$$

Then $P\left(\left|\frac{\sum X_i}{n}\right|>\varepsilon\right) \leq \frac{E\left(\frac{\sum X_i}{n}\right)^2}{\varepsilon^2}\to 0 \Rightarrow \frac{\sum X_i}{n}\xrightarrow{P} 0$