

Statistical Theory Homework 1

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1.
 - (a) $\{(0, g), (0, f), (0, s), (1, g), (1, f), (1, s)\}$
 - (b) $\{(0, s), (1, s)\}$
 - (c) $\{(0, g), (0, f), (0, s)\}$
 - (d) $\{(0, s), (1, s), (1, g), (1, f)\}$
 - (e) $\{(1, s)\}$
2.
 - (a) $A \cap B^c \cap C^c$
 - (b) $A \cap B \cap C^c$
 - (c) $A \cap B \cap C$
 - (d) $A \cup B \cup C$
 - (e) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$
 - (f) $A^c \cap B^c \cap C^c$
3.
 - (a) $\mathbf{P}(\text{at least a 6}) = \frac{6+6-1}{6 \times 6} = \frac{11}{36}$
 - (b) $\mathbf{P}(\text{same number}) = \frac{6}{6 \times 6} = \frac{1}{6}$
4. It's the same number as to choose 2 people out of 20 to shake hands, and that's $\binom{20}{2} = 190$.
5.
 - (a) $\frac{\binom{3 \times 4}{2} \times 2!}{\binom{5}{2} \times 2!} = \frac{3}{5}$
 - (b) $\frac{2 \times 3 + 3 \times 2}{\binom{5}{2} \times 2!} = \frac{3}{5}$
 - (c) $\frac{\binom{3}{2} \times 2!}{\binom{5}{2} \times 2!} = \frac{3}{10}$
6. I think it's to choose n ordered character from a-z, where n is the sum of the numbers of one's last and given name. Then:
 - (a) $26 \times 26 \times 26 = 17576$
 - (b) $26 \times 26 + 26 \times 26 \times 26 = 18252$
 - (c) $26 \times 26 + 26 \times 26 \times 26 + 26 \times 26 \times 26 \times 26 = 475228$
7. We can deal $\{1, 2, 3\}$ as one number and arrange the $n - 2$ numbers in order first, then we arrange 1, 2, 3 in order. Therefore: $\mathbf{P} = \frac{(n-2)! \times 3!}{n!} = \frac{6}{n(n-1)}$, where n must no less than 3.

8. It's equivalent to choose 7 children from 10 to have one and only one gift, therefore the number of results is $\binom{10}{3} = 120$.
9. It's the sum of ways to choose t elements in (x_1, x_2, \dots, x_n) to be 1, where t ranges from k to n . And that's $\sum_{t=k}^n \binom{n}{t}$
10. $\mathbf{P} = \frac{|\text{choose } r \text{ spaces occupied out of the rest } N-2 \text{ spaces}|}{|\text{choose } r \text{ occupied out of } N|} = \frac{\binom{N-2}{r}}{\binom{N}{r}} = \frac{(N-r)(N-r-1)}{N(N-1)}$
11. I'm a little confused about the meaning of "double" here, I think it means 2 dices are in same number.
- (a) $\mathbf{P} = \frac{2}{6 \times 6} / \frac{6}{6 \times 6} = \frac{1}{3}$
- (b) $\mathbf{P} = \frac{5+5}{6 \times 6} / \frac{6 \times 6 - 6}{6 \times 6} = \frac{1}{3}$
12. $\mathbf{P}(\text{coin with heads and tails} | \text{show up heads}) =$
 $\mathbf{P}(\text{choosed third coin and tossed to be heads}) / \mathbf{P}(\text{tossed to be heads}) =$
 $\frac{1/3 \times 1/2}{1/3 \times 1/2 + 1/3 \times 1 + 1/3 \times 0} = \frac{1}{3}$
13. if the n^{th} stamp is of type i , then it's new means the first $n-1$ stamps are of types other than i , whose probability is $(1-p_i)^{n-1}$. Therefore,

$$\mathbf{P}(\text{new}) = \sum_1^m \mathbf{P}(\text{new} | n^{\text{th}} \text{ of type } i) \mathbf{P}(n^{\text{th}} \text{ of type } i) = \sum_1^m p_i (1-p_i)^{n-1}$$

Extra Credit:

Here and in the question, I think, "has k balls" means having exactly k balls.

Denote $A_{n,r,k}$ to be the number of different results of putting n balls in r urns such that exactly one urn has k balls; $B_{n,r,k}$ to be that of putting n balls in r urns such that no urn has k balls; $C_{n,r,k}$ to be that of putting n balls in r urns such that at least one urn has k balls; and $N_{n,r}$ to be the number of methods to put n balls in r urns.

Then, first it's easy to see:

$$B_{n,r,k} = N_{n,r} \quad \text{if } n < k \quad (1)$$

$$B_{n,r,k} = 1 \quad \text{if } n \neq k \text{ and } r = 1 \quad (2)$$

$$B_{n,r,k} = 0 \quad \text{if } n = k \text{ and } r = 1 \quad (3)$$

$$B_{n,r,k} = N_{n,r} - r \quad \text{if } n = k \quad (4)$$

$$N_{n,r} = \binom{n+r-1}{n} \quad (5)$$

and

$$B_{n,r,k} + C_{n,r,k} = N_{n,r} \quad (6)$$

Also consider situations in $A_{n,r,k}$, it's obviously equivalent to choose one urn out of r and put k balls there, and then put the rest $n-k$ balls in rest $r-1$ urns such that no one has k balls. Therefore we have:

$$A_{n,r,k} = \binom{r}{1} B_{n-k,r-1,k} \quad (7)$$

and then we give an iterative expression of $C_{n,r,k}$. We can discuss which urn firstly has k balls. If the 1^{st} urn firstly have k balls then it's equivalent to put rest $n - k$ balls in rest $r - 1$ urns without restriction. If it's the last urn to firstly have k balls, then it's equivalent to put rest $n - k$ balls in first $r - 1$ urns such that no one has k balls.

Also, if b^{th} urn firstly have k balls (where $2 \leq b \leq r - 1$), then we discuss how many balls are in first $b - 1$ urns. Therefore it's the sum of "put t balls in $b - 1$ urns such that no one has k balls and put $n - k - t$ balls in rest $r - b$ urns without restriction " where t range from 0 to $n - k$. Therefore:

$$C_{n,r,k} = N_{n-k,r-1} + \sum_{b=2}^{r-1} \sum_{t=0}^{n-k} B_{t,b-1,k} N_{n-k-t,r-b} + B_{n-k,r-1,k} \quad (8)$$

Initial values in equations 1 to 5 and iterative equations 6 to 8 form a sufficient way to calculate $A_{n,r,k}$ and finally the probability will be:

$$\mathbf{P} = A_{n,r,k} / N_{n,r}$$