Probability Homework 2

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Ex 5.3.12: We need $Z_0=1$ here

If $P(\xi_i^m=1)=1$, then it's abvious

Else if $\mu\le 1$: $P[\lim_n Z_n/\mu^n=0]=1$

So it can only be the situation $\mu>1$

Then $P(Z_n=0 \text{ for some } n)=\rho$

Also let $r=P(\lim Z_n/\mu^n=0)$

Since Z_n/μ^n is a martingale that ≥ 0

$Z_n/\mu^n \xrightarrow{a.s.} Z$ as $n\to+\infty$

Then $r=P(Z=0)=\sum_k P(Z=0\mid Z_1=k)P(Z_1=k)$

Consider $P(Z=0\mid Z_1=k)=P(Z^{(1)}+Z^{(2)}+\cdots Z^{(k)}=0\mid Z_1^{(1)}=1, Z_1^{(2)}=1\cdots)$

where $Z_n^{(i)}$ is k independent copy of same branching process

and $Z^{(i)}=\lim_n Z_n^{(i)}/\mu^n$

Since $Z^{(i)}\ge 0$ and $P(Z^{(i)}\mid Z_1^{(i)}=1)=P(Z^{(i)}\mid Z_0^{(i)}=1)$

$\Rightarrow P(Z=0\mid Z_1=k)=P(Z=0)^k$

Therefore $r = \sum_k p_k \, P(Z=0)^k = \varphi[P(Z=0)] = \varphi(r)$

and since $r<1$

therefore $r = \rho$

We have $\{\lim Z_n/\mu^n > 0\} \subset \{Z_n > 0 \ \forall n\}$

and now they all have probability $1-\rho$

therefore they are equal a.s.

Ex. 5.3.13.

$$\varphi = p_0 + p_1 z + p_2 z^2 + p_3 z^3 = \frac{1}{8} + \frac{3}{8} z + \frac{3}{8} z^2 + \frac{1}{8} z^3$$

then solve $\varphi(z) = z$

$$\Rightarrow \quad z^3 + 3z^2 - 5z + 1 = 0$$

$$\Rightarrow \quad (z^2+4z-1)(z-1) = 0$$

So the root in $[0,1)$ is $\frac{\sqrt{20}-4}{2} = \sqrt{5}-2$

$\doteq 0.236068$

So the die out rate is 0.236068

Ex 5.4.5: $(X_n+C)^2$ is a submartingale $\forall c$.

$$P\left(\max_{1\le m\le n} X_m > \lambda\right) = P\left(\max_{1\le m\le n} X_m + C > \lambda + C\right)$$

$$\le P\left(\max_{1\le m\le n} (X_m+C)^2 > (\lambda+C)^2\right) \le \frac{E(X_n+C)^2}{(\lambda+C)^2}$$

let $\lambda + C = -\frac{E(X_n-\lambda)^2}{E(X_n-\lambda)} \Rightarrow P\left(\max_{1\le m\le n} X_m > \lambda\right) \le \frac{Var(X_n)}{E(X_n-\lambda)^2} = \frac{EX_n^2}{EX_n^2+\lambda^2}$

since $EX_n = EX_0 = 0$

Ex 5.4.6.

(i) $$E(\bar{X}_n \wedge M) = \int_0^{+\infty} P(\bar{X}_n \wedge M \geq \lambda)\, d\lambda$$

$$= \int_0^1 P(\bar{X}_n \wedge M \geq \lambda)\, d\lambda + \int_1^{+\infty} P(\bar{X}_n \wedge M \geq \lambda)\, d\lambda$$

$$\leq 1 + \int_1^{+\infty} \frac{1}{\lambda} \int X_n^+ 1_{(\bar{X}_n \wedge M) \geq \lambda}\, dP\, d\lambda$$

$$= 1 + \int X_n^+ \int_1^{+\infty} \frac{1}{\lambda} 1_{(\bar{X}_n \wedge M) \geq \lambda}\, d\lambda\, dP$$

$$= 1 + \int X_n^+ \log^+ (\bar{X}_n \wedge M)\, dP$$

(ii) for $a \log^+ b$: $a > 0$, $b > 0$

if $b \leq 1$ then $a \log^+ b \leq 0 \leq a \log^+ a + \frac{b}{e}$

if $b > 1$ $a \log^+ b = a \log b = a \log b - a \log a + a \log a$

for $a \log b - a \log a$ use derivative we find.

$$a \log b - a \log a \leq b/e \log b - \frac{b}{e} \log \frac{b}{e} = \frac{b}{e}$$

therefore $a \log^+ b \leq a \log a + \frac{b}{e} \leq a \log^+ a + \frac{b}{e}$

$\Rightarrow$ $a \log^+ b \leq a \log^+ a + b/e$

$\Rightarrow$ $\int X_n^+ \log^+ (\bar{X}_n \wedge M)\, dP \leq \int X_n^+ \log^+ X_n^+\, dP + \frac{1}{e} \int \bar{X}_n \wedge M\, dP$

$\Rightarrow$ $E(\bar{X}_n \wedge M) \leq (1 - \frac{1}{e})^{-1} E[X_n^+ (\log^+ X_n^+]$

let $M \to +\infty \Rightarrow E[\bar{X}_n] \leq (1 - e^{-1})^{-1} E[X_n^+ \log^+ X_n^+]$

5: If X_n is not a $\mathcal{F}_n$-martingale, there must exist N such that

$$E[X_{N+1} \mid \mathcal{F}_N] \neq X_N$$

Choose N to be inf of those such value

$$N = \inf\{n : E[X_{n+1} \mid \mathcal{F}_n] \neq X_n\}$$

Since $E[X_{N+1} \mid \mathcal{F}_N] \neq X_N$, without any loss we suppose

$$P[E[X_{N+1} \mid \mathcal{F}_N] > X_N] > 0$$

and let $\tau = \begin{cases} N+1 & E(X_{N+1} \mid \mathcal{F}_N) > X_N. \\ N & E(X_{N+1} \mid \mathcal{F}_N) \leq X_N \end{cases}$

Then $\{\tau \leq n\} = \Omega$ if $n \geq N+1$

$\{\tau \leq n\} = \phi$ if $n < N$

and $\{\tau \leq N\} = \{\tau = N\} = \{E[X_{N+1} \mid \mathcal{F}_N] \leq X_N\} \in \mathcal{F}_N$

therefore τ is a $\mathcal{F}_n$-stopping time.

Now $E X_\tau = E X_{N+1} 1_{E(X_{N+1} \mid \mathcal{F}_N) > X_N} + X_N 1_{E(X_{N+1} \mid \mathcal{F}_N) \leq X_N}$

$= E[E[X_{N+1} \mid \mathcal{F}_N] 1_{\cdots > X_N} + X_N 1_{\cdots \leq X_N}]$

Since $P(E(X_{N+1} \mid \mathcal{F}_N) > X_N) > 0$ $E\{E(X_{N+1} \mid \mathcal{F}_N) 1_{E(X_{N+1} \mid \mathcal{F}_N) > X_N}\}$

$> E\{X_N 1_{E(X_{N+1} \mid \mathcal{F}_N) > X_N}\}$

therefore $EX_\tau > E\{X_N[1_{\cdots > X_N} + 1_{\cdots \leq X_N}]\} = EX_N$

Since N is the inf, we have $E[X_{i+1} \mid \mathcal{F}_i] = X_i$ $\forall i = 0, 1, \cdots N-1$

$\Rightarrow EX_N = EX_0 \Rightarrow EX_\tau > EX_0$

but τ here is bounded, $\mathcal{F}_n$-stopping time, contradiction

$\Rightarrow X_n$ is $\mathcal{F}_n$-martingale

Ex 5.5.1

$\varphi(x)/x \to +\infty \Rightarrow \forall G,\ \exists M\ \ \forall x > M\ \ \varphi(x) > G \cdot x$

Therefore $\forall \varepsilon > 0$ Let G be large enough such that $\frac{C}{G} < \varepsilon$
and M is related to G.

Then

$$E\left[|X_i| 1_{|X_i| > M}\right] \le E\left[\frac{\varphi|X_i|}{G} 1_{|X_i| > M}\right]$$

$$\le \frac{1}{G} E\,\varphi|X_i| \le \frac{C}{G} < \varepsilon \quad \forall i$$

$$\Rightarrow \sup_i E\left[|X_i|;\ |X_i| > M\right] < \varepsilon$$

$$\Rightarrow \lim_M \sup_i E\left[|X_i|;\ |X_i| > M\right] = 0$$

$\Rightarrow \{X_i : i \in I\}$ is uniformly integrable.