BST 140.751
Problem Set 3

Due: October 17, 2017

1 Linear models

1. Let Σ be a known matrix. Consider the model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim N_p(\mathbf{0}, \Sigma)$. Derive the ML estimate of $\boldsymbol{\beta}$.

2. Let $\mathbf{P}$ be a orthogonal transformation matrix and consider the model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim N_p(\mathbf{0}, \sigma^2 I)$. Suppose someone gave you the ML estimates for $\tilde{\boldsymbol{\beta}}$ and $\tilde{\sigma}^2$ obtained from fitting the model $\tilde{\mathbf{y}} = \tilde{\mathbf{X}}\tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\varepsilon}}$ where $\tilde{\mathbf{y}} = \mathbf{P}\mathbf{y}$ and $\tilde{\mathbf{X}} = \mathbf{P}\mathbf{X}$ and $\tilde{\boldsymbol{\varepsilon}} \sim N_p(\mathbf{0}, \tilde{\sigma}^2\mathbf{I})$. Relate these estimates to the ML estimates of $\boldsymbol{\beta}$ and σ^2.

2 Multivariate normals

1. Let $\mathbf{X} \sim N_p(\mathbf{0}, \mathbf{I})$. Argue that $\mathbf{a}\mathbf{X}/\sqrt{\mathbf{a}'\mathbf{a}} \sim N(0, 1)$ for any non-zero vector $\mathbf{a}$.

2. Let $\mathbf{X} \sim N_p(\mathbf{0}, \mathbf{I})$. Argue that if $\mathbf{A}\mathbf{A}' = \mathbf{I}$ then $\mathbf{A}\mathbf{X} \sim N_p(\mathbf{0}, \mathbf{I})$. Argue geometrically why this occurs.

3. Argue that if $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$, the quadratic form $(\mathbf{y} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu})$ is χ^2_p.

3 Inference and estimation in linear models

1. Let $Y_{ij} = \mu_i + \epsilon_{ij}$ for $i = 1, 2$ and $j = 1, \ldots, J_i$ where the $\epsilon_{ij} \sim N(0, \sigma^2)$ are iid. Show that the unbiased estimate of σ^2 is the so-called pooled variance estimate,

$$S_p^2 = \frac{1}{J_1 + J_2 - 2}\{(J_1 - 1)S_1^2 + (J_2 - 1)S_2^2\}$$

   where S_i^2 is the standard variance estimate within group i.

2. Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed as $N(\mu, \sigma^2)$. Define

$$Q = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2.$$

(a) Show Q is an unbiased estimator of σ^2.

(b) Find the variance of Q.

3. Suppose

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \sim N_4 \left(\begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right) = N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Let $X_1 = Y_1 + Y_2 + Y_3 + Y_4,$ and $X_2 = Y_1 - Y_2 - Y_3 + Y_4.$

(a) Find the joint distribution of $(X_1, X_2)'$.

(b) Find the conditional distribution of X_1 given X_2.