Homework 2, 140.721, due 9/27/17 at 5PM.

1. The algebra $f(\mathcal{A})$ generated by a class $\mathcal{A}$ of subsets of S is defined as the intersection of all algebras on S containing $\mathcal{A}$.
   (a) Prove that $f(\mathcal{A})$ is indeed an algebra. This requires showing that the intersection of an arbitrary collection of algebras is an algebra, and that there is at least one algebra containing $\mathcal{A}$.
   (b) Consider the special case of $S = (0, 1]$ and $\mathcal{A} = \{(x, y] : 0 \leq x < y \leq 1\}$. Characterize $f(\mathcal{A})$ and prove your claim.
   (c) Again, consider the special case of $S = (0, 1]$ and $\mathcal{A} = \{(x, y] : 0 \leq x < y \leq 1\}$. Characterize $\sigma(\mathcal{A})$ (the σ-algebra generated by $\mathcal{A}$) and prove your claim.
2. For each claim below, tell if it is true or false, and prove your answer.
   (a) For any σ-algebras Σ_1, Σ_2 on a set S, we have $\Sigma_1 \cap \Sigma_2$ is a σ-algebra on S.
   (b) For any σ-algebras Σ_1, Σ_2 on a set S, we have $\Sigma_1 \cup \Sigma_2$ is a σ-algebra on S.
   (c) For any σ-algebras Σ_1, Σ_2 on a set S, we have $\Sigma_1 \times \Sigma_2$ is a σ-algebra on $S \times S$ (where $\times$ is the Cartesian product).
   (d) For $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$, we have $\mathcal{B} \times \mathcal{B}$ is the σ-algebra on $\mathbb{R}^2$ generated by the closed rectangles $\{[a, b] \times [c, d] : a, b, c, d \in \mathbb{R}$ such that $a < b, c < d\}$.
3. For each claim below, tell if it is true or false, and prove your answer.
   (a) For any collections $\mathcal{F}$ and $\mathcal{G}$ of subsets of S, we have $\sigma(\mathcal{F} \cap \mathcal{G}) \subseteq (\sigma(\mathcal{F}) \cap \sigma(\mathcal{G}))$.
   (b) For any collections $\mathcal{F}$ and $\mathcal{G}$ of subsets of S, we have $\sigma(\mathcal{F} \cap \mathcal{G}) \supseteq (\sigma(\mathcal{F}) \cap \sigma(\mathcal{G}))$.