Probability Theory Take home Final

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1: With measure theory, you can define uniform distribution (over $[0,1]$) rigorously, and then define every probability distribution rigorously. Without measure theory, there's only a small class of sets you can measure (like finite combination of rectangles)

2: Without general theory of integration, you can't do Riemann integral to even a simple function:

$$\phi(x) = \begin{cases} 0 & x \in [0,1] \setminus Q \\ 1 & x \in [0,1] \cap Q \end{cases}$$

With general theory of integration, you can also easilier compute the limit of integral (in some situation)

3: (a) Denote $\mathcal{B}$ be the Borel σ-algebra of R, then:

$$Y^{-1}(\mathcal{B}) = (f \circ X)^{-1}(\mathcal{B}) = X^{-1} \circ f^{-1}(\mathcal{B}) \subset X^{-1}(\mathcal{B}) = \sigma(X)$$

so Y is measurable with respect to $\sigma(X)$

(since f is measurable: $f^{-1}(\mathcal{B}) \subset \mathcal{B}$)

3. (b) First, show $\{A_{m,n}\}_m$ are disjoint:

$\forall\, m_1 \neq m_2 \in \mathbb{Z}$, without lossing generality, we suppose $m_1 < m_2$.

then $A_{m_1,n} \cap A_{m_2,n} = \{\omega \in \Omega : m_1 2^{-n} \leq Y(\omega) < (m_1+1)2^{-n}$ &

$m_2 2^{-n} \leq Y(\omega) < (m_2+1)2^{-n}\}$

since $m_1 < m_2 \in \mathbb{Z}$, $m_1 + 1 \leq m_2$.

for if $A_{m_1,n} \cap A_{m_2,n} \neq \emptyset$, there exists $\omega_0 \in A_{m_1,n} \cap A_{m_2,n}$

then $Y(\omega_0) < (m_1+1)2^{-n} \leq m_2 2^{-n}$

and $Y(\omega_0) \geq m_2 2^{-n}$

which is a contradiction, then $A_{m_1,n} \cap A_{m_2,n}$

Since m_1, m_2 are ~~abr~~ arbitrary, we have $\{A_{m,n}\}_m$ are disjoint.

Second show $\bigcup_{m\in\mathbb{Z}} A_{m,n} = \Omega$:

$\forall\, \omega \in \Omega$: $Y(\omega)$ is a Real number, ~~so there must exists a m_0~~

consider $m = \lfloor 2^n Y(\omega) \rfloor$ (the greatest number ~~that~~ among those integers smaller or equal to $2^n Y(\omega)$)

then $m \leq 2^n Y(\omega) < m+1 \Rightarrow m 2^{-n} \leq Y(\omega) < (m+1)2^{-n}$

$\Rightarrow \omega \in A_{m,n} \Rightarrow \omega \in \bigcup_{m\in\mathbb{Z}} A_{m,n}$

Since ω arbitary $\Rightarrow$ $\Omega \subset \bigcup_{m\in\mathbb{Z}} A_{m,n}$ the opposite is obvious

So we have $\bigcup_{m\in\mathbb{Z}} A_{m,n} = \Omega$

Proof:

3. (c): $A_{m,n}$ is just $Y^{-1}\{[m2^{-n}, (m+1)2^{-n})\}$

since Y is $\sigma[X]$-measurable, we have that

$A_{m,n} = Y^{-1}\{[m2^{-n}, (m+1)2^{-n})\} \in \sigma(X) = X^{-1}(\mathcal{B})$

So there exists a Borel set $B_{m,n}$ such that $A_{m,n} = X^{-1}\{B_{m,n}\}$

which is the same as $A_{m,n} = \{\omega \in \Omega : X(\omega) \in B_{m,n}\}$

$\forall m_1 \neq m_2 \in \mathbb{Z}$ since $X^{-1}(B_{m_1,n} \cap B_{m_2,n})$

$$= X^{-1}(B_{m_1,n}) \cap X^{-1}(B_{m_2,n})$$

$$= A_{m_1,n} \cap A_{m_2,n} = \emptyset$$

and the range of X is R

we have that $B_{m_1,n} \cap B_{m_2,n} = \emptyset$; $\forall m_1 \neq m_2 \in \mathbb{Z}$.

and also $X^{-1}\left(\bigcup_{m\in\mathbb{Z}} B_{m,n}\right) = \bigcup_{m\in\mathbb{Z}} X^{-1}(B_{m,n}) = \bigcup_{m\in\mathbb{Z}} A_{m,n} = \Omega$

and the range of X is R

we have that $\bigcup_{m\in\mathbb{Z}} B_{m,n} = R$

Therefore $\{B_{m,n}\}_{m\in\mathbb{Z}}$ partition R.

3. (d): We have $f_n = \sum_{m\in\mathbb{Z}} m2^{-n} 1_{B_{m,n}}$

$$= \lim_{L\to+\infty} \sum_{m=-L}^{L} m2^{-n} 1_{B_{m,n}}$$

Since $\sum_{m=-L}^{L} m2^{-n} 1_{B_{m,n}}$ is a simple function, it's measurable

and since f_n is pointwise limit of a sequence of measurable functions,

f_n is also measurable.

$\forall x_0 \in R$, consider $f_n(x_0)$ and $f_{n+1}(x_0)$

suppose $f_n(x_0) = m2^{-n}$, then $x_0 \in B_{m,n}$

Since $A_{m,n} = \{ m2^{-n} \le Y(\omega) < (m+1)2^{-n} \}$

$$= \{ (2m)2^{-(n+1)} \le Y(\omega) < (2m+1)2^{-(n+1)} \} \cup \{ (2m+1)2^{-(n+1)} \le Y(\omega) < (2m+2)2^{-(n+1)} \}$$

$$= A_{2m,n+1} \cup A_{2m+1,n+1}$$

we have $X(A_{m,n}) = X(A_{2m,n+1}) \cup X(A_{2m+1,n+1})$

$$\Rightarrow \quad B_{m,n} = B_{2m,n+1} \cup B_{2m+1,n+1}$$

So $x_0 \in B_{2m,n+1}$ or $x_0 \in B_{2m+1,n+1}$

So $f_{n+1}(x_0) = (2m)2^{-(n+1)}$ or $(2m+1)2^{-(n+1)}$

$\Rightarrow f_{n+1}(x_0) \ge f_n(x_0)$, so $f_n(x) \uparrow \quad \forall x \in R$

therefore $\lim_{n\to+\infty} f_n(x)$ exists for all $x \in R$

and since $f = \lim_{n\to+\infty} f_n$, f_n are measurable, f will also be measurable.

3(e):

Consider $\|Y - f(X)\|_\infty$ where $\|g\|_\infty = \sup_{w\in\Omega} |g(w)|$

$$\|Y-f(X)\|_\infty \leq \|Y-f_N(X)\|_\infty + \|f_N(X)-f(X)\|_\infty$$

For $Y - f_N(X)$: $\forall w$, $\exists m \in \mathbb{Z}$ $w \in A_{m,N}$ ($\{A_{m,N}\}_m$ partition Ω)

then $m2^{-N} \leq Y(w) < (m+1)2^{-N}$

and $X(w) \in B_{m,N} \Rightarrow f_N \circ X(w) = m2^{-N}$

$\Rightarrow |Y - f_N(X)| < 2^{-N}$

$\Rightarrow \|Y - f_N(X)\|_\infty < 2^{-N}$

For $\|f_N(X) - f(X)\|_\infty$: in 3(d) we actually proved that

$|f_{N+1}(x) - f_N(x)| \leq 2^{-(N+1)}$

$\Rightarrow \|f_{N+1}(X) - f_N(X)\|_\infty \leq 2^{-(N+1)}$

and since $f = \sum_{l=N}^{+\infty}(f_{l+1} - f_l) + f_N$. (this is equivlent to $\lim_{l\to+\infty} f_l$)

we have that $\|f(X) - f_N(X)\|_\infty = \|\sum_{l=N}^{+\infty}(f_{l+1}-f_l)(X)(\cdot)\|_\infty$

$\leq \sum_{l=N}^{+\infty} \|f_{l+1}(X) - f_l(X)\|_\infty \leq \sum_{l=N}^{+\infty} 2^{-(l+1)} = 2^{-N}$

therefore $\|Y - f(X)\|_\infty \leq 2 \cdot 2^{-N}$ $\forall N$.

so $\|Y - f(X)\|_\infty = 0$

so $Y = f(X)$