# Advanced Methods Homework 4

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## I Linear Mixed Model

1. (a)

$$\vec{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{31} \\ y_{32} \\ y_{33} \\ y_{41} \\ y_{42} \\ y_{43} \end{pmatrix} = \vec{J}_{12}\, \mu + \begin{pmatrix} \vec{J}_3 & & & 0 \\ & \vec{J}_3 & & \\ & & \vec{J}_3 & \\ 0 & & & \vec{J}_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \vec{\varepsilon}$$

so $X = \vec{J}_{12}$

$\vec{\beta} = \mu$

$$Z = \begin{pmatrix} \vec{J}_3 & & & 0 \\ & \vec{J}_3 & & \\ & & \vec{J}_3 & \\ 0 & & & \vec{J}_3 \end{pmatrix} = I_4 \otimes \vec{J}_3$$

$$\vec{w} = \cancel{\phantom{x}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

(b) BLUP of $\vec{u}$ is $E[\vec{w}|\vec{y}] = E[\vec{u}] + \operatorname{cov}(\vec{u},\vec{y})\, \operatorname{var}(\vec{y})^{-1} \{\vec{y} - E[\vec{y}]\}$

Here $E(\vec{w}) = \vec{0}$ $\quad \operatorname{cov}(\vec{w},\vec{y}) = \sigma_w^2 I_4 Z' = \sigma_u^2 Z'$

$$\operatorname{var}(\vec{y}) = Z \sigma_u^2 I_4 Z' + \sigma^2 I_{12} = \sigma^2 I_{12} + \sigma_w^2 ZZ'$$

$$\operatorname{var}[\vec{y}] = I_4 \otimes \left(\sigma^2 I_3 + \sigma_u^2 \vec{J}_3 \vec{J}_3'\right)$$

recall $(aI + b J_n J_n')^{-1}$

$$= \frac{1}{a}\left(I - \frac{b}{a+nb} J_n J_n'\right)$$

we have that $var[\vec{y}]^{-1} = I_4 \otimes \frac{1}{\sigma^2}\left[I_3 - \frac{\sigma_u^2}{\sigma^2+3\sigma_u^2}\vec{J}_3\vec{J}_3'\right]$

So that $E[\vec{u}|\vec{y}] = \sigma_u^2 z' \, var[\vec{y}]^{-1} (\vec{y} - \mu\vec{J}_{12})$

$= I_4 \otimes \frac{\sigma_u^2}{\sigma^2}\left[\vec{J}_3' - \frac{3\sigma_u^2}{\sigma^2+3\sigma_u^2}J_3'\right][\vec{y} - \mu\vec{J}_{12}]$

$= I_4 \otimes \frac{\sigma_u^2}{\sigma^2+3\sigma_u^2}\vec{J}_3' (\vec{y} - \mu\vec{J}_{12}) = \frac{3\sigma_u^2}{\sigma^2+3\sigma_u^2}\begin{pmatrix}\bar{y}_1-\mu\\ \bar{y}_2-\mu\\ \bar{y}_3-\mu\\ \bar{y}_4-\mu\end{pmatrix}$

where $\bar{y}_i = \frac{\sum_{j=1}^{3} y_{ij}}{3}$

So BLUP of $u_i$ is $\frac{3\sigma_u^2}{\sigma^2+3\sigma_u^2}(\bar{y}_i - \mu)$

(C) BLUE of $\mu$ is $\hat{\mu} = (X'V^{-1}X)^{-1}X'V^{-1}\vec{y}$ where $V = var(\vec{y})$

$\Rightarrow \hat{\mu} = \left(\frac{12}{\sigma^2+3\sigma_u^2}\right)^{-1} \frac{1}{\sigma^2+3\sigma_u^2} J_{12}' \vec{y} = \frac{\Sigma\vec{y}}{12} = \bar{y}$

So the BLUE for $\mu$ is still the sample mean $\bar{y}$

2. (a) For every unbiased predictor $\theta(y)$

$E\{[\theta(y) - \vec{u}]^2\} = E[(\theta(y) - E[\vec{u}|\vec{y}] + E[\vec{u}|\vec{y}] - \vec{u})^2]$

$= E\{[\theta(y) - E(\vec{u}|\vec{y})]^2\} + E\{[E[\vec{u}|\vec{y}] - \vec{u}]^2\}$

$+2\underbrace{E[(\theta(y) - E[\vec{u}|\vec{y}])(E[\vec{u}|\vec{y}] - \vec{u})]}_{①}$

$① = E[E[(\theta(y) - E(\vec{u}|\vec{y}))(E(\vec{u}|\vec{y}) - \vec{u})|\vec{y}]]$

$= E[(\theta(y) - E(\vec{u}|\vec{y}))E[(E[\vec{u}|\vec{y}] - \vec{u})|\vec{y}]] = E[(\theta(y) - E[\vec{u}|\vec{y}])\cdot 0] = 0$

$\Rightarrow E[\theta(y) - \vec{u}]^2 = E[\theta(y) - E(\vec{u}|\vec{y})]^2 + E[E[\vec{u}|\vec{y}] - \vec{u}]^2$.

So when $\theta(y) \overset{a.s.}{=} E(\vec{u}|\vec{y})$ $E[\theta(y) - \vec{u}]^2$ archieves its minimum, also $E[\vec{u}|\vec{y}]$ is unbiased

since normality $E[\vec{u}|\vec{y}]$ is linear for $\vec{y}$ ∴ So $E[\vec{u}|\vec{y}]$ is BLUP.

(b) $\hat{u} = E[\vec{u}|\vec{y}] = E[\vec{u}] + cov[\vec{u}, \vec{y}) \, var[\vec{y}]^{-1} (\vec{y} - E(\vec{y})]$

$$= \Sigma_u Z' \cdot [Z\Sigma_u Z' + \sigma^2 I]^{-1} (\vec{y} - X\vec{\beta})$$

(c) $var(\hat{u}) = \Sigma_u z' [z\Sigma_u z' + \sigma^2 I]^{-1} var(\vec{y} - X\vec{\beta}) [z\Sigma_u z' + \sigma^2 I]^{-1} z \Sigma_u$.

$$= \Sigma_u z' [z \Sigma_u z' + \sigma^2 I]^{-1} z \Sigma_u.$$

(d) $var(\hat{u} - \vec{u}) \cancel{= E(\vec{u} - E[\vec{u}|\vec{y}])^2 - E\vec{u}^2}$

$$= E (\hat{u} - \vec{u})(\hat{u}' - \vec{u}')$$

$$= var[\hat{u}] + var[\vec{u}] - E[\vec{u}\,\hat{u}'] - E[\hat{u} \cdot \vec{u}']$$

$$= \Sigma_u z' [z \Sigma_u z' + \sigma^2 I]^{-1} z\Sigma_u + \Sigma_u$$

$$- \Sigma_u Z' [Z \Sigma_u z' + \sigma I]' z \Sigma_u - \Sigma_u z' [Z \Sigma_u z' + \sigma^2 I]' z \Sigma_u$$

$$= \Sigma_u - \Sigma_u z' [z \Sigma_u z' + \sigma^2 I]^{-1} z \Sigma_u.$$

3. Denote $A = (KVK')^{-1} K Z_{(1)} Z'_{(1)} K'$ and $\vec{\omega} = K\vec{y}$

Then $E[\text{right side}] = E[\vec{\omega}' A (KVK')^{-1} \vec{\omega}] = tr[A (KVK')^{-1} E[\vec{\omega}\vec{\omega}']]$

$$= tr[A (KVK')^{-1} K E(\vec{y}\vec{y}') K']$$

$$= tr[A (KVK')^{-1} KVK'] = tr(A)$$ which is the left side.