Probability Theory II - Homework #2

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1. Assume that $P(\limsup A_n) = 1$ and $P(\liminf B_n) = 1$. Prove that $P(\limsup(A_n \cap B_n)) = 1$. What happens if the condition on $\{B_n\}$ is weakened to $P(\limsup B_n) = 1$?
2. Suppose events $A_1, A_2, ...$ are such that $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$. Prove that $P(A_n\ i.o.) = 0$.
3. Let $Y_1, Y_2, ...$ be i.i.d. Find necessary and sufficient conditions for (i) $Y_n/n \to 0$ almost surely, (ii) $(\max_{m \le n} Y_m)/n \to 0$ almost surely, (iii) $(\max_{m \le n} Y_m)/n \to 0$ in probability, and (iv) $Y_n/n \to 0$ in probability.
4. (From Comps 2015) Let $X_1, X_2, \ldots,$ be i.i.d. random variables with an Exp(1) distribution. Prove that, with probability 1,

$$\frac{\sup_{j \le n} X_j}{\ln n} \longrightarrow 1.$$