Homework 2, 140.721, due 9/27/17 at 5PM.

1. The algebra $f(\mathcal{A})$ generated by a class $\mathcal{A}$ of subsets of S is defined as the intersection of all algebras on S containing $\mathcal{A}$.
   (a) Prove that $f(\mathcal{A})$ is indeed an algebra. This requires showing that the intersection of an arbitrary collection of algebras is an algebra, and that there is at least one algebra containing $\mathcal{A}$.
   (b) Consider the special case of $S = (0, 1]$ and $\mathcal{A} = \{(x, y] : 0 \leq x < y \leq 1\}$. Characterize $f(\mathcal{A})$ and prove your claim.
   (c) Again, consider the special case of $S = (0, 1]$ and $\mathcal{A} = \{(x, y] : 0 \leq x < y \leq 1\}$. Characterize $\sigma(\mathcal{A})$ (the σ-algebra generated by $\mathcal{A}$) and prove your claim.
2. For each claim below, tell if it is true or false, and prove your answer.
   (a) For any σ-algebras Σ_1, Σ_2 on a set S, we have $\Sigma_1 \cap \Sigma_2$ is a σ-algebra on S.
   (b) For any σ-algebras Σ_1, Σ_2 on a set S, we have $\Sigma_1 \cup \Sigma_2$ is a σ-algebra on S.
   (c) For any σ-algebras Σ_1, Σ_2 on a set S, we have $\Sigma_1 \times \Sigma_2$ is a σ-algebra on $S \times S$ (where $\times$ is the Cartesian product).
   (d) For $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$, we have $\mathcal{B} \times \mathcal{B}$ is the σ-algebra on $\mathbb{R}^2$ generated by the closed rectangles $\{[a, b] \times [c, d] : a, b, c, d \in \mathbb{R} \text{ such that } a < b, c < d\}$.
3. For each claim below, tell if it is true or false, and prove your answer.
   (a) For any collections $\mathcal{F}$ and $\mathcal{G}$ of subsets of S, we have $\sigma(\mathcal{F} \cap \mathcal{G}) \subseteq (\sigma(\mathcal{F}) \cap \sigma(\mathcal{G}))$.
   (b) For any collections $\mathcal{F}$ and $\mathcal{G}$ of subsets of S, we have $\sigma(\mathcal{F} \cap \mathcal{G}) \supseteq (\sigma(\mathcal{F}) \cap \sigma(\mathcal{G}))$.

Probability Theory Homework 2

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1. (a) *Proof.* Since the power set of S is a algebra and contains $\mathcal{A}$, $f(\mathcal{A})$ is well defined if the intersection of an arbitrary collection of algebras $\mathcal{A}_i, i \in I$: $\cap_{i\in I}\mathcal{A}_i$, is also an algebra.

   i. Since $\mathcal{A}_i$ are all algebras, for all i: $S \in \mathcal{A}_i$, therefore $S \in \cap_{i\in I}\mathcal{A}_i$.

   ii. $\forall A \in \cap_{i\in I}\mathcal{A}_i$, A must contains in every $\mathcal{A}_i$, then A^c contains in every $\mathcal{A}_i$ because they are all algebras. Therefore, $A^c \in \cap_{i\in I}\mathcal{A}_i$.

   iii. For all A, B in $\cap_{i\in I}\mathcal{A}_i$, we have $\forall i$, $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_i \Rightarrow \forall i : A \cup B \in \mathcal{A}_i \Rightarrow A \cup B \in \cap_{i\in I}\mathcal{A}_i$.

   Therefore, $\cap_{i\in I}\mathcal{A}_i$ is indeed an algebra. □

   (b) Suppose $\mathcal{C} = \{\emptyset\} \cup \{\cup_{i=1}^{k}(x_{2k-1}, x_{2k}] : k \in \mathbb{N}^+ \ \& \ 0 \le x_1 < x_2 < x_3 < \cdots < x_{2k-1} < x_{2k} \le 1\}$. We prove that $f(\mathcal{A}) = \mathcal{C}$.

   Proof. Since every element in $\mathcal{C}$ is just a finite union of sets in $\mathcal{A} \cup \{\emptyset\}$, we have $\mathcal{C} \subset f(\mathcal{A})$. So we only need to prove $\mathcal{C}$ is a algebra, then according to the defnition of $f(\mathcal{A})$, we have $f(\mathcal{A}) \subset \mathcal{C}$. And then $\mathcal{C} = f(\mathcal{A})$. Now we prove this.

   i. By defnition $(0, 1] \in \mathcal{C}$.

   ii. For every set $A = \cup_{i=1}^{k}(x_{2k-1}, x_{2k}] \in \mathcal{C}$, then $A^c = C_1 \cup (x_2, x_3] \cup (x_4, x_5] \cup (x_6, x_7] \cup \cdots \cup (x_{2k-2}, x_{2k-1}] \cup C_2$, where $C_1 = (0, x_1]$ if $x_1 > 0$ else $\emptyset$ and $C_2 = (x_{2k}, 1]$ if $x_{2k} < 1$ else $\emptyset$. It's easy to see that A^c also obey the rules in the definition of $\mathcal{C}$. So $A^c \in \mathcal{C}$.

   iii. For every two sets A, B in $\mathcal{C}$, $A \cup B$ is a finite union of sets of form $(x, y]$, we prove now that every this kind of set is in $\mathcal{C}$.
   Suppose $W = \cup_{i=1}^{n}(x_i, y_i]$, then:
   If $(x_i, y_i]$ are pairwise disjoint or $n = 1$, then we just need to put the end points in order to see that W is in $\mathcal{C}$.
   If $n > 1$ and $(x_i, y_i]$ are not pairwise joint, then we can find some $(x_{i_0}, y_{i_0}]$ and some $(x_{i_1}, y_{i_1}], (x_{i_2}, y_{i_2}], \cdots, (x_{i_k}, y_{i_k}], k \ge 1$, such that for any j: $(x_{i_0}, y_{i_0}] \cap (x_{i_j}, y_{i_j}] \ne \emptyset$. Then is easy to see that (by induction) $\cup_{j=1}^{k}(x_{i_j}, y_{i_j}] \cup (x_{i_0}, y_{i_0}] = (\min_{0\le i\le k} x_i, \max_{0\le i\le k} y_i]$. So we can replace these $k+1$ sets by one set of form $(x, y]$ and make W unchanged. Then the number n of total sets in the union will decrease at least 1 (decrease k and $k > 1$).
   And we can continuously do this when $W = \cup_{i=1}^{n}(x_i, y_i]$ for some not pairwise disjoint $(x_i, y_i]$ and $n > 1$, since n is finite. We finally will reduce n to 1 or reduce $\{(x_i, y_i]\}$ to a pairwise disjoint collection but leave W unchanged. Therefore W is in $\mathcal{C}$.
   So that $A \cup B \in \mathcal{C}$.

   Therefore $\mathcal{C}$ is an algebra, which ends the proof. □

   (c) $\sigma(\mathcal{A})$ is the Borel σ-algebra of $(0, 1]$, denote it by $\mathcal{B}_{(0,1]}$.

   Proof. For all $(x, y] \in \mathcal{A}$: If $y = 1$, $(x, 1]$ is open in $(0, 1]$, so $(x, 1] \in \mathcal{B}_{(0,1]}$; Else $(x, y] = \cap_{n=1}^{\infty}(x, y + \frac{1-y}{n})$.
   Therefore $(x, y] \in \mathcal{B}_{(0,1]} \Rightarrow \mathcal{A} \subset \mathcal{B}_{(0,1]} \Rightarrow \sigma(\mathcal{A}) \subset \mathcal{B}_{(0,1]}$
   On the other side, for all A open interval relative to $(0, 1]$: $A = (x, 1] \in \mathcal{A}$ or $A = (x, y) = \cup_{n=1}^{\infty}(x, y - \frac{y-x}{n}] \in \sigma(\mathcal{A})$. Therefore $A \in \sigma(\mathcal{A}) \Rightarrow \mathcal{B}_{(0,1]} \subset \sigma(\mathcal{A})$.
   So $\sigma(\mathcal{A}) = \mathcal{B}_{(0,1]}$. □

2. (a) True.

Proof. Given Σ_1 and Σ_2 be σ-algebra:

i. $S \in \Sigma_1$ and $S \in \Sigma_2$, so $S \in \Sigma_1 \cap \Sigma_2$.
ii. For all A in $\Sigma_1 \cap \Sigma_2$, we have $A \in \Sigma_1$ and $A \in \Sigma_2$. Therefore $A^c \in \Sigma_1$ and $\Sigma_2 \Rightarrow A^c \in \Sigma_1 \cap \Sigma_2$.
iii. For all $\{A_i\}$ in $\Sigma_1 \cap \Sigma_2$, we have all A_i are in Σ_1 and Σ_2. Therefore $\cup_{i=1}^{\infty} A_i \in \Sigma_1$ and $\cup_{i=1}^{\infty} A_i \in \Sigma_2 \Rightarrow \cup_{i=1}^{\infty} A_i \in \Sigma_1 \cap \Sigma_2$.

So, $\Sigma_1 \cap \Sigma_2$ is also σ-algebra. $\square$

(b) False.
In $\{1, 2, 3, 4\}$, $\{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ is a σ-algebra and $\{\emptyset, \{1, 3\}, \{2, 4\}, \{1, 2, 3, 4\}\}$ is a σ-algebra. But there union is not a σ-algebra, because $\{1, 3\}$ and $\{1, 2\}$ are in it but $\{1\} = \{1, 3\} \cap \{1, 2\}$ is not in it.

(c) False.
If $\Sigma_1 \times \Sigma_2$ is defined by $\{A \times B \ : A \in \Sigma_1, B \in \Sigma_2\}$. Then it may not be a σ-algebra. For example, denote $\mathcal{B}_R$ be the Borel σ-algebra of $\mathbb{R}$. Then $\mathcal{B}_R \times \mathcal{B}_R$ contains all open rectangle. Therefore $\sigma(\mathcal{B}_R \times \mathcal{B}_R)$ is at least as large as $\mathcal{B}_{R^2}$. So if $\mathcal{B}_R \times \mathcal{B}_R$ is σ-algebra, then open unit circle will be in it. However, this is not the case since open cirle is not a Cartrsian product of any set. So $\mathcal{B}_R \times \mathcal{B}_R$ is not a σ-algebra.

(d) I think is to prove $\sigma(\{[a, b] \times [c, d] : a \leq b \ \& \ c \leq d \ \& \ a, b, c, d \in \mathcal{R}\}) = \mathcal{B}_{R^2}$. (If $\mathcal{B} \times \mathcal{B}$ is defined like in 2.c., then it will not be a σ-algebra).

Proof. $\mathcal{B}_{R^2}$ contains all open sets, hence it also contains all closed sets (it's complementary of open sets) $\Rightarrow$ all $[a, b] \times [c, d] \in \mathcal{B}_{R^2} \Rightarrow \sigma(\{\text{All Closed Rectangles}\}) \subset \mathcal{B}_{R^2}$.
On the other side we only need to prove that all open sets are in $\sigma(\{\text{All Closed Rectangles}\})$.
For every open set O, set $S = O \cap \mathbb{Q}^2$. Consider $D = \cup_{C \in \mathcal{A}} C$. (we use $\{p, q\}$ to denote the point $(p, q) \in R^2$ and (a, b) to denote open interval to avoid confusing) where:

$$\mathcal{A} = \{[p, q] \times [r, s] \subset O : \{p, r\} \in S; \{q, s\} \in S\}$$

Then we have $\mathcal{A}$ is a countable collection (since $\mathbb{Q}^4$ is countable and $\mathcal{A}$ is indiced by a subset of it, therefore $\mathcal{A}$ is countable) in $\{\text{All Closed Rectangles}\}$ and therefore $D \in \sigma(\{\text{All Closed Rectangles}\})$.
Finally, we argue that $D = O$. In the definition of D, we have obviously $D \subset O$ (every set in $\mathcal{A}$ is contained in O).
On the other hand, for every point $x = \{u, v\} \in O$, since O is open, there is a open rectangle $x \in (a, b) \times (c, d) \subset O$. Then we have $a < u < b$ and $c < v < d$. Since $\mathbb{Q}$ is dense, we can find rational number p, q, r, s such that $a < p < u < q < b$ and $c < r < v < s < d$. Therefore $[p, q] \times [r, s] \subset (a, b) \times (c, d) \subset O \Rightarrow [p, q] \times [r, s] \in \mathcal{A}$. Since $\{u, v\} \in [p, q] \times [r, s] \Rightarrow \{u, v\} \in D$ and the arbitary of $\{u, v\}$, we get $O \subset D$. Therefore $D = O$.
So every open set is in $\sigma(\{\text{All Closed Rectangles}\}) \Rightarrow \mathcal{B}_{R^2} \subset \sigma(\{\text{All Closed Rectangles}\})$.
Then $\mathcal{B}_{R^2} = \sigma(\{[a, b] \times [c, d] : a \leq b \ \& \ c \leq d \ \& \ a, b, c, d \in \mathcal{R}\})$. $\square$

3. (a) True.

Proof. $\mathcal{F} \cap \mathcal{G} \subset \mathcal{F} \subset \sigma(\mathcal{F})$, and the same way we have $\mathcal{F} \cap \mathcal{G} \subset \sigma(\mathcal{G})$. Therefore $\mathcal{F} \cap \mathcal{G} \subset \sigma(\mathcal{F}) \cap \sigma(\mathcal{G})$. Since $\sigma(\mathcal{F}) \cap \sigma(\mathcal{G})$ is a σ-algebra (we proved it in 2.a.), we have $\sigma(\mathcal{F} \cap \mathcal{G}) \subset \sigma(\mathcal{F}) \cap \sigma(\mathcal{G})$. $\square$

(b) False.
Suppose $\mathcal{F}$ is the collection of all open sets in $\mathbb{R}$ and $\mathcal{G}$ is the collection of all closed set in $\mathbb{R}$. Then $\sigma(\mathcal{F}) = \sigma(\mathcal{G}) = \mathcal{B}_R$. But $\sigma(\mathcal{F} \cap \mathcal{G}) = \sigma(\emptyset) \neq \mathcal{B}_R$.