# Probability Theory Homework 3

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1.1.6 $\mathcal{A}$ is not an algebra, hence not a $\sigma$-algebra. We raise an example that the union of two sets whose asymptotic density exist may not have asymptotic density, therefore $\mathcal{A}$ is not an algebra:

Consider $A_1$ is the set of all odd numbers. And $A_2$ is the set that contains all even number in the closed interval $[2^{2k}, 2^{2k+1}]$, and all odd number in $[2^{2k+1}, 2^{2k+2}]$, where $k \in \mathbb{N}$. First we prove that $A_1$ and $A_2$ all have asymptotic density $\frac{1}{2}$:

For $A_1$, consider the parity of $n$, when $n = 2k, k \in \mathbb{N}$, $|A_1 \cap \{1, 2, \cdots, n\}|/n = \frac{1}{2}$, when $n = 2k+1, k \in \mathbb{N}$, $|A_1 \cap \{1, 2, \cdots, n\}|/n = \frac{n+1}{2n}$. Therefore we have that $\frac{1}{2} \leq |A_1 \cap \{1, 2, \cdots, n\}|/n \leq \frac{n+1}{2n} \to \frac{1}{2}$. By approximation theory, we have $\lim |A_1 \cap \{1, 2, \cdots, n\}|/n = \frac{1}{2}$.

For $A_2$, there are $1+\lfloor 2^{2k-1} \rfloor$ ($\lfloor \cdot \rfloor$ is round down function) even numbers in the interval $[2^{2k}, 2^{2k+1}]$, and $2^{2k}$ odd numbers in $[2^{2k+1}, 2^{2k+2}]$.

If $n = 2^{2k} + l$, where $0 \leq l < 2^{2k}$ (suppose $k > 1$, we are considering the limit, so the situation when $n$ is small will not influence), then:

$$|A_1 \cap \{1, 2, \cdots, 2^{2k}\}| = \sum_{m=0}^{k-1} (1 + \lfloor 2^{2m-1} \rfloor) + \sum_{m=0}^{k-1} 2^{2m} = 2^{2k-1} + k - 1 \tag{1}$$

And for $k$ we have $(\log_2 n)/2 \geq (\log_2 2^{2k})/2 = k$ and $\log_2 n < \log_2 2^{2k+1} = 2k+1 \Rightarrow k > \frac{\log_2 n - 1}{2}$. Consider the parity of $l$, we get:

$$2^{2k-1} + k - 1 + \frac{l}{2} \leq |A_1 \cap \{1, 2, \cdots, n\}| \leq 2^{2k-1} + k - 1 + \frac{l+2}{2} \tag{2}$$

$$|A_1 \cap \{1, 2, \cdots, n\}|/n \leq \frac{2^{2k-1} + l/2 + k}{2^{2k} + l} \leq \frac{n + \log_2 n}{2n} \tag{3}$$

$$|A_1 \cap \{1, 2, \cdots, n\}|/n \geq \frac{2^{2k-1} + l/2 + k - 1}{2^{2k} + l} \geq \frac{n + \log_2 n - 3}{2n} \tag{4}$$

If $n = 2^{2k+1} + l$, where $0 \leq l < 2^{2k+1}$. After totally similiar computation, we have:

$$|A_1 \cap \{1, 2, \cdots, n\}|/n \leq \frac{2^{2k} + l/2 + k}{2^{2k+1} + l} \leq \frac{n + \log_2 n - 1}{2n} \tag{5}$$

$$|A_1 \cap \{1, 2, \cdots, n\}|/n \geq \frac{2^{2k} + (l-1)/2 + k}{2^{2k+1} + l} \geq \frac{n + \log_2 n - 3}{2n} \tag{6}$$

Therefore for every $n > 17$, we have:

$$\frac{1}{2} \leftarrow \frac{n + \log_2 n - 3}{2n} \leq |A_1 \cap \{1, 2, \cdots, n\}| \leq \frac{n + \log_2 n}{2n} \to \frac{1}{2}$$

By approximation theory, we have $\lim |A_1 \cap \{1, 2, \cdots, n\}|/n = \frac{1}{2}$.

Now, consider the union of $A_1, A_2$. First $A_1 \cup A_2$ contains and only contains all numbers in interval $[2^{2k}, 2^{2k+1}]$ and all odd numbers in $[2^{2k+1}, 2^{2k+2}]$, where $k \in \mathbb{N}$. We denote $p_k =$

$|(A_1 \cup A_2) \cap \{1, 2, \cdots, 2^{2k}\}|/2^{2k}$ and $q_k = |(A_1 \cup A_2) \cap \{1, 2, \cdots, 2^{2k+1}\}|/2^{2k+1}$. Then for $k > 1$:

$$p_k = \frac{1}{2^{2k}} \left\{ 1 + \sum_{l=0}^{k-1} (2^{2l} + 1) + \sum_{l=0}^{k-1} 2^{2l} \right\} = \frac{2^{2k+1} - 2 + 3k}{3 \times 2^{2k}} \to \frac{2}{3} \tag{7}$$

$$q_k = \frac{1}{2^{2k+1}} \left\{ \sum_{l=0}^{k} (2^{2l} + 1) + \sum_{l=0}^{k-1} 2^{2l} \right\} = \frac{5^{2k+1} - 4 + 6k}{6 \times 2^{2k+1}} \to \frac{5}{6} \tag{8}$$

$\lim p_k \neq \lim q_k$ and $p_k, q_k$ are just two subarrays of $|(A_1 \cup A_2) \cap \{1, 2, \cdots, n\}|/n$. Therefore, $\lim |(A_1 \cup A_2) \cap \{1, 2, \cdots, n\}|/n$ does not exist. Hence $A_1 \cup A_2$ is not in $\mathcal{A}$.

1.2.2 $\mathbf{P}(\chi \geq 4) = \int_4^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$. Since Theorem 1.2.3:

$$(\frac{1}{x} - \frac{1}{x^3}) e^{-\frac{x^2}{2}} \leq \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \leq \frac{1}{x} e^{-\frac{x^2}{2}}$$

We have:

$$\frac{15}{64} e^{-8} \leq \mathbf{P}(\chi \geq 4) \leq \frac{1}{4} e^{-8}$$

1.2.3 First we notice that distribution function $F$ is non-decreasing, if $t > s$ then $\{X \leq s\} \subset \{X \leq t\} \Rightarrow \mathbf{P}(X \leq t) \geq \mathbf{P}(X \leq s)$.

Then for every discontinuity $x$, since $F$ is monotonic, the left limit and right limit of $F$ at the point $x$ will always exist. Denote them by $F(x-), F(x+)$. Since $F$ is non-decreasing, $F(x-) \leq F(x+)$ and since $x$ is a discontinuity, $F(x-) < F(x+)$. There is always a rational number in non-empty open set $(F(x-), F(x+))$, we choose one and denote by $r_x$.

Then we get a map from discontinuity set $D$ to rational number $\mathbb{Q}$, call it $\phi$: $\phi(x) = r_x$ for $x \in D$. We now proof that $\phi$ is an injection so that $D$ is countable (since $\mathbb{Q}$ is conutable).

*Proof.* $\forall x_1, x_2 \in D : x_1 < x_2$, we assert that $r_{x_1} < r_{x_2}$. If $r_{x_1} \geq r_{x_2}$ then $F(x_1+) > r_{x_1} \geq r_{x_2} > F(x_2-)$. Therefore we can choose a $y_1 \in (x_1, \frac{x_1+x_2}{2})$ and $F(y_1) > r_{x_1}$($F(y_1)$ can be arbitary close to $F(x_1+)$!). And also we can find a $y_2 \in (\frac{x_1+x_2}{2}, x_2)$, such that $F(y_2) < r_{x_2}$. Then we get that $y_1 < y_2$ but $F(y_1) > F(y_2)$, which is contradict to the non-decreasing of $F$. So $r_{x_1} < r_{x_2}$ and therefore $\phi$ is an injection. □

1.2.5 I think this problem is to prove that the density of $g(X)$ is $f(g^{-1}(y))/g'(g^{-1}(y))$, and the support is in $[g(\alpha), g(\beta)]$.

*Proof.* Suppose the distribution function of $X$ is $F$. Then because $g$ is strictly increasing, we have that $\mathbf{P}(g(\alpha) \leq g(X) \leq g(\beta)) = \mathbf{P}(\alpha \leq X \leq \beta) = 1$. So the density outside $(g(\alpha), g(\beta))$ is 0. For $y$ in $(g(\alpha), g(\beta))$, we have that:

$$\mathbf{P}(g(X) \leq y) = \mathbf{P}\left(X \leq g^{-1}(y)\right) = F(g^{-1}(y)) \tag{9}$$

Since $g$ is differentiable and strictly increasing, we have $\frac{dg^{-1}}{dx}|_y = \frac{1}{g'(g^{-1}(y))}$. Along with $F' = f$, we differentiate equation 9 to get the density of $g(X)$ at point $y$ is $f(g^{-1}(y))/g'(g^{-1}(y))$. □