BST 140.751
Problem Set 3
Due: October 5, 2017

1 Generalized inverse

1. Let $\mathbf{G}$ be a generalized inverse of $\mathbf{X}'\mathbf{X}$. Show that
   (a) $\mathbf{G}'$ is also a generalized inverse of $\mathbf{X}'\mathbf{X}$,
   (b) $\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X} = \mathbf{X}$, i. e. $\mathbf{G}\mathbf{X}'$ is a generalized inverse of $\mathbf{X}$,
   (c) $\mathbf{X}\mathbf{G}\mathbf{X}'$ is invariant to $\mathbf{G}$,
   (d) $\mathbf{X}\mathbf{G}\mathbf{X}'$ is symmetric, whether $\mathbf{G}$ is or not.

   Hint for (b): Show first that $\mathbf{X}'\mathbf{X} = \mathbf{0}$ implies $\mathbf{X} = \mathbf{0}$, and second that $\mathbf{P}\mathbf{X}'\mathbf{X} = \mathbf{Q}\mathbf{X}'\mathbf{X}$ implies $\mathbf{P}\mathbf{X}' = \mathbf{Q}\mathbf{X}'$.

2 Inference under incorrectly specified models

1. Let Model 1 be $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}$ and Model 2 be $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \tilde{\boldsymbol{\varepsilon}}$. Suppose that Model 2 is fit while Model 1 is true. Compute the bias and variance of the estimated $\boldsymbol{\beta}_1$. Compute the expected value of s^2.

3 Multivariate means and variances

1. Let $\mathbf{a}$ is a $p \times 1$ vector of constants and $\mathbf{y}$ is a $p \times 1$ random vector with variance-covariance matrix Σ. Show that the variance of $\mathbf{a}'\mathbf{y}$ is given by $\mathbf{a}'\Sigma\mathbf{a}$.

2. Let $\mathbf{x}_i$ for $i = 1, \ldots, I$ be iid k dimensional vectors from a distribution with mean μ and variance Σ. What is the mean and variance of the multivariate pointwise sample average of the vectors?

3. Let $\mathbf{x}_i$ for $i = 1, \ldots, I$ be iid k dimensional vectors from a distribution with mean μ and variance Σ. Give an unbiased estimate of Σ when μ is known.

4. Consider a covariance matrix that is of the form

$$\sigma^2\mathbf{I} + \theta\mathbf{1}\mathbf{1}'$$

   where σ^2 and θ are positive constants and $\mathbf{1}$ is a vector of ones. Argue that this matrix describes random vectors where every pair of elements of the vector are equally correlated and every elemant has the same variance. Give this correlation and variance.

5. For a linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, $E(\boldsymbol{\varepsilon}) = \mathbf{0}$, $\mathrm{var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$, the residuals are given by $\mathbf{e} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}} = (\mathbf{I} - \mathbf{H})\mathbf{y}$. Find

   (a) $E(\mathbf{e})$

   (b) $\mathrm{var}(\mathbf{e})$

   (c) $\mathrm{cov}(e, \mathbf{Hy})$

   (d) $E(\mathbf{e}'\mathbf{e})$