Advanced Methods in Biostatistics I

Lecture 14

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Quadratic Forms - continued

- Last time we discussed the distributional properties of quadratic forms and how they apply to linear models.
- Today we will expand upon these results.

Quadratic Forms

Definition

A quadratic form is a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ of the form:

$$f(\mathbf{y}) = \mathbf{y}'\mathbf{A}\mathbf{y} = \sum_{i,j} A_{ij} y_i y_j.$$

Quadratic Forms

- Note that $\mathbf{y}'\mathbf{A}\mathbf{y} = \mathbf{y}'\mathbf{A}'\mathbf{y}$, so if we let $\mathbf{B} = (\mathbf{A} + \mathbf{A}')/2$, then $\mathbf{B}' = \mathbf{B}$, and $\mathbf{y}'\mathbf{A}\mathbf{y} = \mathbf{y}'\mathbf{B}\mathbf{y}$.
- Thus when studying a quadratic form $\mathbf{y}'\mathbf{A}\mathbf{y}$, we will assume that $\mathbf{A}$ is symmetric.

Distribution of quadratic form

Theorem

If $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$ then

$$\mathbf{y}'\mathbf{A}\mathbf{y} \sim \chi^2_r(\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}/2))$$

if and only if $\mathbf{A}\Sigma$ is idempotent of rank r

t distribution

Theorem

If $Z \sim N(0,1)$ and $U \sim \chi_r^2$ are independent, then the random variable

$$T = \frac{Z}{\sqrt{U/r}}$$

follows t distribution with $r > 0$ degrees of freedom.

We write $T \sim t_r$.

Non-central t distribution

Theorem

If $Z \sim N(\mu, 1)$ and $U \sim \chi^2_r$ are independent, then the random variable

$$T = \frac{Z}{\sqrt{U/r}}$$

follows a noncentral t distribution with $r > 0$ degrees of freedom and noncentrality parameter μ.

We write $T \sim t_r(\mu)$.

F distribution

Theorem

If U_1 and U_2 are independent central χ^2 random variables with degrees of freedom m_1 and m_2, respectively, then the random variable

$$W = \frac{U_1/m_1}{U_2/m_2}$$

follows a (central) F distribution with $m_1 > 0$ and $m_2 > 0$ degrees of freedom.

We write $W \sim F_{m_1,m_2}$.

Noncentral F distribution

Theorem

If $U_1 \sim \chi^2_{m_1}(\lambda)$ and $U_2 \sim \chi^2_2$ be independent, then the random variable

$$W = \frac{U_1/n_1}{U_2/n_2} \sim F_{n_1,n_2}(\lambda).$$

follows a noncentral F distribution with $m_1 > 0$ and $m_2 > 0$ degrees of freedom and noncentrality parameter $\lambda > 0$

We write $W \sim F_{m_1,m_2}(\lambda)$.

Independence

- Let's continue by discussing some independence results related to quadratic forms.
- In particular, we seek conditions for when $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{B}\mathbf{y}$ are independent and when $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{y}'\mathbf{B}\mathbf{y}$ are independent.

Independence

Theorem

Let $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$, $\mathbf{A}$ be symmetric idempotent matrix, and $\mathbf{B}$ a matrix of constants, and suppose $\mathbf{B}\Sigma\mathbf{A} = \mathbf{0}$. Then $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{B}\mathbf{y}$ are independent.

Example

- Assume $\mathbf{y} \sim N_n(\mu \mathbf{J}_n, \sigma^2 \mathbf{I})$.
- Recall, we can write the sample variance as follows:

$$
\begin{aligned}
(n-1)s^2 &= \mathbf{y}'(\mathbf{I} - \mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n')\mathbf{y} \\
&= \mathbf{y}'\mathbf{A}\mathbf{y}
\end{aligned}
$$

where $\mathbf{A} = (\mathbf{I} - \mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n')$.

Example

- Also recall, we can write

$$\begin{aligned}\bar{y} &= (\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n'\mathbf{y} \\ &= \mathbf{B}\mathbf{y}\end{aligned}$$

where $\mathbf{B} = (\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n'$.

Example

- These two statistics are independent because

$$\begin{aligned}\textbf{BVA} &= (\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n'\sigma^2\mathbf{I}(\mathbf{I}-\mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n') \\ &= \mathbf{0}\end{aligned}$$

  since $\mathbf{J}_n'(\mathbf{I}-\mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n') = 0$

- Since functions of independent statistics are also independent, it must hold that $\bar{y}$ and s^2 are independent.

Example

- Last time we showed that

$$(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}.$$

- In addition, it is easy to see that

$$\bar{y} \sim N(\mu, \sigma^2/n).$$

Example

- Note that $\bar{y}$ and s^2 are independent as shown above.
- Thus,

$$
\begin{aligned}
T &= \frac{(\bar{y}-\mu)/(\sigma/\sqrt{n})}{\sqrt{s^2/\sigma^2}} \\
&= \frac{\bar{y}-\mu}{s/\sqrt{n}} \\
&\sim t_{n-1}
\end{aligned}
$$

Independence

Theorem

Let $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$, $\mathbf{A}$ be symmetric idempotent of rank m, and $\mathbf{B}$ be symmetric idempotent of rank s, and suppose $\mathbf{B}\Sigma\mathbf{A} = \mathbf{0}$. Then $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{y}'\mathbf{B}\mathbf{y}$ are independent.

Example

- Consider the linear model: $\mathbf{y} = \mathbf{X}\beta + \varepsilon$, where $\varepsilon \sim N_n(\mathbf{0}, \sigma^2\mathbf{I})$.

- Previously, we showed that

$$\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}/\sigma^2 \sim \chi^2_{n-p}$$

and

$$\mathbf{y}'\mathbf{H}\mathbf{y}/\sigma^2 \sim \chi^2_p(\beta\mathbf{X}'\mathbf{X}\beta/(2\sigma^2)).$$

Example

- Let
$$\mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y}/\sigma^2 = \mathbf{y}'\mathbf{A}\mathbf{y}$$
where $\mathbf{A} = \sigma^{-2}(\mathbf{I}-\mathbf{H})$.
- Also, let
$$\mathbf{y}'\mathbf{H}\mathbf{y}/\sigma^2 = \mathbf{y}'\mathbf{B}\mathbf{y}$$
where $\mathbf{B} = \sigma^{-2}\mathbf{H}$.

Example

- Note,

$$\begin{aligned} \mathbf{B}\Sigma\mathbf{A} &= \sigma^{-2}\mathbf{H}\sigma^{2}\mathbf{I}\sigma^{-2}(\mathbf{I}-\mathbf{H}) \\ &= \mathbf{0} \end{aligned}$$

- Thus, $\mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y}/\sigma^2$ and $\mathbf{y}'\mathbf{H}\mathbf{y}/\sigma^2$ are independent quadratic forms.

Example

- Thus,

$$
\begin{aligned}
F &= \frac{\mathbf{y}'\mathbf{H}\mathbf{y}/p}{\mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y}/(n-p)} \\
&= \frac{\sigma^{-2}\mathbf{y}'\mathbf{H}\mathbf{y}/p}{\sigma^{-2}\mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y}/(n-p)} \\
&\sim F_{p,n-p}(\beta'\mathbf{X}'\mathbf{X}\beta/2\sigma^2)
\end{aligned}
$$

Cochran's Theorem

Theorem

Let $\mathbf{y} \sim N_p(\mu, \sigma^2\mathbf{I})$ and suppose that $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_k$ are $n \times n$ symmetric idempotent matrices with $rank(\mathbf{A}_i) = s_i$. If

$$\mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_k = \mathbf{I},$$

then

$$\mathbf{y}'\mathbf{A}_1\mathbf{y}/\sigma^2, \mathbf{y}'\mathbf{A}_2\mathbf{y}/\sigma^2, \ldots \mathbf{y}'\mathbf{A}_k\mathbf{y}/\sigma^2$$

follow independent $\chi^2_{s_i}(\lambda_i)$ distributions, where $\lambda_i = \boldsymbol{\mu}'\mathbf{A}_i\boldsymbol{\mu}/2\sigma^2$, for $i = 1, 2, \ldots k$ and $\sum_{i=1}^{k} s_i = n$.

Cochran's Theorem

- Cochran's Threreom can be used to determine the distributions of partitioned sums of squares of normally distributed random variables.
- It allows us to split the sum of the squares of observations into a number of quadratic forms where each corresponds to some cause of variation.

Example

- Consider the model:

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon,$$

where

$$\varepsilon \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}).$$

- Suppose we can write: $\mathbf{X} = (\mathbf{X}_1 \ \mathbf{X}_2)$ and $\beta = (\beta_1' \ \beta_2')'$.
- Further suppose $\mathbf{X}_i$ is a $n \times p_i$ matrix where $p_1 + p_2 = p$.

Example

- Now consider both the full model:

$$\mathbf{y} = \mathbf{X_1}\boldsymbol{\beta}_\mathbf{1} + \mathbf{X_2}\boldsymbol{\beta}_\mathbf{2} + \varepsilon$$

and the submodel

$$\mathbf{y} = \mathbf{X_1}\boldsymbol{\beta}_\mathbf{1} + \varepsilon.$$

Example

- Let us define the following projections matrices:

$$\begin{aligned}\mathbf{A}_1 &= \mathbf{P}_{\mathbf{X}_1}\\ \mathbf{A}_2 &= \mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_1}\\ \mathbf{A}_3 &= \mathbf{I} - \mathbf{P}_{\mathbf{X}}\end{aligned}$$

Example

- All $\mathbf{A}_i$ are symmetric and idempotent.
- Further
$$\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$$
and
$$\begin{aligned} r(\mathbf{A}_1) + r(\mathbf{A}_2) + r(\mathbf{A}_3) &= p_1 + (p - p_1) + (n - p) \\ &= n \end{aligned}$$
where $r(\mathbf{A}_i)$ is the rank of $\mathbf{A}_i$.

Example

- Therefore Cochran's theorem applies, and

$$\begin{aligned}
\sigma^{-2}\mathbf{y}'\mathbf{P}_1\mathbf{y} &\sim \chi^2_{p_1}(\beta'\mathbf{X}'\mathbf{P}_{\mathbf{X}_1}\mathbf{X}\beta) \\
\sigma^{-2}\mathbf{y}'(\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_1})\mathbf{y} &\sim \chi^2_{p_2}(\beta'\mathbf{X}'(\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_1})\mathbf{X}\beta) \\
\sigma^{-2}\mathbf{y}'(\mathbf{I} - \mathbf{P}_{\mathbf{X}})\mathbf{y} &\sim \chi^2_{n-p}
\end{aligned}$$

- Note, the last term follows a central χ^2 distribution because

$$\beta'\mathbf{X}'(\mathbf{I} - \mathbf{P}_{\mathbf{X}})\mathbf{X}\beta = 0.$$

Example

- Note, it also holds that

$$\sigma^{-2}\mathbf{y}'(\mathbf{P_X} - \mathbf{P_{X_1}})\mathbf{y}$$

  and

$$\sigma^{-2}\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y}$$

  are independent

- In addition,

$$F = \frac{\mathbf{y}'(\mathbf{P_X} - \mathbf{P_{X_1}})\mathbf{y}/p_2}{\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y}/(n-p)} \sim F_{p_2, n-p}(\beta'\mathbf{X}'(\mathbf{P_X} - \mathbf{P_{X_1}})\mathbf{X}\beta)$$

Example

- This is an example of sequential sums of squares.
- It allows us to study the contribution of adding additional variables to a model.