Probability Homework I

Bohao Tang

Exercise 3.4.3: ~~(If $|X_i|=+\infty$ a.s, then $S_n/\sqrt{n} \Rightarrow$ a limit obviously A)~~

Suppose $EX_i^2=\infty$, we first prove that $E\,Y_i^2=+\infty$

$$X_i^2=[(X_i-X_i')+X_i']^2 \le 2(X_i-X_i')^2+2X_i'^2$$

$$\Rightarrow (X_i-X_i')^2 \ge \frac{X_i^2}{2} - X_i'^2$$

Then we consider $E[(X_i-X_i')^2 \mid X_i'=t]$, since X_i, X_i' are independent and $|t|<+\infty$

we have $0\le E[(X_i-X_i')^2 \mid X_i'=t] = E_{X_i}[(X_i-t)^2] \ge E_{X_i}\left(\frac{X_i^2}{2}-t^2\right] = E_{X_i}[\frac{X_i^2}{2}]-t^2=+\infty$

Then $E(X_i-X_i')^2 = E[E[(X_i-X_i')^2|X_i'=t]] \ge E_t\,(E[(X_i-X_i')^2/X_i'=t]\,1_{|t|<+\infty}] = +\infty$

[Here we need that X_i have mass on $(-\infty,+\infty)$, but if $|X_i|=+\infty$ a.s, then $S_n/\sqrt{n}=\pm\infty$ not have a limit]

Then Let $Y_i=X_i-X_i'$, $U_i=Y_i 1_{(|Y_i|\le A)}$, $V_i=Y_i 1_{(|Y_i|>A)}$, then $\forall K$.

$$P\left(\sum_{m=1}^n Y_m \ge K\sqrt{n}\right) = P\left(\sum_{m=1}^n U_m+\sum_{m=1}^n V_m \ge K\sqrt{n}\right) \ge P\left[\sum_{m=1}^n U_m \ge K\sqrt{n};\ \sum_{m=1}^n V_m \ge 0\right]$$

$$= \sum_{\substack{\text{All possible}\\ \text{case for } 1_{(|Y|_i\le A)}}} P\left(\Sigma U_m \ge K\sqrt{n},\ \Sigma V_m\ge 0 \,\Big|\, |Y_{l_i}|\le A,\ l_i\in I,\ |Y_j|>A,\ j\in J \ \&\ I+J=\{1,2\cdots n\}\right)\cdot P\left(|Y_i|\le A,\ i\in I;\ |Y_j|>A,\ j\in J\right).$$

$$= \sum_{\substack{\text{All possible}\\ I,J}} P\left(\sum_{i\in I} U_i \ge K\sqrt{n},\ \sum_{j\in J} V_j\ge 0 \,\Big|\, |Y_I|\le A, |Y_J|>A\right) P(|Y_I|\le A, |Y_J|>A) \qquad \text{since } I\cap J=\emptyset$$

$$= \sum_{I,J} P\left(\sum_{i\in I} U_i\ge K\sqrt{n} \,\Big|\, |Y_I|\le A, |Y_J|>A\right)\cdot P\left(\sum_{j\in J} V_j\ge 0 \,\Big|\, \cdot\right)\cdot P(|Y_I|\le A, |Y_J|>A)$$

$$= P(\Sigma U_m \ge K\sqrt{n})\cdot P(\Sigma V_m\ge 0) \quad \text{since } V_m \text{ is symmetric} \ \ge \frac{1}{2} P(\Sigma U_m \ge K\sqrt{n})$$

Then we deal with $P(\sum U_m \geq K\sqrt{n})$

notice that $U_i = Y_i 1_{(|Y_i| \leq A)}$ is bounded and symmetric.

Therefore $E U_i = 0$ exists and $E|U_i|^2 < +\infty$ exists

But since $E Y_i^2 = +\infty$ and $E U_i^2 = E Y_i^2 1_{(|Y_i| \leq A)}$

Therefore when $A \to +\infty$ $\quad E U_i^2 \to +\infty$

Here, since $E U_i = 0$ and $E U_i^2 < +\infty$ and U_i i.i.d, CLT holds

Therefore $P\left(\frac{\sum_{m=1}^{n} U_m}{\sqrt{n}} \geq K\right) \to 1 - \Phi(K/\sigma_A)$ where $\sigma_A^2 = E U_i^2$ and $\Phi(\cdot)$ is the cdf for normal.

Then since K is fixed, we choose A big enough to let K/σ_A small enough

So that $1 - \Phi(K/\sigma_A) \geq \frac{2.1}{5}$

Then for this fixed K and A, we can find N, when $n > N$

$$P\left(\sum_{m=1}^{n} U_m \geq K\sqrt{n}\right) \geq \frac{2}{5}$$

Therefore $\frac{1}{2} P(\sum U_m \geq K\sqrt{n}) \geq \frac{1}{5}$

we have

$$P\left(\sum_{m=1}^{n} Y_m \geq K\sqrt{n}\right) \geq \frac{1}{5}.$$

Since $S_n/\sqrt{n} \Rightarrow$ a limit F $\quad$ let F' be a independent copy for F

Then obviously $\sum_{m=1}^{n} Y_m/\sqrt{n} \Rightarrow F - F'$ which means that

$$P\left(\frac{\sum_{m=1}^{n} Y_m}{\sqrt{n}} \geq K\right) \to 0 \quad \text{when } K \to +\infty$$

this is a contradiction $\Rightarrow E X_i^2 < +\infty$

Exercise 3.4.6

Let $Y_n = S_{N_n} / \sigma\sqrt{a_n}$ and $Z_n = S_{a_n} / \sigma\sqrt{a_n}$

Then $Y_n - Z_n = \dfrac{S_{N_n} - S_{a_n}}{\sigma\sqrt{a_n}}$

$\forall \varepsilon, \delta > 0$ we have : $P(|N_n - a_n| > \delta a_n) \to 0$ since $\frac{N_n}{a_n} \xrightarrow{P} 1$

and $P(|Y_n - Z_n| > \varepsilon) = P[|S_{N_n} - S_{a_n}| > \sigma\sqrt{a_n}\,\varepsilon]$

$= P(|S_{N_n} - S_{a_n}| > \sigma\sqrt{a_n}\,\varepsilon ; |N_n - a_n| \le \delta a_n) + P(|S_{N_n} - S_{a_n}| > \sigma\sqrt{a_n}\,\varepsilon ; |N_n - a_n| > \delta a_n)$

$\le P[\sup_{\substack{t \in \mathbb{N} \\ a_n - \delta a_n \le t \le a_n + \delta a_n}} |S_t - S_{a_n}| > \sigma\sqrt{a_n}\,\varepsilon] + P(|N_n - a_n| > \delta a_n)$

$\le 2P[\sup_{t \le [\delta a_n]} |S_t| > \sigma\sqrt{a_n}\,\varepsilon] + P(|N_n - a_n| > \delta a_n)$ By Kolmogorov's inequality

$\le 2 \cdot \dfrac{1}{\sigma^2 a_n \varepsilon^2} \cdot \mathrm{var}(X_1) \cdot \delta a_n + P(|N_n - a_n| > \delta a_n)$

Let $n \to +\infty$ we have

$\overline{\lim_n} P(|Y_n - Z_n| > \varepsilon) \le \dfrac{2\delta}{\varepsilon^2}$, since δ is abitary

we have $\lim_n P(|Y_n - Z_n| > \varepsilon) = 0 \Rightarrow Y_n - Z_n \xrightarrow{P} 0$

Therefore by CLT and Slutsky theory

$Y_n = Y_n - Z_n + Z_n \xrightarrow{P} \chi$

Exercise 3.6.3

Notice that $\tau_k^n - \tau_{k-1}^n$ are independent and have geometric distribution with parameter $1-(k-1)/n$.

and $\tau_0^n = 0$

Then we got that $\tau_k^n - k = \sum_{l=1}^{k} (Y_l^n - 1)$

where Y_l^n is independent (in l) geometric variable with parameter $1-\frac{l-1}{n}$

Then we calculate the characteristic function for $\tau_k^n - k$., denote by $\hat{F}_{\tau_k^n - k}(t)$

$$\hat{F}_{\tau_k^n-k}(t) = \prod_{l=1}^{k} \hat{F}_{(Y_l^n - 1)}(t) = \prod_{l=1}^{k} E\, e^{i(Y_l^n-1)t} = \prod_{l=1}^{k} \left[\sum_{s=0}^{+\infty} e^{its} (1-P_l)^s \cdot P_l \right] \quad \text{where } P_l = 1-\frac{l-1}{n}$$

$$= \prod_{l=1}^{k} P_l \cdot \frac{1}{1-(1-P_l)e^{it}}$$

We change the discussion to use moment generating function since log is not analytic in complex field, there will be much more writing to justify this problem in $\mathbb{C}$.

Denote $M_{\tau_k^n-k}(t)$ to be moment generating function for $\tau_k^n - k$

Then

$$M_{\tau_k^n-k}(t) = \prod_{l=1}^{k} M_{Y_l^n - 1}(t) = \prod_{l=1}^{k} E\, e^{(Y_l^n-1)t} = \prod_{l=1}^{k} \left[\sum_{s=0}^{+\infty} e^{ts} (1-P_l)^s P_l \right] \quad \text{where } P_l = 1-\frac{l-1}{n}$$

$$= \prod_{l=1}^{k} \frac{1-\frac{l-1}{n}}{1-\frac{l-1}{n}e^{t}}$$

where $t \in$ s.t. $\frac{l-1}{n}e^t < 1 \Leftrightarrow t < \log\frac{n}{l-1}$

Since $l \le k$ and $k/\sqrt{n} \to \lambda < +\infty$, we have $\frac{n}{l} \to +\infty$

therefore we consider larger enough n so that $M_{\tau_k^n-k}(t)$ is defined in $t \in (-1,1)$

Then:

$$M_{T_k^n-k}(t) = \prod_{l=1}^{k}\left[1-(1-e^t)\frac{\frac{l-1}{n}}{1-\frac{l-1}{n}e^t}\right]$$

$$= e^{\sum_{l=1}^{k}\log\left[1+(e^t-1)\frac{\frac{l-1}{n}}{1-\frac{l-1}{n}e^t}\right]}$$

we have that for $-1<x<1$:

$$\frac{x}{1+x} \leq \log 1+x \leq x \qquad (\text{use derivitive to justify this})$$

~~and for $0<x<1$ $\frac{1}{1-x} \geq 1+x$ $(1 \geq 1-x^2)$~~

Therefore

$$\sum_{l=1}^{k}\log\left[1+(e^t-1)\frac{\frac{l-1}{n}}{1-\frac{l-1}{n}e^t}\right] \leq \sum(e^t-1)\frac{l-1}{n}\Big/\left(1-\frac{l-1}{n}e^t\right) \leq \sum(e^t-1)\frac{l-1}{n}\cdot\frac{1}{1-\frac{k-1}{n}e^1}$$

$$= (e^t-1)\frac{1}{n}\frac{1}{1-\frac{k-1}{n}e^1}\frac{k(k-1)}{2} \qquad \left(\frac{k^2}{n}\to\lambda^2 \text{ and } \frac{k}{n}\to 0\right)$$

$$\to (e^t-1)\cdot\frac{\lambda^2}{2}$$

On the other side.

$$\sum_{l=1}^{k}\log\left[1+(e^t-1)\frac{\frac{l-1}{n}}{1-\frac{l-1}{n}e^t}\right] \geq \sum_{l=1}^{k}\frac{(e^t-1)\frac{\frac{l-1}{n}}{1-\frac{l-1}{n}e^t}}{1+(e^t-1)\frac{\frac{l-1}{n}}{1-\frac{l-1}{n}e^t}} = \sum_{l=1}^{k}(e^t-1)\frac{l-1}{n}\frac{1}{1-\frac{l-1}{n}}$$

$$\geq \sum_{l=1}^{k}(e^t-1)\frac{l-1}{n} \qquad \text{since } 0\leq\frac{l-1}{n}<1$$

$$= (e^t-1)\frac{1}{n}\cdot\frac{k(k-1)}{2} \to (e^t-1)\cdot\frac{\lambda^2}{2}$$

therefore $\overline{\lim}\, M_{T_k^n-k}(t) \leq e^{\frac{\lambda^2}{2}(e^t-1)}$ and $\underline{\lim}\, M_{T_k^n-k}(t) \geq e^{\frac{\lambda^2}{2}(e^t-1)}$

$$\Rightarrow \lim M_{T_k^n-k}(t) = e^{\frac{\lambda^2}{2}(e^t-1)} \qquad \forall t\in(-1,1) \quad \text{exists}$$

and this is the MGF for Poisson$(\frac{\lambda^2}{2})$, So by the continous theory

$$T_k^n-k \Rightarrow \text{Poisson}\left(\frac{\lambda^2}{2}\right)$$