Advanced Methods Homework 3

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一: Generalized inverse

1: (a) G is the g-inverse of $X'X$, then $X'XGX'X = X'X$

do transpose each side we get $X'XG'X'X = X'X$

$\Rightarrow G'$ is also a g-inverse of $X'X$

(b) First, we prove that for a vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, if $X'X\vec{v} = \vec{0}$, then $X\vec{v} = \vec{0}$

Proof: suppose $X = (x_1, x_2, \cdots x_n)$ then $X'X\vec{v} = \vec{0}$

$\Rightarrow (\sum v_i x_i) \cdot x_i = 0 \quad \forall i$

$\Rightarrow (\sum v_i x_i) \cdot (\sum v_i x_i) = 0 \Rightarrow \sum v_i x_i = \vec{0} \Rightarrow X\vec{v} = 0$

Second, we prove if $PX'X = QX'X$, then $PX' = QX'$

Proof: suppose $(P-Q) = \begin{pmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{pmatrix}$, then $(P-Q)X'X = 0$

$\Rightarrow X'X(u_1, u_2 \cdots u_n) = 0 \Rightarrow X'X\vec{u_i} = \vec{0} \quad \forall i$

$\Rightarrow X\vec{u_i} = \vec{0} \quad \forall i \Rightarrow X(P-Q)^T = 0 \Rightarrow PX' = QX'$

Finally, in (a) we have $X'XG'X'X = X'X$

$\Rightarrow X'XG'X' = X'$

$\Rightarrow XGX'X = X$

(c) use (b), for G_1, G_2 two choice of g-inverse, we have:

$XG_1X'X = X$; $XG_2X'X = X \Rightarrow X(G_1 - G_2)X'X = 0$ $\cancel{\Rightarrow X'X(G_1' - G_2')X' = 0}$

$\Rightarrow X(G_1 - G_2)X'X = 0 \cdot X'X$

$\Rightarrow X(G_1 - G_2)X' = 0 \Rightarrow XG_1X' = XG_2X'$

So XGX' is invariant.

1: (d) G' is a g-inverse of $X'X$ according to (a)

and since (c) we have $XGX' = XG'X'$

therefore $(XGX')' = XG'X' = XGX' \Rightarrow XGX'$ is symmetric.

二: Inference under incorrectly specified models:

1: we have that $\begin{pmatrix} \hat\beta_1 \\ \hat\beta_2 \end{pmatrix} = \left[\begin{pmatrix} X_1' \\ X_2' \end{pmatrix}(X_1, X_2)\right]^{-1} \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} \vec{y}$ $\quad \begin{pmatrix} X_1: n\times p_1 \\ X_2: n\times p_2. \end{pmatrix}$

$$= \begin{pmatrix} (X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1)^{-1}, & -(X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1)^{-1}X_1'X_2(X_2'X_2)^{-1} \\ -(X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2)^{-1}X_2'X_1(X_1'X_1)^{-1}; & (X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2)^{-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} X_1'\vec{y} \\ X_2'\vec{y} \end{pmatrix} \Rightarrow \hat\beta_1 = (X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1)^{-1}X_1'\left[I - X_2(X_2'X_2)^{-1}X_2'\right]\vec{y}$$

$$\Rightarrow E\hat\beta_1 = (X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1)^{-1}(X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1)\beta_1$$
$$= \beta_1$$

Denote $H = I - X_2(X_2'X_2)^{-1}X_2'$ (we have $H^2 = H$)

Then $\hat\beta_1 = (X_1'HX_1)^{-1}X_1'H\vec{y}$

So $var(\hat\beta_1) = (X_1'HX_1)^{-1}X_1'H(\sigma^2 I)HX_1(X_1'HX_1)^{-1}$

$$= \sigma^2(X_1'HX_1)^{-1}$$

Denote $X = (X_1, X_2)$ then $\hat\beta = (X'X)^{-1}X'\vec{y}$

$$E[(\vec{y}-X\hat\beta)'(\vec{y}-X\hat\beta)] = E[y'[I - X(X'X)^{-1}X']y] = \sigma^2 tr[I - X(X'X)^{-1}X'] + \beta'X'X\beta - \beta'X'X\beta.$$

$$= \sigma^2(n - p_1 - p_2) \quad \text{so} \quad E[S^2] = \sigma^2.$$

since (X_1, X_2) full rank.

三、Multivariate means and variances

1: $var(\vec{a}'\vec{y}) = E[a'\cdot y\cdot y'\cdot a] - E(a'\vec{y})\cdot[E(a'\vec{y})]'$

$= a'\left[E[y\cdot y'] - E(y)\cdot E(y)'\right]a$

$= a'\Sigma a$

2: $T=\frac{1}{n}\sum_{i=1}^{n}\vec{X}_i$ then T is the sample average. (Here $n=I$)

$E[T]=\frac{1}{n}\sum E[\vec{X}_i] = \mu$.

$var[T] = E\left[\left(\frac{\sum\vec{X}_i}{n}\right)\left(\frac{\sum\vec{X}_i}{n}\right)'\right] - \mu\mu'$

$= \frac{\sum_{i\neq j}E(X_i)E(X_j)' + \cancel{\sum E X_i E X} + \sum_{i=1}^{n}E X_i X_i' - n\mu\mu' - (n^2-n)\mu\mu'}{n^2}$

$= \frac{n\Sigma}{n^2} = \frac{\Sigma}{n} = \frac{\Sigma}{I}$

3: Consider the sample variance with μ known.

$T = \frac{1}{I}\sum_{i=1}^{I}(X_i-\mu)(X_i-\mu)'$

then $E[T] = \frac{1}{I}\cdot I\cdot E(X-\mu)(X-\mu)' = \Sigma$

So T is an unbiased estimator of Σ

4: When $\sigma^2 I + \theta 1 1'$ is positive-semi-definite, it will be a covariance matrix of some random vector $X = (x_1, x_2 \cdots x_n)'$

Then we have $var(X_i) = \sigma^2+\theta$

$corr(X_i, X_j) = \begin{cases} 1 & i=j \\ \frac{\theta}{\sigma^2+\theta} & i\neq j \end{cases}$

5: (a) $E(e) = (I-H)Ey = [I - X(X'X)^{-1}X']X\beta$
$= [X - X(X'X)^{-1}X'X]\beta$
$= 0$

(b) $var(e) = (I-H)\, var(y)\, (I-H)' = \sigma^2 (I-H)(I-H)$
$= \sigma^2 (I-H)$

(c) $cov(e, Hy) = E[(I-H)yy'H'] - E(e) \cdot E[Hy]'$
$= (I-H)[X\beta\beta'X' + \sigma^2 I]H$
$= \sigma^2 (I-H)H = 0$

(d) $E(e'e) = E[tr[ee']] = tr[E[ee']]$
$= tr[var(e)]$ (given $E(e)=0$)
$= \sigma^2\, tr(I-H)$
$= \sigma^2 (n-p)$

where p is the dimension of β
and X should be full rank.