Probability Theory Homework I

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1 (i) *Proof.* We have $a_1 > 0$ and since $a_n \to 0$, there exists N such that for every $n > N$, $a_n < a_1$. Therefore $\max_{1\le n}\{a_n\}$ is $\max_{1\le n\le N}\{a_n\}$, and then the max exits since $N < \infty$ $\square$

(ii) *Proof.* Suppose A is uncountable, then $A \setminus \{0\}$ is also uncountable. Since $A \setminus \{0\} = \cup_{n=1}^{\infty} A_n$ where $A_n = \{a \in A : a \ge 1/n\}$, then there must exits a N such that A_N is uncountable. Then there must exist $N+1$ elements $\{a_1, \cdots, a_{N+1}\}$ in $A_N \subset A$ such that $\forall i :\ a_i \ge 1/N$. And therefore we have $\sum_{i=1}^{N+1} a_i \ge \frac{N+1}{N} > 1$. A contradiction, so A must be countable. $\square$

2 (i) *Proof.* Suppose in (M, d_1): $x_n \to x$ and in (M, d_2): $x_n \to y$ and $x \ne y$, which infers $d_2(x, y) > 0$. Then we consider an array y_n, where $y_{2k+1} = x$ and $y_{2k} = x_k$. It's easy to see that y_n has limit x in (M, d_1), so according to the assumption in problem, y_n must tend to some $z \in M$. Therefore $d_2(y_{2k+1}, y2k) \le d_2(y_{2k+1}, z) + d_2(z, y2k)$ tends to 0. However $d_2(y_{2k+1}, y2k) \ge d_2(y_{2k+1}, y) - d_2(y, y2k) = d_2(x, y) - d_2(y, x_k) > d_2(x, y)/2 > 0$ when k is large enough (since $x_n \to y$ in (M, d_2)). A contradiction, therefore $x = y$. $\square$

(ii) *Proof.* For every sequence (x_n, y_n) in the set, if $(x_n, y_n) \to (x_0, y_0)$. Then since function $y - x^2$ is continuous with (x, y), we have $\lim y_n - x_n^2 = y_0 - x_0^2$, and since $(x_n, y_n) \in \{(x, y) : y \ge x^2\}$, we have $\lim y_n - x_n^2 \ge 0$ and therefore $y_0 \ge x_0^2$ and $(x_0, y_0) \in \{(x, y) : y \ge x^2\}$, which means the set is closed. $\square$

(iii) (a) $\cap_{n=1}^{\infty}(0, 1 + \frac{1}{n}) = (0, 1]$ which is not open.
(b) $\{(x, x^2) : x \in [0, 1]\} \cap \{(x, x) : x \in [0, 1]\} = \{(0, 0), (1, 1)\}$ which is not connected.

3 (i) *Proof.* Suppose M is infinite, since M is compact, an infinite subarray of M must have an accumulation point, which means we can find an array $\{x_n\}$ of distinct elements in M such that $x_n \to x$ and $x \notin \{x_n\}$. Therefore $\{x_n\}$ is a subset of M hence compact, but not closed. However, since M is a metric space, hence T_2 space, every compact set in M must be closed. A contradiction, which means M is finite $\square$

(ii) *Proof.* For every open cover $\{O_\alpha\}$ of $f(K)$, $\{f^{-1}(O_\alpha)\}$ will be an open cover of K. Since K is compact, there exists an finite subcover $\{f^{-1}(O_{\alpha_i})\}_1^N$. Then it's easy to see $\{O_{\alpha_i}\}_1^N$ is an finite open subcover of $f(K)$. Therefore, $f(K)$ is compact. $\square$

4 (i) *Proof.* By using derivative, $f_n(x)$ is increase in $[0, \frac{1}{\sqrt{3n}}]$ and decrease in $[\frac{1}{\sqrt{3n}}, 1]$. So we have $\sup_{x\in[0,1]} |f_n(x) - 0| = f_n(\frac{1}{\sqrt{3n}}) > \frac{1}{2} > 0$, therefore f_n don't uniformly convergent to 0 in $[0, 1]$. But in $[\delta, 1]$, when n is large enough such that $\frac{1}{\sqrt{3n}} < \delta$, $\sup_{x\in[\delta,1]} |f_n(x) - 0| = f_n(\delta) = \frac{\sqrt{n\delta}}{n^2\delta^2+1} \to 0$, therefore, f_n uniformly convergent to 0 in every $[\delta, 1]$ where $\delta > 0$. $\square$

(ii) $|(1 + n^2)^{-2}\sin(nx)|, |n(1 + n^2)^{-2}\cos(nx)|, |(-1)^n n^2(1 + n^2)^{-2}\sin(nx)|$ are all less than $\frac{1}{1+n^2}$, where $\sum_1^{\infty} \frac{1}{1+n^2} < \infty$. Therefore (a), (b), (c) are all uniformly convergent to some function.

Since $(\frac{\sin(nx)}{(1+n^2)^2})' = n(1 + n^2)^{-2}\cos(nx)$, which is continuous and (a) and (b) convergent uniformly, we have $\left(\sum_{n=1}^{\infty}(1 + n^2)^{-2}\sin(nx)\right)' = \sum_{n=1}^{\infty} n(1 + n^2)^{-2}\cos(nx)$. And in the same way, we can prove that $\left(\sum_{n=1}^{\infty} n(1 + n^2)^{-2}\cos(nx)\right)' = \sum_{n=1}^{\infty}(-1)n^2(1 + n^2)^{-2}\sin(nx)$. So, $f'' = \sum_{n=1}^{\infty}(-1)n^2(1 + n^2)^{-2}\sin(nx)$ exists and it's continuous since uniform convergence and the continuousness of $\sin(nx)$.

5 (i) Since every nonempty interval in $\mathbb{R}$ contains rational and irrational number. Therefore every upper sum is 1 and every lower sum is 0, hence f is not Riemann integrable.

(ii) *Proof.* In this situation we have that f is uniformly continuous in $[a, b]$, which means $\forall \epsilon > 0, \exists \delta > 0\ \forall |x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. For this ϵ and δ, for every partition $0 = x_0 < x_1 < \cdots < x_N = 1$ with $\max_{0 \leq i \leq N-1}(x_{i+1} - x_i) < \epsilon$, we have that upper sum minus lower sum will less than $\epsilon \times (1 - 0) = \epsilon$, which means $|\text{upper sum} - \text{lower sum}| \to 0$ when $\max_{0 \leq i \leq N-1}(x_{i+1} - x_i) \to 0$. Therefore f is Riemann integrable. □

(iii) *Proof.* We have $0 \leq f_n(x) < 1, \forall x \in [0, 1]$ and $\forall \delta > 0$, f_n uniformly convergent to 0, hence $\lim \int_\delta^1 f_n(x)dx = 0, \forall \delta > 0$. Therefore we have:

$$\int_0^1 f_n(x)dx = \int_0^\delta f_n(x)dx + \int_\delta^1 f_n(x)dx < \delta + \int_\delta^1 f_n(x)dx$$

then:

$$\overline{\lim_{n\to\infty}} \int_0^1 f_n(x)dx < \delta, \qquad \forall \delta > 0$$

which means $\overline{\lim} \int_0^1 f_n(x)dx = 0$, together with $f_n \geq 0$, we have $\lim \int_0^1 f_n(x)dx = 0$. □

Bonus:

Suppose the strategy is to switch the number with probability $p(x)$ when we get x, then calculate the probability that we end with a larger number if two numbers are x and $2x$. It's:

$$\mathbf{P}(\text{end with larger number}) = \frac{1}{2}p(x) + \frac{1}{2}(1 - p(2x)) = \frac{1}{2} + \frac{p(2) - p(2x)}{2}$$

So we only need $p(x) > p(2x)$, for example we can switch the number with probability $\frac{1}{1+x}$ when we get x, and then we will have more chance than 1/2 to get the best number for every x.