Homework 6

1. You are called on to analyze the results of an experiment in which cells are exposed to low level radiation and scientists are interested in the rate of resulting mutations. The experiment is conducted using a 96-well plate. Let X_i be the number of mutated cells in well i. Suppose that $X_i \sim Poi(\lambda)$. It's not possible to count the number of mutated cells in each well, but it is possible to detect, for each well, whether there are **any** mutated cells in that well.

   (a) Find the MLE ($\hat{\lambda}$) and MME ($\tilde{\lambda}$) for λ.

   (b) Find their asymptotic distributions.

2. Let $X_1, \ldots, X_n$ be iid random variables with pdf

$$f(x \mid \theta) = \theta(1+\theta)x^{\theta-1}(1-x) \ for \ x \in (0,1), \theta > 0.$$

   Find the method of moments estimator of θ. Is it unbiased? Does it achieve the CRLB?

3. Assume that $X_1, \ldots, X_n$ are iid random variables following the truncated exponential distribution with parameters λ and β. The density function is $f(x \mid \lambda, \beta) = \lambda e^{-\lambda(x-\beta)} 1\{x > \beta\}$.

   (a) Find the MLE for λ and β.

   (b) What is its limiting distribution?

4. Suppose $X_1, ..., X_n$ are iid and follow a uniform distribution on$[0, \theta]$. Show that $2\bar{X}$ and $(n+1)\frac{X_{(n)}}{n}$ are both consistent estimators of θ and compare their variances.

Statistical Theory Homework 6

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1. (a) Suppose we have n wells, here $n = 96$. According to the setting in the question, we can only observe $Y_i = \mathbf{1}_{X_i>0}$. Its reasonable to suppose X_i are i.i.d, therefore Y_i are also i.i.d random variables follow Bernoulli distribution will success probability $p = 1 - e^{-\lambda}$.

   Therefore, the likellihood of samples is $\Pi_{i=1}^n (1-e^{-\lambda})^{Y_i}(e^{-\lambda})^{1-Y_i}$. Maximize this function we get the MLE of λ is $\hat{\lambda} = -\log(1 - \frac{\sum_1^n Y_i}{n})$.

   Also, the mean of Y_i is $1 - e^{-\lambda}$, so the MME for λ is the solution of equation $\frac{\sum_1^n Y_i}{n} = 1 - e^{-\lambda}$. Therefore $\tilde{\lambda} = -\log(1 - \frac{\sum_1^n Y_i}{n})$ as well.

   (b) Since the CLT, we have $\sqrt{n}(\frac{\sum_1^n Y_i}{n} - (1-e^{-\lambda})) \xrightarrow{D} N(0, (1-e^{-\lambda})e^{-\lambda})$. By delta method we have the asymptotic distribution of $\hat{\lambda}$ and $\tilde{\lambda}$ is:

$$\sqrt{n}(-\log(1 - \frac{\sum_1^n Y_i}{n}) - \lambda) \xrightarrow{D} N(0, (e^{\lambda})^2(1-e^{-\lambda})e^{-\lambda}) = N(0, e^{\lambda} - 1)$$

2. Given θ, X follows beta distribution of parameter $\alpha = \theta$ and $\beta = 2$. Therefore the mean of X is $\frac{\alpha}{\alpha+\beta} = \frac{\theta}{\theta+2}$. The moments estimator is the solution of $\frac{\theta}{\theta+2} = \sum_1^n X_i/n$, then we have $\hat{\theta} = \frac{\bar{X}}{1-\bar{X}}$, where $\bar{X}$ is the sample mean.

   It is biased, for example when $n = 1$, $\hat{\theta}$ becomes $\frac{2X_1}{1-X_1}$. The expectation of it is $\int_0^1 \frac{2x}{1-x}\theta(\theta+1)x^{\theta-1}(1-x)dx = \int_0^1 2\theta(\theta+1)x^{\theta}dx = 2\theta \neq \theta$. So the estimator is biased.

   Also, it will not achieve the CRLB. Recall the sufficient and necessary condition for a θ estimator $w(x)$ to achieve CLRB, it is that there exists a function $a(\theta)$, such that $\frac{\partial}{\partial\theta}\log f(x|\theta) = a(\theta)(w(x) - \theta)$. We write out the likelihood:

$$f(x|\theta) = \theta^n(1+\theta)^n \exp\{(\theta-1)\sum \log x_i + \sum \log(1-x_i)\}$$

   Then the $\frac{\partial}{\partial\theta}\log f(x|\theta) = \frac{n}{\theta} + \frac{n}{\theta+1} + \sum \log x_i$, which can not be writen in form of $a(\theta)(\frac{2\bar{x}}{1-\bar{x}} - \theta)$. Therefore the $\hat{\theta}$ will not achieve the CRLB.

3. (a) the joint likelihood is $f(x|\lambda,\beta) = \lambda^n e^{-\lambda(\sum x_i - n\beta)}\mathbf{1}_{\{\forall i:\ x_i>\beta\}}$. Maximize this function we get the MLE estimator for λ and β:

$$\hat{\lambda} = \frac{1}{\bar{X} - X_{(1)}} \quad (1)$$

$$\hat{\beta} = X_{(1)} \quad (2)$$

   Where $\bar{X}$ is the sample mean and $X_{(1)}$ is the first order statistics.

(b) The distribution does not satisfy the CR regularity, so we compute the limiting distribution directly.

First we compute the joint distribution of $\sum_1^n X_i$ and $nX_{(1)}$. It's known that the joint distribution of all order statistics here is $f = n!\lambda^n e^{-\lambda(\sum x_i - n\beta)}\mathbf{1}_{\{x_n > \cdots > x_1 > \beta\}}$

Then we do a transformation as:

$$\begin{cases} y_1 = nx_1 \\ y_2 = x_1 + (n-1)x_2 \\ y_3 = x_1 + x_2 + (n-2)x_3 \\ \quad\vdots \\ y_n = x_1 + x_2 + x_3 + \cdots + x_n \end{cases}$$

Then the density for $y_1 \cdots y_n$ is $f/|\frac{\partial(y_1 \cdots y_n)}{\partial(x_1 \cdots x_n)}| = \lambda^n e^{-\lambda(y_n - n\beta)}$, and the support is $\{y_n > y_{n-1} > \cdots > y_2 > y_1 > n\beta\}$.

Marginalize the density on y_1 and y_n, we get that the joint distribution of $X_{(1)}$ and $\sum X_i$ is:

$$f(y_1, y_n) = \lambda^n \frac{(y_n - y_1)^{n-2}}{(n-2)!} e^{-\lambda(y_n - n\beta)}\mathbf{1}_{\{y_n > y_1 > n\beta\}}$$

Then do the transformation:

$$\begin{cases} \hat{\lambda} = \dfrac{n}{y_n - y_1} \\ \hat{\beta} = \dfrac{y_1}{n} \end{cases}$$

We get that the joint distribution of $(\hat{\lambda}, \hat{\beta})$ is:

$$f(z_1, z_2) = \frac{\lambda^n n^n}{(n-2)! z_1^n} e^{-\lambda n/z_1}\mathbf{1}_{z_1 > 0} \times e^{-n\lambda(z_2 - \beta)}\mathbf{1}_{z_2 > \beta}$$

Since $f(z_1, z_2)$ can be factorized into production of function of z_1 and function of z_2, we know that $\hat{\lambda}$ and $\hat{\beta}$ are independent.

By the factorization, we know that $n(\hat{\beta} - \beta) \sim \exp(\lambda)$, therefore $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{p} 0$. Recall the CLT we have that $\sqrt{n}\left\{\bar{X} - (\frac{1}{\lambda} + \beta)\right\} \xrightarrow{p} N(0, \frac{1}{\lambda^2})$, by Slutsky's Theorem we have:

$$\sqrt{n}(\bar{X} - X_{(1)} - \frac{1}{\lambda}) = \sqrt{n}(\bar{X} - \frac{1}{\lambda} - \beta) - \sqrt{n}(X_{(1)} - \beta) \xrightarrow{p} N(0, \frac{1}{\lambda^2})$$

By delta method we have:

$$\sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{p} N(0, \lambda^2)$$

In conclusion, the estimators $\hat{\lambda}$ and $\hat{\beta}$ are consistent and mutual independent. But they are not in the same scale.

$$\begin{aligned} n(\hat{\beta} - \beta) &\xrightarrow{p} \exp(\lambda) && (3) \\ \sqrt{n}(\hat{\lambda} - \lambda) &\xrightarrow{p} N(0, \lambda^2) && (4) \end{aligned}$$

4. By the WLLN we have that $\bar{X} \xrightarrow{p} \mathbb{E}[X] = \theta/2$, so $2\bar{X} \xrightarrow{p} \theta$, which means $2\bar{X}$ is consistent.

   For every $\gamma < \theta$, we have that $\mathbf{P}(\gamma < X_{(n)} \leq \theta) = 1 - (\frac{\gamma}{\theta})^n \to 1$, therefore $X_{(n)} \xrightarrow{p} \theta$. By Slutsky's Theorem, we have $(n+1)\frac{X_{(n)}}{n} \xrightarrow{p} \theta$, so it's also consistent.

   Now we compute their variance:

$$var\left(2\bar{X}\right) = \frac{4}{n} var\left(X_1\right) = \frac{\theta^2}{3n}$$

   And:

$$\begin{aligned} var\left(\frac{n+1}{n}X_{(n)}\right) &= (\frac{n+1}{n})^2 \left\{ (\int_0^\theta t^2 \frac{nt^{n-1}}{\theta^n} dt) - (\int_0^\theta t \frac{nt^{n-1}}{\theta^n} dt)^2 \right\} && (5) \\ &= (\frac{n+1}{n})^2 \left(\frac{n}{n+2}\theta^2 - (\frac{n}{n+1}\theta)^2 \right) && (6) \\ &= \frac{\theta^2}{n(n+2)} && (7) \end{aligned}$$

   We get that $var\left(\frac{n+1}{n}X_{(n)}\right) < var\left(2\bar{X}\right)$ when $n > 1$, so estimator $var\left(\frac{n+1}{n}X_{(n)}\right)$ is more efficient.