Statistical Theory Homework 1

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1. (a) $\{(0,g),(0,f),(0,s),(1,g),(1,f),(1,s)\}$
   (b) $\{(0,s),(1,s)\}$
   (c) $\{(0,g),(0,f),(0,s)\}$
   (d) $\{(0,s),(1,s),(1,g),(1,f)\}$
   (e) $\{(1,s)\}$

2. (a) $A \cap B^c \cap C^c$
   (b) $A \cap B \cap C^c$
   (c) $A \cap B \cap C$
   (d) $A \cup B \cup C$
   (e) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$
   (f) $A^c \cap B^c \cap C^c$

3. (a) $\mathbf{P}(\text{ at least a 6 }) = \frac{6+6-1}{6\times 6} = \frac{11}{36}$
   (b) $\mathbf{P}(\text{ same number }) = \frac{6}{6\times 6} = \frac{1}{6}$

4. It's the same number as to choose 2 people out of 20 to shake hands, and that's $\binom{20}{2} = 190$.

5. (a) $\frac{3\times 4}{\binom{5}{2}\times 2!} = \frac{3}{5}$
   (b) $\frac{2\times 3+3\times 2}{\binom{5}{2}\times 2!} = \frac{3}{5}$
   (c) $\frac{\binom{3}{2}\times 2!}{\binom{5}{2}\times 2!} = \frac{3}{10}$

6. I think it's to choose n ordered characters from a-z , where n is the sum of the numbers of one's last and given names. Then:
   (a) $26 \times 26 \times 26 = 17576$
   (b) $26 \times 26 + 26 \times 26 \times 26 = 18252$
   (c) $26 \times 26 + 26 \times 26 \times 26 + 26 \times 26 \times 26 \times 26 = 475228$

7. We can deal $\{1,2,3\}$ as one number and arrange the $n-2$ numbers in order first, then we arrange $1,2,3$ in order. Therefore: $\mathbf{P} = \frac{(n-2)!\times 3!}{n!} = \frac{6}{n(n-1)}$, where n must no less than 3.

8. It's equivalent to choose 7 children from 10 to have one and only one gift, therefore the number of results is $\binom{10}{3} = 120$.

9. It's the sum of ways to choose t elements in $(x_1, x_2, \cdots, x_n)$ to be 1 , where t ranges from k to n. And that's $\sum_{t=k}^{n} \binom{n}{t}$

10. $\mathbf{P} = \frac{|\text{choose } r \text{ spaces occupied out of the rest } N-2 \text{ spaces}|}{|\text{choose } r \text{ occupied out of N}|} = \frac{\binom{N-2}{r}}{\binom{N}{r}} = \frac{(N-r)(N-r-1)}{N(N-1)}$, where $r \leq N-2$. And obviously $\mathbf{P} = 0$ when $r > N-2$.

11. I'm a little confused about the meaning of "doubles" here, I think it means 2 dices are in same number.

    (a) $\mathbf{P} = \frac{|\{(1,1),(2,2)\}|}{6\times 6} / \frac{|\{(1,1),(1,2),(2,1),(1,3),(3,1),(2,2)\}|}{6\times 6} = \frac{1}{3}$

    (b) $\mathbf{P} = \frac{5+5}{6\times 6} / \frac{6\times 6-6}{6\times 6} = \frac{1}{3}$

12. $\mathbf{P}(\text{coin with heads and tails}|\text{show up heads}) =$
    $\mathbf{P}(\text{choosed third coin and tossed to be heads})/\mathbf{P}(\text{tossed to be heads}) =$
    $\frac{1/3\times 1/2}{1/3\times 1/2+1/3\times 1+1/3\times 0} = \frac{1}{3}$

13. if the n^{th} stamp is of type i, then it's new means the first $n-1$ stamps are of types other than i, whose probability is $(1-p_i)^{n-1}$. Therefore,

$$\mathbf{P}(\text{new}) = \sum_1^m \mathbf{P}(\text{new}|n^{th} \text{ is of type i})\mathbf{P}(n^{th} \text{ is of type i}) = \sum_1^m p_i(1-p_i)^{n-1}$$

Extra Credit:

Here and in the question, I think, "has k balls" means having exactly k balls.

Denote $A_{n,r,k}$ to be the number of different results of putting n balls in r urns such that exactly one urn has k balls; $B_{n,r,k}$ to be that of putting n balls in r urns such that no urn has k balls; $C_{n,r,k}$ to be that of putting n balls in r urns such that at least one urn has k balls; and $N_{n,r}$ to be the number of methods to put n balls in r urns.

Then, first it's easy to see:

$$\begin{aligned} B_{n,r,k} &= N_{n,r} \quad \text{if } n < k & (1) \\ B_{n,r,k} &= 1 \quad \text{if } n \neq k \text{ and } r = 1 & (2) \\ B_{n,r,k} &= 0 \quad \text{if } n = k \text{ and } r = 1 & (3) \\ B_{n,r,k} &= N_{n,r} - r \quad \text{if } n = k & (4) \end{aligned}$$

$$N_{n,r} = \binom{n+r-1}{n} \tag{5}$$

and

$$B_{n,r,k} + C_{n,r,k} = N_{n,r} \tag{6}$$

Also consider situations in $A_{n,r,k}$, it's obviously equivalent to choose one urn out of r and put k balls there, and then put the rest $n-k$ balls in rest $r-1$ urns such that no one has k balls. Therefore we have:

$$A_{n,r,k} = \binom{r}{1} B_{n-k,r-1,k} \tag{7}$$

and then we give an iterative expression of $C_{n,r,k}$. We can discuss which urn firstly has k balls. If the 1^{st} urn firstly have k balls then it's equivalent to put rest $n-k$ balls in rest $r-1$ urns without restriction. If it's the last urn to firstly have k balls, then it's equivalent to put rest $n-k$ balls in first $r-1$ urns such that no one has k balls.

Also, if b^{th} urn firstly have k balls (where $2 \leq b \leq r-1$), then we discuss how many balls are in first $b-1$ urns. Therefore it's the sum of "put t balls in $b-1$ urns such that no one has k balls and put $n-k-t$ balls in rest $r-b$ urns without restriction " where t range from 0 to $n-k$. Therefore:

$$C_{n,r,k} = N_{n-k,r-1} + \sum_{b=2}^{r-1} \sum_{t=0}^{n-k} B_{t,b-1,k} N_{n-k-t,r-b} + B_{n-k,r-1,k} \tag{8}$$

Initial values in equations 1 to 5 and iterative equations 6 to 8 form a sufficient way to calculate $A_{n,r,k}$ and finally the probability will be:

$$\mathbf{P} = A_{n,r,k}/N_{n,r}$$