Probability Theory II - Homework 2

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1: We prove first: ~~(limsup A_n) ∩ (limsup~~

$\limsup (A_n \cap B_n) \supset (\limsup A_n) \cap (\liminf B_n)$

Proof: $\forall \omega \in (\limsup A_n) \cap (\liminf B_n)$

By their equivalent definition, we can find $n_i \in N^*$, $n_i \uparrow +\infty$ as $i \uparrow +\infty$

and $\omega \in A_{n_i}$ $\forall i$ and $\exists K \in N^*$ $n > K \Rightarrow \omega \in B_n$.

Since $n_i \uparrow +\infty$, there exists I, $n_I > K$

then $\omega \in B_{n_I}$, $\omega \in B_{n_{I+1}}$ ......

So $\omega \in (A_{n_I} \cap B_{n_I}) \cap (A_{n_{I+1}} \cap B_{n_{I+1}}) \cap \cdots (A_{n_{I+j}} \cap B_{n_{I+j}}) \cdots$

Which means ω happens infinitely often in sets $A_n \cap B_n$

$\Rightarrow \omega \in \limsup (A_n \cap B_n)$

$\Rightarrow ((\limsup A_n) \cap (\liminf B_n) \subset \limsup (A_n \cap B_n)$

Then $1 \geq p[\limsup(A_n \cap B_n)] \geq p(\limsup A_n \cap \liminf B_n) \geq 1 - [1 - p[(\limsup A_n)^*]] - [1 - p([\liminf B_n]^*)]$

$= 1$

So $p[\limsup (A_n \cap B_n)] = 1$

If $p[\limsup B_n] = 1$, then $p[\limsup(A_n \cap B_n)]$ can be 0

Think $A_n = \begin{cases} [0, \frac{1}{2}) & n, \text{odd} \\ [\frac{1}{2}, 1) & n: \text{even} \end{cases}$ and $B_n = \begin{cases} [\frac{1}{2}, 1) & n: \text{odd} \\ [0, \frac{1}{2}) & n: \text{even} \end{cases}$

~~Then lim~~ And P ~~uniform~~ the lebesgue measure on $[0, 1)$

Then $\limsup A_n = [0, 1) = \limsup B_n$ which with measure 1

and $A_n \cap B_n = \phi$ $\forall n$ $\Rightarrow p[\limsup (A_n \cap B_n)] = 0$

2: First we prove $\forall K, L \in \mathbb{N}^*$, $L > 1$

$$(A_K \cup A_{K+1} \cup A_{K+2} \cup \cdots \cup A_{K+L}) \setminus A_{K+L+1} \subset \bigcup_{i=K}^{K+L} (A_i \setminus A_{i+1})$$

Proof: $\forall \omega \in (A_K \cup A_{K+1} \cup A_{K+2} \cdots \cup A_{K+L}) \setminus A_{K+L+1} = \left(\bigcup_{i=K}^{K+L} A_i\right) \setminus A_{K+L+1}$

we have $\omega \in \bigcup_{i=K}^{K+L} A_i$ and $\omega \notin A_{K+L+1}$

denote $I = \{K \le i \le K+L : \omega \in A_i\}$ then $\omega \in \bigcup_{i=K}^{K+L} A_i \Rightarrow I \neq \emptyset$

since $|I| \le L+1$, I have a maximum element

suppose $i_0 \in I$ and i_0 is the max in I

if $i_0 = K+L$, we have $\omega \in A_{K+L}$ and $\omega \notin A_{K+L+1}$ by the chosen of ω

$$\Rightarrow \omega \in A_{K+L} \setminus A_{K+L+1} \Rightarrow \omega \in \bigcup_{i=K}^{K+L} (A_i \setminus A_{i+1})$$

if $K \le i_0 < K+L$, then $\omega \in A_{i_0}$ and since i_0 is the biggest in I

$\omega \notin A_{i_0+1}$ (else $i_0+1 \in I$)

$\Rightarrow \omega \in A_{i_0} \setminus A_{i_0+1} \in \bigcup_{i=K}^{K+L} (A_i \setminus A_{i+1})$ since $K \le i_0 < K+L$

so ω will always $\in \bigcup_{i=K}^{K+L} (A_i \setminus A_{i+1})$

$$\Rightarrow (A_K \cup A_{K+1} \cup A_{K+2} \cdots \cup A_{K+L}) \setminus A_{K+L+1} \subset \bigcup_{i=K}^{K+L} (A_i \setminus A_{i+1})$$

Then $\forall K, L \in \mathbb{N}^*$ we have

$$P\left(\bigcup_{i=K}^{K+L+1} A_i\right) = P\left[\left(\bigcup_{i=K}^{K+L} A_i\right) \setminus A_{K+L+1}\right] + P(A_{K+L+1})$$
$$\le \sum_{i=K}^{K+L} P[A_i \setminus A_{i+1}] + P(A_{K+L+1})$$

Let $L \to +\infty$ $\Rightarrow$

$$P\left(\bigcup_{i=K}^{+\infty} A_i\right) \le \sum_{i=K}^{+\infty} P(A_i \cap A_{i+1}^c) + \overline{\lim_{L}}\, P(A_{K+L+1})$$
$$= \sum_{i=K}^{+\infty} P(A_i \cap A_{i+1}^c)$$

then since $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < +\infty$

we have $P(A_n\ i.o.) = \lim_{K \to +\infty} P\left(\bigcup_{i=K}^{+\infty} A_i\right) \le \lim \sum_{i=K}^{+\infty} P(A_i \cap A_{i+1}^c) = 0$

which ends the proof.

(since every sequence of P here is monotone, $\lim$ always exists

so I just use $\lim$ instead of $\overline{\lim}$)

3:

(i) I: " $\forall K \in N^*$: $\sum_{n=1}^{+\infty} P(|Y_n| > \frac{n}{K}) < +\infty$ "

Proof: " $\Rightarrow$ ": if I holds, $\forall K \in N^*$; $K>0$

$$\sum_{n=1}^{+\infty} P(|Y_n| > \frac{n}{K}) < +\infty \Rightarrow P\left[\{w : |Y_n|/n\,(w) \leq \frac{1}{K}\text{ . eventually happens}\}\right] = 1$$

$$\Rightarrow \exists A_K,\ P(A_K)=1,\ \forall w \in A_K \quad \overline{\lim} \frac{|Y_n|(w)}{n} \leq \frac{1}{K}$$

let $A = \bigcap_{K=1}^{+\infty} A_K$, then $P(A)=1$ since $1 \geq P(A) \geq 1 - \sum_{K=1}^{+\infty}(1-P(A_K)) = 1$

$$\forall w \in A \Rightarrow w \in A_K : \forall K \Rightarrow \overline{\lim} \frac{|Y_n|(w)}{n} \leq \frac{1}{K} \quad \forall K \Rightarrow \overline{\lim_n} \frac{|Y_n|(w)}{n} \leq 0$$

$$\Rightarrow \lim_n \frac{Y_n}{n}(w) = 0$$

$\Rightarrow Y_n/n \to 0$ a.s.

" $\Leftarrow$ ": if I not hold, then $\exists K_0 \in N^*$, $\sum_{n=1}^{+\infty} P(|Y_n| > \frac{n}{K}) = +\infty$

since Y_n are i.i.d. $\{|Y_n| > \frac{n}{K}\}$ are mutual independent

use B-C second Lemma $\Rightarrow P\{|Y_n| > \frac{n}{K_0},\ \text{i.o.}\} = 1$

$\forall w \in \{|Y_n| > \frac{n}{K_0}\ .\ \text{i.o.}\}$ $\quad \frac{|Y_n|(w)}{n} > \frac{1}{K_0}$ infinitly often.

$\Rightarrow \frac{Y_n(w)}{n}$ can't tend to 0

$\Rightarrow Y_n/n$ not a.s. tend to 0.

(ii) I: " $P(Y_1 > -\infty) > 0$ and $\forall \varepsilon > 0$: $\sum_{n=1}^{\infty} P(Y_1 > n\varepsilon) < +\infty$ and $P(Y_1 = +\infty) = 0$ "

Proof: First we notice that $\lim_n \frac{\max_{m \leq n} Y_m(w)}{n} = 0$

$$\Leftrightarrow \overline{\lim_n} \frac{\max_{m \leq n} Y_m(w)}{n} \leq 0 \quad \text{and} \quad \underline{\lim_n} \frac{\max_{m \leq n} Y_m}{n} \geq 0$$

So we just need to prove:

(It's obvious that I_1 and I_2 a.s. holds iff I_1 a.s and I_2 a.s.)

1° $P(Y_1>-\infty)>0 \iff \underline{\lim}_{n} \frac{\max_{m\le n} Y_m}{n} \geq 0$ a.s.

2° $\forall \varepsilon>0$: $\underbrace{\sum_{n=1}^{+\infty} P(Y_n>n\varepsilon)<+\infty}_{\text{and } P(Y_1=+\infty)=0} \iff \overline{\lim} \frac{\max_{m\le n} Y_m}{n} \leq 0$ a.s.

For 1°: "$\Rightarrow$": Denote $A=\{\omega: \exists N<+\infty, Y_N(\omega)>-\infty\}$

Then $A=\bigcup_{i=1}^{+\infty}\{Y_i>-\infty\}$, $P(A)=1-\prod_{i=1}^{+\infty}P(Y_i \neq +\infty) = 1-[1-P(Y_1>-\infty)]^{+\infty}$

$=1$

$\forall \omega\in A$, for $n>N$ $\max_{m\le n} Y_m \geq Y_N > \cancel{+\infty}-\infty$

So $\underline{\lim}_{n} \frac{\max_{m\le n} Y_m}{n} \geq \underline{\lim}_{n} \frac{Y_N}{n} = 0$

"$\Leftarrow$": If $P(Y_1>-\infty)=0$ Then Y_1 a.s. $=-\infty$ then $\frac{\max_{m\le n} Y_m}{n} = -\infty$ a.s

So $\underline{\lim}_{n} \frac{\max_{m\le n} Y_m}{n} = -\infty < 0$, contradiction.

For 2°: "$\Rightarrow$": Denote $A_n^K=\{\omega: Y_n>\frac{n}{K}\}$ $\forall K$ fixed

Then $\sum_n P(A_n^K)<+\infty \Rightarrow P\{A_n^K \text{ i.o}\}=0$

$\Rightarrow \exists B_K$, $P(B_K)=1$, $\forall \omega\in B_K$ $Y_n^{(\omega)}\leq \frac{n}{K}$ eventually happens

$\Rightarrow \exists L$, $\forall n>L$ $Y_n(\omega)\leq\frac{n}{K}$

Since $P(Y_i=+\infty)=0$ we have $P[\bigcup_{i=1}^{+\infty}\{Y_i=+\infty\}]=0$

Let $C_K=B_K\cap\left[\bigcup_{i=1}^{+\infty}\{Y_i=+\infty\}\right]^c$ we have $P(C_K)=1$

Then $\forall \omega\in C_K$, $Y_i(\omega)<+\infty$ $\forall i$ and $\exists L$, $\forall n>L$ $Y_n(\omega)\leq\frac{n}{K}$.

Let $M=\max_{1\le i\le L} Y_i(\omega)$, then $M<+\infty$ since $Y_i(\omega)<+\infty$ and L finite

It's easy to see $\forall n>L$ $\max_{m\le n} Y_m(\omega) \leq \max_{L<m\le n} Y_m^{(\omega)} + \max_{1\le m\le L} Y_m(\omega)$

$\leq \frac{n}{K}+M$

$\Rightarrow \overline{\lim}_{n} \frac{\max_{m\le n} Y_m(\omega)}{n} \leq \frac{1}{K}$

Let $D = \bigcap_{k=1}^{+\infty} C_k$, we have $P(D) = 1$

and $\forall \omega \in D$: $\overline{\lim_n} \frac{\max_{m \le n} Y_m(\omega)}{n} \le \frac{1}{k}$, $\forall k$

$\Rightarrow \overline{\lim_n} \frac{\max_{m \le n} Y_m(\omega)}{n} \le 0$

$\Rightarrow \overline{\lim_n} \frac{\max_{m \le n} Y_m(\omega)}{n} \le 0$ a.s.

"$\Leftarrow$": If "$\forall \varepsilon > 0$: $\sum_{n=1}^{+\infty} P(Y_n > n\varepsilon) < +\infty$ and $P(Y_1 = +\infty) = 0$" not hold

Then either ①: $P(Y_1 = +\infty) > 0$ $\forall \omega \in \{Y_1 = +\infty\}$

obviously $\frac{\max_{m \le n} Y_m(\omega)}{n} = +\infty$

$\Rightarrow \overline{\lim} \frac{\max_{m \le n} Y_m(\omega)}{n} = +\infty > 0$ contradiction.

②: $\exists \varepsilon_0 > 0$, $\sum_{n=1}^{+\infty} P(Y_n > n\varepsilon_0) = +\infty$

since $\{Y_n > n\varepsilon\}$ are mutual independent.

use B-C second lemma $P[\{Y_n > n\varepsilon_0 \text{ e.v.}\}] = 1$

$\forall \omega \in \{Y_n > n\varepsilon_0, \text{ e.v.}\}$ ~~state~~ $\overline{\lim_n} \frac{Y_n(\omega)}{n} \ge \varepsilon_0$

then $\overline{\lim_n} \frac{\max_{m \le n} Y_m(\omega)}{n} \ge \overline{\lim_n} \frac{Y_n(\omega)}{n} \ge \varepsilon_0 > 0$. Contradiction.

Therefore "$\Leftarrow$" holds.

combine ~~"$\Rightarrow$"~~ 1° and 2° ends the proof.

(iii) I: " $P(Y_1 > -\infty) > 0$ and $\forall \varepsilon > 0$, $[P(Y_1 \le n\varepsilon)]^n \to 1$ $_{n\to+\infty}$ "

Proof: $(\max_{m\le n} Y_m)/n \xrightarrow{P} 0$

$\Leftrightarrow$ $\forall \varepsilon > 0$ $P(|\max_{m\le n} Y_m/n| > \varepsilon) \to 0$ $n \to +\infty$

$\Leftrightarrow$ $\forall \varepsilon > 0$ $P[\max_{m\le n} Y_m/n < -\varepsilon] + P(\max_{m\le n} Y_m/n > \varepsilon) \to 0$ $n \to +\infty$

$\Leftrightarrow$ $\forall \varepsilon > 0$ $P[\bigcap_{m=1}^{n}\{Y_m < -n\varepsilon\}] + P[\bigcup_{m=1}^{n}\{Y_m > n\varepsilon\}] \to 0$ $n \to +\infty$

$\Leftrightarrow$ $\forall \varepsilon > 0$ $P^n(Y_1 < -n\varepsilon) + 1 - P[\bigcap_{m=1}^{n}\{Y_m \le n\varepsilon\}] \to 0$ $n \to +\infty$

$\Leftrightarrow$ $\forall \varepsilon > 0$ $P^n(Y_1 < -n\varepsilon) + 1 - P^n(Y_1 \le n\varepsilon) \to 0$ $n \to +\infty$

since $0 \le P(Y_1 < -n\varepsilon) \le 1$, $0 \le P(Y_1 \le n\varepsilon) \le 1$, the upper $\Leftrightarrow$

$\forall \varepsilon > 0$ $P^n(Y_1 < -n\varepsilon) \to 0$ and $P^n(Y_1 \le n\varepsilon) \to 1$.

and since $\forall \varepsilon > 0$ $P^n(Y_1 < -n\varepsilon) \to 0 \Leftrightarrow P(Y_1 > -\infty) > 0$

("$\Rightarrow$": if $P(Y_1 > -\infty) = 0$ then $P(Y_1 < -n\varepsilon) = 1$ $\forall \varepsilon \Rightarrow P^n(Y_1 < -n\varepsilon) \to 1$

"$\Leftarrow$" if $P(Y_1 > -\infty) > 0$ then $\exists K \in \mathbb{R}$, $P(Y_1 < K) < 1$

$\forall \varepsilon$, $\exists L$ $\forall n > L$, $-n\varepsilon < K$, for these n.

$P(Y_1 < -n\varepsilon) < P(Y_1 < K) < 1$

$\Rightarrow \overline{\lim} P^n(Y_1 < -n\varepsilon) \le \overline{\lim}_n P^n(Y_1 < K) = 0$

$\Rightarrow P^n(Y_1 < -n\varepsilon) \to 0$ $\forall \varepsilon$.)

we have finally $(\max_{m\le n} Y_m)/n \xrightarrow{P} 0 \Leftrightarrow$ I holds

(iv) I: "$P(|Y_1|<+\infty)=1$"

$Y_n/n \xrightarrow{P} 0 \iff \forall \varepsilon>0: P(|\frac{Y_n}{n}|>\varepsilon)\to 0$

$\iff \forall \varepsilon>0: P(|Y_1|\leq n\varepsilon)\to 1$. since i.i.d.

Since $P(|Y_1|\leq n\varepsilon)\to P(|Y_1|<+\infty)$, $\forall\varepsilon>0$ the upper

$\iff P(|Y_1|<+\infty)=1$

So $Y_n/n \xrightarrow{P} 0 \iff$ I holds

4: $\forall \varepsilon>0$, $\varepsilon<\frac{1}{2}$:

First calculate probability of $A_n: \{\omega: \frac{\sup_{j\leq n} X_j}{\ln n} < 1-\varepsilon\}$.

$P(A_n) = P\{\sup_{j\leq n} X_j < (1-\varepsilon)\ln n\} = P[\bigcap_{j=1}^{n}\{X_j < (1-\varepsilon)\ln n\}]$

$= P^n[X_1<(1-\varepsilon)\ln n] = [1-e^{-(1-\varepsilon)\ln n}]^n = [1-(\frac{1}{n})^{1-\varepsilon}]^n$.

since $x\leq e^{x-1}$ $\forall x \Rightarrow [1-(\frac{1}{n})^{1-\varepsilon}]\leq e^{-n^{\varepsilon-1}}$

$\Rightarrow [1-(\frac{1}{n})^{1-\varepsilon}]^n \leq e^{-n^{\varepsilon}}$

Since $\varepsilon>0$ $\lim_{n\to+\infty}\frac{e^{-n^\varepsilon}}{\frac{1}{n^2}} = \lim_{n\to+\infty}\frac{n^2}{e^{n^\varepsilon}} = 0$

along ~~and~~ with $\sum_{i=1}^{+\infty}\frac{1}{i^2}<+\infty \Rightarrow \sum_{n=1}^{+\infty}e^{-n^\varepsilon}<+\infty$

$\Rightarrow \sum_{n=1}^{+\infty}P(A_n)<\infty$ By B-C Lemma. we have $P(A_n \text{ i.o.})=0$, $\forall\varepsilon$

$\Rightarrow P\{A_n^c, \text{ e.v.}\}=1$, $\forall\varepsilon \Rightarrow \forall\varepsilon, \exists\Omega_\varepsilon$, $P(\Omega_\varepsilon)=1$. s.t. $\forall\omega\in\Omega_\varepsilon$.

$\frac{\sup_{j\leq n}X_j}{\ln n}(\omega)\geq 1-\varepsilon$ eventually happen

$\Rightarrow \forall\omega\in\Omega_\varepsilon: \underline{\lim}_{n}\frac{\sup_{j\leq n}X_j(\omega)}{\ln n}\geq 1-\varepsilon$.

Let $\Omega_0 = \bigcap_{n=1}^{+\infty} \Omega_{\frac{1}{n}}$ then $1 \geq P(\Omega_0) \geq 1 - \sum_n P(\Omega^c_{\frac{1}{n}}) = 1$

$\Rightarrow P(\Omega_0) = 1$

$\forall \omega \in \Omega_0 \quad \underline{\lim}_n \dfrac{\sup_{j \leq n} X_j(\omega)}{\ln n} \geq 1 - \frac{1}{k} \quad \forall k.$

$\Rightarrow \underline{\lim}_n \dfrac{\sup_{j \leq n} X_j(\omega)}{\ln n} \geq 1$ a.s.

Sencond calculate probability of $A'_n : \{\omega : \frac{X_n}{\ln n} > 1+\varepsilon\}$

$P(A'_n) = e^{-(1+\varepsilon)\ln n} = \dfrac{1}{n^{1+\varepsilon}}$

since $\varepsilon > 0 \quad \sum_{n=1}^{+\infty} P(A'_n) < +\infty$

$\Rightarrow \forall \varepsilon > 0, \exists \Omega'_\varepsilon ; \forall \omega \in \Omega'_\varepsilon, \ \frac{X_n(\omega)}{\ln n} \leq 1+\varepsilon$ eventually happens
$P(\Omega'_\varepsilon) = 1$
$\Rightarrow \overline{\lim}_n \dfrac{X_n(\omega)}{\ln n} \leq 1+\varepsilon.$

use the same argument as in Problem 3: (ii) 2° "$\Rightarrow$" part
we have $\overline{\lim}_n \dfrac{\sup_{j \leq n} X_j(\omega)}{n} \leq 1+\varepsilon$

Let $\Omega'_0 = \bigcap_{n=1}^{+\infty} \Omega'_{\frac{1}{n}}$ then $P(\Omega'_0) = 1$

$\forall \omega \in \Omega'_0 \quad \overline{\lim}_n \dfrac{\sup_{j \leq n} X_j(\omega)}{n} \leq 1 + \frac{1}{k}$

$\Rightarrow \overline{\lim}_n \dfrac{\sup_{j \leq n} X_j(\omega)}{n} \leq 1$ a.s.

consider ω in $\Omega_0 \cap \Omega'_0$ where $P(\Omega_0 \cap \Omega'_0) = 1$

We have that $\lim_n \dfrac{\sup_{j \leq n} X_j}{n} = 1$ a.s.