1. (i) ($3\frac{1}{2}$ points) If $\{a_n\}_{n=1,2,\ldots,}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$, prove that $\max\{a_n : n \in \mathbf{N}\}$ exists.

(ii) ($3\frac{1}{2}$ points) If A is an infinite subset of $(0,1)$ with the property that $\sum_{j=1}^{q} a_j \le 1$ whenever $q \in \mathbf{N}$ and $a_1, \ldots, a_q$ are distinct elements of A, prove that A is countable. Hint: For $n \in \mathbf{N}$, let $A_n = \{a \in A : a \ge 1/n\}$.

2. (i) (3 points) Suppose d_1, d_2 are two metrics for a space M, and suppose that any sequence which converges in (M, d_1) also converges in (M, d_2). Prove that a sequence $\{x_n\}$ which has limit x in (M, d_1) also has the same limit x in (M, d_2).

(ii) (2 points) If $\mathbf{R}^2$ is equipped with its usual metric, prove that $\{(x,y) : y \ge x^2\}$ is a closed subset of $\mathbf{R}^2$.

(iii) (2 points) Give (a) an example to show that the intersection of countable many open sets need not be open, and (b) an example to show that the intersection of two connected sets need not be connected.

3. (i) (4 points) Prove that if (M, d) is a metric space such that every subset of M is compact, then M is a finite set.

(ii) (3 points) Suppose M, d and N, ρ are metric spaces, $K \subset M$ is compact and $f : K \to N$ is continuous. Give the proof that $f(K)$ is a compact subset of N.

4. (i) (3 points) If $f_n(x) = (nx)^{1/2}/(1+(nx)^2)$, $x \in [0,1]$, $n = 1, 2, \ldots,$ prove that, for any given $\delta \in (0,1)$, f_n converges uniformly to zero on the interval $[\delta, 1]$, but f_n does not converge uniformly on $[0,1]$.

(ii) (4 points) Discuss pointwise and uniform convergence of the following series: (a) $f(x) = \sum_{n=1}^{\infty}(1+n^2)^{-2}\sin nx$, (b) $g(x) = \sum_{n=1}^{\infty} n(1+n^2)^{-2}\cos nx$, and (c) $h(x) = \sum_{n=1}^{\infty}(-1)^n n^2(1+n^2)^{-2}\sin nx$ on $\mathbf{R}$. Hence (using the appropriate theorem from lecture) show that f is twice continuously differentiable on $\mathbf{R}$.

5. (i) (2 points) Suppose $f : [0,1] \to \mathbf{R}$ is such that $f(x) = 1$ when x is rational and $f = 0$ when x is irrational. Discuss upper and lower sums of f with respect to a partition $P : 0 = x_0 < x_1 < \cdots < x_N = 1$, and hence show that f is not Riemann integrable.

(ii) (2 points) Give the proof that a continuous function $f : [a,b] \to \mathbf{R}$ is Riemann integrable.

(iii) (3 points) If $\{f_n\}$ is as in 4(i), prove that $\lim \int_0^1 f_n(x)\,dx = 0$.
Note: You can use the result of 4(i) (whether you managed to prove it or not!)