1. (i) ($3\frac{1}{2}$ points) If $\{a_n\}_{n=1,2,\ldots,}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$, prove that $\max\{a_n : n \in \mathbf{N}\}$ exists.

(ii) ($3\frac{1}{2}$ points) If A is an infinite subset of $(0,1)$ with the property that $\sum_{j=1}^{q} a_j \le 1$ whenever $q \in \mathbf{N}$ and $a_1, \ldots, a_q$ are distinct elements of A, prove that A is countable. Hint: For $n \in \mathbf{N}$, let $A_n = \{a \in A : a \ge 1/n\}$.

2. (i) (3 points) Suppose d_1, d_2 are two metrics for a space M, and suppose that any sequence which converges in (M, d_1) also converges in (M, d_2). Prove that a sequence $\{x_n\}$ which has limit x in (M, d_1) also has the same limit x in (M, d_2).

(ii) (2 points) If $\mathbf{R}^2$ is equipped with its usual metric, prove that $\{(x,y) : y \ge x^2\}$ is a closed subset of $\mathbf{R}^2$.

(iii) (2 points) Give (a) an example to show that the intersection of countable many open sets need not be open, and (b) an example to show that the intersection of two connected sets need not be connected.

3. (i) (4 points) Prove that if (M, d) is a metric space such that every subset of M is compact, then M is a finite set.

(ii) (3 points) Suppose M, d and N, ρ are metric spaces, $K \subset M$ is compact and $f : K \to N$ is continuous. Give the proof that $f(K)$ is a compact subset of N.

4. (i) (3 points) If $f_n(x) = (nx)^{1/2}/(1+(nx)^2)$, $x \in [0,1]$, $n = 1, 2, \ldots,$ prove that, for any given $\delta \in (0,1)$, f_n converges uniformly to zero on the interval $[\delta, 1]$, but f_n does not converge uniformly on $[0,1]$.

(ii) (4 points) Discuss pointwise and uniform convergence of the following series: (a) $f(x) = \sum_{n=1}^{\infty}(1+n^2)^{-2}\sin nx$, (b) $g(x) = \sum_{n=1}^{\infty} n(1+n^2)^{-2}\cos nx$, and (c) $h(x) = \sum_{n=1}^{\infty}(-1)^n n^2(1+n^2)^{-2}\sin nx$ on $\mathbf{R}$. Hence (using the appropriate theorem from lecture) show that f is twice continuously differentiable on $\mathbf{R}$.

5. (i) (2 points) Suppose $f : [0,1] \to \mathbf{R}$ is such that $f(x) = 1$ when x is rational and $f = 0$ when x is irrational. Discuss upper and lower sums of f with respect to a partition $P : 0 = x_0 < x_1 < \cdots < x_N = 1$, and hence show that f is not Riemann integrable.

(ii) (2 points) Give the proof that a continuous function $f : [a,b] \to \mathbf{R}$ is Riemann integrable.

(iii) (3 points) If $\{f_n\}$ is as in 4(i), prove that $\lim \int_0^1 f_n(x)\,dx = 0$.

Note: You can use the result of 4(i) (whether you managed to prove it or not!)

Probability Theory Homework I

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September 2, 2017

1 (i) *Proof.* We have $a_1 > 0$ and since $a_n \to 0$, there exists N such that for every $n > N$, $a_n < a_1$. Therefore $\max_{1\leq n}\{a_n\}$ is $\max_{1\leq n\leq N}\{a_n\}$, and then the max exits since $N < \infty$ $\square$

(ii) *Proof.* Suppose A is uncountable, then $A \setminus \{0\}$ is also uncountable. Since $A \setminus \{0\} = \cup_{n=1}^{\infty} A_n$ where $A_n = \{a \in A : a \geq 1/n\}$, then there must exits a N such that A_N is uncountable, and therefore $N+1$ elements $\{a_1, \cdots, a_{N+1}\}$ in $A_N \subset A$ such that $\forall i: \ a_i \geq 1/N$. We'll have $\sum_{i=1}^{N+1} a_i \geq \frac{N+1}{N} > 1$. A contradiction, so A must be countable. $\square$

2 (i) *Proof.* Suppose in (M, d_1): $x_n \to x$ and in (M, d_2): $x_n \to y$ and $x \neq y$, which infers $d_2(x, y) > 0$. Then we consider an array y_n, where $y_{2k+1} = x$ and $y_{2k} = x_k$. It's easy to see that y_n has limit x in (M, d_1), so according to the assumption in problem, y_n must tend to some $z \in M$. Therefore $d_2(y_{2k+1}, y_{2k}) \leq d_2(y_{2k+1}, z) + d_2(z, y_{2k})$ tends to 0. However when k is large enough:

$$d_2(y_{2k+1}, y_{2k}) \geq d_2(y_{2k+1}, y) - d_2(y, y_{2k}) = d_2(x, y) - d_2(y, x_k) > d_2(x, y)/2 > 0$$

A contradiction shows up, therefore $x = y$. $\square$

(ii) *Proof.* For every sequence (x_n, y_n) in the set, if $(x_n, y_n) \to (x_0, y_0)$. Then since function $y - x^2$ is continuous with (x, y), we have $\lim(y_n - x_n^2) = (y_0 - x_0^2)$, and since $(x_n, y_n) \in \{(x, y) : y \geq x^2\}$, we have $\lim(y_n - x_n^2) \geq 0$ and therefore $y_0 \geq x_0^2$ and $(x_0, y_0) \in \{(x, y) : y \geq x^2\}$, which means the set is closed. $\square$

(iii) (a) $(0, 1 + \frac{1}{n})$ are open but $\cap_{n=1}^{\infty}(0, 1 + \frac{1}{n}) = (0, 1]$ is not open.
(b) $\{(x, x^2) : x \in [0, 1]\} \cap \{(x, x) : x \in [0, 1]\} = \{(0, 0), (1, 1)\}$ is not connected.

3 (i) *Proof.* Suppose M is infinite, since M is compact, an infinite subarray of M must have an accumulation point, which means we can find an array $\{x_n\}$ of distinct elements in M such that $x_n \to x$ and $x \notin \{x_n\}$. Therefore $\{x_n\}$ is a subset of M, hence compact, but not closed. However, since M is a metric space, hence T_2 space, every compact set in M must be closed [1]. A contradiction, which means M is finite $\square$

(ii) *Proof.* For every open cover $\{O_\alpha\}$ of $f(K)$, $\{f^{-1}(O_\alpha)\}$ will be an open cover of K. Since K is compact, there exists an finite subcover $\{f^{-1}(O_{\alpha_i})\}_1^N$. Then:

$$f(K) \subset f\left(\cup_1^N f^{-1}(O_{\alpha_i})\right) = \cup_1^N f(f^{-1}(O_{\alpha_i})) = \cup_1^N O_{\alpha_i}$$

So $\{O_{\alpha_i}\}_1^N$ is an finite open subcover of $f(K)$. Therefore, $f(K)$ is compact. $\square$

4 (i) *Proof.* By using derivative, $f_n(x)$ is increase in $[0, \frac{1}{\sqrt{3}n}]$ and decrease in $[\frac{1}{\sqrt{3}n}, 1]$. So we have:

$$\sup_{x\in[0,1]} |f_n(x) - 0| = f_n(\frac{1}{\sqrt{3}n}) > \frac{1}{2} > 0$$

therefore f_n don't uniformly convergent to 0 in $[0, 1]$. But in $[\delta, 1]$, when n is large enough such that $\frac{1}{\sqrt{3}n} < \delta$, we have:

$$\sup_{x\in[\delta,1]} |f_n(x) - 0| = f_n(\delta) = \frac{\sqrt{n\delta}}{n^2\delta^2 + 1} \to 0$$

therefore, f_n uniformly convergent to 0 in every $[\delta, 1]$ where $\delta > 0$. $\square$

(ii) $|(1+n^2)^{-2}\sin(nx)|, |n(1+n^2)^{-2}\cos(nx)|, |(-1)^n n^2(1+n^2)^{-2}\sin(nx)|$ are all less than $\frac{1}{1+n^2}$, where $\sum_1^\infty \frac{1}{1+n^2} < \infty$. Therefore (a), (b), (c) are all uniformly convergent to some function [3]. Since $(\frac{\sin(nx)}{(1+n^2)^2})' = n(1+n^2)^{-2}\cos(nx)$, which is continuous and (a) and (b) convergent uniformly, we have [6]:

$$\left(\sum_{n=1}^{\infty}(1+n^2)^{-2}\sin(nx)\right)' = \sum_{n=1}^{\infty} n(1+n^2)^{-2}\cos(nx)$$

And in the same way, we can prove that:

$$\left(\sum_{n=1}^{\infty} n(1+n^2)^{-2}\cos(nx)\right)' = \sum_{n=1}^{\infty}(-1)n^2(1+n^2)^{-2}\sin(nx)$$

So, $f'' = \sum_{n=1}^{\infty}(-1)n^2(1+n^2)^{-2}\sin(nx)$ exists and it's continuous since uniform convergence and the continuousness of $\sin(nx)$ [4].

5 (i) Since every nonempty interval in $\mathbb{R}$ contains rational and irrational number. Therefore every upper sum is 1 and every lower sum is 0, hence f is not Riemann integrable.

(ii) *Proof.* In this situation we have that f is uniformly continuous [2] in $[a,b]$, which means $\forall \epsilon > 0, \exists \delta > 0\ \forall |x-y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.
For this (but arbitary) ϵ and δ, $\forall$ partition $0 = x_0 < x_1 < \cdots < x_N = 1$ with $\max_{0\le i\le N-1}(x_{i+1} - x_i) < \epsilon$. Denote U, L be the upper sum and the lower sum, then we have:

$$|U - L| \le \epsilon \cdot \sum_{i=0}^{N-1}(x_{i+1} - x_i) = \epsilon$$

which means $|U-L| \to 0$ when $\max_{0\le i\le N-1}(x_{i+1}-x_i) \to 0$. Therefore f is Riemann integrable. □

(iii) *Proof.* We have $0 \le f_n(x) < 1, \forall x \in [0,1]$ and $\forall \delta > 0$, f_n uniformly convergent to 0, hence $\lim \int_\delta^1 f_n(x)dx = 0, \forall \delta > 0$ [5]. Therefore we have:

$$\int_0^1 f_n(x)dx = \int_0^\delta f_n(x)dx + \int_\delta^1 f_n(x)dx < \delta + \int_\delta^1 f_n(x)dx$$

then:

$$\overline{\lim_{n\to\infty}} \int_0^1 f_n(x)dx < \delta, \qquad \forall \delta > 0$$

which means $\overline{\lim} \int_0^1 f_n(x)dx = 0$, together with $f_n \ge 0$, we have $\lim \int_0^1 f_n(x)dx = 0$. □

Bonus:

Suppose the strategy is to switch the number with probability $p(x)$ when we get x, then calculate the probability that we end with a larger number if two numbers are x and $2x$. It's:

$$\mathbf{P}(\text{end with larger number}) = \frac{1}{2}p(x) + \frac{1}{2}(1 - p(2x)) = \frac{1}{2} + \frac{p(x) - p(2x)}{2}$$

So we only need $p(x) > p(2x)$, for example we can switch the number with probability $\frac{1}{1+x}$ when we get x, and then we will have more chance than $1/2$ to get the biggest number for every x.

References

[1] **Theorem 26.3. "Every compact subspace of a Hausdorff space is closed"**. Munkres, James R. *Topology.* Prentice Hall, 2000.

[2] **Theorem 4.19. "Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X "**. Rudin, Walter. *Principles of mathematical analysis.* Vol. 3. New York: McGraw-hill, 1964.

[3] **Theorem 7.10. "$\cdots$ suppose $|f_n(x)| \leq M_n \cdots$ Then $\sum f_n$ converges uniformly $\cdots$ if $\sum M_n$ converges "**. Rudin, Walter. *Principles of mathematical analysis.* Vol. 3. New York: McGraw-hill, 1964.

[4] **Theorem 7.12. "If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E"**. Rudin, Walter. *Principles of mathematical analysis.* Vol. 3. New York: McGraw-hill, 1964.

[5] **Theorem 7.16. "$\cdots f_n \to f$ uniformly on $[a,b]$. Then $\cdots$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$ "**. Rudin, Walter. *Principles of mathematical analysis.* Vol. 3. New York: McGraw-hill, 1964.

[6] **Theorem 7.17. "Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$"**. Rudin, Walter. *Principles of mathematical analysis.* Vol. 3. New York: McGraw-hill, 1964.