Advanced Methods in Biostatistics I

Lecture 12

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October 10, 2017

Quadratic Forms

- We are often interested in working with random vectors that are combined into quadratic forms.
- The result is a function of random variables which is a scalar, and itself a random variable.
- We have previously discussed how to compute the expected value of quadratic forms.
- Here we discuss their distributional properties and how they apply to linear models.

Quadratic Forms

Definition

A quadratic form is a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ of the form:

$$f(\mathbf{y}) = \mathbf{y}'\mathbf{A}\mathbf{y} = \sum_{i,j} A_{ij} y_i y_j.$$

Expected value of quadratic forms

Theorem

Let $\mathbf{y}$ be a random vector with $E[\mathbf{y}] = \mu$ and $\mathrm{cov}(\mathbf{y}) = \Sigma$, and let $\mathbf{A}$ be a constant symmetric matrix. Then

$$E[\mathbf{y}'\mathbf{A}\mathbf{y}] = \mathrm{tr}(\mathbf{A}\Sigma) + \mu'\mathbf{A}\mu.$$

Variance of quadratic forms

Theorem

Let $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$, then

$$\mathrm{var}[\mathbf{y}'\mathbf{A}\mathbf{y}] = 2\mathrm{tr}[(\mathbf{A}\Sigma)^2] + 4\boldsymbol{\mu}'\mathbf{A}\Sigma\mathbf{A}\boldsymbol{\mu}.$$

Covariance of quadratic forms

Theorem

Let $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$, then

$$\operatorname{cov}[\mathbf{By}, \mathbf{y}'\mathbf{Ay}] = 2\mathbf{B}\Sigma\mathbf{A}\boldsymbol{\mu}.$$

Projections

Definition

A matrix $\mathbf{P}$ is idempotent if $\mathbf{P}^2 = \mathbf{P}$. A symmetric idempotent matrix is called a projection matrix.

Projections

Properties of a projection matrix $\mathbf{P}$:

- If $\mathbf{P}$ is an $n \times n$ matrix and $\text{rank}(\mathbf{P}) = r$, then $\mathbf{P}$ has r eigenvalues equal to 1 and $n - r$ eigenvalues equal to 0.
- $\text{tr}(\mathbf{P}) = \text{rank}(\mathbf{P})$.
- $\mathbf{P}$ is positive semidefinite.

Distributions

- As a next step we want to discuss the distributional properties of quadratic forms.
- Let's begin by reviewing both the central and noncentral χ^2-distribution.

χ^2 distribution

Definition

A random variable U has a (central) χ^2 distribution with $n > 0$ degrees of freedom if it has a pdf given by

$$f_U(u|n) = \frac{1}{\Gamma(\frac{n}{2})2^{n/2}} u^{\frac{n}{2}-1} e^{-u/2} I(u > 0)$$

We write $U \sim \chi^2_n$.

Properties of the χ^2 distribution

Properties

Let $U \sim \chi^2_n$. Then, the moment generating function of U is

$$M_U(t) = (1 - 2t)^{-n/2}$$

for $t < 1/2$.

Properties of the χ^2 distribution

Properties

Suppose $U \sim \chi^2_n$, then

- $E(U) = n$
- $var(U) = 2n$

χ^2 distribution

Theorem

If $Z \sim N(0, 1)$, then $U = Z^2 \sim \chi^2(1)$.

χ^2 distribution

Theorem

If $\mathbf{y} \sim N_p(\mathbf{0}, \mathbf{I})$, then $\mathbf{y}'\mathbf{y} \sim \chi^2(p)$.

χ^2 distribution

Theorem

If $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$, then

$$(\mathbf{y} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu}) \sim \chi^2(p).$$

Noncentral χ^2 distribution

Definition

A random variable V has a noncentral χ^2 distribution with $n > 0$ degrees of freedom and noncentrality parameter $\lambda > 0$ if it has a pdf given by

$$f_V(v|n,\lambda) = \sum_{j=0}^{\infty} \left(\frac{e^{-\lambda}\lambda^j}{j!} \right) f_U(v|n+2j)$$

We write $V \sim \chi^2_n(\lambda)$.

Noncentral χ^2 distribution

- This is an example of a mixture distribution involving a central χ^2 and a Poisson distribution.
- If $V|W \sim \chi^2_{n+2W}$ and $W \sim Poisson(\lambda)$, then $V \sim \chi^2_n(\lambda)$.

Properties of the noncentral χ^2 distribution

Properties

Let $V \sim \chi_n^2(\lambda)$. Then, the moment generating function of V is

$$M_V(t) = (1-2t)^{-n/2} \exp\left(\frac{2t\lambda}{1-2t}\right)$$

for $t < 1/2$.

Properties of the noncentral χ^2 distribution

Properties

Suppose $V \sim \chi_n^2(\lambda)$, then

- $E(U) = n + 2\lambda$
- $var(U) = 2n + 8\lambda$

Noncentral χ^2 distribution

Theorem

If $Y \sim N(\mu, 1)$, then $U = Y^2 \sim \chi_1^2(\lambda)$ where $\lambda = \mu^2/2$.

Noncentral χ^2 distribution

Theorem

If $U_1, U_2, \ldots U_m$ are independent random variables, where $U_i \sim \chi^2(n_i, \lambda_i)$ for $i = 1, 2, \ldots m$, then $U = \sum_i U_i \sim \chi^2(n, \lambda)$, where $n = \sum_i n_i$ and $\lambda = \sum_i \lambda_i$.

Noncentral χ^2 distribution

Theorem

If $\mathbf{y} \sim N_p(\mathbf{0}, \mathbf{I})$ and $\mathbf{A}$ is symmetric idempotent of rank r then

$$\mathbf{y}'\mathbf{A}\mathbf{y} \sim \chi^2(r).$$

Noncentral χ^2 distribution

Theorem

If $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \mathbf{I})$ then

$$\mathbf{y}'\mathbf{A}\mathbf{y} \sim \chi^2(r, \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}/2))$$

if and only if $\mathbf{A}$ is symmetric idempotent of rank r

Noncentral χ^2 distribution

Theorem

If $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \Sigma)$ then

$$\mathbf{y}'\mathbf{A}\mathbf{y} \sim \chi^2(r, \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}/2))$$

if and only if $\mathbf{A}\Sigma$ is idempotent of rank r

Example

- Assume $y_i \sim N(\mu, \sigma^2)$ for $i = 1, 2, \ldots n$.
- Consider the sample variance:

$$
\begin{aligned}
s^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \\
&= \frac{1}{n-1} \mathbf{y}'(\mathbf{I} - \mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n')\mathbf{y}.
\end{aligned}
$$

Example

- Let
$$\mathbf{A} = \sigma^{-2}(\mathbf{I} - \mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n').$$

- Note, the matrix
$$\mathbf{AV} = (\mathbf{I} - \mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n')$$
is symmetric idempotent with rank $n-1$.

- In addition, note that
$$\begin{aligned} \mathbf{AJ}_n &= \sigma^{-2}(\mathbf{I} - \mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n')\mathbf{J}_n \\ &= 0. \end{aligned}$$

Example

- Hence, it follows that

$$\begin{aligned}\lambda &= \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} \\ &= \mu^2\mathbf{J}_n'\mathbf{A}\mathbf{J}_n \\ &= 0\end{aligned}$$

- Therefore,

$$\mathbf{y}'(\mathbf{I} - \mathbf{J}_n(\mathbf{J}_n'\mathbf{J}_n)^{-1}\mathbf{J}_n')\mathbf{y}/\sigma^2 = (n-1)s^2/\sigma^2 \sim \chi^2(n-1).$$

General Linear Model

- Next consider the linear model:

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon,$$

where

$$\varepsilon \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}).$$

- Let $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.
- Consider the following partition of the sum of squares:

$$\mathbf{y}'\mathbf{y} = \mathbf{y}'\mathbf{H}\mathbf{y} + \mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}.$$

General Linear Model

- Consider the term: $\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}$
- Dividing this term by σ^2 we obtain

$$\mathbf{y}'\mathbf{A}\mathbf{y}$$

where

$$\mathbf{A} = \sigma^{-2}(\mathbf{I} - \mathbf{H}).$$

General Linear Model

- Note that $\mathbf{AV} = (\mathbf{I} - \mathbf{H})$ is idempotent with rank $n - p$.
- Also note that

$$\lambda = \frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} = \frac{1}{2}(\mathbf{X}\boldsymbol{\beta})'(\mathbf{I} - \mathbf{H})\mathbf{X}\boldsymbol{\beta} = 0.$$

- Hence,

$$\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}/\sigma^2 \sim \chi^2(n - p).$$

General Linear Model

- Now consider the term: $\mathbf{y}'\mathbf{H}\mathbf{y}$
- Dividing this term by σ^2 we obtain

$$\mathbf{y}'\mathbf{A}\mathbf{y}$$

where

$$\mathbf{A} = \sigma^{-2}\mathbf{H}.$$

General Linear Model

- Note that $\mathbf{AV} = \mathbf{H}$ is idempotent with rank p.
- Also note that

$$\lambda = \frac{1}{2}\mu'\mathbf{A}\mu = \frac{1}{2}(\mathbf{X}\beta)'\mathbf{H}\mathbf{X}\beta = \beta\mathbf{X}'\mathbf{X}\beta/(2\sigma^2).$$

- Hence,

$$\mathbf{y}'\mathbf{H}\mathbf{y} \sim \chi^2(p, \beta\mathbf{X}'\mathbf{X}\beta/(2\sigma^2)).$$