Statistical Theory Problem Set 2

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(i) Then: (since the meaning of y, $f(y;\theta)=0$ for $y<0$)

$$Pr(Y^{obs}, I \mid \theta, \alpha) = \prod_{i:I_i=1} f(Y_i,\theta)\,\pi(Y_i,\alpha) \prod_{i:I_i=0} \int f(Y_i,\theta)(1-\pi(Y_i,\alpha))\,dY_i$$

$$= \prod_{i:I_i=1} f(Y_i,\theta)\frac{Y_i}{\alpha} \prod_{i:I_i=0} \int_0^\alpha f(Y_i,\theta)(1-\frac{Y_i}{\alpha})\,dY_i$$

$$= \left(\prod_{i:I_i=1} \frac{Y_i f(Y_i,\theta)}{\alpha}\right)\cdot\left[\int_0^\alpha f(y,\theta)(1-\frac{y}{\alpha})\,dy\right]^{(n-\sum_{i=1}^n I_i)}$$

$$= \left[\prod_{i:I_i=1} \frac{Y_i f(Y_i,\theta)}{\alpha}\right]\cdot\left[1-\frac{E[Y_i\mid\theta]}{\alpha}\right]^{n-\sum_{i=1}^n I_i}$$

(ii) $Pr(Y^{obs} \mid I^{obs}, n_1=500, \theta, \alpha)$

$$\cancel{= Pr(Y^{obs} \mid I, \theta, \alpha)}$$

$$= \frac{Pr(Y_1, Y_2 \cdots Y_{n_1}, I_1=1, I_2=1 \cdots I_{n_1}=1, I_{n_1+1}=0, I_{n_1+2}=0 \cdots I_n=0 \mid \theta,\alpha)}{Pr(I_1=1, I_2=1 \cdots I_{n_1}=1, I_{n_1+1}=0, I_{n_1+2}=0, \cdots, I_n=0 \mid \theta,\alpha)}$$

$$= \frac{\left(\prod_{i=1}^{n_1} \frac{Y_i f(Y_i,\theta)}{\alpha}\right)\cdot\left[\int_0^\alpha f(y,\theta)(1-\frac{y}{\alpha})dy\right]^{n-n_1}}{\left(\prod_{i=1}^{n_1}\int_0^\alpha \frac{y f(y,\theta)}{\alpha}dy\right]\cdot\left[\int_0^\alpha f(y,\theta)(1-\frac{y}{\alpha})\,dy\right]^{n-n_1}}$$

$$= \frac{\left[\prod_{i=1}^{n_1} \frac{Y_i f(Y_i,\theta)}{\alpha}\right]\cdot\left[\int_0^\alpha f(y,\theta)(1-\frac{y}{\alpha})dy\right]^{n-n_1}}{\left[\int_0^\alpha f(y,\theta)\frac{y}{\alpha}\,dy\right]^{n_1}\left[\int_0^\alpha f(y,\theta)(1-\frac{y}{\alpha})\,dy\right]^{n-n_1}} = \frac{\prod_{i=1}^{n_1}[Y_i f(Y_i,\theta)]}{\left[\int_0^\alpha f(y,\theta)\,y\,dy\right]^{n_1}}$$

where $n_1=500$

(iii) Actually here α will be $+\infty$ since Gamma is not bounded

then $\pi(Y_i, \alpha) = \frac{Y_i}{\alpha}$ will not be a "regular" density

But If we ignore this and regard α a normal constant.

Then: $Pr[Y_i \mid I_i=1, \theta, \alpha] = \dfrac{Pr[Y_i, I_i=1 \mid \theta, \alpha]}{Pr[I_i=1 \mid \theta, \alpha]}$

$$= \frac{f(Y_i,\theta)\frac{Y_i}{\alpha}}{\int_0^{+\infty} f(Y_i,\theta)\frac{Y_i}{\alpha}\,dY_i} = \frac{f(Y_i,\theta)\cdot Y_i}{E[Y]}$$

So $E[Y_i \mid I_i=1, \theta, \alpha] = \dfrac{E[Y_i^2]}{E[Y_i]} = \dfrac{\theta_2+\theta_1^2}{\theta_1}$

(iv) In the likelihood in (ii)

$$L = \frac{\prod_{i=1}^{n_1}[Y_i \cdot f(Y_i,\theta)]}{[\int_0^{\alpha} f(y,\theta)\,y\,dy]^{n_1}}$$ where $n_1 = 500$ is considered as a constant.

Since there's no further factorization (not rigorous enough)

the minimal sufficient statistic is just $(Y_1, Y_2 \cdots Y_{500})$

In the assumption in (iii) again consider α a normal constant $\in \mathbb{R}$

(and likelihood in (i))

$$\cancel{L = \prod_{i:I_i=1}\left[\frac{(\theta_1/\theta_2)^{\theta_1^2/\theta_2}}{\Gamma[\theta_1^2/\theta_2]} \cdot Y_i^{\theta_1^2/\theta_2}\, e^{-\frac{\theta_1 Y_i}{\theta_2}}\right] / \theta_1^{n_1}}$$

$$L = \prod_{I_i=1}\left[\frac{(\theta_1/\theta_2)^{\theta_1^2/\theta_2}}{\Gamma(\theta_1^2/\theta_2)} Y_i^{\theta_1^2/\theta_2} e^{-\frac{\theta_1 Y_i}{\theta_2}}\right] \cdot \frac{1}{\alpha^{\sum_{i=1}^n I_i}} \cdot \left[1 - \frac{\theta_1}{\alpha}\right]^{n - \sum_{i=1}^n I_i}$$

$$= h(\theta_1, \theta_2, \alpha) \cdot \left(\prod_{I_i=1} Y_i\right)^{\theta_1^2/\theta_2} e^{-\frac{\theta_1 \sum_{I_i=1} Y_i}{\theta_2}} \cdot (\alpha-\theta_1)^{-\sum_{i=1}^n I_i}$$

There's no further factorization, so minimal sufficient statistic is

$$\left(\prod_{i=1}^n Y_i^{I_i},\ \sum_{i=1}^n Y_i \cdot I_i,\ \sum_{i=1}^n I_i\right)$$

(V) Again assume α is a constant $\in R$.

then $Pr\left(Y^{obs} \middle| I^{obs}, n_1=500, \theta, \alpha\right)$

$$= \prod_{i=1}^{n_1} \left[\frac{(\theta_1/\theta_2)^{\theta_1^2/\theta_2}}{\Gamma(\theta_1^2/\theta_2)} Y_i^{\theta_1^2/\theta_2} e^{-\frac{\theta_1 Y_i}{\theta_2}} \right] \Big/ \theta_1^{n_1}$$

$$= h(\theta_1,\theta_2) \cdot \left[\prod_{i=1}^{n_1} Y_i\right]^{\theta_1^2/\theta_2} e^{-\frac{\theta_1}{\theta_2}\sum_{i=1}^{n_1} Y_i} \Big/ \theta_1^{n_1}$$ $n_1=500$ is a constant

There's no futher factorization

so minimal statistic is $\left(\sum_{i=1}^{n_1} Y_i, \prod_{i=1}^{n_1} Y_i\right)$

If $\pi(Y_i;\alpha)$ is not a function of Y_i $\pi(Y_i;\alpha)=\pi(\alpha)$

consider also the situation ~~in~~ in (ii) and assumption in (iii)

Then $Pr\left(Y^{obs} \middle| I^{obs}, n_1=500, \theta, \alpha\right)$

$$= \frac{\prod_{i=1}^{n_1} [f(Y_i,\theta)\, \pi(\alpha)]}{\prod_{i=1}^{n_1}\left[\int_0^{+\infty} f(y,\theta)\cdot \pi(\alpha)\, dy\right]} = \prod_{i=1}^{n_1} f(Y_i,\theta)$$

$$= h(\theta_1,\theta_2) \cdot \left[\prod_{i=1}^{n_1} Y_i\right]^{\theta_1^2/\theta_2 - 1} e^{-\frac{\theta_1}{\theta_2}\sum_{i=1}^{n_1} Y_i}$$

The likelihood ratio of two situation is $\left[\prod_{i=1}^{n_1} Y_i\right] / \theta_1^{n_1}$

So they will give the same inference for θ_2

but will not necessary give the same inference for θ_1

consider their specific method.