Probability Theory II - Homework 3

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1. Since X_i's and N are independent, the distribution

$$P\left(\sum_{j=1}^{N} X_j \in A \mid N=n\right) = P\left(\sum_{j=1}^{n} X_j \in A\right) \quad \forall A \text{ measuarable.}$$

$$\Rightarrow E\left(\sum_{j=1}^{N} X_j \mid N=n\right) = E\left(\sum_{j=1}^{n} X_j\right) = n\,E(X_1)$$

Therefore $$E\left(\sum_{j=1}^{N} X_j\right) = E\left[E\left[\sum_{j=1}^{N} X_j \mid N\right]\right] = E\left[N\,E(X_1)\right] = E(N)\,E(X_1)$$

2. (i) We have

$$\frac{P(X_i > x+\theta/x)}{\frac{1}{\sqrt{2\pi}\,(x+\theta/x)}\, e^{-(x+\theta/x)^2/2}} \to 1$$

and

$$\frac{P(X_i > x)}{\frac{1}{\sqrt{2\pi}\,x}\, e^{-x^2/2}} \to 1$$

So $$\lim_{x\to+\infty} \frac{P(X_i > x+\theta/x)}{P(X_i > x)} = \lim \frac{\frac{1}{\sqrt{2\pi}(x+\theta/x)}\, e^{-\frac{x^2}{2}-\theta-\frac{\theta^2}{2x^2}}}{\frac{1}{\sqrt{2\pi}\,x}\, e^{-x^2/2}} = e^{-\theta}$$

(ii) $$P\left[b_n(M_n - b_n) \le x\right] = P\left[M_n \le b_n + \frac{x}{b_n}\right] = \left[1 - P\left(X_1 > b_n + \frac{x}{b_n}\right)\right]^n$$

$$= \left[1 - \frac{P(X_1 > b_n + \frac{x}{b_n})}{P(X_1 > b_n)} P(X_1 > b_n)\right]^n = \left[1 - \frac{P(X_1 > b_n + \frac{x}{b_n})}{P(X_1 > b_n)}\frac{1}{n}\right]^n$$

Since $P(X_i > b_n) = \frac{1}{n}$ we have $b_n \to +\infty$ as $n \to +\infty$

therefore $\dfrac{P(X_1 > b_n + \frac{x}{b_n})}{P(X_1 > b_n)} \to e^{-x}$ (use (i))

$$\Rightarrow \lim_n \left[1 - \frac{P(X_1 > b_n + \frac{x}{b_n})}{P(X_1 > b_n)} \frac{1}{n}\right]^n = \lim_n \left[1 - \frac{e^{-x}}{n} + \frac{o(1)}{n}\right]^n$$

$$= \exp[-e^{-x}]$$

So $P(b_n(M_n - b_n) \leq x) \to \exp[-e^{-x}]$

(iii) If $b_n \nsim (2\log n)^{1/2}$, the we can find a subarray n_k and a $1 > \varepsilon > 0$

either $\quad b_{n_k} > (1+\varepsilon)(2\log n_k)^{1/2}$ for all k. ①

or $\quad b_{n_k} < (1-\varepsilon)(2\log n_k)^{1/2}$ for all k ②

for situation ①, since $b_{n_k} \to +\infty$ ($P(X_1 > b_n) = \frac{1}{n} \to 0$)

we have $\lim_k \dfrac{P(X_i > b_{n_k})}{\frac{1}{\sqrt{2\pi}\, b_{n_k}} e^{-b_{n_k}^2/2}} = 1$

use derivative $\frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \downarrow \quad \Rightarrow \quad \dfrac{P(X_i > b_{n_k})}{\frac{1}{\sqrt{2\pi}\, b_{n_k}} e^{-b_{n_k}^2/2}} > \dfrac{\frac{1}{n_k}}{\frac{1}{\sqrt{2\pi}(1+\varepsilon)(2\log n_k)^{\frac{1}{2}}} e^{-\log(n_k)^{(1+\varepsilon)^2}}}$

$$= O(1) \log^{-\frac{1}{2}} n_k \cdot n_k^{(1+\varepsilon)^2 - 1} \to +\infty$$

$\Rightarrow \lim \dfrac{P(X_i > b_{n_k})}{\frac{1}{\sqrt{2\pi}\, b_{n_k}} e^{-b_{n_k}^2/2}} \to +\infty$ contradiction.

for situation ②, it's the same, $b_{n_k} \to +\infty$

$$1 = \lim_k \frac{P(X_i > b_{n_k})}{(\sqrt{2\pi}\, b_{n_k})^{-1} e^{-b_{n_k}^2/2}} < \overline{\lim_k}\, O(1) \log^{-\frac{1}{2}} n_k \cdot n_k^{(1-\varepsilon)^2 - 1} = 0$$

contradiction

so $\quad b_n \sim (2\log n)^{1/2}$.

Then $\forall \varepsilon>0$: $P\left(\left|\frac{M_n}{(2\log n)^{1/2}}-1\right|>\varepsilon\right)$

$\leq P\left[M_n>(1+\varepsilon)(2\log n)^{1/2}\right]+P\left[M_n\leq(1-\varepsilon)(2\log n)^{1/2}\right]$

$\leq \sum_{i=1}^{n} P\left(X_i>(1+\varepsilon)(2\log n)^{1/2}\right)+P\left[b_n(M_n-b_n)\leq b_n\left[(1-\varepsilon)(2\log n)^{1/2}-b_n\right]\right]$

$=\underbrace{n P\left[X_1>(1+\varepsilon)(2\log n)^{1/2}\right]}_{\text{I}}+\underbrace{P\left[b_n(M_n-b_n)\leq b_n^2\left[(1-\varepsilon)\frac{(2\log n)^{1/2}}{b_n}-1\right]\right]}_{\text{II}}$

for I: $P\left[X_1>(1+\varepsilon)(2\log n)^{1/2}\right]\sim\frac{1}{\sqrt{2\pi}\,(1+\varepsilon)(4\log n)^{1/2}}e^{-(1+\varepsilon)^2\log n}$

$=O(1)\log^{-\frac{1}{2}}n\,\frac{1}{n^{(1+\varepsilon)^2}}$

therefor $n\,P\left[X_1>(1+\varepsilon)(2\log n)^{1/2}\right]\sim O(1)\log^{-\frac{1}{2}}n\cdot n^{1-(1+\varepsilon)^2}\to 0$

for II: Since $b_n^2\to+\infty$ and $\frac{(2\log n)^{1/2}}{b_n}\to 1$

$b_n^2\left[(1-\varepsilon)\frac{(2\log n)^{1/2}}{b_n}-1\right]\to-\infty$

therefore for arbitany $x<0$, there's a N, $n>N\Rightarrow b_n^2\left[(1-\varepsilon)\frac{(2\log n)^{1/2}}{b_n}-1\right]<x$

$\Rightarrow 0\leq\overline{\lim_n}\,P\left[b_n(M_n-b_n)\leq b_n^2\left[(1-\varepsilon)\frac{(2\log n)^{1/2}}{b_n}-1\right]\right]$

$\leq\lim_n P\left[b_n(M_n-b_n)\leq x\right]=\exp\left[-e^{-x}\right]\quad\forall x$

Let $x\to-\infty$ we get that $\text{II}\to 0$.

therefore $\lim_n P\left(\left|\frac{M_n}{(2\log n)^{1/2}}-1\right|>\varepsilon\right)=0\quad\forall\varepsilon>0$

Which means $\frac{M_n}{(2\log n)^{1/2}}\xrightarrow{P}1$

3: Here we state a Lemma and ~~proof~~ prove it in the last page

Lemma 1: For monotone increase function F and G and $a, b \in [-\infty, +\infty]$

suppose $F(-\infty) = \lim_{x \to -\infty} F(x) > -\infty$ $F(+\infty) = \lim_{x \to +\infty} F(x) < +\infty$

$G(-\infty) > -\infty$ and $G(+\infty) < +\infty$

Then $\sup_{a \le x \le b} |F(x) - G(x)| \le |F(a) - G(a)| + |F(b) - G(b)| + 2(G(b) - G(a))$

Now we prove Ex 3.2.9:

$\forall \varepsilon > 0$ since F is continuous and monotone increase we can find N and points $-\infty = x_0 < x_1 < x_2 \cdots < x_N < x_{N+1} = +\infty$ such that $F(x_i) \le F(x_{i+1}) < F(x_i) + \varepsilon$ $\forall i$

Since $F_n \Rightarrow F$ and F continuous on x_i, for this ε and N, we can find K.

$n > K \Rightarrow \sup_{0 \le i \le N+1} |F_n(x_i) - F(x_i)| < \varepsilon$.

Then for this ε, N and K, for $n > K$:

$$\sup_x |F_n(x) - F(x)| = \max_{0 \le i \le N} \left\{ \sup_{x_i \le x \le x_{i+1}} \left(|F_n(x) - F(x)| \right) \right\} \quad \text{--- use Lemma 1}$$

$$\le \max_{0 \le i \le N} \left\{ |F_n(x_i) - F(x_i)| + |F_n(x_{i+1}) - F(x_{i+1})| + 2(F(x_{i+1}) - F(x_i)) \right\}$$

$$< \max (4\varepsilon) = 4\varepsilon$$

Since $\varepsilon > 0$ is arbitrary and K only depends on ε (N depends only on ε)

we have $\lim_n \sup_x |F_n(x) - F(x)| = 0$

4: Use strong Law of Large Number and continuity of $\frac{x^4}{x^2+1}$

we got that $\frac{\bar{X}_n^4}{\bar{X}_n^2+1} \xrightarrow{a.s} \frac{5^4}{5^2+1} = \frac{625}{26}$

Also, since $\left|\frac{\bar{X}_n^4}{\bar{X}_n^2+1}\right| \leq \bar{X}_n^4 \leq 9^4 < +\infty$

we use bounded convergence theorem

and get that $\lim_{n\to\infty} E \frac{\bar{X}_n^4}{\bar{X}_n^2+1} = E \lim_n \frac{\bar{X}_n^4}{\bar{X}_n^2+1} = E \frac{625}{26} = \frac{625}{26}$

Proof of Lemma 1:

$$\sup_{a\leq x\leq b} |F(x)-G(x)| \leq \sup_{a\leq x\leq b} \{|F(x)-G(a)| + |G(x)-G(a)|\}$$

since $F(x)$ monotone increase $|F(x)-G(a)|$ either archieve its supreme on point a or point b

therefore
$$\sup_{a\leq x\leq b} |F(x)-G(x)| \leq \sup_{a\leq x\leq b} \{|F(a)-G(a)| + |F(b)-G(a)| + |G(b)-G(a)|\}$$
$$\leq \sup_{a\leq x\leq b} \{|F(a)-G(a)| + |F(b)-G(b)| + |G(a)-G(b)| + |G(b)-G(a)|\}$$
$$= \sup_{a\leq x\leq b} \{|F(a)-G(a)| + |F(b)-G(b)| + 2(G(b)-G(a))\}$$

Which ends the proof.