You can work together, but you must write up your solutions independently. Be sure to show how you derived your solutions.

For the simple linear regression model $Y_i = \alpha + \beta X_i + \epsilon_i$,

1. Find the method of moments estimator of β.
2. Find the MLE of β.
3. To test $H_0 : \beta = 0$,
   (a) derive a LRT test.
   (b) derive a Wald test.
   (c) derive a score test.

   Be explicit about the assumptions you are making and include any equations you solve in addition to the final answer.

Statistical Theory Final Solution

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1. Suppose α, β is parameters and ϵ_i, X_i are samples from two uncorrelated random variables, hence Y_i are also samples from a random variables. Also we suppose ϵ_i have mean 0 and $\frac{\sum X_i^2}{n} > (\frac{\sum X_i}{n})^2$.

   Then we have moments:

$$\begin{aligned} \mathbb{E}[Y] &= \alpha + \beta\mathbb{E}[X] + \mathbb{E}[\epsilon] = \alpha + \beta\mathbb{E}[X] & (1)\\ \mathbb{E}[XY] &= \alpha\mathbb{E}[X] + \beta\mathbb{E}\left[X^2\right] + \mathbb{E}[\epsilon X] = \alpha\mathbb{E}[X] + \beta\mathbb{E}\left[X^2\right] & (2) \end{aligned}$$

   Replace the expectation by the sample mean we have the MME of α, β are the solution of equation (n is the sample size):

$$\begin{aligned} \frac{\sum Y_i}{n} &= \alpha + \beta\frac{\sum X_i}{n} & (3)\\ \frac{\sum X_i Y_i}{n} &= \alpha\frac{\sum X_i}{n} + \beta\frac{\sum X_i^2}{n} & (4) \end{aligned}$$

   Then we solve that $\hat{\beta} = \frac{\overline{XY} - \overline{X}\cdot\overline{Y}}{\overline{X^2} - \overline{X}^2}$, where $\overline{X}, \overline{Y}, \overline{XY}, \overline{X^2}$ are the sample means of X, Y, XY, X^2.

2. Suppose the sample size is n and $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, where $\mathbf{I}$ is the identity matrix of size $n \times n$. And X_i are given numbers, σ is known, α, β are parameters. Also suppose that $\frac{\sum X_i^2}{n} > (\frac{\sum X_i}{n})^2$.

   Then we have the likelihood of the sample is:

$$L(\alpha, \beta) = \prod_1^n \frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{(Y_i - \alpha - \beta X_i)^2}{2\sigma^2}\}$$

   Calculate the derivtives of $-\log L(\alpha, \beta)$, we get the regular equation for the MLE:

$$\begin{aligned} \sum_1^n Y_i - \alpha - \beta X_i &= 0 & (5)\\ \sum_1^n X_i(Y_i - \alpha - \beta X_i) &= 0 & (6) \end{aligned}$$

   And the unique solution is indeed the minimal point of $-\log L$. Then we get that $\hat{\beta} = \frac{\overline{XY} - \overline{X}\cdot\overline{Y}}{\overline{X^2} - \overline{X}^2}$, where $\overline{X}, \overline{Y}, \overline{XY}, \overline{X^2}$ are the sample means of X, Y, XY, X^2.

3. Here, we use the same assumption and notation as in question 2.

   (a) In this situation, the likelihood ratio is:

   $$\Lambda = \frac{\sup_\alpha L(\alpha, 0)}{\sup_{\alpha,\beta} L(\alpha, \beta)}$$

   The rejection field will be $\{\Lambda < C\}$, where C is choosen by solve:

   $$\{\sup_\alpha \mathbf{P}(\Lambda < C|\alpha = \alpha, \beta = 0)\} = l_\alpha$$

   Where l_α is the level of the test you need.

   (b) In this situation, the test statistics will be $T = \frac{(\hat{\beta}-0)^2}{var(\hat{\beta})}$. And the rejection field will be $\{T > C\}$, where C is choosen by solve:

   $$\{\sup_\alpha \mathbf{P}(T > C|\alpha = \alpha, \beta = 0)\} = l_\alpha$$

   Where l_α is the level of the test you need.

   (c) In this situation, denote $\hat{\alpha}_0$ be the MLE of α when $\beta = 0$. And $U(\alpha, \beta)$ be the derivtive vector $\frac{\partial \log L(\alpha,\beta)}{\partial(\alpha,\beta)}$. And $I(\alpha, \beta)$ be the Fisher information matrix of this model.

   Then the test statistics is

   $$S = U^T(\hat{\alpha}_0, 0)I^{-1}(\hat{\alpha}_0, 0)U(\hat{\alpha}_0, 0)$$

   The rejection field is $\{S > C\}$, where C is choosen by solve:

   $$\{\sup_\alpha \mathbf{P}(S > C|\alpha = \alpha, \beta = 0)\} = l_\alpha$$

   Where l_α is the level of the test you need.