Statistical Theory Take home Exam Solution

Bohao Tang

1. MME of β:

We have two ways to deal with it, First is to suppose ε_i are i.i.d random variables with mean 0 and variance σ^2. And α, β, X_i ~~#~~ are all constant ~~given~~, then we have two moment equations: (n is sample size)

$$0 = E\varepsilon = \overline{\varepsilon} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \alpha - \beta X_i)$$

$$\sigma^2 = E\varepsilon^2 = \overline{\varepsilon^2} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \alpha - \beta X_i)^2$$

This ~~funct~~ group of equations often yield two possible β estimators and this idea requires the knowledge of σ^2, which is not practicle. So we may deal it in another way:

Suppose ε_i are i.i.d random variables with mean 0, X_i are i.i.d random variables $\sim X$ ~~with~~, where X is uncorrelated with ε. And α, β are constant, then we also have two moment equations:

$$0 = E\varepsilon = \overline{\varepsilon} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \alpha - \beta X_i)$$

$$0 = E\varepsilon X = \overline{\varepsilon X} = \frac{1}{n}\sum_{i=1}^{n} X_i(Y_i - \alpha - \beta X_i)$$

The solution of β will be the estimator:

$$\hat{\beta} = \frac{\overline{XY} - \overline{X}\cdot\overline{Y}}{\overline{X^2} - \overline{X}^2}$$

where $\overline{X} = \frac{\sum X_i}{n}$, $\overline{Y} = \frac{\sum Y_i}{n}$, $\overline{XY} = \frac{\sum X_i Y_i}{n}$ and $\overline{X^2} = \frac{\sum X_i^2}{n}$

And we add an assumption of $\overline{X^2} > \overline{X}^2$.

2: Here we suppose the sample size is n and ε_i i.i.d $\sim N(0,\sigma^2)$, where $N(0,\sigma^2)$ is the normal distribution with mean 0 and variance σ^2. Also here X_i are given numbers (it has no influence to consider X_i are i.i.d from a distribution since then the likelihood will only times some values that don't contain α, β or σ), α, β, σ are parameters. Also suppose $\frac{\sum_1^n X_i^2}{n} > \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2$

Then we have the likelihood of the sample is:

$$L(\alpha,\beta,\sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(Y_i-\alpha-\beta X_i)^2}{2\sigma^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}^{\,n}\sigma^n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\alpha-\beta X_i)^2\right\}$$

To maximize $L(\alpha,\beta,\sigma)$ you need to minimize $\sum_{i=1}^{n}(Y_i-\alpha-\beta X_i)^2$ use the partial derivtives we get the normal equations:

$$\sum_{i=1}^{n}(Y_i-\alpha-\beta X_i) = 0$$

$$\sum_{i=1}^{n} X_i(Y_i-\alpha-\beta X_i) = 0$$

Then we get the MLE of β is $\hat{\beta} = \frac{\overline{XY} - \bar{X}\bar{Y}}{\overline{X^2} - \bar{X}^2}$, the notation is the same as in question 1.

(Here it's obviously that the solution of normal equations are indeed the minimal point)

3: use the assumptions and notations in question 2!

(a) Here the likelihood ratio test statistic for testing H_0 is

$$\lambda(\vec{X}) = \frac{\sup_{(\alpha,\beta,\sigma)\in H_0} L(\alpha,\beta,\sigma)}{\sup_{(\alpha,\beta,\sigma)\in R^2\times R^+} L(\alpha,\beta,\sigma)}$$

$$= \frac{\sup_{\{\alpha\in R,\sigma>0\}} L(\alpha,0,\sigma)}{L(\hat{\alpha},\hat{\beta},\hat{\sigma})}$$

Where $(\hat{\alpha},\hat{\beta},\hat{\sigma})$ is the MLE of (α,β,σ). After calculation, ~~$\lambda(X)$ is equal to:~~ we have:

$$\sup_{(\alpha,\beta)\in R\times R^+} L(\alpha,0,\sigma) = L\left(\bar{Y},0,\sqrt{\overline{Y^2}-\bar{Y}^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}^{\,n}}\cdot\left(\overline{Y^2}-\bar{Y}^2\right)^{-\frac{n}{2}}\exp\{-\tfrac{n}{2}\}$$

where $\overline{Y^2} = \frac{\sum Y_i^2}{n}$, and

$$\cancel{L(\hat{\alpha},\hat{\beta},\hat{\sigma}) = \frac{1}{\sqrt{2\pi}^{\,n}}\cdot\left(\overline{Y^2}-\bar{Y}^2\right)^{-\frac{n}{2}}\cdot(1-\hat{\beta}^2)}$$

$$L(\hat{\alpha},\hat{\beta},\hat{\sigma}) = \frac{1}{\sqrt{2\pi}^{\,n}}\cdot\left(\overline{Y^2}-\bar{Y}^2\right)^{-\frac{n}{2}}\cdot(1-\hat{\rho}^2)^{-\frac{n}{2}}\exp\{-\tfrac{n}{2}\}$$

Where $\hat{\rho} = \frac{\overline{XY}-\bar{X}\bar{Y}}{\sqrt{\overline{X^2}-\bar{X}^2}\cdot\sqrt{\overline{Y^2}-\bar{Y}^2}}$ is the sample correlation coefficient.

Then $\lambda(\vec{X}) = \left\{(1-\hat{\rho}^2)^{-\frac{n}{2}}\right\}^{-1} = (1-\hat{\rho}^2)^{\frac{n}{2}}$

By the theory in slide 13 we have $-2\log\lambda(X) \xrightarrow{d} \chi^2_1$

where χ^2_1 is a χ^2 distribution with degree of freedom 1. So the Rejection field of LRT with level α will be

$$R = \left\{X,Y:\ \lambda(\vec{X}) < e^{-\frac{q_{1,1-\alpha}}{2}}\right\}$$

where $q_{1,1-\alpha}$ is the $1-\alpha$ quantile of χ^2_1 distribution

3. (b) Here we have multiple parameters, so the form in the Slide won't work. But we have that for MLE: $(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ and Hypothesis $H_0: R\begin{pmatrix}\alpha\\\beta\\\sigma\end{pmatrix} = r \quad H_1: H_0^C$

where R is $Q \times P$ given matrix, ~~we~~ under null hypothesis:

$$\left(R\begin{pmatrix}\hat{\alpha}\\\hat{\beta}\\\hat{\sigma}\end{pmatrix} - r\right)'\left(R\hat{I}_n^{-1}R'\right)^{-1}\left(R\begin{pmatrix}\hat{\alpha}\\\hat{\beta}\\\hat{\sigma}\end{pmatrix} - r\right) \xrightarrow{d} \chi^2_Q$$

Where $\hat{I}_n$ is the Fisher Information matrix replace (α, β, σ) to $(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$

And this is also called Wald test statistics. So here we set the statistic T to be

$$T = \hat{\beta}\left\{(0,1,0)\,\hat{I}_n^{-1}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}^{-1}\hat{\beta}$$

Calculate the I_n, we have (we deal σ as parameter, not σ^2)

$$I_n = \begin{pmatrix} \frac{n}{\sigma^2}, & \frac{n\Sigma X_i}{\sigma^2}, & 0 \\ \frac{n\Sigma X_i}{\sigma^2}, & \frac{n\Sigma X_i^2}{\sigma^2}, & 0 \\ 0, & 0, & \frac{2n}{\sigma^2} \end{pmatrix}$$

so

$$T = \hat{\beta}^2 \cdot \frac{n(\overline{X^2} - \bar{X}^2)}{(\overline{Y^2} - \bar{Y}^2)(1-\hat{\rho}^2)} = n \cdot \frac{\hat{\rho}^2}{1-\hat{\rho}^2}$$

and $T \xrightarrow{d} \chi_1^2$

so the rejection field will be

$$R = \{X, Y : T > q_{1,1-\alpha}\}$$

where $q_{1,1-\alpha}$ is the $1-\alpha$ quantile of χ_1^2.

3. (C): Also here we have multiple parameters, the method in slide won't work. But we have for MLE: $(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ and MLE: $(\tilde{\alpha}, 0, \tilde{\sigma})$ under null hypothesis $H_0: \beta = 0$:

$$S_n^T(\tilde{\alpha}, 0, \tilde{\sigma})\, I_n^{-1}(\tilde{\alpha}, 0, \tilde{\sigma})\, S_n(\tilde{\alpha}, 0, \tilde{\sigma}) \sim \chi^2_k$$

where k is the number of constraints imposed by null hypothesis

and $S_n(\alpha, \beta, \sigma) = \dfrac{\partial L(\alpha, \beta, \sigma)}{\partial(\alpha, \beta, \sigma)}$. And this is also called score test

So we here choose the test statistics

$$Z_s = S_n^T(\tilde{\alpha}, 0, \tilde{\sigma})\, I_n^{-1}(\tilde{\alpha}, 0, \tilde{\sigma})\, S_n(\tilde{\alpha}, 0, \tilde{\sigma})$$

We have that

$$I_n^{-1} = \begin{pmatrix} \frac{\sigma^2}{n} \frac{\overline{X^2}}{\overline{X^2} - \overline{X}^2}, & \frac{-\overline{X}}{\overline{X^2} - \overline{X}^2} \cdot \frac{\sigma^2}{n}, & 0 \\ \frac{-\sigma^2}{n} \frac{\overline{X}}{\overline{X^2} - \overline{X}^2}, & \frac{\sigma^2}{n} \frac{1}{\overline{X^2} - \overline{X}^2}, & 0 \\ 0, & 0, & \frac{\sigma^2}{2n} \end{pmatrix}$$

and

$$S_n = \begin{pmatrix} \frac{n}{\sigma^2}(\overline{Y} - \alpha - \beta \overline{X}) \\ \frac{n}{\sigma^2}(\overline{XY} - \alpha \overline{X} - \beta \overline{X^2}) \\ -\frac{n}{\sigma} + \frac{n}{\sigma^3} \overline{(Y - \alpha - \beta X)^2} \end{pmatrix}$$

where

$$\overline{(Y - \alpha - \beta X)^2} = \frac{\sum_{i=1}^n (Y_i - \alpha - \beta X_i)^2}{n}$$

so $Z_s = n\hat{\rho}^2 \xrightarrow{d} \chi^2_1$

and the rejection field will be

$R: \{X, Y: Z_s > q_{1,1-\alpha}\}$ where $q_{1,1-\alpha}$ is the $1-\alpha$ quantile of χ^2_1