Probability Theory Final Exam

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1: Denote $\varphi_{X_n}(t)$ be the characteristic function of X_n

Then recall the ~~co~~ Continuity theorem, we have that

$X_n \Rightarrow X_\infty \Leftrightarrow \varphi_{X_n}(t) \to \varphi_{X_\infty}(t)$ for all t, given that $\varphi_{X_\infty}(t)$ is a characteristic function.

Proof:

1° if $Y_n \perp\!\!\!\perp X_n$ $\forall n$, $1 \le n \le +\infty$

Then we have $\varphi_{X_n+Y_n}(t) = \varphi_{X_n}(t)\,\varphi_{Y_n}(t)$ $\forall n \in N^*$

and $\varphi_{X_\infty+Y_\infty}(t) = \varphi_{X_\infty}(t)\,\varphi_{Y_\infty}(t)$

Since $X_n \Rightarrow X_\infty$ $Y_n \Rightarrow Y_\infty$, we have

$\forall t$: $\lim_n \varphi_{X_n}(t) = \varphi_{X_\infty}(t)$
$\lim_n \varphi_{Y_n}(t) = \varphi_{Y_\infty}(t)$, so by the continuity of multiplication in R, we have

$$\forall t \quad \lim_n \varphi_{X_n+Y_n}(t) = \lim_n \varphi_{X_n}(t)\,\varphi_{Y_n}(t) = \varphi_{X_\infty}(t)\,\varphi_{Y_\infty}(t) = \varphi_{X_\infty+Y_\infty}(t)$$

so by continuity theorem

$$X_n + Y_n \Rightarrow X_\infty + Y_\infty$$

2^0 If $Y_\infty = C$ a.s.

Then then distribution function of $X_\infty + Y_\infty$ is $F_{X_\infty+Y_\infty}(t) =$

$$P(X_\infty + Y_\infty \le t) = P(X_\infty \le t - C) = F_{X_\infty}(t-C)$$

So the continuous point of $F_{X_\infty+Y_\infty}$ is just a shift from those of F_{X_∞}

Then $\forall\, t$ be continuous point of $F_{X_\infty+Y_\infty}$, $t-C$ is ~~the~~ a continuous point of F_{X_∞}

We have $P(X_n + Y_n \le t) \le P(X_n \le t - C + \varepsilon) + P(|Y_n - C| \ge \varepsilon)$ $\quad \forall \varepsilon > 0$

Since there are at most countable discontinuous point of F_{X_∞},

we can find a sequence of $\underbrace{\varepsilon_k \downarrow 0}_{\varepsilon_k > 0}$ so that $t - C + \varepsilon_k$ are all continuous point for F_{X_∞}

Then
$$\overline{\lim_n}\, P(X_n + Y_n \le t) \le \overline{\lim_n}\, P(X_n \le t - C + \varepsilon_k) + \overline{\lim_n}\, P(|Y_n - C| \ge \varepsilon_k)$$
$$\le P(X_\infty \le t - C + \varepsilon_k) + \overline{\lim_n}\, P(Y_n \le C - \varepsilon_k) + \overline{\lim_n}\, [1 - P(Y_n \le C + \varepsilon_{k+1})]$$

Since $Y_n \Rightarrow C$ and $\varepsilon_k > 0$, $C - \varepsilon_k$, $C + \varepsilon_{k+1}$ will be continuous points for $F_C(t)$, the we have:

$$\overline{\lim_n}\, P(X_n + Y_n \le t) \le P(X_\infty \le t - C + \varepsilon_k) + 0 \quad \forall k,$$ ~~since~~

Since $\varepsilon_k \downarrow 0$ and F_{X_∞} continuous at $t - C$, we have

$$\overline{\lim_n}\, P(X_n + Y_n \le t) \le P(X_\infty \le t - C) = F_{X_\infty + Y_\infty}(t)$$

On the other side:

$$P(X_n+Y_n \leq t) \geqslant P[|Y_n-c|\leq \varepsilon;\ X_n+Y_n\leq t]$$

$$\geqslant P[\,|Y_n-C|\leq \varepsilon\,;\ X_n \leq t-C-\varepsilon\,]$$

$$\geqslant P\,[X_n\leq t-C-\varepsilon] - P[|Y_n-c|>\varepsilon]$$

For the same reason we chose $\varepsilon_k \downarrow 0$ $\varepsilon_k>0$, $t-C-\varepsilon_k$ be F_{X_∞} continuous point

Then we have

$$\underline{\lim}\ P(X_n+Y_n \leq t) \geqslant P(X_\infty \leq t-C-\varepsilon_k)+0 \quad \forall k$$

Since $\varepsilon_k \downarrow 0$ and $t-C$ be continous point for F_{X_∞}, we have

$$\underline{\lim}\ P(X_n+Y_n\leq t) \geqslant P(X_\infty \leq t-C) = F_{X_\infty+Y_\infty}(t)$$

Therefore $\lim_n P(X_n+Y_n \leq t) = F_{X_\infty+Y_\infty}(t)$ $\forall t$ be continuous point of $F_{X_\infty+Y_\infty}$

$\Rightarrow X_n+Y_n \Rightarrow X_\infty+Y_\infty$

So if $Z_n \Rightarrow X$ and $Z_n-X_n \Rightarrow 0$, then $X_n = Z_n+X_n-Z_n \Rightarrow X+0=X$.

and if $Z_n-X_n\Rightarrow 0$, $X_n\Rightarrow X$, then $Z_n = Z_n-X_n+X_n \Rightarrow X+0=X$.

(Here we use that if $X_n \Rightarrow 0$ then $-X_n \Rightarrow 0$, it's obvious since in this question we actually proved that $X_n \Rightarrow C \Leftrightarrow P(|X_n-C|\geq \varepsilon) \to 0$ $\forall \varepsilon>0$ that is $X_n \Rightarrow C \Leftrightarrow X_n \xrightarrow{P} C$ for C constant.

Then notice that $P(|X_n|\geq \varepsilon) = P(|-X_n|\geq\varepsilon)$ we got ~~the~~ what we need)

2: Y_n is tight so that $\forall \varepsilon>0\ \exists M_\varepsilon<+\infty\quad P(|Y_n|>M_\varepsilon)<\varepsilon\quad \forall n$

In question 1 part 2 we commented that $X_n \Rightarrow C \Leftrightarrow X_n \xrightarrow{P} C$ for constant C

So here we need to prove $\forall \varepsilon>0,\ P(|X_n Y_n|>\varepsilon) \to 0$

Proof: $\forall \varepsilon>0$, $\forall \varepsilon'>0$

$$P(|X_n Y_n|>\varepsilon) \le P\left(|X_n|>\frac{\varepsilon}{M_{\varepsilon'}}\right) + P(|Y_n|>M_{\varepsilon'})$$

$$\Rightarrow \overline{\lim_n}\ P(|X_n Y_n|>\varepsilon) \le 0+\varepsilon'. \qquad \forall \varepsilon'>0.$$

$$\Rightarrow \overline{\lim_n}\ P(|X_n Y_n|>\varepsilon) \le 0 \Rightarrow P(|X_n Y_n|>\varepsilon)\to 0$$

$$\Rightarrow |X_n Y_n| \xrightarrow{P} 0 \Rightarrow X_n Y_n \Rightarrow 0$$

3: $$\varphi_Z(t) = E e^{it(X-Y)} = E e^{itX} E e^{-itY} = \varphi_X(t)\cdot\overline{\varphi_Y(t)}$$

$$= |\varphi_X(t)|^2$$ is non-negative and real-valued.

Denote U be Uniform $[-1,1]$ random variable, then

$$\varphi_U(t) = \int_{-1}^{1} \frac{1}{2} e^{itu}\,du = \frac{\sin t}{t}$$ which has negative values

so U won't be $X-Y$ of i.i.d X, Y.