# Advanced Methods Homework 4

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## I Linear Mixed Model

1. (a)

$$\vec{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{31} \\ y_{32} \\ y_{33} \\ y_{41} \\ y_{42} \\ y_{43} \end{pmatrix} = \vec{J}_{12}\,\mu + \begin{pmatrix} \vec{J}_3 & & & 0 \\ & \vec{J}_3 & & \\ & & \vec{J}_3 & \\ 0 & & & \vec{J}_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \vec{\varepsilon}$$

so $X = \vec{J}_{12}$

$\vec{\beta} = \mu$

$$Z = \begin{pmatrix} \vec{J}_3 & & & 0 \\ & \vec{J}_3 & & \\ & & \vec{J}_3 & \\ 0 & & & \vec{J}_3 \end{pmatrix} = I_4 \otimes \vec{J}_3$$

$$\vec{w} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

(b) BLUP of $\vec{u}$ is $E[\vec{w}\mid\vec{y}] = E[\vec{u}] + cov(\vec{u},\vec{y})\, var(\vec{y})^{-1}\{\vec{y} - E[\vec{y}]\}$

Here $E(\vec{w}) = \vec{0}$ $\quad cov(\vec{w},\vec{y}) = \sigma_w^2 I_4 Z' = \sigma_u^2 Z'$

$var(\vec{y}) = Z \sigma_u^2 I_4 Z' + \sigma^2 I_{12} = \sigma^2 I_{12} + \sigma_w^2 ZZ'$

$var[\vec{y}] = I_4 \otimes (\sigma^2 I_3 + \sigma_u^2 \vec{J}_3\vec{J}_3')$ recall $(aI + bJ_nJ_n')^{-1}$

$= \frac{1}{a}\left(I - \frac{b}{a+nb} J_nJ_n'\right)$

we have that $var[\vec{y}]^{-1} = I_4 \otimes \frac{1}{\sigma^2}\left[I_3 - \frac{\sigma_u^2}{\sigma^2+3\sigma_u^2}\vec{J}_3\vec{J}_3'\right]$

So that $E[\vec{u}|\vec{y}] = \sigma_u^2 z' \ var[\vec{y}]^{-1} (\vec{y} - \mu\vec{J}_{12})$

$= I_4 \otimes \frac{\sigma_u^2}{\sigma^2}\left[\vec{J}_3' - \frac{3\sigma_u^2}{\sigma^2+3\sigma_u^2}J_3'\right][\vec{y}-\mu\vec{J}_{12}]$

$= I_4 \otimes \frac{\sigma_u^2}{\sigma^2+3\sigma_u^2}\vec{J}_3' (\vec{y}-\mu\vec{J}_{12}) = \frac{3\sigma_u^2}{\sigma^2+3\sigma_u^2}\begin{pmatrix}\bar{y}_1-\mu\\ \bar{y}_2-\mu\\ \bar{y}_3-\mu\\ \bar{y}_4-\mu\end{pmatrix}$

where $\bar{y}_i = \frac{\sum_{j=1}^{3} y_{ij}}{3}$

So BLUP of $u_i$ is $\frac{3\sigma_u^2}{\sigma^2+3\sigma_u^2}(\bar{y}_i-\mu)$

(C) BLUE of $\mu$ is $\hat{\mu} = (X'V^{-1}X)^{-1}X'V^{-1}\vec{y}$ where $V = var(\vec{y})$

$\Rightarrow \hat{\mu} = \left(\frac{12}{\sigma^2+3\sigma_u^2}\right)^{-1} \frac{1}{\sigma^2+3\sigma_u^2}\vec{J}_{12}'\vec{y} = \frac{\sum\vec{y}}{12} = \bar{y}$

So the BLUE for $\mu$ is still the sample mean $\bar{y}$

2. (a) For every unbiased predictor $\theta(y)$

$E\{[\theta(y)-\vec{u}]^2\} = E[(\theta(y) - E[\vec{u}|\vec{y}] + E(\vec{u}|\vec{y}] - \vec{u})^2]$

$= E\{[\theta(y)-E(\vec{u}|\vec{y})]^2\} + E\{[E[\vec{u}|\vec{y}]-\vec{u}]^2\}$
$+ 2\underbrace{E[(\theta(y)-E[\vec{u}|\vec{y}])(E[\vec{u}|\vec{y}]-\vec{u})]}_{①}$

$① = E[E[(\theta(y)-E(\vec{u}|\vec{y}))(E(\vec{u}|\vec{y})-\vec{u})|\vec{y}]]$

$= E[(\theta(y)-E[\vec{u}|\vec{y}])E[(E[\vec{u}|\vec{y}]-\vec{u})|\vec{y}]] = E[(\theta(y)-E[\vec{u}|\vec{y}])\cdot 0]$
$= 0$

$\Rightarrow E[\theta(y)-\vec{u}]^2 = E[\theta(y)-E(\vec{u}|\vec{y})]^2 + E[E[\vec{u}|\vec{y}]-\vec{u}]^2$

So when $\theta(y) \overset{a.s.}{=} E(\vec{u}|\vec{y})$ $E[\theta(y)-\vec{u}]^2$ archieves its minimum, also $E[\vec{u}|\vec{y}]$ is unbiased

since normality
$E[\vec{u}|\vec{y}]$ is linear for $\vec{y}$ $\therefore$ So $E[\vec{u}|\vec{y}]$ is BLUP.

(b) $\hat{u} = E[\vec{u}|\vec{y}] = E[\vec{u}] + cov[\vec{u}, \vec{y}) \, var[\vec{y}]^{-1}(\vec{y} - E(\vec{y})]$

$$= \Sigma_u Z' \cdot [Z\Sigma_u Z' + \sigma^2 I]^{-1}(\vec{y} - X\vec{\beta})$$

(c) $var(\hat{u}) = \Sigma_u Z' [Z\Sigma_u Z' + \sigma^2 I]^{-1} var(\vec{y} - X\vec{\beta}) \, [Z\Sigma_u Z' + \sigma^2 I]^{-1} Z \Sigma_u$.

$$= \Sigma_u Z' [Z\Sigma_u Z' + \sigma^2 I]^{-1} Z \Sigma_u .$$

(d) $var(\hat{u} - \vec{u}) \cancel{= E(\vec{u} - E[\vec{u}|\vec{y}])^2 - E\vec{u}^2}$

$= E(\hat{u} - \vec{u})(\hat{u}' - \vec{u}')$

$= var[\hat{u}] + var[\vec{u}] - E[\vec{u}\,\hat{u}'] - E[\hat{u} \cdot \vec{u}']$

$= \Sigma_u Z' [Z\Sigma_u Z' + \sigma^2 I]^{-1} Z\Sigma_u + \Sigma_u$

$- \Sigma_u Z' [Z\Sigma_u Z' + \sigma I]' Z\Sigma_u - \Sigma_u Z' [Z\Sigma_u Z' + \sigma^2 I]' Z\Sigma_u$

$= \Sigma_u - \Sigma_u Z' [Z\Sigma_u Z' + \sigma^2 I]^{-1} Z\Sigma_u$.

3. Denote $A = (KVK')^{-1} K Z_{(1)} Z'_{(1)} K'$ and $\vec{w} = K\vec{y}$

Then $E[\text{right side}] = E[\vec{w}' A (KVK')^{-1} \vec{w}] = tr[A(KVK')^{-1} E[\vec{w}\vec{w}']]$

$= tr[A(KVK')^{-1} K E(\vec{y}\vec{y}') K']$

$= tr[A(KVK')^{-1} KVK'] = tr(A)$ which is the left side.

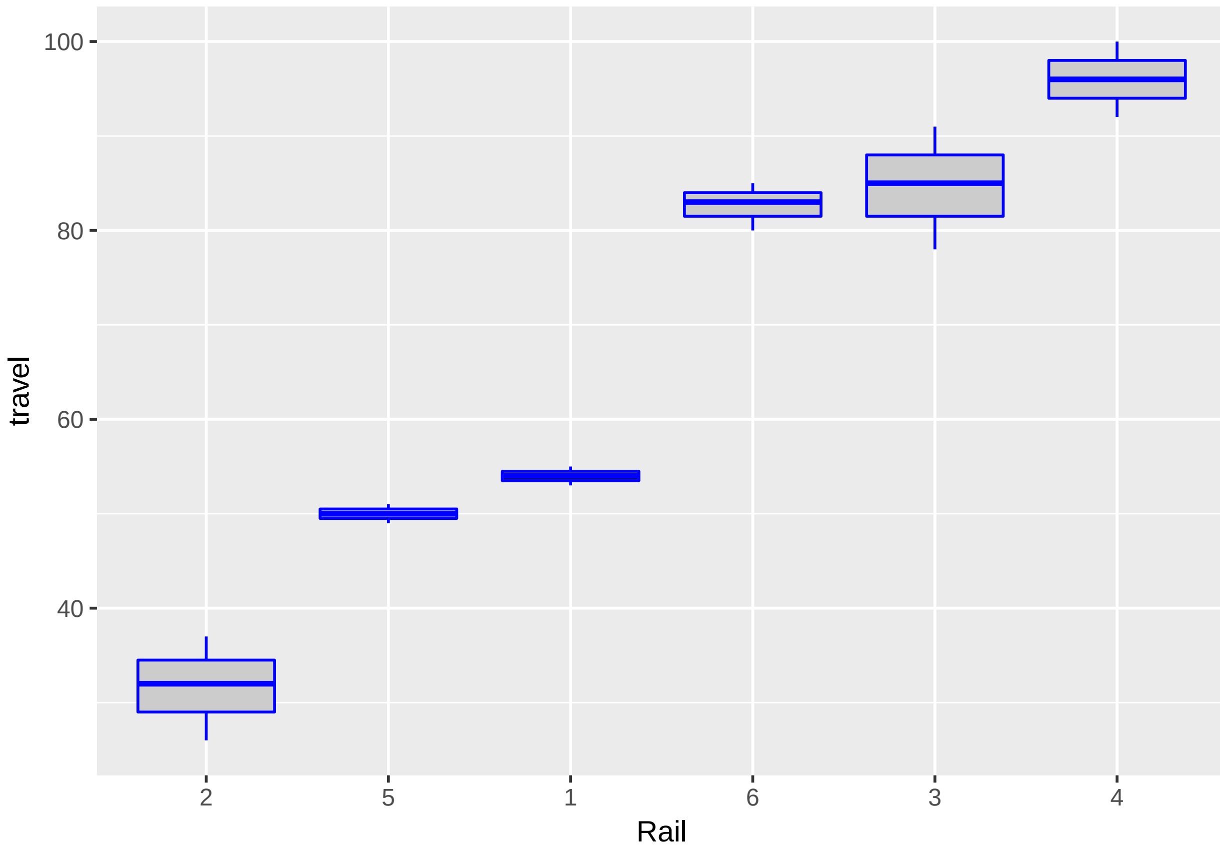
# Coding

## 1

Visualize the data.

```
library("nlme")
library("ggplot2")

Raildata = Rail
ggplot(Raildata, aes(x=Rail, y=travel)) +
    geom_boxplot(fill="grey80", color="blue")
```




(a). Here's the mean `u`, effects `effects` and test information of fixed effect model:

```
library(dplyr)
u = mean(Raildata$travel)
effects = Raildata %>%
        group_by(Rail) %>%
        summarise(means=mean(travel)-u)
print(u)
```

```
## [1] 66.5
```

```
print(effects)
```

```
## # A tibble: 6 x 2
```

```
##    Rail     means
##   <ord>     <dbl>
## 1      2 -34.83333
## 2      5 -16.50000
## 3      1 -12.50000
## 4      6  16.16667
## 5      3  18.16667
## 6      4  29.50000
```

```
summary(aov(travel~Rail, Raildata))
```

```
##             Df Sum Sq Mean Sq F value   Pr(>F)    
## Rail         5   9310  1862.1   115.2 1.03e-09 ***
## Residuals   12    194    16.2                     
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(b). Fitted model:

```
library(lme4)
Rail.mixed = lmer(travel ~ 1 + (1|Rail), Raildata)
summary(Rail.mixed)
```

```
## Linear mixed model fit by REML ['lmerMod']
## Formula: travel ~ 1 + (1 | Rail)
##    Data: Raildata
## 
## REML criterion at convergence: 122.2
## 
## Scaled residuals: 
##      Min       1Q   Median       3Q      Max 
## -1.61883 -0.28218  0.03569  0.21956  1.61438 
## 
## Random effects:
##  Groups   Name        Variance Std.Dev.
##  Rail     (Intercept) 615.31   24.805  
##  Residual              16.17    4.021  
## Number of obs: 18, groups:  Rail, 6
## 
## Fixed effects:
##             Estimate Std. Error t value
## (Intercept)    66.50      10.17   6.538
```

(c).Use the fomula like in 1.2(b) to calculate the BLUPs and compare with results in 2.1(a), we can find that BLUPs is a little shrinked towards zero.

```
sigma_u = 24.805
sigma = 4.021
u = 66.50
BLUPs = Raildata %>%
            group_by(Rail) %>%
            summarise(effects = 3*sigma_u^2/(sigma^2+3*sigma_u^2) * (mean(travel)-u))
print(BLUPs)
```

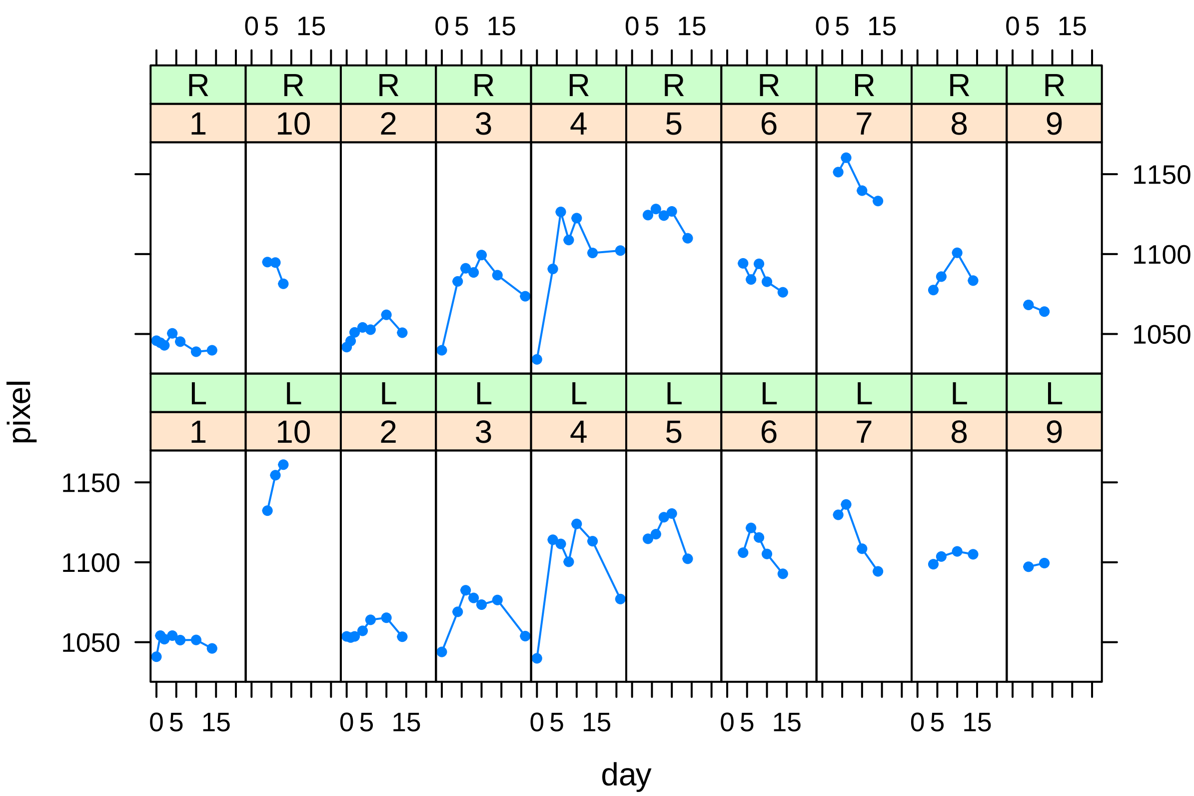
```
## # A tibble: 6 x 2
##    Rail   effects
##   <ord>     <dbl>
```

```
## 1      2 -34.53087
## 2      5 -16.35673
## 3      1 -12.39146
## 4      6  16.02629
## 5      3  18.00892
## 6      4  29.24384
```

## 2

(a). Visualize the data.

```
library(lattice)
data("Pixel")
xyplot(pixel ~ day | Dog + Side, data=Pixel, type='o', pch=20)
```




(b). We regard the two $\beta_1$ in this question as a typo, we fit three fixed effects $\beta_0, \beta_1, \beta_2$. Here's the result, it says that the estimates for $\beta_0, \beta_1, \beta_2$ are 1074.496, 4.87216 and -0.24739. And for the random effect, for `Dog` level, the standard deviation is 22.82; for `Side` within `Dog`, std is 15.70 and for the random error, the std is `12.92`.

```
D = mutate(Pixel,day2=day^2)
fit = lmer(pixel~day+day2+(1|Dog/Side), data=D)
summary(fit)
```

```
## Linear mixed model fit by REML ['lmerMod']
## Formula: pixel ~ day + day2 + (1 | Dog/Side)
##    Data: D
```

```
##
## REML criterion at convergence: 864.8
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -3.8639 -0.4810  0.0653  0.5333  1.9336
##
## Random effects:
##  Groups   Name        Variance Std.Dev.
##  Side:Dog (Intercept) 246.5    15.70
##  Dog      (Intercept) 520.8    22.82
##  Residual             166.8    12.92
## Number of obs: 102, groups:  Side:Dog, 20; Dog, 10
##
## Fixed effects:
##               Estimate Std. Error t value
## (Intercept) 1074.49600    8.77583  122.44
## day            4.87216    0.82537    5.90
## day2          -0.24739    0.04222   -5.86
##
## Correlation of Fixed Effects:
##      (Intr) day
## day  -0.353
## day2  0.294 -0.945
```