Statistical Theory Problem set 5

Bohao Tang

Question 1:

(i) Regard X_i, T_i as given data, then since the model is normal, the MLE $\hat{\beta}$ of β can be derived by OLS.

$$\Rightarrow \hat{\beta} = (D^TD)^{-1}D^TY \qquad \text{where } Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \quad D = \begin{pmatrix} 1 & T_1 & X_1 \\ \vdots & \vdots & \vdots \\ 1 & T_n & X_n \end{pmatrix}$$

$$\Rightarrow \hat{\beta} = \begin{pmatrix} n & \Sigma T_i & \Sigma X_i \\ \Sigma T_i & \Sigma T_i^2 & \Sigma T_i X_i \\ \Sigma X_i & \Sigma T_i X_i & \Sigma X_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma Y_i \\ \Sigma T_i Y_i \\ \Sigma X_i Y_i \end{pmatrix} = \begin{pmatrix} 1 & \frac{\Sigma T_i}{n} & \frac{\Sigma X_i}{n} \\ \frac{\Sigma T_i}{n} & \frac{\Sigma T_i^2}{n} & \frac{\Sigma T_i X_i}{n} \\ \frac{\Sigma X_i}{n} & \frac{\Sigma T_i X_i}{n} & \frac{\Sigma X_i^2}{n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\Sigma Y_i}{n} \\ \frac{\Sigma T_i Y_i}{n} \\ \frac{\Sigma X_i Y_i}{n} \end{pmatrix}$$

Then we do the (ii) and (iii) together, ~~notice that~~ we only assume EX and ET, EY, ETY, EXY and ETX, ET^2, EX^2 exist and finite

Then WLLN holds. Since matrix inverse is continous in its defined domain and matrix multiply and indexing are also continous

Therefore By continous mapping theory, If $\begin{pmatrix} 1 & ET & EX \\ ET & ET^2 & ETX \\ EX & ETX & EX^2 \end{pmatrix}^{-1}$ exists

Then $\hat{\beta}_2 \xrightarrow{P} \begin{pmatrix} 1 & ET & EX \\ ET & ET^2 & ETX \\ EX & ETX & EX^2 \end{pmatrix}^{-1} \begin{pmatrix} EY \\ ETY \\ EXY \end{pmatrix} [2,1]$

Since T and X are independent, $\hat{\beta}_2 \xrightarrow{P} \begin{pmatrix} 1 & ET & EX \\ ET & ET^2 & ET\,EX \\ EX & ET\,EX & EX^2 \end{pmatrix}^{-1} \begin{pmatrix} EY \\ ETY \\ EXY \end{pmatrix} [2,1]$

whenever this operation can be done.

Then we can use softuare like Mathematica ~~to~~ simply do this symbolic computation

Call in Mathematica:

$$H = \begin{pmatrix} 1 & ET & EX \\ ET & ETT & ET*EX \\ EX & ET*EX & EXX \end{pmatrix};$$

$$L = \begin{pmatrix} EY \\ ETY \\ EXY \end{pmatrix};$$

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Simplify[(Inverse[H] . L)[[2,1]]]
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Then we get $\hat{\beta}_2 \xrightarrow{P} \dfrac{ET \cdot EY - ETY}{(ET)^2 - ET^2} = \dfrac{cov(T,Y)}{var(T)}$

whenever $var\, T > 0$

We rewrite the ~~write~~ right side, since T is coin flip variable, we have

$$\frac{cov(T,Y)}{var\, T} = 4 \cdot \left\{ E\left[E[TY|X,T]\right] - \frac{1}{2} E\left[E[Y|X,T]\right] \right\}$$

$$= 4 \cdot \left[\frac{1}{2} E_X[E(Y|X,T=1)] - \frac{1}{2} E_X\left[\frac{1}{2} E[Y|X,T=1] + \frac{1}{2} E[Y|X,T=0]\right] \right]$$

$$= E_X\left[E(Y|X,T=1) - E(Y|X,T=0) \right]$$

$$= \int_{\mathbb{R}} \left[E(Y|x,T=1) - E(Y|x,T=0)\right] f(x)\, dx \qquad \text{where } f(\cdot) \text{ is density for } X$$

~~[illegible]~~ Reasonable result, therefore ~~whenev~~ whether or not $E(X_i) = 0$ we [under condition all means are finite] have

$$\hat{\beta}_2 \xrightarrow{P} \int_{\mathbb{R}} f(x) \left[E(Y|x,T=1) - E(Y|x,T=0)\right] dx$$

Question 2:

(i) use the model in question 1 suppose that we ignore the difference in empirical std, we have $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ is given by regular equation

$$\begin{cases} \sum_{i=1}^{60}(Y_i - \hat{\beta}_1 - \hat{\beta}_2 T_i - \hat{\beta}_3 X_i) = 0 \\ \sum_{i=1}^{60}(Y_i - \hat{\beta}_1 - \hat{\beta}_2 T_i - \hat{\beta}_3 X_i) T_i = 0 \\ \sum_{i=1}^{60}(Y_i - \hat{\beta}_1 - \hat{\beta}_2 T_i - \hat{\beta}_3 X_i) X_i = 0 \end{cases}$$

do the sum by group of T_i and X_i we have:

$$\begin{cases} 10.0\times12 - 12\hat{\beta}_1 + 11.5\times18 - 18\hat{\beta}_1 - 18\hat{\beta}_3 + 9.2\times14 - 14\hat{\beta}_1 - 14\hat{\beta}_2 + 10.4\times16 - 16\hat{\beta}_1 - 16\hat{\beta}_2 - 16\hat{\beta}_3 = 0 \\ 9.2\times14 + 10.4\times16 - 30\hat{\beta}_1 - 30\hat{\beta}_2 - 16\hat{\beta}_3 = 0 \\ 11.5\times18 + 10.4\times16 - 34\hat{\beta}_1 - 34\hat{\beta}_3 - 16\hat{\beta}_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 60\hat{\beta}_1 + 30\hat{\beta}_2 + 34\hat{\beta}_3 = 622.2 \\ 30\hat{\beta}_1 + 30\hat{\beta}_2 + 16\hat{\beta}_3 = 295.2 \\ 34\hat{\beta}_1 + 16\hat{\beta}_2 + 34\hat{\beta}_3 = 373.4. \end{cases}$$

Then we can solve that $\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{pmatrix} = \begin{pmatrix} 10.0916 \\ -0.97 \\ 1.3473 \end{pmatrix}$

(ii) Use bootstrap, resample ~~12, 18, 14, 16~~ $a, 30-a, b, 30-b$ samples for group (X,T): $(0,0)$ $(1,0)$ $(0,1)$ $(1,1)$

and $y_{X,T}$ is driven from $N(\bar{y}_{X,T}, \widehat{var}(y)_{X,T})$

and a, b independent $\sim$ binomial$(30, \frac{1}{2})$

the we get a new estimate for $\hat{\beta}_2$, redo the resample N times and use the standard error of these N $\hat{\beta}_2$ to be the true std of $\hat{\beta}_2$

(iii) We can also use the jackknife method:

For 60 observation, each turn we drop an observation (in order)

the droped $y_{X,T}$ value can be $\bar{y}_{X,T}$ or a sample from $N(\bar{y}_{X,T}, \widehat{var}(y|_{X,T}))$

the we use the remaining 59 observation to get a estimate of β_2, say $\hat{\beta}_2^{(i)}$

The the std can be estimate as

$$SE(\hat{\beta}_2)_{jack} = \left[\frac{60-1}{60}\sum_{i=1}^{60}\left(\hat{\beta}_2^{(i)} - \hat{\beta}_2^{(\cdot)}\right)^2\right]^{\frac{1}{2}}$$

where $\hat{\beta}_2^{(\cdot)} = \frac{1}{60}\sum_{i=1}^{60}\hat{\beta}_2^{(i)}$

(iv) We use the method in (iii) ~~both~~ with the dropped $y_{X,T} = \bar{y}_{X,T}$ ~~and and $y_{X,T} \sim N(\bar{y}_{X,T}, \widehat{var}(y_{X,T}))$~~

Then we have $SE(\hat{\beta}_2)_{jack}$

$= 0.02026$.

code is in appendix

Extra Credit

Bohao Tang

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(iv)

Prepare for the data and run jacknife.

```
Ti = c(rep(0,30), rep(1,30))
Xi = c(rep(0,12), rep(1,18), rep(0,14), rep(1,16))
Y = c(rep(10,12), rep(11.5,18), rep(9.2,14), rep(10.4,16))

D = cbind(1, Ti, Xi)

beta2 = c()
for(i in 1:60){
    L = D[-i,]
    W = Y[-i]
    be = solve( t(L) %*% L ) %*% t(L) %*% W
    beta2 = c(beta2, be[2,1])
}

SE = sqrt( 59/60 * sum( (beta2 - mean(beta2))^2 ) )
print(SE)
```

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## [1] 0.02026134
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