

## Homework 4

Before you do problems 5 and 6, read Section 4.6 of Casella & Berger.

1. Does a distribution exist for which  $M_X(t) = \frac{t}{1-t}$ ,  $|t| < 1$ ? If yes, find it; if not, explain why not.
2. Let  $X \sim \exp(1)$  and let  $Y$  be  $X + 1$  rounded down to the nearest integer.
  - (a) What is the distribution of  $Y$ ?
  - (b) What is the distribution of  $X - 4$  conditional on  $Y \geq 5$ ?
3. Let  $X$  and  $Y \sim f(x, y)$ , and let  $U = aX + b$  and  $V = cY + d$  where  $a$  and  $c$  are positive. What is  $f_{U,V}(u, v)$ ?
4. Let  $X, Y$ , and  $Z$  be independent  $\text{unif}(0, 1)$  random variables. Find  $P(\frac{XY}{Z} < t)$ .
5. Let  $X_1, \dots, X_n$  be independent  $\exp(\lambda)$  random variables. Find the distribution of  $Y = X_1 + X_2 + \dots + X_n$  by finding the joint distribution of  $Z_1 = X_1$ ,  $Z_2 = X_1 + X_2$ ,  $\dots$ ,  $Z_n = Y$  and then finding the marginal of  $Z_n$ .
6. The random vector  $X = (X_1, X_2, X_3)$  is distributed according to  $f_{\mathbf{X}}(\mathbf{x}) = \frac{2}{2e-5}x_1^2 \cdot x_2 \cdot e^{x_1x_2x_3}I\{0 < x_1, x_2, x_3 < 1\}$ . What is the marginal distribution of  $X_1$ ? What is the conditional distribution of  $X_2$  and  $X_3$  given  $X_1$ ? Are  $X_1$ ,  $X_2$ , and  $X_3$  mutually independent? Are  $X_2$  and  $X_3$  independent conditional on  $X_1$ ?

# Statistical Theory Homework 4

Bohao Tang

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1.  $M_X(t)$  is not a moment generating function of some random variable. Because for every  $X$ ,  $M_X(t) = \mathbb{E}[e^{tX}]$ , so  $M_X(0)$  will always be 1. However here  $M_X(0) = 0$ , a contradiction.
2. (a)  $Y$  is an integer valued random variable from 1 to  $\infty$ . And its pmf is

$$\mathbf{P}(Y = k) = \mathbf{P}(k - \frac{1}{2} \leq X + 1 < k + \frac{1}{2}) = \begin{cases} 1 - e^{-\frac{1}{2}} & \text{if } k = 1 \\ (e^{\frac{3}{2}} - e^{\frac{1}{2}})e^{-k} & \text{if } k > 1 \end{cases}$$

- (b) We derive the cdf of  $X - 4$  condition on  $Y \geq 5$ .

$$\mathbf{P}(X - 4 \leq t | Y \geq 5) = \frac{\mathbf{P}(X - 4 \leq t; Y \geq 5)}{\mathbf{P}(Y \geq 5)} \quad (1)$$

$$= \frac{\mathbf{P}(X - 4 \leq t; X + 1 \geq 4.5)}{\mathbf{P}(X + 1 \geq 4.5)} \quad (2)$$

$$= \frac{\mathbf{P}(3.5 \leq X \leq 4 + t)}{\mathbf{P}(X \geq 3.5)} \quad (3)$$

$$= 1 - e^{-(t+\frac{1}{2})} \quad (4)$$

where  $t > -\frac{1}{2}$ .

3. Then we have  $X = \frac{1}{a}(U - b)$ ,  $\frac{1}{c}(V - d)$  and  $|\frac{\partial(X,Y)}{\partial(U,V)}| = \frac{1}{ac}$ . Therefore

$$f_{U,V}(u, v) = \frac{1}{ac} f(\frac{1}{a}(u - b), \frac{1}{c}(v - d))$$

4. We have:

$$\mathbf{P}(\frac{XY}{Z} < t) = \iiint_{xy < tz; 0 < x, y, z < 1} 1 \, dx dy dz$$

Of course  $t$  should be bigger than 0. If  $t \leq 1$ , we have:

$$\mathbf{P}(\frac{XY}{Z} < t) = \iiint_{xy < tz; 0 < x, y, z < 1} 1 \, dx dy dz \quad (5)$$

$$= \int_0^1 dz \int_0^1 dy \int_0^{\min(1, \frac{tz}{y})} 1 \, dx \quad (6)$$

$$= \int_0^1 dz (\int_0^{tz} + \int_{tz}^1) \min(1, \frac{tz}{y}) dy \quad (7)$$

$$= \int_0^1 (tz - tz \log tz) dz \quad (8)$$

$$= \frac{3t^2}{4} - \frac{t^2}{2} \log t \quad (9)$$

And if  $t > 1$ , we have:

$$\mathbf{P}\left(\frac{XY}{Z} < t\right) = \iiint_{xy < tz; 0 < x, y, z < 1} 1 \, dx dy dz \quad (10)$$

$$= \int_0^1 dz \int_0^1 dy \int_0^{\min(1, \frac{tz}{y})} 1 \, dx \quad (11)$$

$$= \int_0^1 dz \int_0^1 \min(1, \frac{tz}{y}) dy \quad (12)$$

$$= \left(\int_0^{\frac{1}{t}} + \int_{\frac{1}{t}}^1\right) \left(\int_0^1 \min(1, \frac{tz}{y}) dy\right) dz \quad (13)$$

$$= \int_{\frac{1}{t}}^1 1 \, dz + \int_0^{\frac{1}{t}} dz \left(\int_0^{tz} + \int_{tz}^1\right) \min(1, \frac{tz}{y}) dy \quad (14)$$

$$= \left(1 - \frac{1}{t}\right) + \frac{3}{4} = \frac{7}{4} - \frac{1}{t} \quad (15)$$

Therefore:

$$\mathbf{P}\left(\frac{XY}{Z} < t\right) = \begin{cases} \frac{3t^2}{4} - \frac{t^2}{2} \log t & 0 < t \leq 1 \\ \frac{7}{4} - \frac{1}{t} & t > 1 \end{cases}$$

5. The joint density of  $X_1, X_2, \dots, X_n$  is  $f(x_1, x_2, \dots, x_n) = \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \mathbf{1}_{\{\text{all } x_i > 0\}}$ . By using the transformation as in the question, we have

$$\left| \frac{\partial(X_1, X_2, \dots, X_n)}{\partial(Z_1, Z_2, \dots, Z_n)} \right| = \left| \frac{\partial(Z_1, Z_2, \dots, Z_n)}{\partial(X_1, X_2, \dots, X_n)} \right|^{-1} = 1$$

So we have the joint density of  $Z_1, Z_2, \dots, Z_n$  is

$$f_{Z_1, Z_2, \dots, Z_n}(z_1, z_2, \dots, z_n) = \lambda^n e^{-\lambda z_n} \mathbf{1}_{\{z_1 < z_2 < \dots < z_n\}}$$

Then marginalizing on  $Z_n$ , we get:

$$f_{Z_n}(z_n) = \int_0^{z_n} dz_{n-1} \int_0^{z_{n-1}} dz_{n-2} \dots \int_0^{z_2} \lambda^n e^{-\lambda z_n} dz_1 = \lambda^n \frac{z_n^{n-1}}{(n-1)!} e^{-\lambda z_n} \mathbf{1}_{z_n > 0}$$

Therefore the pdf of  $Y$  is  $f_Y(y) = \frac{\lambda^n y^{n-1}}{(n-1)!} e^{-\lambda y} \mathbf{1}_{y > 0}$ .

6. We have:

$$f_{X_1}(x_1) = \frac{2}{2e-5} x_1 \int_0^1 dx_2 \int_0^1 x_1 x_2 e^{x_1 x_2 x_3} dx_3 \quad (16)$$

$$= \frac{2}{2e-5} \int_0^1 x_1 e^{x_1 x_2} - x_1 dx_2 = \frac{2}{2e-5} (e^{x_1} - x_1 - 1) \quad (17)$$

where  $0 < x_1 < 1$ . Then the conditional density:

$$f_{X_2, X_3}(x_2, x_3 | x_1) = \frac{x_1^2}{e^{x_1} - x_1 - 1} x_2 e^{x_1 x_2 x_3} \mathbf{1}_{0 < x_2, x_3 < 1}$$

$X_1, X_2, X_3$  are not mutually independent, because  $e^{x_1 x_2 x_3}$  can not be factorized.

Also  $X_2$  and  $X_3$  are not independent conditional on  $X_1$ , because  $e^{x_1 x_2 x_3}$  can not be factorized.