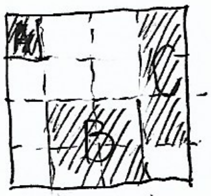
Probability Theory Homework 2

Bohao Tang

1: 

We prove that $E(X|\mathcal{V}) = \frac{1}{2}\mathbb{1}_{[0,\frac{1}{4})} + 5\,\mathbb{1}_{[\frac{1}{4},\frac{3}{4})} + 3\,\mathbb{1}_{[\frac{3}{4},1]}$

$$= \frac{1}{2}\mathbb{1}_{[0,\frac{1}{4})\times[0,1]} + 5\,\mathbb{1}_{[\frac{1}{4},\frac{3}{4})\times[0,1]} + 3\,\mathbb{1}_{[\frac{3}{4},1]\times[0,1]}$$

denote by g

Frist g is $\mathcal{V}$ measurable because g is simple function and $[0,\frac{1}{4})\times[0,1]$, $[\frac{1}{4},\frac{3}{4})\times[0,1]$ and $[\frac{3}{4},1]\times[0,1]$ all belong to $\mathcal{V}$

Second $\mathcal{V}$ obviously $= \{A\times[0,1] : A\in\mathcal{B}[0,1]\}$

Therefore $\forall\, A\times[0,1]\in\mathcal{V}$ $\displaystyle\int_{A\times[0,1]} g\,dp = \frac{1}{2}m(A\cap[0,\tfrac{1}{4})) + 5\cdot m(A\cap[\tfrac{1}{4},\tfrac{3}{4})) + 3\cdot m(A\cap[\tfrac{3}{4},1]))$

where $m(\cdot)$ is lebesgue measure on $([0,1],\mathcal{B}[0,1])$

We also have
$$\int_{A\times[0,1]} x\,dp = 2\cdot P[(A\cap[0,\tfrac{1}{4}))\times[\tfrac{3}{4},1]] + 10\,P[(A\cap[\tfrac{1}{4},\tfrac{3}{4}))\times[0,\tfrac{1}{2}]] + 4\,P((A\cap[\tfrac{3}{4},1])\times[\tfrac{1}{4},1])$$
$$= \frac{2}{4}\cdot m(A\cap[0,\tfrac{1}{4})) + \frac{10}{2}m(A\cap[\tfrac{1}{4},\tfrac{3}{4})) + 4\cdot\frac{3}{4}\cdot P(A\cap[\tfrac{3}{4},1]))$$
$$= \int_{A\times[0,1]} g\,dp \quad\Rightarrow\quad E(X|\mathcal{V}) = g$$

2: Since $\forall t\in\mathbb{R}$ $(Y-tX)^2\geq 0$ and since X, Y have finite second moment

$\Rightarrow \forall t\in\mathbb{R}$ $E[(Y-tX)^2|\mathcal{G}]\geq 0$ a.s. $\Rightarrow E(Y^2|\mathcal{G}) - 2t\,E(XY|\mathcal{G}) + t^2E(X^2|\mathcal{G})\geq 0$ a.s

use the well-known propert for quadratic function: if $at^2+bt+c\geq 0$ $\forall t$ then $a\geq 0$ $b^2-4ac\leq 0$

$\Rightarrow 4E(XY|\mathcal{G})^2 \leq 4\,E(X^2|\mathcal{G})\,E(Y^2|\mathcal{G}) \Rightarrow E(XY|\mathcal{G})^2\leq E(X^2|\mathcal{G})E(Y^2|\mathcal{G})$.