Homework 4

Before you do problems 5 and 6, read Section 4.6 of Casella & Berger.

1. Does a distribution exist for which $M_X(t) = \frac{t}{1-t}, |t| < 1$? If yes, find it; if not, explain why not.

2. Let $X \sim exp(1)$ and let Y be $X + 1$ rounded down to the nearest integer.

   (a) What is the distribution of Y?

   (b) What is the distribution of $X - 4$ conditional on $Y \geq 5$?

3. Let X and $Y \sim f(x, y)$, and let $U = aX + b$ and $V = cY + d$ where a and c are positive. What is $f_{U,V}(u, v)$?

4. Let $X, Y,$ and Z be independent $unif(0, 1)$ random variables. Find $P(\frac{XY}{Z} < t)$.

5. Let $X_1, ..., X_n$ be independent $exp(\lambda)$ random variables. Find the distribution of $Y = X_1 + X_2 + ... + X_n$ by finding the joint distribution of $Z_1 = X_1$, $Z_2 = X_1 + X_2$, ..., $Z_n = Y$ and then finding the marginal of Z_n.

6. The random vector $X = (X_1, X_2, X_3)$ is distributed according to $f_{\mathbf{X}}(\mathbf{x}) = \frac{2}{2e-5} x_1^2 \cdot x_2 \cdot e^{x_1 x_2 x_3} I\{0 < x_1, x_2, x_3 < 1\}$. What is the marginal distribution of X_1? What is the conditional distribution of X_2 and X_3 given X_1? Are X_1, X_2, and X_3 mutually independent? Are X_2 and X_3 independent conditional on X_1?

Statistical Theory Homework 4

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1. $M_X(t)$ is not a moment generating function of some random variable. Becuase for every X, $M_X(t) = \mathbb{E}[e^{tX}]$, so $M_X(0)$ will always be 1. However here $M_X(0) = 0$, a contradiction.

2. (a) Y is an integer valued random variable from 1 to ∞. And its pmf is

$$\mathbf{P}(Y = k) = \mathbf{P}(k - \frac{1}{2} \leq X + 1 < k + \frac{1}{2}) = \begin{cases} 1 - e^{-\frac{1}{2}} & \text{if } k = 1 \\ (e^{\frac{3}{2}} - e^{\frac{1}{2}})e^{-k} & \text{if } k > 1 \end{cases}$$

   (b) We derive the cdf of $X - 4$ condition on $Y \geq 5$.

$$\begin{aligned} \mathbf{P}(X - 4 \leq t | Y \geq 5) &= \frac{\mathbf{P}(X - 4 \leq t; Y \geq 5)}{\mathbf{P}(Y \geq 5)} && (1) \\ &= \frac{\mathbf{P}(X - 4 \leq t; X + 1 \geq 4.5)}{\mathbf{P}(X + 1 \geq 4.5)} && (2) \\ &= \frac{\mathbf{P}(3.5 \leq X \leq 4 + t)}{\mathbf{P}(X \geq 3.5)} && (3) \\ &= 1 - e^{-(t + \frac{1}{2})} && (4) \end{aligned}$$

   where $t > -\frac{1}{2}$.

3. Then we have $X = \frac{1}{a}(U - b)$, $\frac{1}{c}(V - d)$ and $|\frac{\partial(X,Y)}{\partial(U,V)}| = \frac{1}{ac}$. Therefore

$$f_{U,V}(u, v) = \frac{1}{ac} f(\frac{1}{a}(u - b), \frac{1}{c}(v - d))$$

4. We have:

$$\mathbf{P}(\frac{XY}{Z} < t) = \iiint_{xy < tz;\ 0 < x,y,z < 1} 1\ dxdydz$$

   Of course t should be bigger than 0. If $t \leq 1$, we have:

$$\begin{aligned} \mathbf{P}(\frac{XY}{Z} < t) &= \iiint_{xy < tz;\ 0 < x,y,z < 1} 1\ dxdydz && (5) \\ &= \int_0^1 dz \int_0^1 dy \int_0^{\min(1, \frac{tz}{y})} 1\ dx && (6) \\ &= \int_0^1 dz (\int_0^{tz} + \int_{tz}^1) \min(1, \frac{tz}{y}) dy && (7) \\ &= \int_0^1 (tz - tz \log tz) dz && (8) \\ &= \frac{3t}{4} - \frac{t}{2} \log t && (9) \end{aligned}$$

And if $t > 1$, we have:

$$\begin{aligned}
\mathbf{P}(\frac{XY}{Z} < t) &= \iiint_{xy<tz;\ 0<x,y,z<1} 1\ dxdydz && (10)\\
&= \int_0^1 dz \int_0^1 dy \int_0^{\min(1,\frac{tz}{y})} 1\ dx && (11)\\
&= \int_0^1 dz \int_0^1 \min(1, \frac{tz}{y})dy && (12)\\
&= (\int_0^{\frac{1}{t}} + \int_{\frac{1}{t}}^1)(\int_0^1 \min(1, \frac{tz}{y})dy)dz && (13)\\
&= \int_{\frac{1}{t}}^1 1\ dz + \int_0^{\frac{1}{t}} dz((\int_0^{tz} + \int_{tz}^1) \min(1, \frac{tz}{y})dy) && (14)\\
&= (1 - \frac{1}{t}) + \frac{3}{4t} = 1 - \frac{1}{4t} && (15)
\end{aligned}$$

Therefore:

$$\mathbf{P}(\frac{XY}{Z} < t) = \begin{cases} \frac{3t}{4} - \frac{t}{2}\log t & 0 < t \leq 1 \\ 1 - \frac{1}{4t} & t > 1 \end{cases}$$

5. The joint density of $X_1, X_2, \cdots, X_n$ is $f(x_1, x_2, \cdots, x_n) = \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \mathbf{1}_{\{\text{all } x_i > 0\}}$. By using the transformation as in the question, we have

$$|\frac{\partial(X_1, X_2, \cdots, X_n)}{\partial(Z_1, Z_2, \cdots, Z_n)}| = |\frac{\partial(Z_1, Z_2, \cdots, Z_n)}{\partial(X_1, X_2, \cdots, X_n)}|^{-1} = 1$$

So we have the joint density of $Z_1, Z_2, \cdots, Z_n$ is

$$f_{Z_1,Z_2,\cdots,Z_n}(z_1, z_2, \cdots, z_n) = \lambda^n e^{-\lambda z_n} \mathbf{1}_{\{z_1<z_2<\cdots<z_n\}}$$

Then marginalizing on Z_n, we get:

$$f_{Z_n}(z_n) = \int_0^{z_n} dz_{n-1} \int_0^{z_{n-1}} dz_{n-2} \cdots \int_0^{z_2} \lambda^n e^{-\lambda z_n} dz_1 = \lambda^n \frac{z_n^{n-1}}{(n-1)!} e^{-\lambda z_n} \mathbf{1}_{z_n>0}$$

Therefore the pdf of Y is $f_Y(y) = \frac{\lambda^n y^{n-1}}{(n-1)!} e^{-\lambda y} \mathbf{1}_{y>0}$.

6. We have:

$$\begin{aligned}
f_{X_1}(x_1) &= \frac{2}{2e-5} x_1 \int_0^1 dx_2 \int_0^1 x_1 x_2 e^{x_1 x_2 x_3} dx_3 && (16)\\
&= \frac{2}{2e-5} \int_0^1 x_1 e^{x_1 x_2} - x_1 dx_2 = \frac{2}{2e-5}(e^{x_1} - x_1 - 1) && (17)
\end{aligned}$$

where $0 < x_1 < 1$. Then the conditional density:

$$f_{X_2,X_3}(x_2, x_3|x_1) = \frac{x_1^2}{e^{x_1} - x_1 - 1} x_2 e^{x_1 x_2 x_3} \mathbf{1}_{0<x_2,x_3<1}$$

X_1, X_2, X_3 are not mutually independent, because $e^{x_1 x_2 x_3}$ can not be factorized.

Also X_2 and X_3 are not independent conditional on X_1, because $e^{x_1 x_2 x_3}$ can not be factorized.