Potentials in Homogeneous Media

Consider a continuous current flowing in an isotropic homogeneous medium. The current density J and the electric field E are related through Ohm's law:

$$J = \sigma E$$

Where sigma is the conductivity of the media in Siemens per meter (S/m).

The electric field is the gradient of a scalar potential

$$E = -\nabla V$$

Thus we have

$$J = -\sigma \nabla V$$

From equation 6.7 $\quad \nabla . J = 0, so$

$$\nabla . (\sigma \nabla V) = 0$$

Using equation A.21, we have

$$\nabla \sigma . \nabla V + \sigma \nabla^2 V = 0$$

If σ is constant throughout, the first term vanishes and we have Laplace's equation, i.e., the potential is harmonic:

$$\nabla^2 V = 0$$

in homogeneous isotropic media, we can express these relations, together with Ohm's law, as

$$\nabla . J = 0 \qquad \nabla . D = \in \in_0 \nabla . E = 0$$

There are two boundary conditions that must hold at any contact between two regions of different conductivity. There are boundary conditions for interfaces where a and /x change abruptly.

As in DC resistivity there are several boundary conditions for EM fields that must hold at interfaces where σ *and* μ change abruptly.

$$n \times (E_1 - E_2)=0 \qquad \text{(i)}$$

Electrical field tangential to interface is continuous

$$n \times (H_1 - H_2)=0 \qquad \text{(ii)}$$

Magnetic field tangential to interface is continuous

$$n.(\sigma_1 E_1 - \sigma_2 E_2)=0 \qquad \text{(iii)}$$

Current density normal to interface is continuous

$$n.(\mu_1 H_1 - \mu_2 H_2)=0 \qquad \text{(iv)}$$

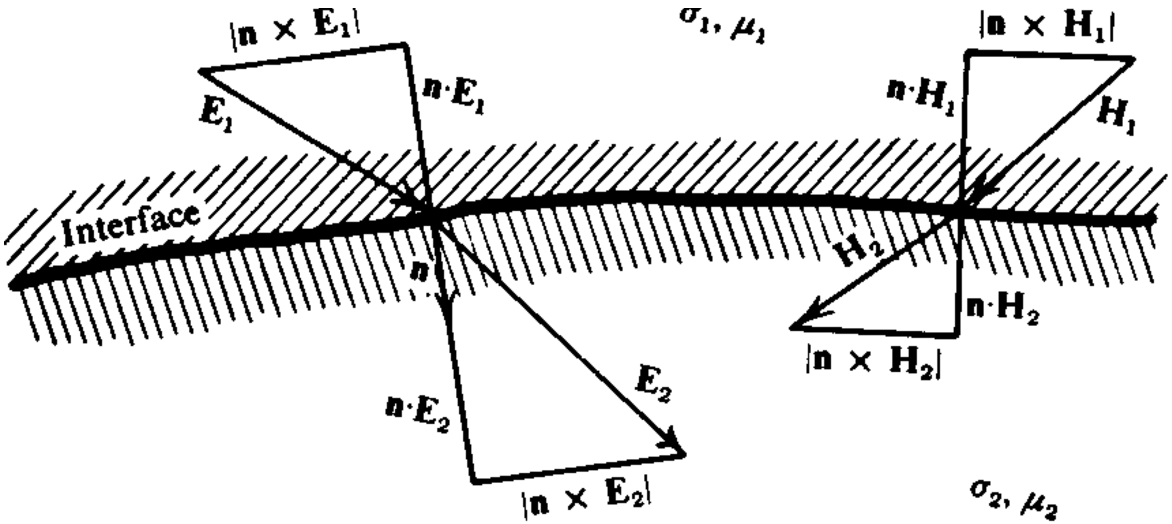
Magnetic flux normal to interface is continuous

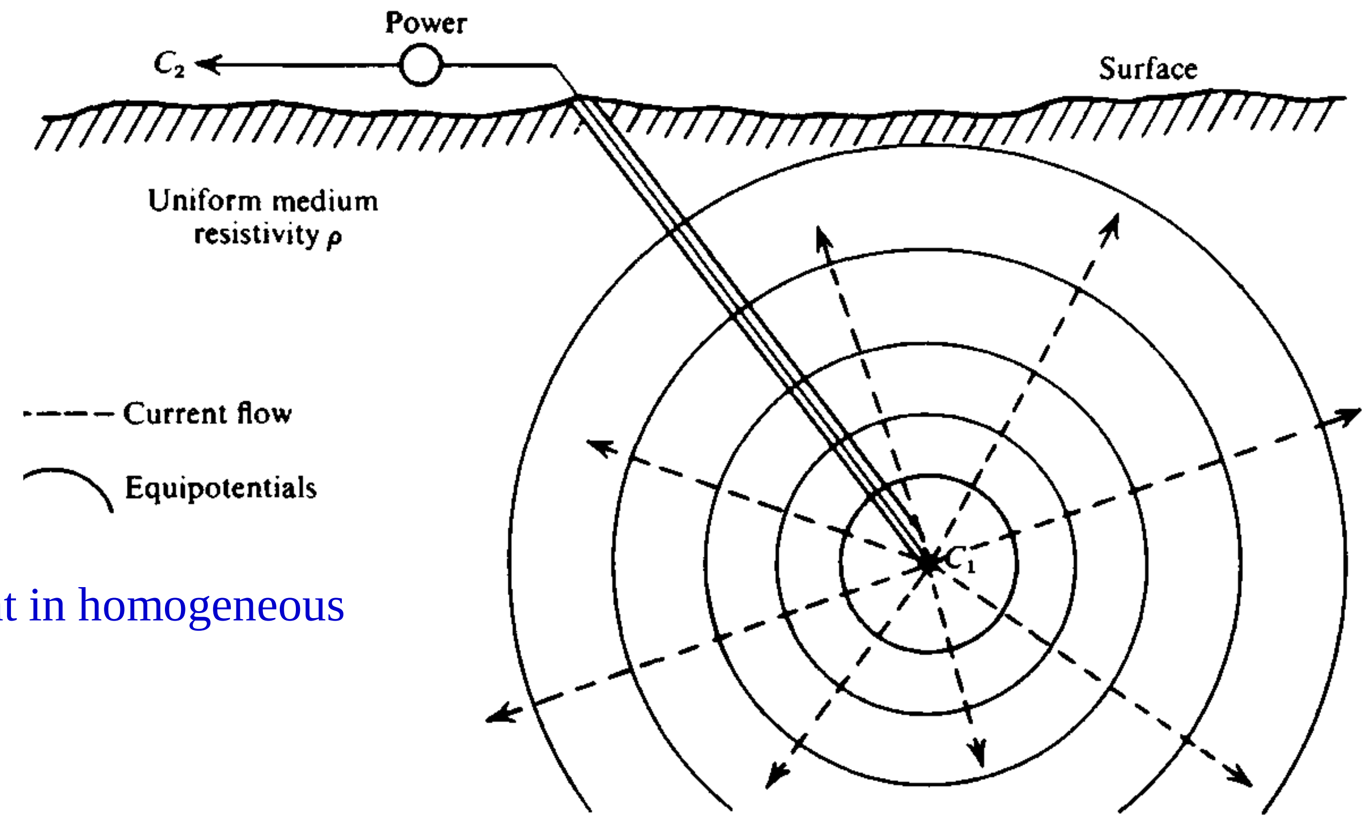
The first and third equations may be written as in the form

$$E_{x_1} = E_{x_2} \text{ and } \sigma_1 E_{z_1}=\sigma_2 E_{z_2}$$

where the *x* and *z* axes are tangential and normal, respectively, to the interface, Ex_1 being the tangential component in medium 1, and so forth. In addition,

$$V_1 = V_2$$





Buried point source of current in homogeneous ground

Single Current Electrode at Depth

There are several field configurations used in resistivity. In the first of these, we have an electrode of small dimensions buried in a homogeneous isotropic medium. This corresponds to the mise-a-la-masse method, where the single electrode is down a drill hole or otherwise under the ground. The current circuit is completed through another electrode keeping at surface. From the symmetry of the system, the potential will be a function of r only, where r is the distance from the first electrode.

Under these conditions Laplace's equation, in spherical coordinates, simplifies to

$$\nabla^2 V = \frac{d^2V}{dr^2} + (2/r)\frac{dV}{dr} = 0$$

Multiplying by r2 and integrating, we get

$$\frac{dV}{dr} = \frac{A}{r^2}$$

integrating again, we have

$$V = -\frac{A}{r} = B$$

where A and B are constants. Because V = 0 when $r \rightarrow \infty$ we get B = 0. In addition, the current flows radially outward in all directions from the point electrode. Thus the total current crossing a spherical surface is given by

$$I = 4\pi r^2 J = -4\pi r^2 \sigma \frac{dV}{dr} = -4\pi\sigma A$$

$$A = -\frac{I\rho}{4\pi}$$

hence,

$$V = \left(\frac{I\rho}{4\pi}\right)\frac{1}{r} \text{ or } \rho = \left(\frac{4\pi r V}{I}\right)$$

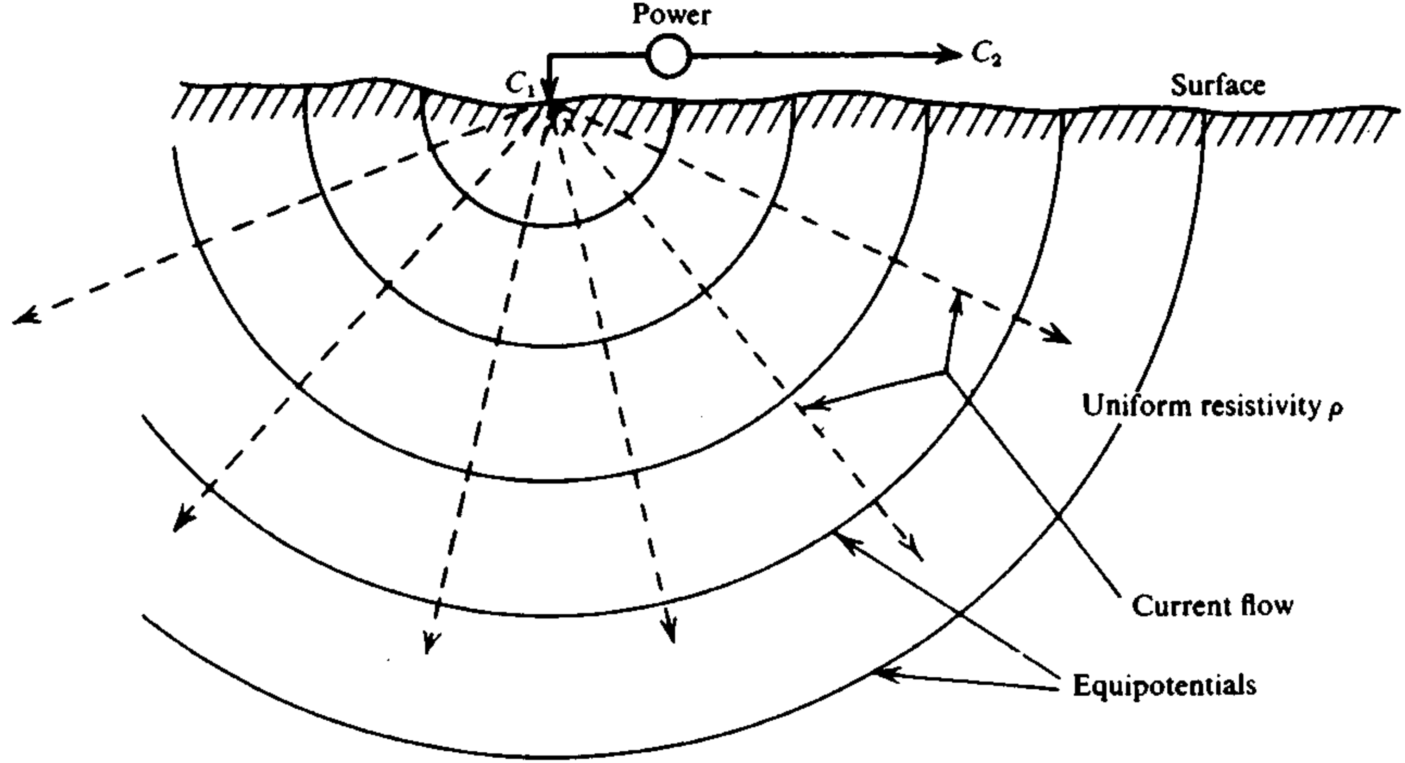
The equipotentials, which are everywhere orthogonal to the current flow lines, will be spherical surfaces given by r = constant

Single Current Electrode at Surface

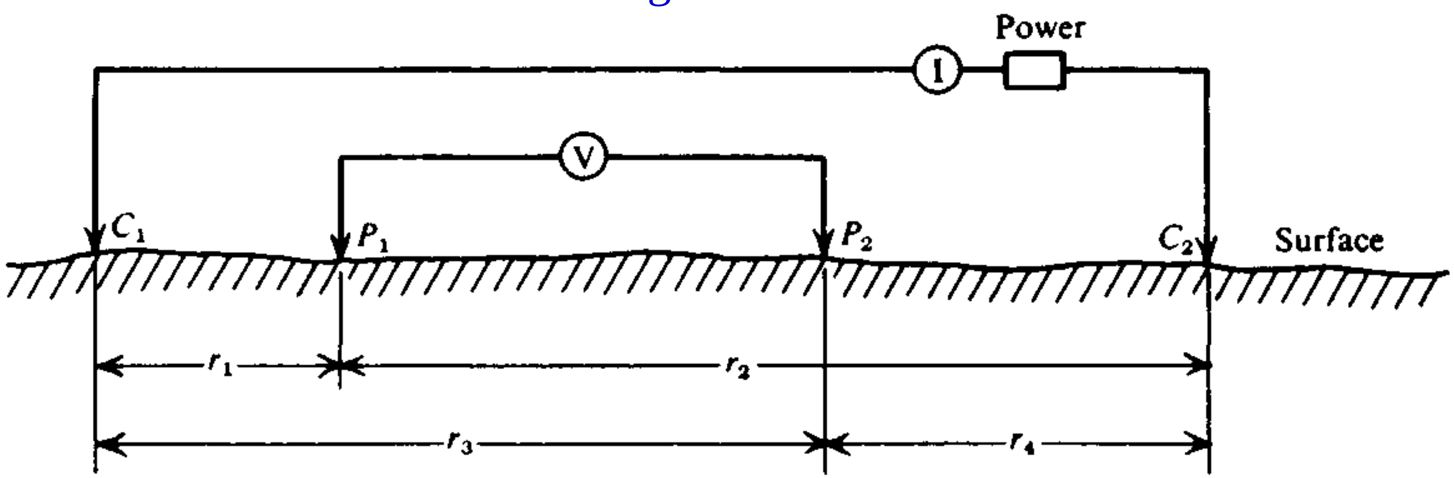
If the point electrode delivering / amperes is located at the surface of the homogeneous isotropic medium and if the air above has zero conductivity, then we have the single probe or three-point system used in surface resistivity layouts. The boundary condition at the surface requires that $E_x = {}^{\partial V}/_{\partial z} = 0 \; at \; z = 0$ (because $\sigma_{air} = 0$). This is already fulfilled because ${}^{\partial V}/_{\partial z} = {}^{d}/_{dr}\,({}^{A}/_{r})({}^{\partial r}/_{\partial z}) = {}^{Az}/_{r^3} = 0$ at z = 0.

In addition all the current now flows through a hemispherical surface in the lower medium, or

$$A = -\frac{I\rho}{2\pi}$$



Point source of current at the surface of a homogeneous medium



Two current and two potential electrodes on the surface of homogeneous isotropic ground of resistivity p.

In this case

$$V = \left(\frac{I\rho}{2\pi}\right)\frac{1}{r} \text{ or } \rho = \left(\frac{2\pi r V}{I}\right)$$

Here the equipotentials are hemispherical surfaces below the ground as shown in Figure

Two Current Electrodes at Surface

When the distance between the two current electrodes is finite as shown in Figure, the potential at any nearby surface point will be affected by both current electrodes. As before, the potential due to C1 at P1 is

$$V_1 = -\frac{A_1}{r_1} \text{ where } A_1 = -\frac{I\rho}{2\pi}$$

Because the currents at the two electrodes are equal and opposite in direction, the potential due to C2 at P1 is

$$V_2 = -\frac{A_2}{r_2} \text{ where } A_2 = -\frac{I\rho}{2\pi} = \text{-}A_1$$

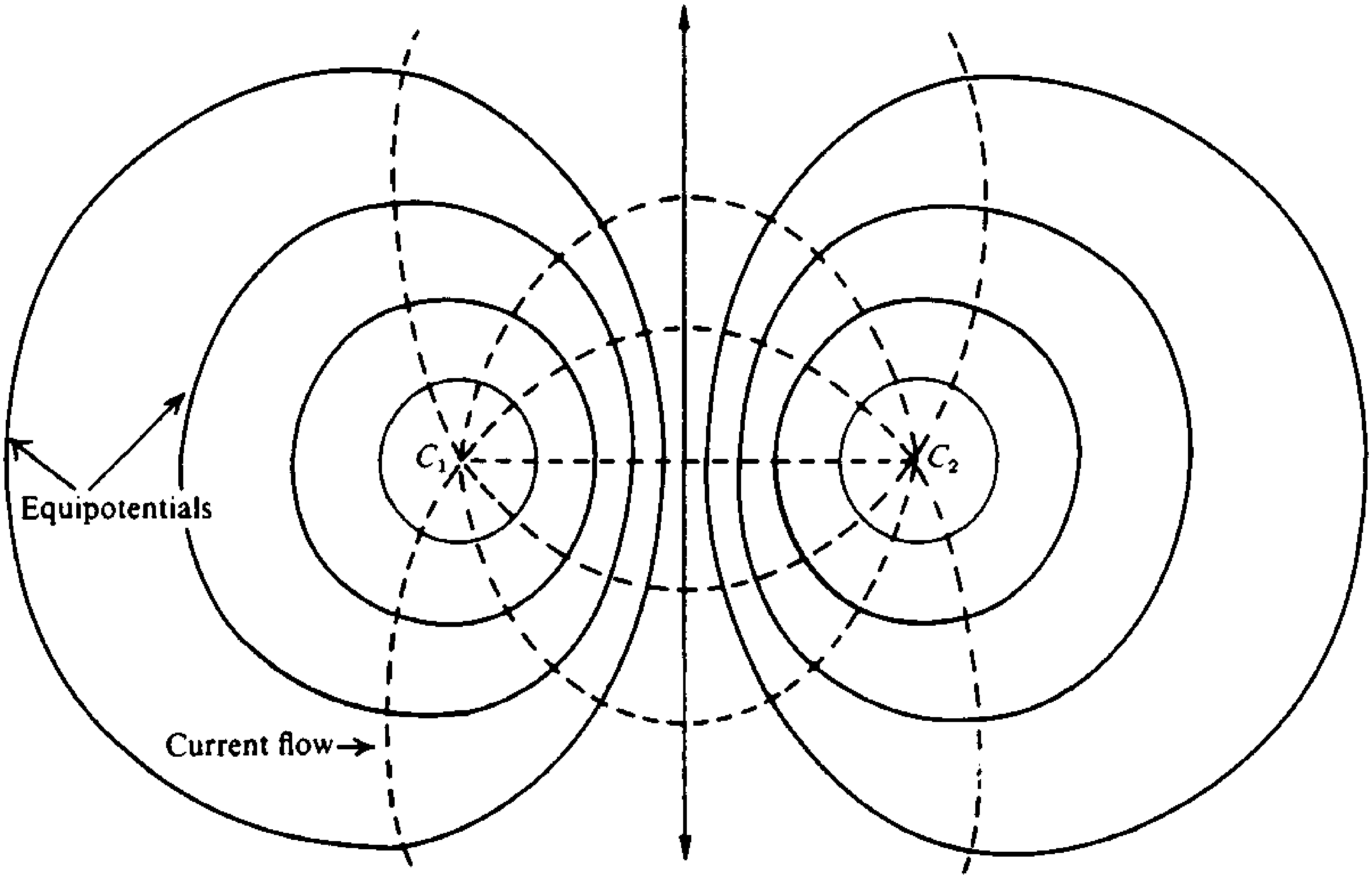
Now we have

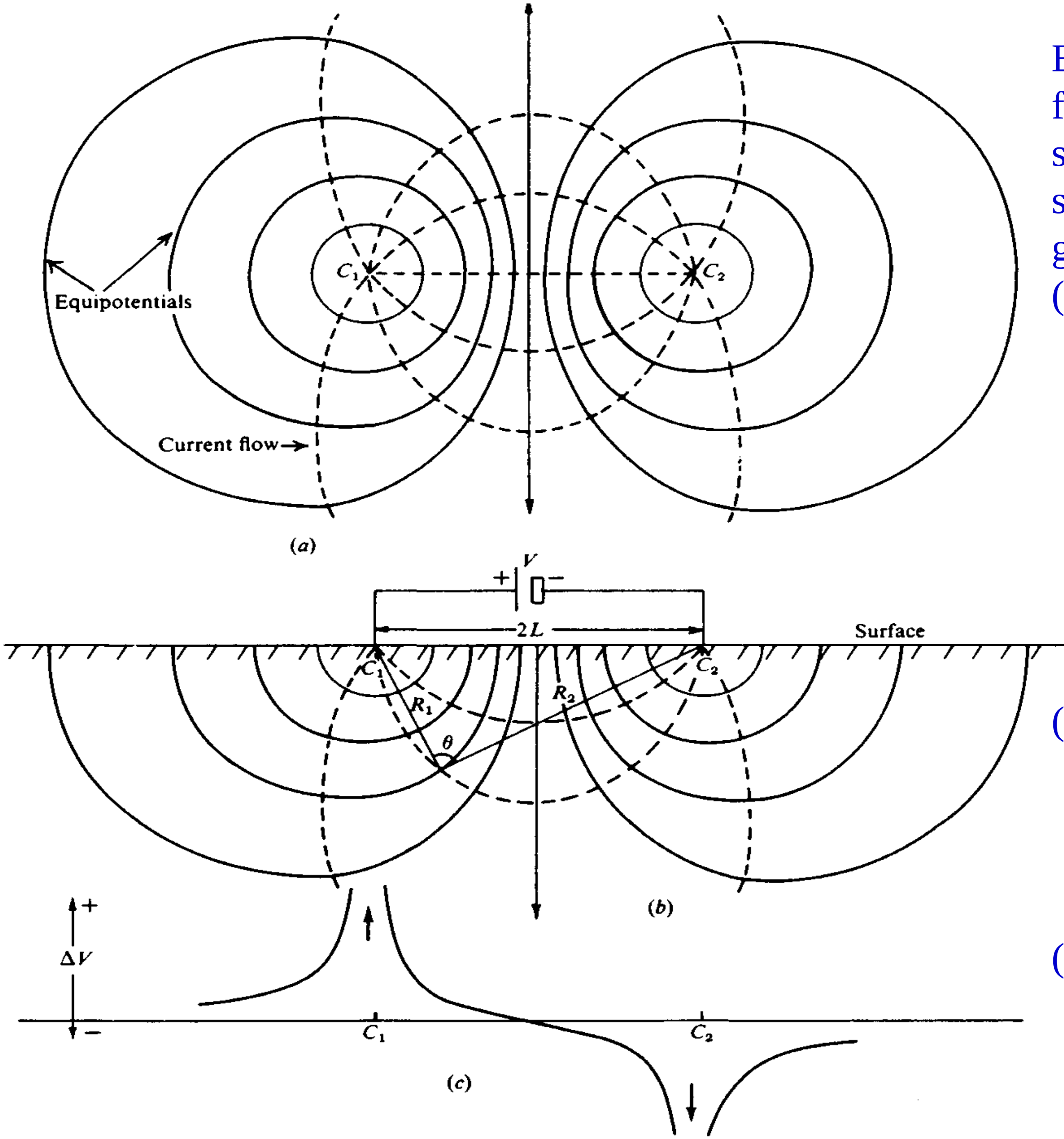
$$V_1 + V_2 = \frac{I\rho}{2\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Finally, by introducing a second potential electrode at P_2 we can measure the difference in potential between P_1 and P_2, which will be

$$\Delta V = \frac{I\rho}{2\pi}\left\{\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \left(\frac{1}{r_3} - \frac{1}{r_4}\right)\right\}$$

Such an arrangement corresponds to the four electrode spreads normally used in resistivity field work. In this configuration the current-flow lines and equipotentials are distorted by the proximity of the second current electrode C2.





Equipotentials and current flow lines for two point sources of current on surface of homogeneous ground.

(a) Plan view,

(b) Vertical section,

(c) Potential variation at the surface along a straight line through the point sources.

Current Distribution

In a general way, the flow of current in homogeneous ground show that increasing the electrode spacing increases the penetration. Consider the current flow in a homogeneous medium between two point electrodes C1 and C2 in Figure. The horizontal current density at point P is

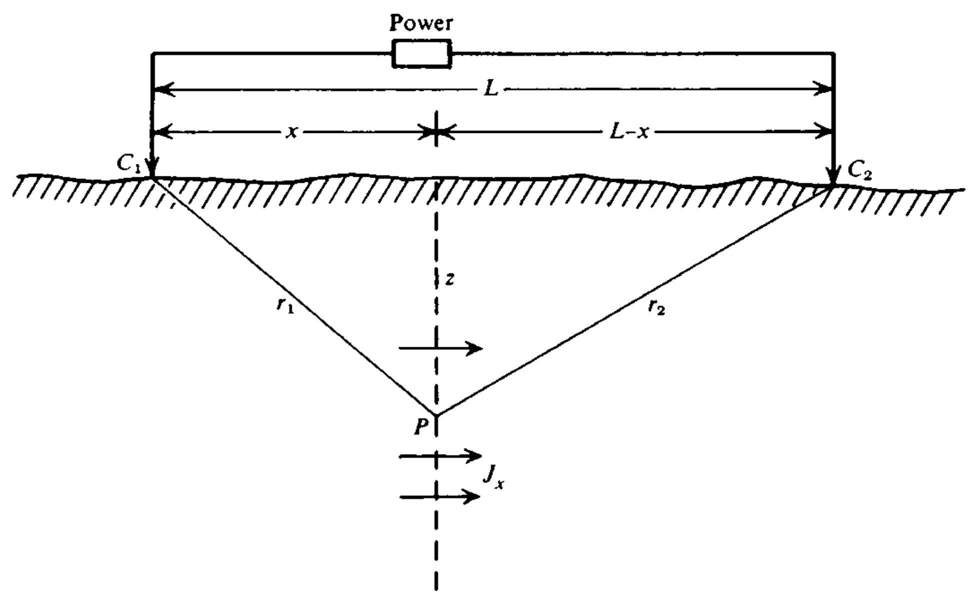
$$J_x = \left(-\frac{1}{\rho}\right)\frac{\partial V}{\partial x}$$

$$= \left(-\frac{I}{2\pi}\right)\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

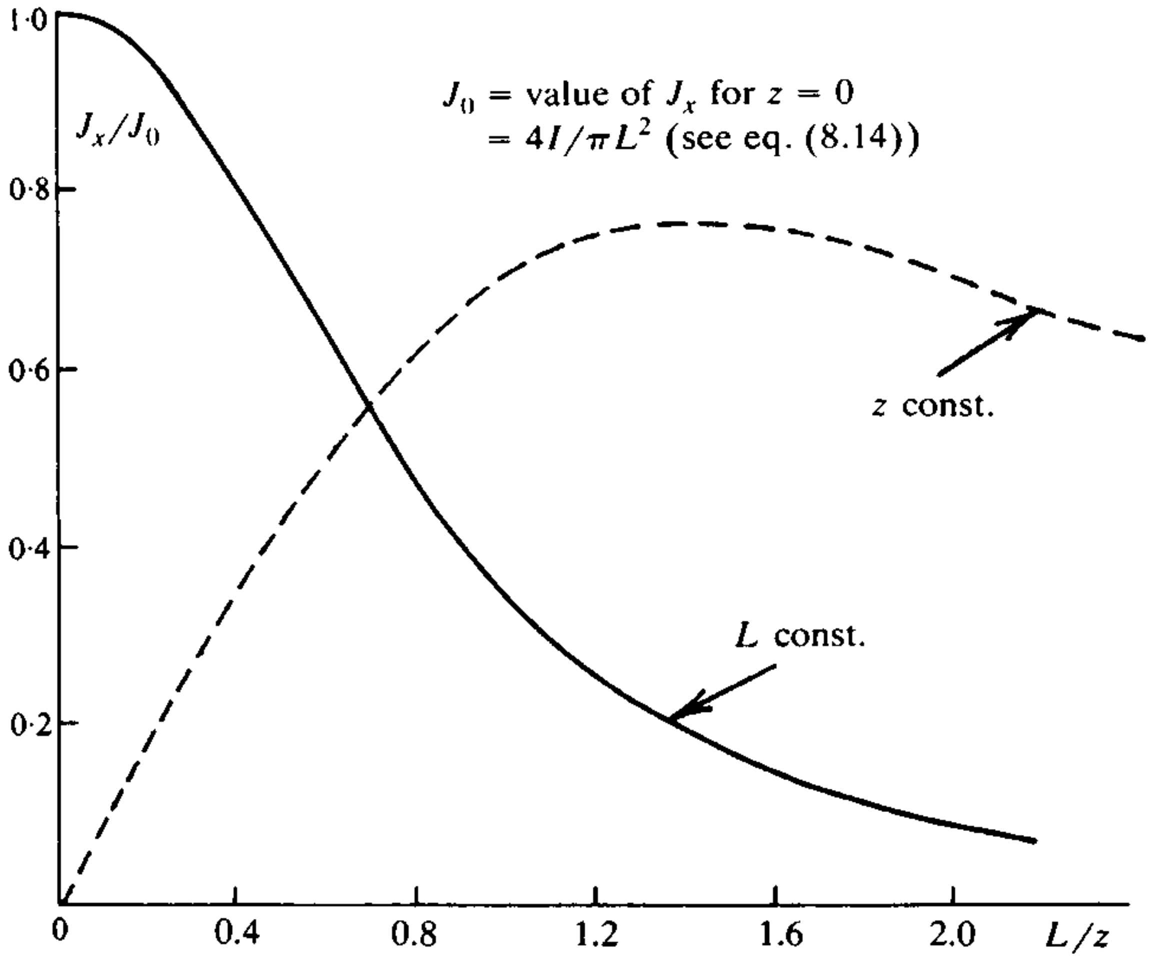
$$= \left(\frac{I}{2\pi}\right)\left\{\frac{x}{r_1^3} - (x-L)/{r^3}_2\right\}$$

$$J_x = \left(\frac{1}{2\pi}\right)\frac{L}{\left(z^2+\frac{L^2}{4}\right)^{3/2}}$$

and if this point is on the vertical plane midway between C_1 and C_2, we have $r_1 = r_2$ = r and



Determining the current density in uniform ground below two surface electrodes.



Current density versus depth (solid line) and electrode spacing (dashed line).

This Figure shows the variation in current density with depth across this plane when the electrode separation is maintained constant.

If, the electrode spacing is varied, it is found that Jx is a maximum when

$$L = \sqrt{2}z$$

The current through an element dydz of the strip is

$$\delta I_z = J_x dydz = \frac{I}{2\pi}\frac{L}{\left\{\left(\frac{L}{2}\right)^2 + (y^2 + z^2)\right\}^{3/2}}dydz$$

the fraction of total current through a long strip (z_2 — z_1) wide will be

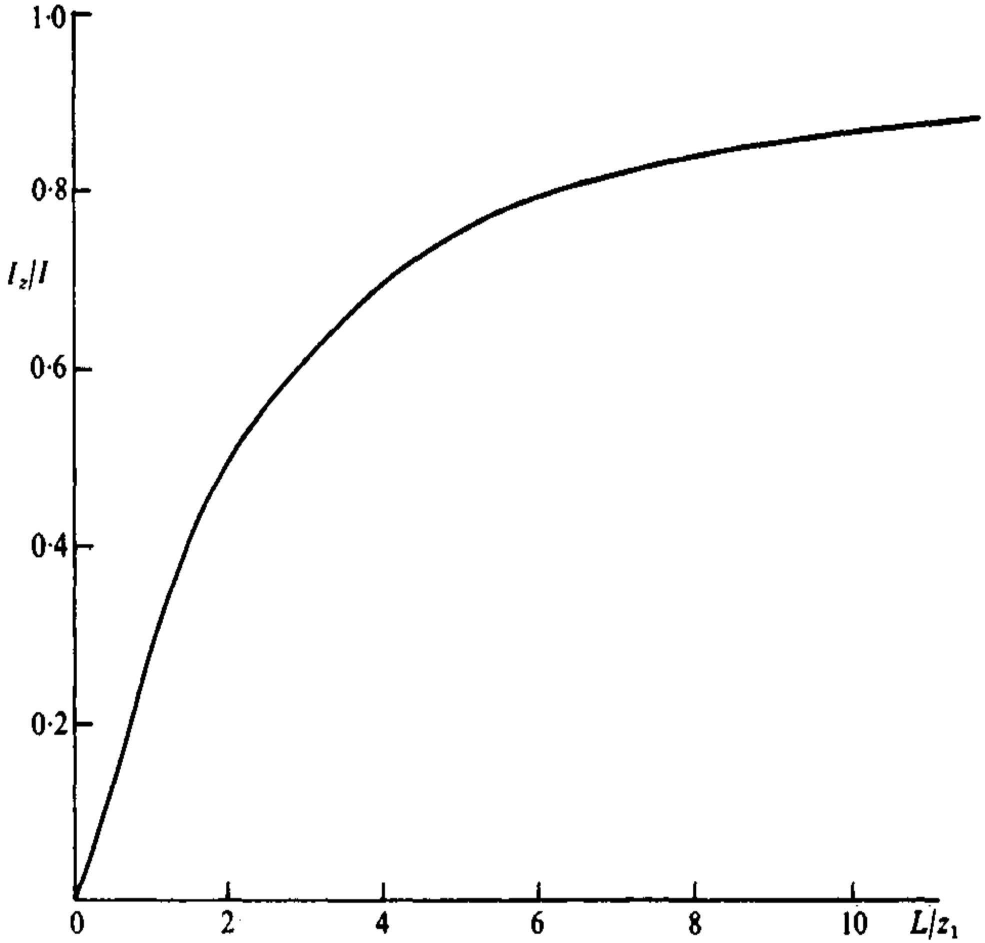
$$\frac{I_x}{I} = \frac{L}{2\pi}\int_{z_2}^{z_1} dz \int_{-\infty}^{\infty} \frac{dy}{\left\{\left(\frac{L}{2}\right)^2 + (y^2 + z^2)\right\}^{3/2}} = \frac{2}{\pi}\left(tan^{-1}\frac{2z_2}{L} - tan^{-1}\frac{2z_1}{L}\right)$$

This fraction has a broad maximum when L$= (z_1 z_2)^{1/2}$. Taking a numerical example, if z1 = 180 m, z_2 = 300 m, the electrode spacing should be 420 m to get the maximum horizontal current density in the slab, which is not significant

Otherwise, if z2 becomes infinity, the above Equation becomes

$$\frac{I_x}{I} = 1 - \frac{2}{\pi} tan^{-1}\frac{2z_1}{L}$$

Figure shows the electrode spacing necessary to force a given fraction of the current into the ground below a depth z_1 From this plot we see that, when $L = 2z_1$, half the current flows in the top layer, half below it.



Fraction of current flowing below depth z1 for an electrode spacing L.

Because the variations in potential, measured at
surface, are proportional to the current flow below, it is desirable to get as much current into the ground as possible.

For good penetration we must use large enough spacing that sufficient current reaches the target depth;

if the latter is 100 m, about one-third of the current will pass below this depth when the spacing is also 100 m.

Compared to magnetotellurics (MT), this places an inherent limitation on the resistivity method.

However the controlled power source provides certain advantages.

EFFECT OF INHOMOGENEOUS GROUND

The current flow and potential have been considered in and over homogeneous ground, which is extremely rare in the field and which would be of no practical significance anyway.

What we want to detect is the presence of anomalous conductivity in various forms, such as three-dimensional bodies, dikes, faults, and vertical or horizontal contacts between geological beds. The resistivity method is most suitable for outlining horizontal beds and vertical contacts, less useful on bodies of irregular shape.

Distortion of Current Flow at a Plane Interface

Consider two homogeneous media of resistivities ρ_1 and ρ_2 separated by a plane boundary considering a current density J_1 is flowing in medium 1 to meet the boundary at an angle θ_1 to the normal. To determine the direction of this current in medium 2, the given conditions $E_{x_1} = E_{x_2}$, $\rho_1 E_{z_1} = \rho_2 E_{z_2}$, and $V_1 = V_2$ can be recalled.

using Ohm's law to express these results in terms of the current density, we obtain

$J_{x_1}\rho_1 = J_{x_2}\rho_2$ and $J_{z_1} = J_{z_2}$

Dividing these expressions, we have

$$\frac{J_{x_1}}{J_{z_1}}\rho_1 = \frac{J_{x_2}}{J_{z_2}}\rho_2$$

Or $\rho_1 tan\theta_1 = \rho_2 tan\theta_2$

Or $\frac{tan\theta_1}{tan\theta_2} = \frac{\rho_2}{\rho_1}$

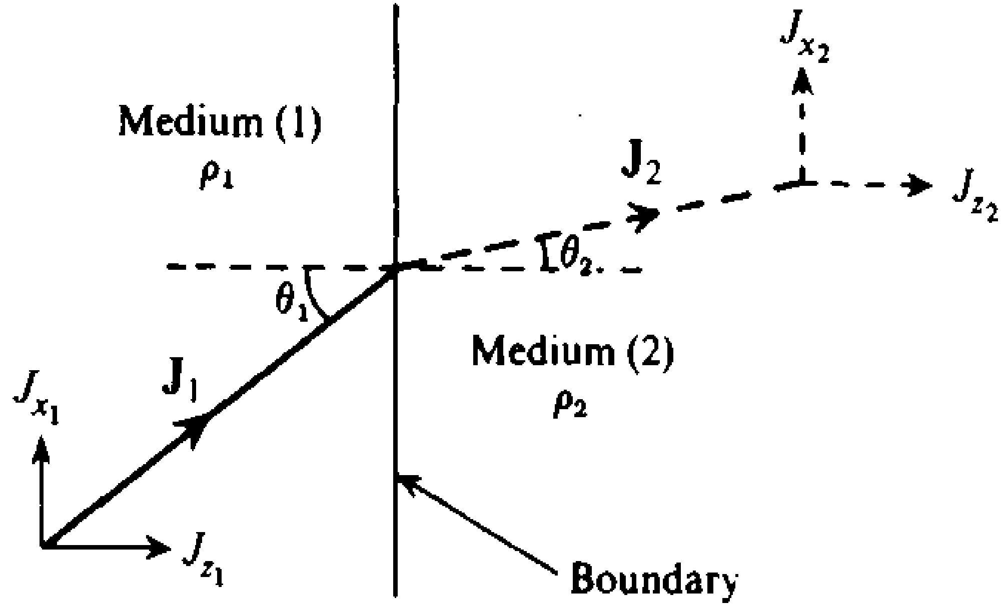


Figure: Distortion of current flow at a plane boundary when $\rho_1 < \rho_2$.

Thus the current lines are bent in crossing the boundary. If $\rho_1 < \rho_2$, they will be bent toward the normal and vice versa.

Distortion of Potential at a Plane Interface

If the current flow is distorted in passing from a medium of one resistivity into another, the equipotentials also will be distorted. It is possible to determine the potential field mathematically by solving Laplace's equation for the appropriate boundary conditions. A much simpler approach employs electrical images, in analogy with geometrical optics.

The analogy between the electrical situation and optics is based on the current density, like light ray intensity, decreases with the inverse square of distance from a point source in a medium of resistivity ρ_1 separated from an adjacent medium ρ_2 by a plane boundary.

In optics, the analogous case would be a point source of light in one medium separated from another by a semi-transparent mirror, having reflection and transmission coefficients k and 1- k. Then the light intensity at a point in the first medium is partly due to the point source and partly to its image in the second medium, the latter effect diminished by reflection from the mirror.

On the other hand, the intensity at a point in the second medium is due only to the source in the first, diminished by transmission through the mirror.

If we replace the point source of light by a point source of current and the light intensity at a point by potential, the problem is now in the electrical domain. From Figure (b) we see that the potential at P in the first medium is

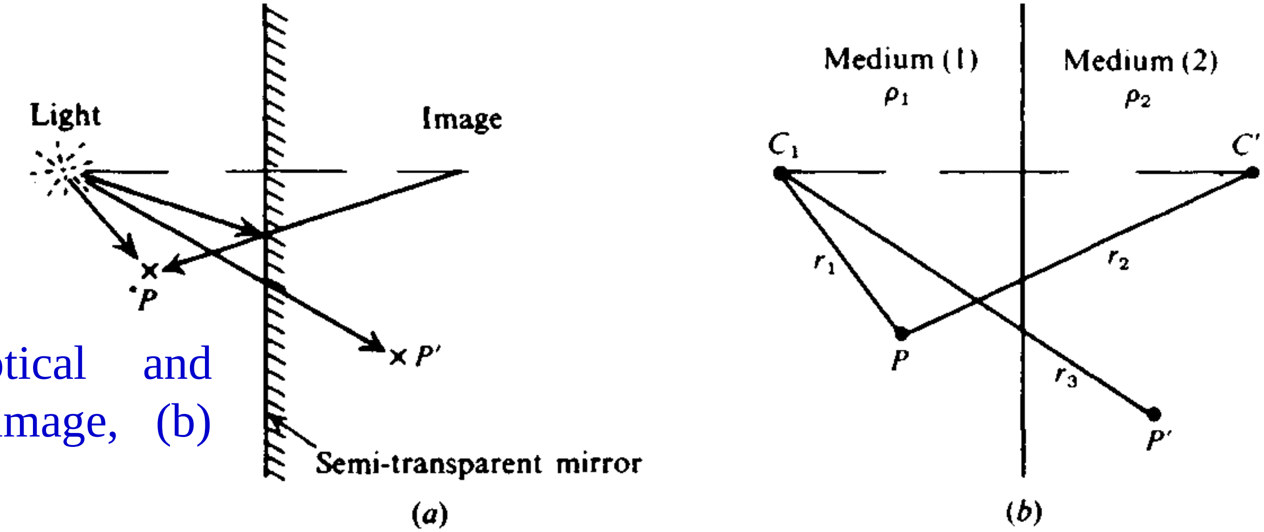


Figure: Analogy between optical and electrical images, (a) Optical image, (b) Electrical image

From Figure b, the potential at P in the first medium is

$$V = \frac{I\rho_1}{2\pi}\left(\frac{1}{r_1} + \frac{k}{r_2}\right)$$

and in the second medium at P' it is

$$V' = \frac{I\rho_2}{2\pi}\left(\frac{1-k}{r_3}\right)$$

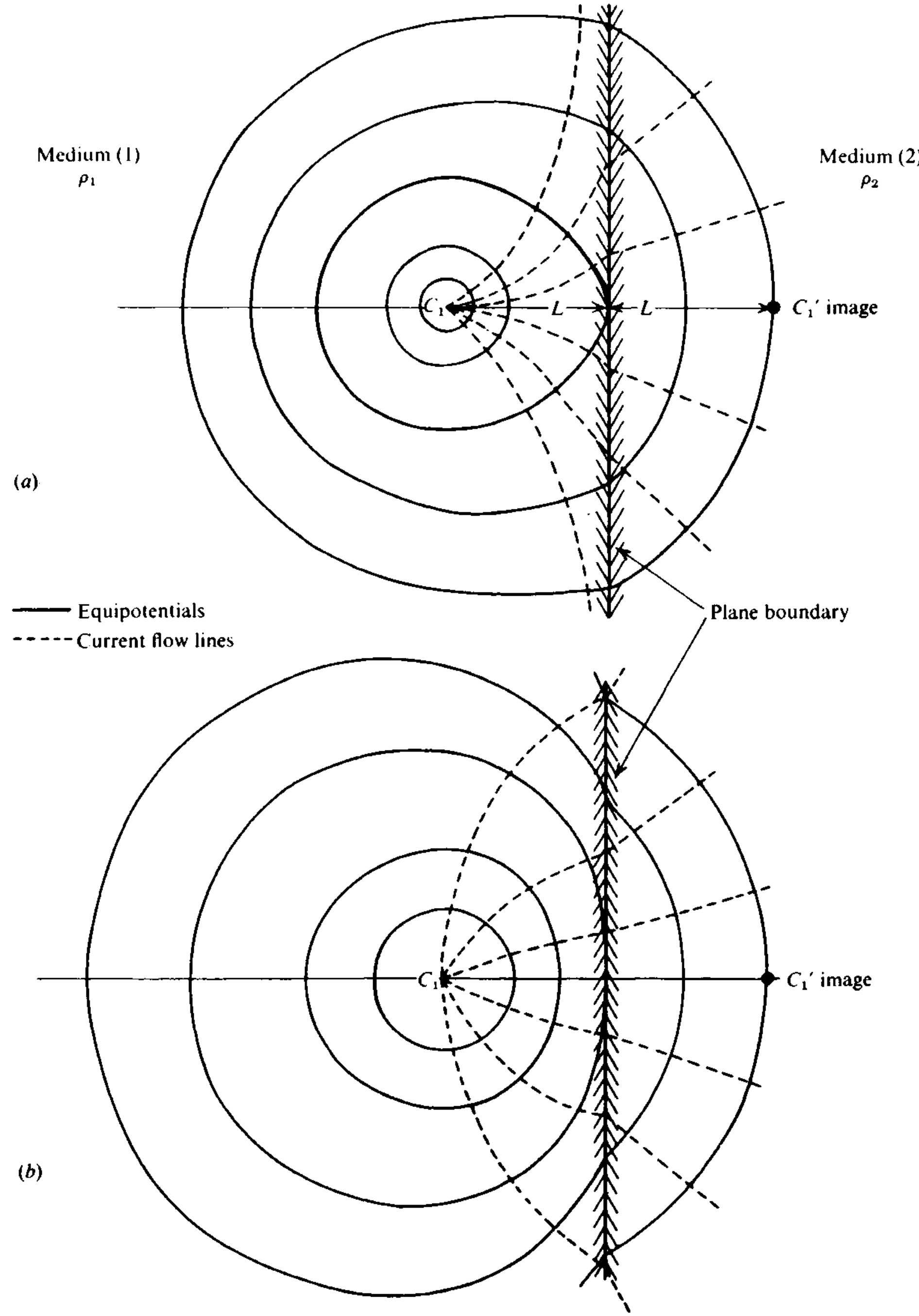
Applying the boundary condition, these potentials must be equal at the interface when r1=r2=r3

$$\frac{\rho_1}{\rho_2} = \frac{1-k}{1+k}$$

$$\text{Or } k = \frac{\rho_2-\rho_1}{\rho_2+\rho_1}$$

Where, k is a reflection coefficient, value lies between ±1, depending on the relative resistivities in the two media.

Figure. Distortion of equipotentials and current flow lines at a boundary between two media of different resistivities, (a) $\frac{\rho_2}{\rho_1}$=3, k = 0.5. (b) $\frac{\rho_2}{\rho_1}$=1/3, k = - 0.5



Surface Potential Due to Horizontal Beds

If the current source and potential point are located on the surface, above a horizontal boundary separating two media, the upper resistivity ρ_1 the lower ρ_2. Because of the ground surface there are now three media, separated by two interfaces.

As a result there is an infinite set of images above and below the current electrode, as shown in Figure. The original image C_1' at depth 2z below surface, is reflected in the surface boundary to give an image C_1'' a distance 2z above C_1 This second image, reflected in the lower boundary, produces a third C_1''' at a depth 4z, and so on.

The effect of each successive image on the potential at P is reduced by the reflection coefficient between the boundaries. For the current source and its first image below ground, the potential,

$$V' = \frac{I\rho_1}{2\pi}\left(\frac{1}{r} + \frac{k}{r_1}\right)$$

The effect of the second image at C", 2z above ground, is

$$V'' = \frac{I\rho_1}{2\pi}\left(\frac{k + k_a}{r_1}\right)$$

where k_a is the reflection coefficient at the surface boundary. Because ρ_a is infinite, this coefficient is unity, Now,

$$V' + V'' = \frac{I\rho_1}{2\pi}\left(\frac{1}{r} + \frac{2k}{r_1}\right)$$

The potential due to the third image C_1''', 4z below ground, will be further reduced, as will that of its image 4z above ground, hence

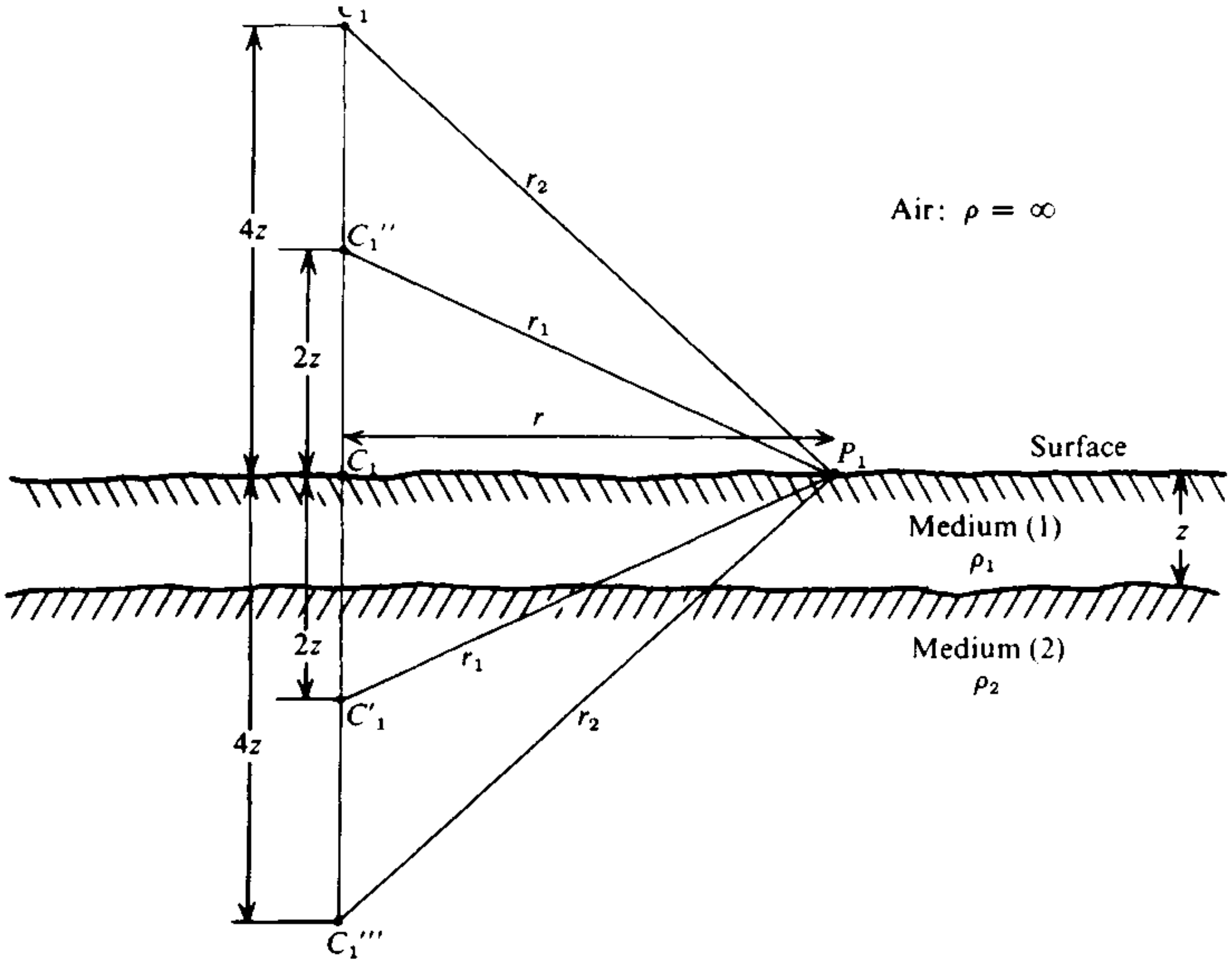
$$V''' + V^{IV} = \frac{I\rho_1}{2\pi}\left(\frac{k \times k}{r_2} + \frac{k \times k \times k_a}{r_2}\right)$$

$$= \frac{I\rho_1}{2\pi}\left(\frac{2k^2}{r_2}\right)$$

The resultant total potential at P can thus be expressed as an infinite series of the form

$$V = \frac{I\rho_1}{2\pi}\left(\frac{1}{r} + \frac{2k}{r_1} + \frac{2k^2}{r_2} + \cdots + \frac{2k^m}{r_2} + \cdots\right)$$

Figure: Images resulting from two horizontal beds



Where, $r_1 = \{r^2 + (2z)^2\}^{1/2}$

$r_2 = \{r^2 + (4z)^2\}^{1/2}$

$r_m = \{r^2 + (2mz)^2\}^{1/2}$

This series can be written in the compact form

$$V = \frac{I\rho_1}{2\pi}\left(\frac{1}{r} + 2\sum_{m=1}^{\propto}\frac{k^m}{\{r^2 + (2mz)^2\}^{1/2}}\right)$$

$$= \frac{I\rho_1}{2\pi}\left(1 + 2\sum_{m=1}^{\propto}\frac{k^m}{\left\{1+(\frac{2mz}{r})^2\right\}^{1/2}}\right) \qquad (1)$$

Potential Due to Buried Sphere

In a 3D spherical body as illustrated by Figure, in which we use spherical coordinates with the sphere center as origin and the polar axis parallel to the x axis. The problem is to find solutions of Laplace's equation for particular boundary conditions; we assume the sphere to be in a uniform field E_0 parallel to the x axis. This is equivalent to having the current electrode at considerable distance from the sphere.

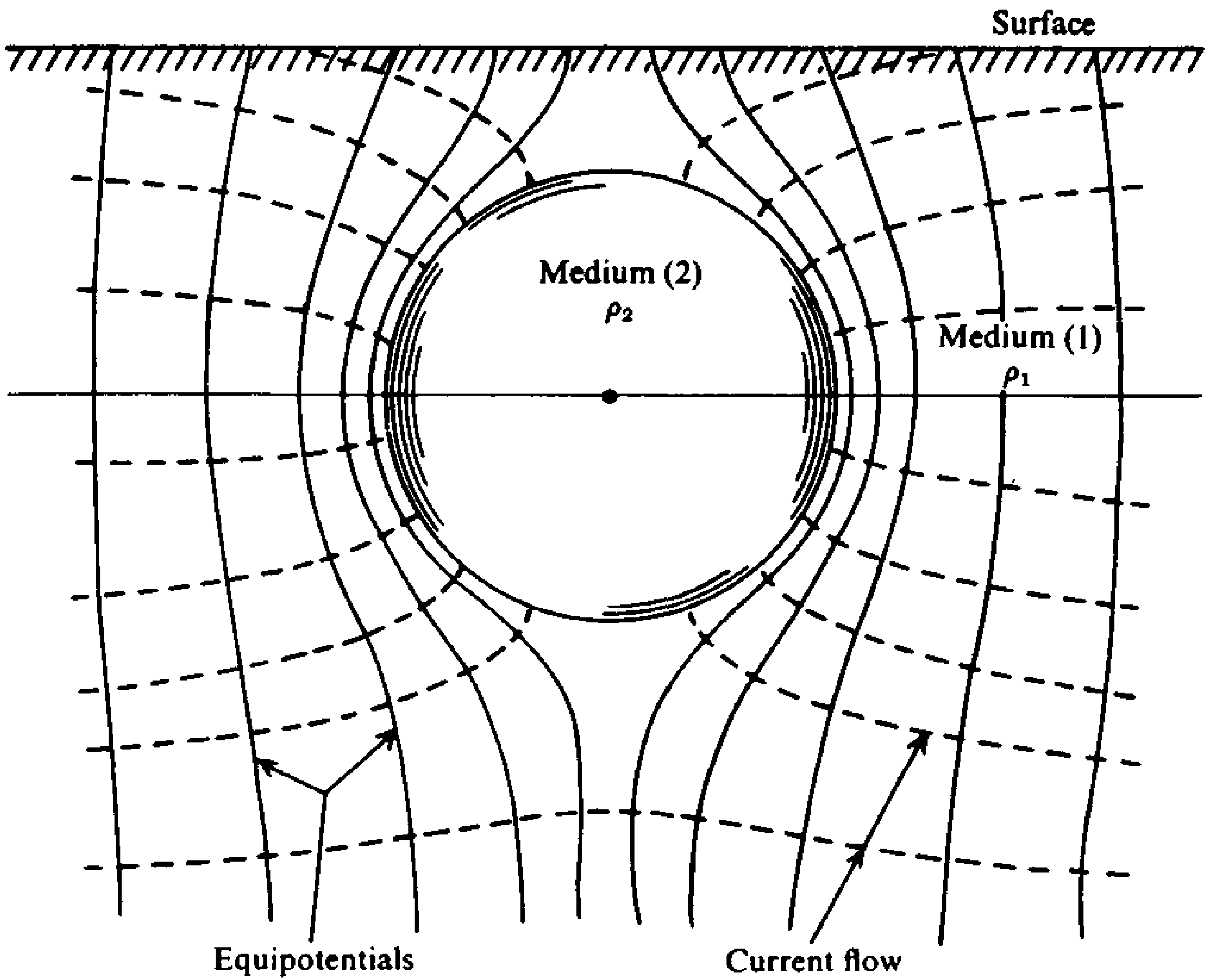


Figure: Equipotentials and current flow lines for buried conductive sphere

Using spherical coordinates and applying the boundary conditions of Equation ($E_{x_1} = E_{x_2}$, $\rho_1 E_{z_1} = \rho_2 E_{z_2}$ and $V_1 = V_2$), we can solve Laplace's equation in the form of a series of Legendre polynomials, satisfying potential relations inside and outside the sphere. For r > a,

we get

$$V_1 = -E_0 r cos\theta \left\{1 - \frac{(\rho_1 - \rho_2)}{(\rho_1 + 2\rho_2)} \left(\frac{a}{r}\right)^3\right\} \quad (2)$$

If the potential is measured at the ground surface, the sphere will have an image that will double the second term. In addition, if we consider the field to be generated by a current source C1 at a distance R from the origin, we can write

$$V_1 = \frac{I\rho_1}{2\pi R^2} \left\{1 - 2\frac{(\rho_1 - \rho_2)}{(\rho_1 + 2\rho_2)} \left(\frac{a}{r}\right)^3\right\} r cos\theta \quad (3)$$

As in Equation (8.21) we have two terms, the first being the normal potential, the second the disturbing potential caused by the sphere as shown above

When the sphere is a great distance from both the current source and surface, in which case the anomaly could not be detected anyway.

However, if the distance between the sphere's center and the surface is not less than 1.3 times the radius, the approximation is reasonably good.

Effect of Anisotropic Ground

Most rock masses are anything but homogeneous and isotropic in the electrical sense because they are full of fractures due to presence of shales, slates, and frequently limestones and schists have a definite anisotropic character with respect to the bedding planes.

In anisotropy case, consider a point source at the surface of a semi-infinite medium in which the resistivity is uniform in the horizontal direction ρ_h and has the value ρ_v in the vertical direction it is also constant and has a different magnitude ρ_v, ρ_v almost invariably being larger than ρ_h.

Proceeding as in case of '**Single Current Electrode at Surface**' with modifications to allow for the difference between horizontal and vertical directions, we find the equipotential surfaces to be ellipsoidal and symmetrical about the z axis. Mathematically this may be expressed by

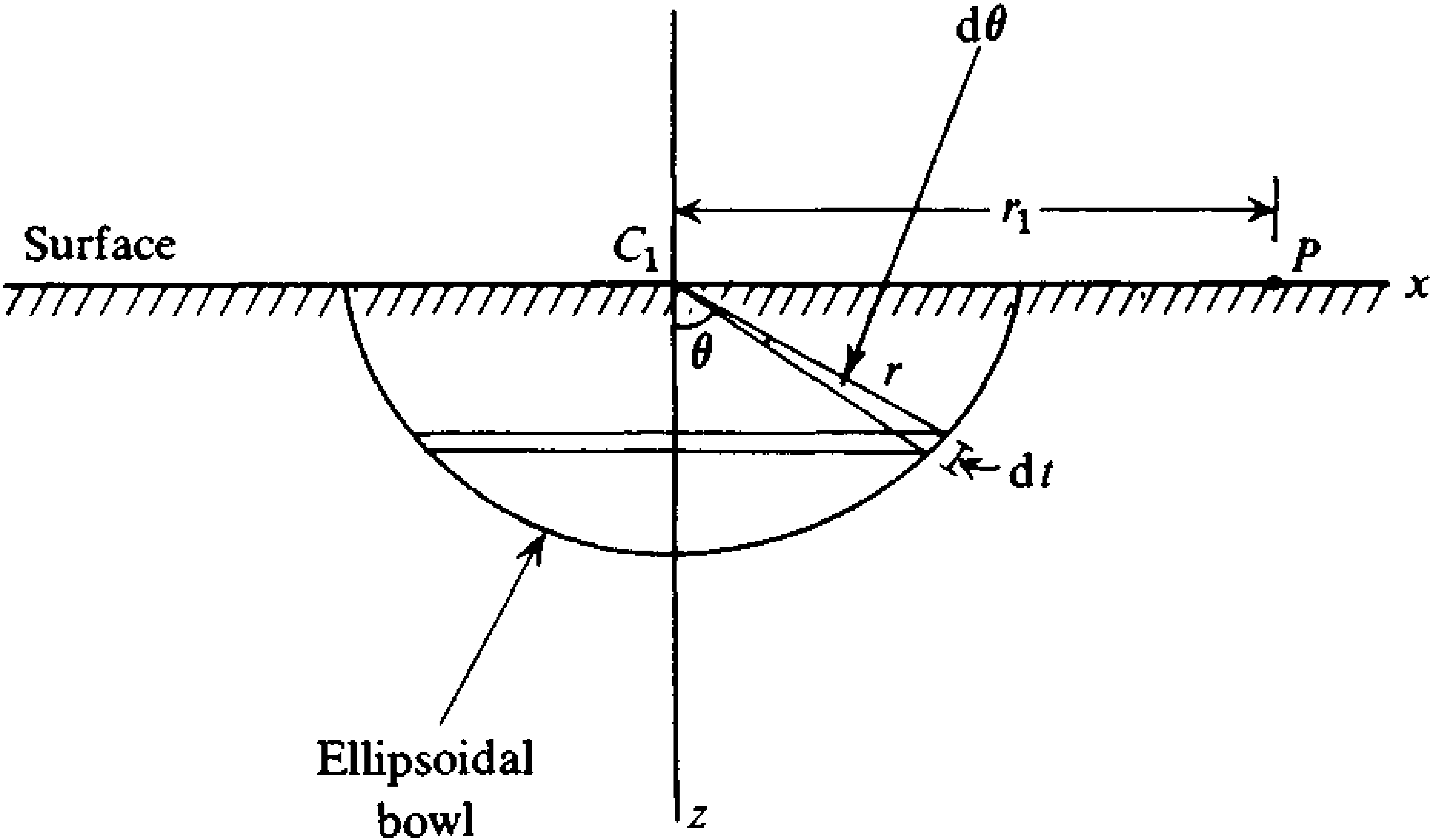


Figure: Point current source at the surface of anisotropic ground having resistivities ρ_h and ρ_v in the horizontal and vertical directions, respectively

$$V = \frac{I\rho_h \lambda}{2\pi}\left(x^2 + y^2 + \lambda^2 z^2\right) \qquad (4)$$

Where $\lambda = \left(\frac{\rho_v}{\rho_h}\right)^{1/2}$ is the coefficient of anisotropy. This relation is similar to the equation $V = \left(\frac{I\rho}{2\pi}\right)\frac{1}{r}$ or $\rho = \left(\frac{2\pi r V}{r}\right)$ with $\frac{\lambda}{\left(x^2+y^2+\lambda^2 z^2\right)}$ replacing r.

The potential at a surface point P, a distance r_1 from the current electrode C_1, will be

$$V = \frac{I\rho_h \lambda}{2\pi r_1} = -\frac{I(\rho_h \rho_v)^{1/2}}{2\pi r_1} \qquad (5)$$

i.e., the potential is equivalent to that for an isotropic medium of resistivity $(\rho_h \rho_v)^{1/2}$. Thus it is not possible to detect this type of anisotropy from field measurements.

From equation 5, the resistivity measured over horizontal beds is larger than the actual horizontal resistivity in the beds, but smaller than the vertical resistivity.

On the other hand,

If the beds have a steep dip and the measurement is made with a spread perpendicular to strike, the apparent resistivity will be smaller than the true resistivity normal to the bedding, just the opposite to the result over horizontal layers; this is known as the "paradox of anisotropy".

If the array is parallel to the strike of the dipping beds, the apparent resistivity may be too large, depending on the current-electrode separation.

Effect of Topography

As mentioned earlier, resistivity measurements are strongly influenced by local variations in surface conductivity, caused by weathering and moisture content. Rugged topography will have a similar effect, because the current flow is concentrated or focused in valleys and dispersed or diverged beneath a hill. The equipotential surfaces are distorted as a result, producing false anomalies due to the topography alone. This effect may distort or mask a real anomaly.

At the surface of homogeneous ground, resistivity is anomalously low on hills and ridges, high in valleys and 3-D depressions. Figure (a) illustrates the finite-element mesh representing a 2-D ridge, whereas Figure (b) shows the distortion of a uniform field produced by the ridge.

The terrain effect increases with surface relief, being insignificant for slopes of less than 10°. Furthermore the resistivity array complicates the effect.

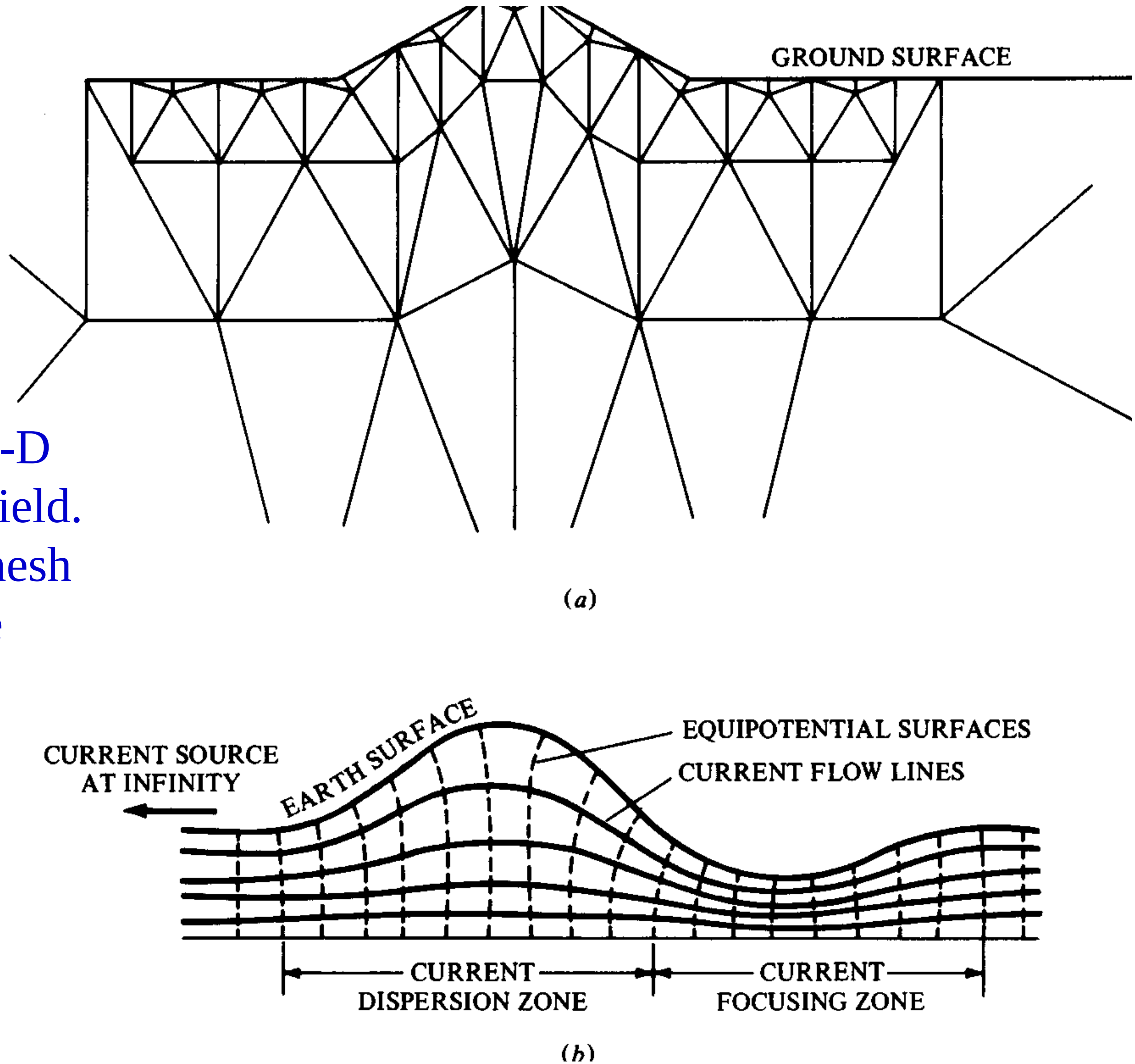


Figure: Effect of a 2-D ridge on a uniform field.

(a) Finite-element mesh used to calculate terrain effect of ridge,

(b) Distortion of uniform field by ridge.

A double-dipole system straddling a hill produces current focusing and a resistivity high, whereas a valley results in a low resistivity.

The response is also sensitive to the direction of the measuring array; for 2-D structures the anomaly is smaller if the spread is parallel, rather than normal, to strike. Analysis of the type described above allows us to reduce the field data to a flat earth by removing or at least minimizing the terrain anomaly.