RELATIONS AND FUNCTIONS

Let A and B be two non-empty sets, let n(A) = m, n(B) = n and $f: A \to B$ be the function from A to B

Each element of A can be associated to n elements of B, so total number of functions that can be formed from A to B is n × n × ... × n (m times), i.e. $\mathbf{n^m}$.

Hence **total number of functions from A to B =** $\mathbf{n^m}$

then number of relations possible from A to B is 2^{mn}

Number of relations from A to B which are
not functions is $2^{mn} - n^m$

number of 1-1 functions from A to B $= \begin{cases} {}^{n}P_{m} & \boldsymbol{if m \leq n} \\ \mathbf{0} & \boldsymbol{m > n} \end{cases}$

number of bijective functions $= \begin{cases} \boldsymbol{n}! & \boldsymbol{if m = n} \\ \mathbf{0} & \boldsymbol{if m \neq n} \end{cases}$

$\boldsymbol{if m \leq n}$ then $n^m - {}^{n}P_{m}$ functions from A to B are many-one functions.

If m > n, then all the n^m functions are many-one functions.

1. Number of relations from A to A is 2^{n^2}.
2. Number of reflexive relations from A to A is 2^{n^2-n}..
3. Number of symmetric relations from A to A is $2^{\frac{n(n+1)}{2}}$.

INVERSE TRIGONOMETRY

S.no.	FUNCTION	DOMAIN	Range (Principal Value Branch)
1	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4	$\text{cosec}^{-1} x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
5	$\sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
6	$\cot^{-1} x$	R	$(0, \pi)$

$sin^{-1}(sin\, x) = x, \;\; x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$cos^{-1}(cos\, x) = x, x \in [0, \pi]$

$sin(sin^{-1} x) = x, \quad x \in [-1, ,1]$

MATRICES & DETERMINANTS

Following are the properties of transpose of a matrix:

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(kA)^T = kA^T$

(iv) $(AB)^T = B^T A^T$

(v) $(AB)^{-1} = B^{-1} \cdot A^{-1}$ Where A and B are invertible matrices of the same order

(vi) $(A^{-1})^{-1} = A$.

A square matrix A is called a symmetric matrix if $A = A^T$

A square matrix A is called a skew symmetric matrix if

$A^T = -A$ or $-A^T = A$

All main diagonal elements of a skew-symmetric matrix are zero.

Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Square Matrix = Symmetric Matrix + Skew-symmetric Matrix.

If A is a square matrix of order n, then $A(\text{adj}A) = |A|I_n = (\text{adj}\, A)A$.

If A and B are square matrices of the same order n, then

(i) $\text{adj}(AB) = (\text{adj}\, B)(\text{adj}\, A)$

(ii) $\text{adj}\, A^T = (\text{adj}\, A)^T$

(iii) $\text{adj}(\text{adj}\, A) = |A|^{n-2}A$

(iv) $|\text{adj}\, A| = |A|^{n-1}$

(v) $|\text{adj}(\text{adj}\, A)| = |A|^{(n-1)^2}$

(vi) If A and B are two invertible matrices of the same order, then

$A^{-1} = \frac{1}{|A|}(\text{ adj}\, A)$

(vii) If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$.

(viii) A is a non-singular matrix, then $|A^{-1}| = \frac{1}{|A|}$.

(ix) $X = A^{-1}B$, if $|A| \neq 0$

(x) If $|A| = 0$ and $(\text{adj}\, A) \neq 0$, then the system is inconsistent.

(xi) $\text{adj}\,(A^{-1}) = (\text{adj}A)^{-1}$

APPLICATION OF DERIVATIVES

Continuity of a function at a point – A real valued function f(x) is said to be continuous at $x = a$ if LHL at a = RHL at a= f (a)

i.e. $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h) = f(a)$

Derivative of a function- The derivative of a function f(x) is defined by $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Left Hand Derivative (LHD) = L$f'(a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$

Right Hand Derivative (RHD) = R$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

A real valued function f(x) is said to be differentiable at x= a if its LHD and RHD at x = a exist and both are equal.

Sl. No	Function	Derivative
1	x^n	nx^{n-1}
2	K (constant)	0
3	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
4	$\sin x$	$\cos x$
5	$\cos x$	$-\sin x$
6	$\tan x$	$\sec^2 x$
7	$\sec x$	$\sec x \tan x$
8	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
9	$\cot x$	$-\operatorname{cosec}^2 x$
10	e^x	e^x
11	$\log_e x$	$\frac{1}{x}$
12	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
13	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
14	$\tan^{-1} x$	$\frac{1}{1+x^2}$
15	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
16	$\operatorname{cosec}^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$
17	$\cot^{-1} x$	$\frac{-1}{1+x^2}$
18	a^x	$a^x \log_e a$

Product rule : If u and v are two differentiable functions of x, then

$$\frac{d}{dx}[u.v] = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

Quotient rule : If u and v are two differentiable functions of x, then

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Chain rule : If y = f(u) is a function of u and u =k(x) is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Methods of Finding Local Maxima and Local Minima

First Derivative Test: Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change significantly as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called a *point of inflection*.

Second Derivative Test: Let f be a function defined on an open interval I and c $\in$ I. Let f be twice differentiable at c. Then

(i) $x = c$, is a point of local maxima if $f'(c) = 0$, and $f''(c) < 0$.
The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$, and $f''(c) > 0$.
In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$, and $f''(c) = 0$.
In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima, or a point of inflexion.

If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then f(c) is a local minimum value of f.

INTEGRATION

Indefinite Integrals: Formulae

1. $\int 1dx = x + c$

2. $\int x\, dx = x^2 + c$

3. $\int sinx\, dx = -cosx + c$

4. $\int cosxdx = sinx + c$

5. $\int tanx\, dx = \log secx + c$

6. $\int cosecx\, dx = \log|(cosecx - cotx)| + c$

7. $\int secx\, dx = \log|secx + tanx| + c$

8. $\int cotxdx = log|sinx| + c$

9. $\int sec^2xdx = tanx + c$

10. $\int cosec^2xdx = -cotx + c$

11. $\int secx.tanxdx = secx + c$

12. $\int cosecx.cotxdx = -cosecx + c$

13. $\int e^xdx = e^x + c$

14. $\int \frac{1}{x}dx = |x| + C$

15. $\int e^xdx = e^x + C$

1. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

2. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

3. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

4. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

5. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

6. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$

S.No	Form of rational function	Form of Partial fraction
1	$\frac{1}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
3	$\frac{px^2+qx+c}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4	$\frac{1}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
5	$\frac{1}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
6	$\frac{px+q}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
8	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorized further	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$

VECTORS AND 3D

- Multiple of a vector by a scalar: $\vec{a}$ is any vector and $\lambda \varepsilon$ R then $\lambda\vec{a}$ is a vector of magnitude $|\lambda|\ |\vec{a}\ |$ in a direction parallel to $\vec{a}$.
- If $|\vec{a}| \neq 0$ then, $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector in direction $\vec{a}$.
- **Scalar product**: $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ & $\vec{b}$.
- Projection of $\vec{a}$ *along* $\vec{b}$ is $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$
- $\vec{a}.\vec{a}=|\vec{a}|^2$
- Vectors $\vec{a}$ & $\vec{b}$ are perpendicular iff $\vec{a}.\vec{b} = 0$.
- **Cross product**: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{n}$,where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ & $\vec{b}$ and θ is the angle between $\vec{a}$ & $\vec{b}$.
- Unit vector perpendicular to $\vec{a}$ & $\vec{b}$ $\pm\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$
- Vectors $\vec{a}$ & $\vec{b}$ are collinear if $\vec{a} \times \vec{b} = 0$.
- Area of triangle whose sides are $\vec{a}$ & $\vec{b}$ is $\frac{1}{2}\ |\vec{a} \times \vec{b}|$.
- Area of parallelogram whose adjacent sides are $\vec{a}$ & $\vec{b}$ is $|\vec{a} \times \vec{b}|$.
- Area of parallelogram whose diagonals are $\vec{p}$ & $\vec{q}$ is $\frac{1}{2}\ |\vec{p} \times \vec{q}|$.

$If\ \vec{a} = x\hat{i} + y\hat{j} + z\hat{k},$ $then\ magnitude\ or\ length\ or\ norm\ or\ absolute\ value\ of\ \vec{a}\ is$ $\left\|\vec{a}\right\| = a = \sqrt{x^2 + y^2 + z^2}$
$A\ vector\ of\ unit\ magnitude\ is\ unit\ vector.$ $If\ \vec{a}\ is\ a\ vector\ then\ unit\ vector\ of\ \vec{a}\ is\ denoted\ by\ \hat{a}\ and\ \hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$
$Important\ unit\ vectors\ are\ \hat{i}, \hat{j}, \hat{k}, where\ \hat{i} = [1,0,0],\ \hat{j} = [0,1,0],\ \hat{k} = [0,0,1]$
$If\ l = \cos\alpha, m = \cos\beta, n = \cos\gamma,$ $then\ \alpha, \beta, \gamma, are\ called\ directional\ angles\ of\ the\ vectors\ \overrightarrow{a}\ and$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Position Vector	$\overrightarrow{OP} = \vec{r} = \sqrt{x^2 + y^2 + z^2}$
Direction Ratios	$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$
Vector Addition	$\vec{PQ} + \vec{QR} = \vec{PR}$
Properties of Vector Addition	Commutative Property: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ Associative Property: $\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$
Vector Joining Two Points	$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$
Skew Lines	$Cos\theta = \left\lvert \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \right\rvert$
Equation of a Line	$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Direction ratios of a line through (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Direction cosines of a line whose direction ratios are a, b, c are given by

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

(i) Vector equation of a line through point $\overline{a}$ and parallel to vector $\overline{b}$ is $\overline{r} = \overline{a} + \lambda\overline{b}$.

(ii) Cartesian equation of a line through point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

Angle 'θ' between lines $\overline{r} = \overline{a_1} + \lambda\overline{b_1}$ and $\overline{r} = \overline{a_2} + \mu\overline{b_2}$ is given by $\cos\theta = \frac{\overline{b_1}\cdot\overline{b_2}}{|\overline{b_1}||\overline{b_2}|}$.

Two lines are perpendicular to each other if

$\overline{b_1}\cdot\overline{b_2} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Cartesian equation of line through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Shortest distance between two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ *and* $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ **is given by**
S.D $= \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$ **. Note S.D = 0 implies lines are intersecting (Coplanar)**

Distance between two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ *and* $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is given by $\left|\frac{(\vec{a}_2-\vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$

Note: Before applying formula for finding distance, identify whether lines are parallel or not by taking ratio of direction vectors, then apply formula appropriately.

Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar iff

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

DIFFERENTIAL EQUATION

- **Homogeneous differential equations**

Steps involved: If a first order first degree differential equation is expressible in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where $f(x, y)$ and $g(x, y)$ are homogenous functions of the same degree, then it is called a homogenous differential equation. Such type of equations can be reduced to variable separable form by the substitution.

$y = vx$ or $x = vy$.

- **Linear differential equations**

Steps involved:

If a differential equation is expressible in the form $\frac{dy}{dx} + py = Q$ where P and Q are functions of x, then it is called a linear differential equation. The solution of this equation is given by $y\left(e^{\int Pdx}\right) = \int \left(Qe^{\int Pdx}\right)dx + C$

Sometimes a linear differential equation is in the form $\frac{dx}{dy} + Rx = S$, where R and S are functions of y. The solution of this equation is given by

$$x\left(e^{\int \mathrm{R}dy}\right) = \int \left(Q^{e^{\int \mathrm{R}dy}}\right)dy + C$$

PROBABILITY

$$P(A) = 1 - P(A')$$

multiplication rules (joint probability)

dependent $P(A \cap B) = P(A) * P(B|A)$

independent $P(A \cap B) = P(A) * P(B)$

mutually exclusive $P(A \cap B) = 0$

addition rules (union of events)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

mutually exclusive $P(A \cup B) = P(A) + P(B)$

conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$