CHAPTER 3

Infinite Series

Most of the functions which are encountered in mathematical applications can be represented in terms of an infinite series. There we need to study their convergence or divergence because unless a series employed in an investigation is convergent, it may lead to absurd conclusions. This makes the concept of convergence and divergence of an infinite series of vital importance to the students of engineering and technology.

3.1 SEQUENCES AND SERIES

We first introduce the concept of sequences which is a prerequisite for the study of infinite series. After this, we will discuss infinite series and their convergence and divergence.

3.1.1 Sequences

A *sequence* is an ordered set of numbers $u_1, u_2, u_3, \ldots$, and is denoted by $\{u_n\}$; here u_n is called the *nth* term, or the *general term* of the sequence $\{u_n\}$. If the number of terms are infinite, then the sequence is said to be an *infinite sequence*. If the number of terms are finite, it is said to be a *finite sequence*.

For example, the set of numbers 2, 4, 8, 10, ..., $2n$, ... is an infinite sequence and is represented by $\{2n\}$. Here $u_n = 2n$ is the *nth term*, or the *general term*.

Similarly, $\{(-1)^n\}$ represents the sequence $-1, 1, -1, 1, -1, \ldots$ with $u_n = (-1)^n$ as its general term.

Limit of a sequence: A number l is called the *limit* of an infinite sequence $u_1, u_2, u_3, \ldots$ if, for given $\epsilon > 0$, no matter how small, we can find a positive integer N, depending upon ϵ, such that $|u_n - l| < \epsilon$ for all $n \geq N$. In such a case we write $\lim_{n\to\infty} u_n = l$, or simply $\{u_n\} \to l$, as $n \to \infty$.

Convergence or divergence of a sequence: If the limit of a sequence exists the sequence is called *convergent*, otherwise, it is called *divergent*, or *oscillating*. For example, the sequence $\left\{\frac{1}{2n-1}\right\}$ is convergent but the sequence $\{2n\}$ is divergent. For any convergent sequence, the limit is always unique. The sequence $\{(-1)^{n-1}\}$ is not convergent, it is oscillating. It oscillates between -1 and $+1$.

Bounded sequence: If $u_n \leq M$ for $n = 1, 2, 3, \ldots$, where M is a constant (not depending on n) we say that the sequence $\{u_n\}$ is *bounded above* and M is called an *upper bound* of $\{u_n\}$. If $u_n \geq m$ for $n = 1, 2, 3, \ldots$, where m is a constant, then we say that the sequence $\{u_n\}$ is *bounded below* and m is called a *lower bound* of $\{u_n\}$.

If $m \leq u_n \leq M$ for all $n \in N$, the sequence $\{u_n\}$ is called *bounded*. We have an important result to state that *every convergent sequence is bounded but its converse is not necessarily true.*

For example the sequence $\{(-1)^n\}$ is bounded but not convergent.

Monotonic sequence: If $u_{n+1} \geq u_n$, the sequence $\{u_n\}$ is said to be *monotonically increasing* and if $u_{n+1} \leq u_n$, the sequence is called *monotonically decreasing*. Both increasing and decreasing sequences are called *monotonic sequences.*

A monotonic sequence always tends to a definite limit, finite or infinite. Thus *a sequence which is monotonic and bounded is always convergent.*

3.1.2 Series

If $u_1, u_2, u_3, \ldots, u_n, \ldots$ is an infinite sequence of real numbers, then

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is called an *infinite series*. It is denoted by $\sum_{n=1}^{\infty} u_n$, or $\sum_{n} u_n$, or simply Σu_n. The sum of its first n terms, denoted by $S_n = u_1 + u_2 + \ldots + u_n$, is called the *nth partial sum* of Σu_n. The sequence $\{S_n\}$ is called the *Sequence of partial sums* of the infinite series Σu_n. Thus to every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

Convergence, divergence or oscillation of a series: The convergence, divergence or oscillation of an infinite series Σu_n is defined according to the sequence $\{S_n\}$ of its partial sum converges, diverges or oscillates.

Consider an infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots$ and let $S_n = u_1 + u_2 + \ldots + u_n$ be the sum of its first n terms. Three cases arise:

(a) If S_n tends to a unique finite limit as $n \to \infty$, then the series Σu_n is said to be *convergent.*

(b) If S_n tends to an infinite limit as $n \to \infty$, then the series Σu_n is said to be *divergent.*

(c) If S_n does not tend to a unique limit, finite or infinite as $n \to \infty$, then the series Σu_n is said to be *oscillatory.*

We must note the following in context with the convergence or divergence of an infinite series.

(a) The nature of an infinite series does not change by the addition or removal of a finite number of terms.

(b) The convergence or divergence of an infinite series remains unaffected by multiplying or dividing each term by a finite non-zero constant.

(c) If two series Σu_n and Σv_n are convergent, then the series $\Sigma (u_n + v_n)$ is also convergent, but the result may not hold otherwise.

Example 3.1: Examine the convergence of the series $1 + 3 + 5 + 7 + \ldots \infty$.

Solution: Here, $S_n = 1 + 3 + 5 + 7 + \ldots + (2n - 1) = \frac{n}{2}\{2 + (n-1)2\} = n^2$.

Consider $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n^2 \to \infty$. Hence the given series is *divergent.*

Example 3.2: Test the convergence of the series $6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + \ldots \infty$.

Solution: Here $S_n = 6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + \ldots = 0, 6, -4$ according as n is of the form $3m, 3m+1, 3m+2$ for any positive integer m. Clearly in this case S_n does not tend to a unique limit. Hence, the given series is *oscillatory*.

Example 3.3: Test the convergence of the series $1^2 + 2^2 + 3^2 + \ldots + n^2 \ldots$

Solution: Here $S_n = 1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Consider $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{n(n+1)(2n+1)}{6} \to \infty$. Therefore, the given series is *divergent.*

Example 3.4: Discuss the convergence of the geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$

Solution: Here, $S_n = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = a\dfrac{(1-r^n)}{1-r}$, provided $r \neq 1$.

We have the following cases.

Case I: $|r| < 1$, then $r^n \to 0$ as $n \to \infty$.

Therefore, $\lim_{n\to\infty} S_n = \dfrac{a}{1-r}$, which is finite and unique, thus the given series *converges* to $\dfrac{a}{1-r}$.

Case II: $|r| > 1$, that is, $r > 1$, or $r < -1$.

When $r > 1$, then $r^n \to \infty$ as $n \to \infty$.

Therefore, $\lim_{n\to\infty} S_n = \dfrac{a(r^n - 1)}{r-1} \to \infty$, thus the given series is *divergent.*

When $r < -1$, then $r^n \to +\infty$ or $-\infty$ as $n \to \infty$. Hence, $\lim_{n\to\infty} S_n$ oscillates either to ∞ or $-\infty$. Thus, the given series *oscillates infinitely.*

Case III: $|r| = 1$, that is, $r = 1$, or -1. When $r = 1$, the series becomes, $a + a + a + \ldots + a \ldots$

Therefore, $S_n = na$. Thus $S_n \to \infty$ as $n \to \infty$ and the series *diverges.*

When $r = -1$, the series becomes, $a - a + a - a + \ldots$ and, therefore,

$$S_n = \begin{cases} 0, & \text{when } n \text{ is even.} \\ a, & \text{when } n \text{ is odd.} \end{cases}$$

Thus the given series *oscillates finitely* between 0 and a.

EXERCISE 3.1

1. Determine the general term of each of the following sequences and show that each is convergent.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots$ (b) $\frac{1^2}{1!}, \frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \frac{5^2}{5!} \ldots$

(c) $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}, \ldots$ (d) $\left(\frac{2}{1}\right)^1, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4 \ldots$

2. Examine the convergence of each of the following sequences, whose general term is

(a) $u_n = (\sqrt[3]{n^3+1} - n)$ C (b) $u_n = \left(1+\frac{1}{n}\right)^{n+1}$ C

(c) $u_n = \frac{\ln(n)}{n+1}$ C (d) n^2 ∞

3. Examine the following series for their convergence:

(a) $1 + 2 + 3 + 4 + \ldots$ D (b) $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \ldots$ C

(c) $\sum_{n=0}^{\infty} (-1)^n$ O (d) $4 - 3 - 1 + 4 - 3 - 1 + 4 - 3 - 1 + \ldots$ O

(e) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 + \ldots$ D (f) $2 + 3 + \frac{9}{2} + \frac{27}{4} + \frac{81}{8} + \ldots$ D

3.2 POSITIVE TERMS SERIES

An infinite series in which all the terms, after some finite number of terms, are positive, is called a positive term series. For example, the series $-5 - 4 - 3 - 2 - 1 + 0 + 1 + 2 + 3 + 4 + 5 + \ldots$

is a positive term series, since by omitting a few initial terms it can be considered as a series of only positive terms. Even a series which consists all the terms after some finite number of terms as negative, may be treated as a series of positive terms.

It is easy to see that a series of positive terms either converges or diverges to $+\infty$, for the sum S_n of first n terms in this case, barring the negative terms, tends either to a finite limit or to $+\infty$.

Cauchy's convergence principle for series: *The necessary and sufficient condition for an infinite series Σu_n to converge is that given $\epsilon > 0$, no matter how small, we can find a positive integer* $N(\epsilon)$, *such that*

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+m}| < \epsilon \text{ for every } n > N \text{ and } m = 1, 2, \ldots \quad \ldots(3.1)$$

In practice, this result becomes difficult to apply to test the convergence of series. We shall study a few specialized tests which are convenient to apply.

A necessary condition for convergence: *A necessary condition for a positive term series Σu_n to be convergent is that*

$$\lim_{n\to\infty} u_n = 0. \quad ...(3.2)$$

To establish this, consider $S_n = u_1 + u_2 + ... + u_n$. Since the series Σu_n is convergent, therefore $\lim_{n\to\infty} S_n = k$, a finite quantity and also $\lim_{n\to\infty} S_{n-1} = k$. Consider $u_n = S_n - S_{n-1}$. This gives

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} [S_n - S_{n-1}] = 0$$

The result (3.2) can also be derived from (3.1) for $m = 1$.

We must note that the converse of the above result is not true. For example, consider the series $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+...+\frac{1}{\sqrt{n}}+...$ Here, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$. But

$$S_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+...+\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+...+\frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n},$$

which tends to infinity as n tends to infinity.

Thus, the series is divergent. So, *$\lim_{n\to\infty} u_n = 0$ is only a necessary condition but not sufficient one for a series to be convergent.* In fact, the importance of the above result is that it leads to a simple test for divergence of an infinite series, that is

If $\lim_{n\to\infty} u_n \neq 0$, the positive terms series Σu_n must be divergent.

For example, consider the series $1 + \frac{2}{3}+\frac{3}{4}+\frac{4}{5}+...+\frac{n}{n+1}+...$

Here $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$, hence the series is divergent.

3.3 TESTS FOR THE CONVERGENCE OF POSITIVE TERMS SERIES

We have observed that the convergence or divergence of a series can be established by taking th limits of S_n, its nth partial sum, as $n \to \infty$. But in actual practice, sometimes it becomes cumbersom to find the sum of the first n terms of an infinite series. In this section, we outline the various tes employed to test for the convergence of the positive terms series.

3.3.1 Comparison Tests

The three comparison tests are:

A. *If two positive terms series Σu_n and Σv_n are such that*

(i) *Σv_n is convergent,*

(ii) *$u_n \le v_n$ for all values of n, then Σu_n is also convergent.*

Since, Σv_n is convergent, thus $\lim_{n\to\infty} (v_1 + v_2 + ... + v_n) = S$; a finite quantity. Also, since $u_1 \le v_1$, $u_2 \le v_2, ..., u_n \le v_n$, thus we have

$$u_1 + u_2 + ... + u_n \le v_1 + v_2 + ... + v_n$$

It gives $\lim_{n\to\infty} (u_1 + u_2 + ... + u_n) \le \lim_{n\to\infty} (v_1 + v_2 + ... v_n) = S$

Thus, the series Σu_n is also convergent.

B. *If two positive terms series Σu_n and Σv_n be such that:*

(i) *Σv_n is divergent,*

(ii) *$u_n \ge v_n$ for all values of n, then Σu_n is also divergent.*

The result follows on the similar lines as in **A.**

C. *If two positive term series Σu_n and Σv_n be such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite quantity ($\ne 0$), then Σu_n and Σv_n converge or diverge together.*

Since $\lim_{n\to\infty} \frac{u_n}{v_n} = l(\ne 0)$, then by definition of limits there exists a positive number $\in > 0$, no matter how small, such that

$$\left|\frac{u_n}{v_n} - l\right| < \in, \text{ for } n \ge m \quad \text{or,} \quad l - \in < \frac{u_n}{v_n} < l + \in, \text{ for } n \ge m.$$

Ignoring the first m terms of both the series, we have

$$l - \in < \frac{u_n}{v_n} < l + \in \text{ for all } n \ge 1 \qquad ...(3.3)$$

Case I: Σv_n is convergent, then

$$\lim_{n\to\infty} (v_1 + v_2 + ... + v_n) = k, \text{ a finite quantity.}$$

Now from (3.3), $\frac{u_n}{v_n} < l + \in$, for all n, which gives $u_n < v_n(l + \in)$

or, $\sum_{n=1}^{n} u_n < (l + \in) \sum_{n=1}^{n} v_n$, or $\lim_{n\to\infty} \sum_{n=1}^{n} u_n < (l + \in) \lim_{n\to\infty} \sum_{n=1}^{n} v_n = (l + \in)k$.

Hence Σu_n is also convergent.

Case II: Σv_n is divergent, then $\lim_{n\to\infty} (v_1 + v_2 + ... + v_n) \to \infty$.

Now from (3.3), $l - \epsilon < \frac{u_n}{v_n}$, for all n, which gives $u_n > (l - \epsilon)v_n$,

or, $\sum_{n=1}^{n} u_n > (l-\epsilon)\sum_{n=1}^{n} v_n$, or $\lim_{n\to\infty} \sum_{n=1}^{n} u_n > (l-\epsilon)\lim_{n\to\infty} \sum_{n=1}^{n} v_n \to \infty$.

Hence Σu_n is divergent.

It should be noted that because of the property that the nature of an infinite series is unaffected by addition or deletion of a finite number of terms, the comparison tests A and B hold good even for $u_n \le v_n$ (or, $u_n \ge v_n$) for $n \ge m$; where m is a finite positive integer. Further to test the nature of a series, the comparison test C is very useful, we choose the series Σv_n in such a way that its nature is already known and after applying the test we conclude the nature of the series Σu_n.

3.3.2 Integral Test

A positive term series $f(1) + f(2) + ... + f(n) + ...$, where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^{\infty} f(x)dx$ is finite or infinite.

From Fig. 3.1, it is clear that

$$f(1) + f(2) + ... + f(n) \ge \int_1^{n+1} f(x)dx \ge f(2) + f(3) + ... + f(n+1),$$

or,
$$S_n \ge \int_1^{n+1} f(x)dx \ge S_{n+1} - f(1) \qquad ...(3.4)$$

From the two terms on the right of (3.4), we have

$$S_{n+1} \le \int_1^{n+1} f(x)dx + f(1)$$

Taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} S_{n+1} \le \int_1^{\infty} f(x)dx + f(1).$$

Hence, if the integral $\int_1^{\infty} f(x)\,dx$ is finite, then so is

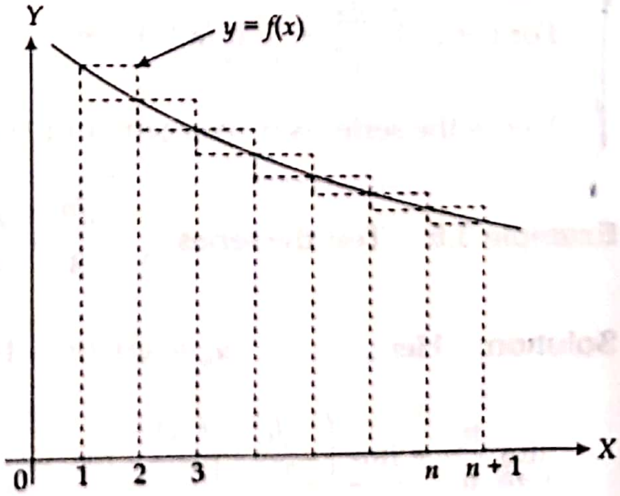

Fig. 3.1

$\lim_{n\to\infty} S_{n+1}$. Similarly, from (3.4), by considering, $S_n \ge \int_1^{n+1} f(x)dx$, we observe that if the integral

$\int_1^\infty f(x)dx$ is infinite, then so is $\lim_{n\to\infty} S_n$. Since the given series, being of positive terms, either converges or diverges, that is, $\lim_{n\to\infty} S_n$ is either finite or infinite, hence the desired result follows from the integral $\int_1^\infty f(x)dx$.

Next, we apply integral test to discuss the convergence of the *harmonic series of order p*, that is, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$. This result is oftenly used in comparison tests.

Example 3.5: Show that the harmonic series of order p, $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ is convergent for $p > 1$, and divergent for $p \le 1$.

Solution: By integral test, this series will converge or diverge according as the integral $\int_1^\infty \frac{dx}{x^p}$ is finite or infinite. For $p \neq 1$,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{n\to\infty} \int_1^n \frac{dx}{x^p} = \lim_{n\to\infty} \left[\frac{n^{1-p} - 1}{1-p}\right],$$

which is finite and is equal to $1/(p-1)$, for $p > 1$; tends to infinity for $p < 1$.

For $p = 1$, $\int_1^\infty \frac{dx}{x^p} = \lim_{n\to\infty} ln\,(n) = \infty$.

Hence the series is convergent for $p > 1$ and divergent for $p \le 1$.

Example 3.6: Test the series $\frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{8} + \ldots + \frac{\sqrt{n}}{n^2 - 1} + \ldots$ for its convergence or divergence.

Solution: Here, $u_n = \sqrt{n}/(n^2 - 1)$. Take $v_n = \sqrt{n}/n^2 = 1/n^{3/2}$. Consider

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2-1}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{1}{1-(1/n^2)} = 1;$$ a finite non-zero quantity.

Thus, by comparison test the series $\Sigma\, u_n$ and $\Sigma\, v_n$ either both converge, or both diverge. But the series $\Sigma v_n = \Sigma(1/n^{3/2})$ is convergent by integral test, since $p = 3/2 > 1$, hence the series $\Sigma\, u_n$ is also convergent.

Example 3.7: Test the series $\sum\left(\sqrt[3]{n^3+1}-n\right)$ for its convergence, or divergence.

Solution: Here, $u_n = (n^3+1)^{1/3} - n = n\left[1+\frac{1}{n^3}\right]^{1/3} - n$

$$= n\left[1+\frac{1}{3n^3}-\frac{1}{9n^6}+\ldots\right]-n = \frac{1}{3n^2}-\frac{1}{9n^5}+\ldots$$

Take $v_n = \frac{1}{n^2}$. Consider,

$$\frac{u_n}{v_n} = n^2\left[\frac{1}{3n^2}-\frac{1}{9n^5}+\ldots\right] = \frac{1}{3}-\frac{1}{9n^2}+\ldots$$

It tends to $\frac{1}{3}$, a finite non-zero number, as n tends to infinity. Hence, by comparison test the series Σu_n and Σv_n are either both convergent or both divergent. But the series Σv_n is convergent by integral test since $p = 2 > 1$, hence the series Σu_n is also convergent.

Example 3.8: Test the series $\sum_{n=2}\frac{1}{n\ln(n)}$ for its convergence or divergence.

Solution: It is easy to check that the series $\sum_{n=2}\frac{1}{n\ln(n)}$ is decreasing one. Here the nth term is

$f(n) = \frac{1}{(n+1)\ln(n+1)}$. Consider, the integral $I = \int_1^{\infty}\frac{1}{(x+1)\ln(x+1)}dx.$

Set $\ln(x+1) = t$, it gives $\frac{dx}{x+1} = dt$, so I becomes

$$I = \int_{\ln 2}^{\infty}\frac{dt}{t} = \lim_{n\to\infty}\int_{\ln 2}^{n}\frac{dt}{t} = \lim_{n\to\infty}[\ln(n)-\ln(\ln 2)] = \infty.$$

Hence, by integral test the given series is divergent.

EXERCISE 3.2

Test the following series for their convergence or divergence.

1. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$ C

2. $\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots$ - D

3. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots$ C

4. $\sum\left(\sqrt{(n^4+1)}-\sqrt{(n^4-1)}\right)$ c

5. $\sum\left[1/n^{\left(1+\frac{1}{n}\right)}\right]$ D

6. $\sum\frac{2n^3+5}{4n^5+1}$ C

7. $\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+\ldots$ D

8. $\sum\left(\sqrt{n^2+1}-n\right)$ D

9. $\sum n^{\ln x}$ $x<\frac{1}{e}$ C $x>\frac{1}{e}$ D

10. $\sum\frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7}}$ D

11. $\sum\frac{\sqrt{n+1}-\sqrt{n}}{n^p}$, $p>\frac{1}{2}$ C

12. $\sum\frac{\sqrt{n+1}-1}{(n+2)^3-1}$ C

13. $\sum\sqrt{\frac{2^n-1}{3^n-1}}$ C

14. $\sum n\tan^{-1}\left(\frac{1}{n^3}\right)$ D

15. $\sum n^p/(n+1)^q$ $q>1+p$ C $q\le 1+p$ d

16. Test for the convergence of harmonic series $\sum 1/n^p$ using comparison tests.

3.3.3 D'Alembert's Ratio Test

"If Σu_n is a positive terms series such that $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=k$, then the series Σu_n is convergent, if $k<1$, divergent, if $k>1$, and for $k=1$, the test fails".

Case I: When $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=k<1$. By definition of the limit, we can find a number $r<1$ such that

$$\frac{u_{n+1}}{u_n}<r,\ \text{for all}\quad n>m.$$

Omitting the first m terms, let the series be $u_1+u_2+u_3+\ldots$, so that

$$\frac{u_2}{u_1}<r,\quad \frac{u_3}{u_2}<r,\quad \frac{u_4}{u_3}<r\ldots\quad \text{and so on.}$$

Then $\quad u_1+u_2+u_3+\ldots$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\right) < u_1(1+r+r^2+r^3+\ldots) = \frac{u_1}{1-r}, \text{ a finite quantity.}$$

Hence $\Sigma\, u_n$ is convergent.

Case II: When $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = k > 1$. By definition of limit, we can find a number m, such that $\frac{u_{n+1}}{u_n} \geq 1$, for all $n > m$.

Omitting the first m terms let the series be $u_1 + u_2 + u_3 + \ldots$, so that

$$\frac{u_2}{u_1} \geq 1, \quad \frac{u_3}{u_2} \geq 1, \quad \frac{u_4}{u_3} \geq 1 \ldots \text{ and so on.}$$

Therefore, $u_1 + u_2 + u_3 + \ldots$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\frac{u_2}{u_1}+\frac{u_4}{u_3}\frac{u_3}{u_2}\frac{u_2}{u_1}+\ldots\right) \geq u_1(1+1+1+1+\ldots),$$

which tends to infinity. Hence, $\Sigma\, u_n$ is divergent.

Remarks

1. The ratio test fails when $k = 1$. Consider, for instance, the series $\sum \frac{1}{n}$. Here $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n}{n+1} = 1$. But we know that $\sum\frac{1}{n}$ is divergent, being harmonic series with $p = 1$. Similarly, in case of $\sum\frac{1}{n^2}$, the $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$, but $\sum\frac{1}{n^2}$ is convergent, being harmonic series with $p = 2 > 1$. Hence, when $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1$, the series $\Sigma\, u_n$ may be convergent or divergent.

2. The test makes no reference to the magnitude of $\frac{u_{n+1}}{u_n}$ for any finite value of n but concerns only with the limit of this ratio as $n \to \infty$.

3. This test is also used in the following form: *If $\Sigma\, u_n$ is a positive term series such that* $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = k$, *then the series $\Sigma\, u_n$ is convergent if $k > 1$, divergent if $k < 1$, and for $k = 1$ the test fails.*

Example 3.9: Test for convergence of the series

(a) $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots\infty$ (b) $\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\cdot\frac{2}{5}\right)^2+\left(\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\right)^2+\ldots\infty.$

Solution: (a) Here $u_n = \dfrac{n^p}{n!}$ and $u_{n+1} = \dfrac{(n+1)^p}{(n+1)!}$.

Consider $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^p}{(n+1)!}\,\dfrac{n!}{n^p} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^p \dfrac{1}{n+1} = 0 < 1.$

Hence by ratio test the series is convergent.

(b) Here, $u_n = \left[\dfrac{n!}{3.5.7.\ldots(2n+1)}\right]^2$, and $u_{n+1} = \left[\dfrac{(n+1)!}{3.5.7.\ldots(2n+3)}\right]^2$

Consider $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)!}{3.5.7.\ldots(2n+3)}\right]^2 \cdot \left[\dfrac{3.5.7.\ldots(2n+1)}{n!}\right]^2$

$$= \lim_{n\to\infty}\left[\frac{(n+1)}{(2n+3)}\right]^2 = \lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right]^2 = \frac{1}{4} < 1.$$

Hence, by ratio test the series is convergent.

Example 3.10: Test for convergence of the series

(a) $\sum \dfrac{n!\,2^n}{n^n}$ (b) $\sum \dfrac{x^n}{n}$ $(x > 0)$

Solution: (a) Here $u_n = \dfrac{n!\,2^n}{n^n}$ and $u_{n+1} = \dfrac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$. Consider

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\,2^n} = \lim_{n\to\infty} 2.\left(\frac{n}{1+n}\right)^n = 2.\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1.$$

Hence, by ratio test the series is convergent.

(b) Here $u_n = \dfrac{x^n}{n}$, and $u_{n+1} = \dfrac{x^{n+1}}{n+1}$. Consider

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n} = \lim_{n\to\infty}\frac{n}{n+1}.x = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}.x = x.$$

Hence, the given series is convergent for $x < 1$ and divergent for $x > 1$. At $x = 1$; the series is $1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots\dfrac{1}{n}+\ldots$ which is divergent, being harmonic series with $p = 1$.

Example 3.11: Discuss the convergence of the series $\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\ldots, (x>0)$

Solution: Here $u_n = \frac{x^n}{1+x^n}$, and $u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$. Consider

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{n+1}}{1+x^{n+1}}\cdot\frac{1+x^n}{x^n} = x\lim_{n\to\infty}\frac{1+x^n}{1+x^{n+1}}.$$

For $x<1$, $\lim_{n\to\infty} x^n = 0 = \lim_{n\to\infty} x^{n+1}$, and hence, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x < 1$.

Thus, the series Σu_n is convergent for $x<1$.

For $x>1$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{x^n}{1+x^n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{x^n}} = 1 \neq 0.$

Hence the series Σu_n is divergent.

For $x=1$, the series becomes $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\infty$, which is divergent; since $S_n = \frac{n}{2}$ tends to infinity as n tends to infinity.

Thus the given series is convergent for $x<1$ and divergent for $x\geq 1$.

EXERCISE 3.3

Test for convergence of the following series:

1. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots$ C

2. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots$ C

3. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots$ $x\leq 1$ C, $x>1$ D

4. $\frac{1^2.2^2}{1!}+\frac{2^2.3^2}{2!}+\frac{3^2.4^2}{3!}+\frac{4^2.5^2}{4!}+\ldots$ C

5. $\sum\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n$ $x<1$ C $x\geq 1$ D

6. $1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots, (\alpha, \beta>0)$
$\beta>\alpha>0$ C
$\alpha\geq\beta>0$ D

7. $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots+\frac{2^n-2}{2^n+1}x^{n-1}+\ldots, (x>0)$

$x<1$ C $x\geq 1$ D

8. $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \ldots + \frac{(n+1)x^n}{n^3} + \ldots$ x≤1 C x>1 D

9. $\sum \frac{\sqrt{(n-1)}}{\sqrt{n^2+1}} x^n$ x<1 C

10. $\sum \frac{1}{x^n + x^{-n}}$ x<1 C x>1 x=1 D

11. $\sum \frac{a^n}{x^n + a^n}$ x>a C x≤a D

12. $\sum \left(\frac{n^2}{2^n} + \frac{1}{n^2}\right)$ Con

13. $\sum \frac{n^3 - n + 1}{n!}$ Con

14. $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots$ Con

15. $\frac{4}{18} + \frac{4.12}{18.27} + \frac{4.12.20}{18.27.36} + \ldots$ Con

3.3.4 Raabe's Test

We have seen that the ratio test fails when $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. In that situation in general, we apply the Raabe's test given as follows:

If Σu_n is a positive term series such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, *then the series converges for* $k > 1$, *diverges for* $k < 1$, *and the test fails for* $k = 1$.

When $k > 1$, let p be a number such that $k > p > 1$. Compare the given series Σu_n with the harmonic series of order p, that is, with $\Sigma 1/n^p$, which is convergent for $p > 1$. The series Σu_n will be convergent, if

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}.$$

$$\text{or if,} \quad \frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \ldots$$

$$\text{or if,} \quad n\left\{\frac{u_n}{u_{n+1}} - 1\right\} > p + \frac{p(p-1)}{2n} + \ldots$$

$$\text{or if,} \quad \lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} > \lim_{n\to\infty}\left\{p + \frac{p(p-1)}{2n} + \ldots\right\} = p$$

that is, if $k > p$, which is true. Hence Σu_n is convergent.

The case $k < 1$ can be proved on the similar lines.

Another test which is generally employed as a substitute to Raabe's test, when the ratio test fails, is the *logarithmic test*. This is employed when either n occurs as an exponent in u_n/u_{n+1}, or evaluation of limit $n \to \infty$ becomes simpler on taking logarithm of $\frac{u_n}{u_{n+1}}$.

3.3.5 Logarithmic Test

If series $\Sigma\, u_n$ is a positive term series such that $\lim\limits_{n\to\infty}\left(n \ln \frac{u_n}{u_{n+1}}\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$, and the test fails for $k = 1$.

The proof is similar to that of Raabe's test. As in Raabe's test for $k > p > 1$, the series $\Sigma\, u_n$ will be convergent, if

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p,$$

$$\text{or if,} \qquad \ln \frac{u_n}{u_{n+1}} > p \ln\left(1+\frac{1}{n}\right) = p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right)$$

$$\text{or if,} \qquad n \ln \frac{u_n}{u_{n+1}} > p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots\right)$$

$$\text{or if,} \qquad \lim_{n\to\infty}\left(n \ln \frac{u_n}{u_{n+1}}\right) > p, \text{ that is if } \quad k > p,$$

which is true. Hence the series $\Sigma\, u_n$ is convergent.

The case $k < 1$ can be proved on the similar lines.

Example 3.12: Test for convergence the series $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \dots$

Solution: Here $u_n = \frac{x^n}{(2n-1).2n}$, and $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$. Consider

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+1)(2n+2)} \frac{(2n-1)2n}{x^n}$$

$$= \lim_{n\to\infty} \frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)} = x.$$

Thus Σu_n is convergent for $x < 1$, divergent for $x > 1$ and test fails for $x = 1$.
When $x = 1$ consider

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{2n(2n-1)} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{8n+2}{2n(2n-1)}\right] = \lim_{n\to\infty} \frac{\left(2+\frac{2}{4n}\right)}{\left(1-\frac{1}{2n}\right)} = 2 > 1.$$

Hence, by Raabe's test the series is convergent for $x = 1$.
Thus, the given series is convergent for $x \le 1$ and divergent for $x > 1$.

Example 3.13: Discuss the convergence of the series $1 + \frac{1!}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \ldots$

Solution: Here $u_n = \frac{n!}{(n+1)^n}x^n$, and $u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$. Consider

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}\,\frac{(n+1)^n}{n!}\,\frac{1}{x^n}$$

$$= \lim_{n\to\infty} n^n\left(1+\frac{1}{n}\right)^n \frac{(n+1)}{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}x = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}x = \frac{ex}{e^2} = \frac{x}{e},$$

since $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$.

Hence, by ratio test the given series is convergent if $x < e$, divergent if $x > e$, and the test fails for $x = e$. At $x = e$,

$$\frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)^n}e^n\,\frac{(n+2)^{n+1}}{(n+1)!}\,\frac{1}{e^{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}\,\frac{1}{e}$$

Since the expression involves the number e, we apply *logarithmic test*. We have

$$\ln\frac{u_n}{u_{n+1}} = (n+1)\ln\left(1+\frac{2}{n}\right) - (n+1)\ln\left(1+\frac{1}{n}\right) - \ln e$$

$$= (n+1)\left[\ln\left(1+\frac{2}{n}\right) - \ln\left(1+\frac{1}{n}\right)\right] - 1$$

$$= [(n+1)\left[\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3}\cdots\right) - \left(\frac{1}{n} - \frac{1}{2}\frac{1}{n^2} + \frac{1}{3}\cdot\frac{1}{n^3}\cdots\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2}\cdot\frac{1}{n^2} + \frac{7}{3}\frac{1}{n^3}\cdots\right] - 1 = -\frac{1}{2n} + \frac{5}{6n^2} + \ldots$$

Therefore, $\lim\limits_{n\to\infty} n \ln \dfrac{u_n}{u_{n+1}} = \lim\limits_{n\to\infty}\left(-\dfrac{1}{2} + \dfrac{5}{6n} + \ldots\right) = -1/2 < 1$

Hence by logarithmic test the series is divergent for $x = e$.

Thus the given series $\Sigma\, u_n$ converges for $x < e$ and diverges for $x \geq e$.

Example 3.14: Discuss the convergence of the series $x^2(\ln 2)^q + x^3(\ln 3)^q + x^4(\ln 4)^q + \ldots$

Solution: $u_n = x^{n+1}[\ln(n+1)]^q$ and $u_{n+1} = x^{n+2}[\ln(n+2)]^q$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{x^{n+1}[\ln(n+1)]^q}{x^{n+2}[\ln(n+2)]^q} = \lim_{n\to\infty}\left[\frac{\ln(n) + \ln\left(1+\frac{1}{n}\right)}{\ln(n) + \ln\left(1+\frac{2}{n}\right)}\right]^q \frac{1}{x}$$

$$= \lim_{n\to\infty}\left[\frac{\ln(n) + \frac{1}{n} - \frac{1}{2n^2} + \ldots}{\ln(n) + \frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \ldots}\right]^q \frac{1}{x}$$

$$= \lim_{n\to\infty}\left[\frac{\ln(n)\left(1 + \frac{1}{n\ln(n)} - \frac{1}{2n^2\ln(n)} + \ldots\right)}{\ln(n)\left(1 + \frac{2}{n\ln(n)} - \frac{2}{n^2\ln(n)} + \ldots\right)}\right]^q \frac{1}{x}$$

$x > 1$

$$= \lim_{n\to\infty}\left[\frac{1 + \frac{1}{n\ln(n)} - \frac{1}{2n^2\ln(n)} + \ldots}{1 + \frac{2}{n\ln(n)} - \frac{2}{n^2\ln(n)} + \ldots}\right]^q \frac{1}{x} = \frac{1}{x}.$$

Hence, by ratio test the series is convergent for $\dfrac{1}{x} > 1$, that is, for $x < 1$ and divergent for $x > 1$ and the test fails for $x = 1$. For $x = 1$,

$$\frac{u_n}{u_{n+1}} = \left[\frac{1+\dfrac{1}{n\ln(n)}-\dfrac{1}{2n^2\ln(n)}+\dots}{1+\dfrac{2}{n\ln(n)}-\dfrac{2}{n^2\ln(n)}+\dots}\right]^q$$

$$= \left(1+\frac{1}{n\ln(n)}-\frac{1}{2n^2\ln(n)}+\dots\right)^q\left(1+\frac{2}{n\ln(n)}-\frac{2}{n^2\ln(n)}+\dots\right)^{-q}$$

$$= \left[1+q\left(\frac{1}{n\ln(n)}-\frac{1}{2n^2\ln(n)}+\dots\right)+\dots\right]\left[1-q\left(\frac{2}{n\ln(n)}-\frac{2}{n^2\ln(n)+\dots}+\dots\right)\right]$$

$$= 1+\frac{1}{n\ln(n)}(q-2q)+\dots,$$

This gives, $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}}-1\right) = \lim\limits_{n\to\infty}\left[\dfrac{-q}{\ln(n)}+\dots\right] = 0 < 1.$

Hence, by Raabe's test, the series is divergent for $x = 1$. Thus the series Σu_n is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 3.15: Test for convergence of the series

forget 1

$$1+\frac{a}{b}x+\frac{a(a+1)}{b(b+1)}x^2+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3+\dots\ (a, b>0, x>0)$$

Solution: Here, $u_n = \dfrac{a(a+1)(a+2)\dots(a+n-1)}{b(b+1)(b+2)\dots(b+n-1)}x^n$ and, $u_{n+1} = \dfrac{a(a+1)(a+2)\dots(a+n)}{b(b+1)(b+2)\dots(b+n)}x^{n+1}.$

Consider $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{a+n}{b+n}.x = \lim\limits_{n\to\infty}\dfrac{\left(1+\dfrac{a}{n}\right)}{\left(1+\dfrac{b}{n}\right)}x = x.$

Hence, by ratio test the series is convergent for $x < 1$ and divergent for $x > 1$, and the test fails at $x = 1$. At $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{b+n}{a+n}, \text{ which gives}$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty} n\left[\frac{b+n}{a+n}-1\right] = \lim_{n\to\infty}\frac{n(b-a)}{n+a} = \lim_{n\to\infty}\frac{b-a}{\left(1+\dfrac{a}{n}\right)} = b-a.$$

Thus, by Raabe's test at $x = 1$, the series is convergent for $b - a > 1$ and divergent for $b - a < 1$ and the test fails at $b - a = 1$.

Further, at $x = 1$ and $b = a + 1$, the given series becomes

$$1 + \frac{a}{a+1} + \frac{a}{a+2} + \frac{a}{a+3} + \frac{a}{a+4} + \dots$$

Obviously, the series is decreasing one with positive terms, because $a > 0$.

Take, $f(x) = \dfrac{a}{a+x}$, and consider

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{a}{a+x}dx = \lim_{n\to\infty} \int_0^n \frac{a}{a+x}dx = a \lim_{n\to\infty} [\ln (a+n) - \ln a] = \infty.$$

Hence, by integral test the series is divergent for $b = a + 1$ (at $x = 1$).

Thus, the given series is convergent for $x < 1$ and divergent for $x > 1$ and at $x = 1$, the series is convergent for $b - a > 1$ and divergent for $b - a \leq 1$.

EXERCISE 3.4

Test the following series for convergence

1. $\dfrac{1}{2} + \dfrac{1.3}{2.4} + \dfrac{1.3.5}{2.4.6} + \dots$

2. $1 + \dfrac{3}{7}x + \dfrac{3.6}{7.10}x^2 + \dfrac{3.6.9}{7.10.13}x^3 + \dots$

3. $\dfrac{(a+x)}{1!} + \dfrac{(a+2x)^2}{2!} + \dfrac{(a+3x)^3}{3!} + \dots$

4. $\dfrac{x}{1} + \dfrac{1.x^3}{2.3} + \dfrac{1.3}{2.4}\dfrac{x^5}{5} + \dfrac{1.3.5}{2.4.6}\dfrac{x^7}{7} + \dots \quad (x > 0)$

5. $\sum \dfrac{(n!)^2}{(2n)!}x^{2n}$

6. $x + \dfrac{2^2x^2}{2!} + \dfrac{3^3x^3}{3!} + \dfrac{4^4x^4}{4!} + \dots$

7. $\dfrac{1^2}{2^2} + \dfrac{1^2.3^2}{2^2.4^2}x + \dfrac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2 + \dots$

8. $\dfrac{1}{1^2} + \dfrac{1+2}{1^2+2^2} + \dfrac{1+2+3}{1^2+2^2+3^2} + \dots$

9. $1 + \dfrac{\alpha.\beta}{1.\gamma}x + \dfrac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \dots$

10. $\dfrac{\sqrt{1}}{\sqrt{2}}x + \dfrac{\sqrt{2}}{\sqrt{5}}x^2 + \dfrac{\sqrt{3}}{\sqrt{10}}x^3 + \dots$

11. $\sum \left(\dfrac{n^2}{2^n} + \dfrac{1}{n^2}\right)$

12. $\sum \left(\sqrt{n^2+1} - n\right)x^{2n}$

x^2 x^2

13. $\sum \dfrac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$

14. $\dfrac{a}{b}+\dfrac{a(a+d)}{b(b+d)}x+\dfrac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2+\ldots$

15. $x \ln x + x^2 \ln (2x) + x^3 \ln (3x) + \ldots$

3.3.6 Cauchy's Root Test

In a positive term series Σu_n, if $\lim_{n\to\infty} (u_n)^{1/n} = k$, *then the series converges for* $k<1$, *diverges for* $k>1$, *and the test fails for* $k=1$.

Case I: When $\lim_{n\to\infty} (u_n)^{1/n} = k < 1$. By definition of limit we can find a positive number p, where $k<p<1$, such that

$$(u_n)^{1/n} < p, \text{ for all } n > m, \quad \text{or} \quad u_n < p^n, \text{ for all } n > m.$$

Since $p<1$, the geometric series Σp^n is convergent, and hence by comparison test, the series Σu_n is also convergent.

Case II: When $\lim_{n\to\infty} (u_n)^{1/n} = k > 1$. By definition of limit we can find a positive number q, where $k>q>1$, such that

$$(u_n)^{1/n} > q, \text{ for all } n > m, \quad \text{or} \quad u_n > q^n, \text{ for all } n > m.$$

Since $q>1$, the geometric series Σq^n is divergent, and hence by comparison test, the series Σu_n is also divergent.

The test fails when $k=1$. For example, consider the series $\sum \dfrac{1}{n^p}$. Here

$$\lim_{n\to\infty}\left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty}\left[n^{1/n}\right]^{-p} = 1$$

irrespective of the values of p, whereas we know that series is convergent for $p>1$ and divergent for $p \le 1$.

3.3.7 Gauss Test

In a positive term series Σu_n, *if* $\dfrac{u_n}{u_{n+1}}$ *can the expanded in the form* $\dfrac{u_n}{u_{n+1}} = 1+\dfrac{k}{n}+o\left(\dfrac{1}{n^2}\right)$, *then* Σu_n *converges, if* $k>1$ *and diverges, if* $k \le 1$.

This test is also applied after the failure of the ratio test when it is possible to expand $\dfrac{u_n}{u_{n+1}}$ in powers of $\dfrac{1}{n}$. Also we note that this test does not fail even if $k=1$.

Example 3.16: Test for convergence the series $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\dots$

Solution: Here, $u_n=\left(\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right)^{-n}$. Consider

$$\lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)\right]^{-1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

$$=\frac{1}{e-1}<1,\text{ since }\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

Hence by Cauchy's root test, the given series is convergent.

Example 3.17: Discuss the convergence of the series

$$1+\frac{\alpha\beta}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\gamma(\gamma+1)(\gamma+2)}+\dots$$

Solution: Neglecting the first term, we have

$$u_n=\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{1.2.3\dots n.\gamma(\gamma+1)\dots(\gamma+n-1)}\quad u_{n+1}=\frac{\alpha(\alpha+1)\dots(\alpha+n)\beta(\beta+1)\dots(\beta+n)}{1.2.3\dots(n+1)\gamma(\gamma+1)\dots(\gamma+n)}$$

Consider, $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}=1$$

Hence, the ratio test fails.

Next consider,

$$\frac{u_n}{u_{n+1}}=\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}=\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1}$$

$$=\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}-\dots\right)\left(1-\frac{\beta}{n}+\frac{\beta^2}{n^2}-\dots\right)$$

$$= \left(1 + \frac{1}{n} + \frac{\gamma}{n} + \frac{\gamma}{n^2}\right)\left(1 - \frac{\alpha}{n} - \frac{\beta}{n} + \frac{\alpha\beta}{n^2} + \frac{\alpha^2}{n^2} + \frac{\beta^2}{n^2} + \ldots\right)$$

$$= 1 + (1 + \gamma - \alpha - \beta)\frac{1}{n} + \text{terms of second and higher order in } \frac{1}{n}.$$

Hence, by Gauss test, the series $\Sigma\, u_n$ converges if $1 + \gamma - \alpha - \beta > 1$, that is, if $\gamma > \alpha + \beta$ and diverges if $\gamma \leq \alpha + \beta$.

EXERCISE 3.5

Examine the convergence of the following series:

1. $\Sigma\, 3^{-n-(-1)^n}$

2. $\sum \frac{1}{(\ln n)^n}$

3. $\sum \left(\frac{n+1}{n+2}\right)^n x^n, \quad (x > 0)$

4. $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

5. $\sum \frac{(1+nx)^n}{n^n}$

6. $\sum \frac{(n - \ln n)^n}{2^n - n^n}$

7. $\frac{1}{2}x + \left(\frac{2}{3}\right)^4 x^2 + \left(\frac{3}{4}\right)^9 x^3 + \ldots, \quad (x > 0)$

8. $\frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \frac{2^2.4^2.6^2.8^2}{3^2.5^2.7^2.9^2} + \ldots$

9. $(a + b) + (a^2 + b^2) + (a^3 + b^3) + \ldots\ (a, b > 0)$

10. $\Sigma\, (n \ln n)^{-1} (\ln \ln n)^{-k}$

3.4 ALTERNATING SERIES

A series in which the terms are alternately positive or negative is called an alternating series. For example, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$ is an alternating series.

The general form of this series is $u_1 - u_2 + u_3 - u_4 + \ldots$

Next we give a test for the convergence of an alternating series called the *Leibnitz's test.*

Leibnitz test: *An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$ is convergent if each term is numerically less than its preceding term, that is if, $u_1 > u_2 > u_3 \ldots$, and* $\lim_{n\to\infty} u_n = 0$.

To establish this test consider S_{2n}, the sum of the first $2n$ terms, as

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots + (u_{2n-1} - u_{2n}) \quad \ldots(3.5)$$

or as,

$$S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - (u_{2n-2} - u_{2n-1}) - u_{2n} \quad \ldots(3.6)$$

Each term in the brackets in (3.5) is positive, since $u_1 > u_2 > u_3 \ldots$, hence S_{2n} is positive and increases as n increases.

Similarly, each term in the brackets in (3.6) is positive and hence S_{2n} is always less than u_1.

Thus sequence $\{S_{2n}\}$ is monotonically increasing and bounded, therefore $\lim_{n\to\infty} S_{2n}$ exists and is finite. Further,

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + u_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty} S_{2n},$$

since $\lim_{n\to\infty} u_{2n+1} = 0$. Thus S_n tends to the same finite limit whether n is odd or even.

Hence, the given series is convergent.

When $\lim_{n\to\infty} u_n \neq 0$, then $\lim_{n\to\infty} S_{2n} \neq \lim_{n\to\infty} S_{2n+1}$.

Thus in this case the given alternating series is *oscillatory*.

Example 3.18: Discuss the convergence of the series.

(a) $1 - \dfrac{1}{2!} + \dfrac{1}{4!} - \dfrac{1}{6!} + \ldots$ (b) $\dfrac{1}{6} - \dfrac{2}{11} + \dfrac{3}{16} - \dfrac{4}{21} + \ldots$

Solution: (a) The terms of the given series are alternately positive and negative; each term being numerically less than its preceding term. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2n!} = 0.$$

Hence, by Leibnitz test the series Σu_n is *convergent*.

(b) Here, $u_n = \dfrac{n}{5n+1}$, and $u_{n+1} = \dfrac{n+1}{5n+6}$. Consider,

$$u_{n+1} - u_n = \frac{n+1}{5n+6} - \frac{n}{5n+1} = \frac{1}{(5n+6)(5n+1)} > 0;$$

thus, the terms are not numerically of decreasing order. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{1}{5+1/n} = \frac{1}{5} \neq 0.$$

Thus, the given alternating series is not *convergent*.

Example 3.19: Discuss the convergence of the series

(a) $\dfrac{x}{1+x} - \dfrac{x^2}{1+x^2} + \dfrac{x^3}{1+x^3} - \dfrac{x^4}{1+x^4} + \ldots,\quad (0 < x < 1)$

(b) $\left(\dfrac{1}{2} - \dfrac{1}{\ln 2}\right) - \left(\dfrac{1}{2} - \dfrac{1}{\ln 3}\right) + \left(\dfrac{1}{2} - \dfrac{1}{\ln 4}\right) - \ldots$

Solution: (a) The terms of the given series are alternately positive and negative. Here

$$u_n - u_{n-1} = \frac{x^n}{1+x^n} - \frac{x^{n-1}}{1+x^{n-1}}$$

$$= x^{n-1}\left[\frac{x+x^n-1-x^n}{(1+x^n)(1+x^{n-1})}\right] = \frac{x^{n-1}(x-1)}{(1+x^n)(1+x^{n-1})} < 0, \quad (0<x<1).$$

Thus, each term is numerically less than its preceding term. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0, \quad (0<x<1).$$

Hence, by Leibnitz test, the given series is *convergent.*

(b) Terms of the given series are alternately positive and negative. Here

$$u_n - u_{n-1} = \left(\frac{1}{2} - \frac{1}{\ln(n+1)}\right) - \left(\frac{1}{2} - \frac{1}{\ln(n)}\right) = \frac{1}{\ln(n)} - \frac{1}{\ln(n+1)} > 0.$$

Thus, each term is not numerically less than its preceding term. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{\ln(n+1)}\right) = \frac{1}{2} \neq 0.$$

Hence, series is not *convergent.*

EXERCISE 3.6

Discuss the convergence of the following series

1. $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \ldots$

2. $1 - 2x + 3x^2 - 4x^3 + \ldots \ (0<x<1)$

3. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \ (0<x<1)$

4. $\sum \frac{(-1)^{n-1} n}{2n-1}$

5. $\sum \frac{\cos n\pi}{n^2+1}$

6. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)}, \quad 0<x<1$

7. $\sum \frac{(-1)^n}{n \ln n}$

8. $\sum \frac{(-1)^{n-1}}{2^{n-1}} \sin \frac{1}{\sqrt{n}}$

9. $\sum \frac{\sqrt{n} \cos n\pi}{\ln n}$

3.5 ABSOLUTE CONVERGENCE OF A SERIES

If a series of arbitrary terms $u_1 + u_2 + u_3 + ... + u_n + ...$ is such that the series $|u_1| + |u_2| + |u_3| + ... + |u_n| + ...$ is convergent, then the series Σu_n is said to be absolutely convergent.

In case the series $\Sigma |u_n|$ is divergent but the series Σu_n is convergent, then the series Σu_n is said to be conditionally convergent.

For example, the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} ...$ is absolutely convergent since the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ is convergent being geometric series with common ratio $\frac{1}{2} < 1$. But the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$ is only conditionally convergent, since this is convergent by Leibnitz test, however, the corresponding series of absolute terms, that is, the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ...$ is divergent, being harmonic series of order 1.

Next we state an important result concerning the absolute convergence of series.

An absolutely convergent series is necessarily convergent but not conversely.

For if $\Sigma |u_n|$ is convergent, then since

$$u_1 + u_2 + ... + u_n + ... \le |u_1| + |u_2| + |u_n| + ...$$

thus Σu_n is also convergent.

To show that converse is not true, consider the series $\sum \frac{(-1)^n}{n}$. It is convergent by alternate series test, but, the corresponding series of absolute terms, the series $\sum \frac{|(-1)^n|}{n} = \sum \frac{1}{n}$ is known to be divergent.

Remarks

1. The convergence of $\Sigma |u_n|$ implies the convergence of Σu_n, but divergence of $\Sigma |u_n|$ does not imply the divergence of Σu_n.
2. Since the series $\Sigma |u_n|$ is of positive terms series, the test already established for positive terms series are applicable to examine the absolute convergence of the series Σu_n.

Example 3.20: Test whether the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - ...$ converges or not.

Solution: The corresponding series of absolute terms is

$$\frac{|\sin x|}{1^3} + \frac{|\sin 2x|}{2^3} + \frac{|\sin 3x|}{3^3} + ...$$

Here, $|u_n| = \frac{|\sin nx|}{n^3}$. Consider $v_n = \frac{1}{n^3}$. Clearly, $\frac{|\sin nx|}{n^3} \le \frac{1}{n^3}$.

By comparison test, the series $\sum \frac{|\sin nx|}{n^3}$ is convergent, since the series $\sum \frac{1}{n^3}$ is convergent, being harmonic series of order $3 > 1$.

Thus the given series is absolutely convergent and hence it is convergent also.

Example 3.21: Examine the convergence and absolute convergence of the series $\sum \frac{(-1)^{n+1} n}{n^2+1}$.

Solution: The series is an alternating series of the form $u_1 - u_2 + u_3 - u_4 + \ldots$ with $u_n = \frac{n}{n^2+1} > 0$, numerically for all n. Here,

$$u_{n+1} - u_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{-(n^2+n)+1}{(n^2+1)[(n+1)^2+1]} < 0.$$

Thus, the terms are of decreasing order. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+(1/n)} = 0.$$

Hence, by Leibnitz's test the given series is convergent.

To discuss the absolute convergence, consider the corresponding series with positive terms, that is, the series $\sum \frac{n}{n^2+1}$. Choose $v_n = \frac{1}{n}$. We have,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1.$$

Hence, by comparison test the series $\sum \frac{n}{n^2+1}$ and $\sum \frac{1}{n}$ both converge or diverge together. But the series $\sum \frac{1}{n}$ is divergent, being harmonic series of order 1, thus, the series $\sum \frac{n}{n^2+1}$ is also divergent. Therefore, the given series is not absolutely convergent; it is only conditionally convergent.

EXERCISE 3.7

Discuss the convergence of the following series:

1. $1 - \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^5} + \frac{1}{2^3} - \frac{1}{3^7} + \ldots$

2. $1-\frac{2}{3}+\frac{3}{3^2}-\frac{4}{3^3}+\ldots$

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$

4. $2\sin\frac{x}{3}+4\sin\frac{x}{9}+8\sin\frac{x}{27}+\ldots$

5. $\sum\frac{(-1)^{n-1}\cos^2 nx}{n\sqrt{n}}$

3.6 POWER SERIES

A series of the form

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots, \qquad \ldots(3.7)$$

where all the a_i's are independent of x, is called a power series in x.

Such a series may converge for some or all values of x. In the power series (3.7) $u_n = a_nx^n$. Therefore,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{a_{n+1}x^{n+1}}{a_nx^n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}x=lx,$$

where $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=l$. Thus, by ratio test, the series converges, when lx is numerically less than one, that is, when $|x|<\frac{1}{l}$ and diverges, when $|x|>\frac{1}{l}$. Hence, the power series converges when x lies in the interval $-\frac{1}{l}<x<\frac{1}{l}$. This interval is called the *interval of convergence* of the power series (3.7).

3.6.1 A Few Special Power Series

We discuss the convergence of the following power series which occur most frequently in applications.

(a) **The exponential series:** *The exponential series*

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots \qquad \ldots(3.8)$$

is convergent for all values of x.

Here, $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^n}{n!}\frac{(n-1)!}{x^{n-1}}=\lim_{n\to\infty}\frac{x}{n}=0<1.$$

Hence, by ratio test the series (3.8) converges for all real values of x.

(b) **The logarithmic series:** *The logarithmic series*

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^n\frac{x^n}{n}+\ldots \qquad \ldots(3.9)$$

is convergent for $-1<x\le 1$.

Here, $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(-1)^{n+1}x^{n+1}}{(n+1)}\frac{n}{(-1)^n x^n}$$

$$=\lim_{n\to\infty}\frac{n}{n+1}(-x)=-x.$$

Hence, by ratio test the series is convergent for $|x|<1$ and divergent for $|x|>1$, and the test fails for $|x|=1$. When $x=+1$, the series becomes

$$1-\frac{1}{2}+\frac{1}{3}-\ldots(-1)^n\frac{1}{n}+\ldots$$

which is convergent by Leibnitz test for alternating series. When $x=-1$, the series becomes

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots\right),$$

which is divergent, being negative of harmonic series of order 1.

Hence the logarithmic series (3.9) converges for $-1<x\le 1$.

(c) The binomial series: *The binomial series*

$$1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots\infty \quad \text{...(3.10)}$$

converges for $|x|<1$.

Here, $$\lim_{r\to\infty}\frac{u_{r+1}}{u_r}=\lim_{r\to\infty}\frac{n(n-1)\ldots(n-r+1)}{r!}x^r\frac{(r-1)!}{n(n-1)\ldots(n-r+2)x^{r-1}}$$

$$=\lim_{r\to\infty}\frac{n-r+1}{r}x=\lim_{r\to\infty}\left(\frac{n+1}{r}-1\right)x=-x.$$

Hence by ratio test the series (3.10) converges for $|x|<1$, diverges for $|x|>1$.

Example 3.22: Examine the convergence of the series $\frac{1}{1-x}+\frac{1}{2(1-x)^2}+\frac{1}{3(1-x)^3}+\ldots (x\neq 1)$.

Solution: Here $$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{|n(1-x)^n|}{|(n+1)(1-x)^{n+1}|}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)|1-x|}=\frac{1}{|1-x|}.$$

Hence, by ratio test the series is convergent for $|1-x|>1$ and divergent for $|1-x|<1$, except at $|1-x|=0$; and the test fails at $|1-x|=1$.

When $(1-x)=1$, that is $x=0$, the series becomes $1+\frac{1}{2}+\frac{1}{3}+\ldots$ which is divergent, and when $(1-x)=-1$, that is at $x=2$, the series becomes $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots$ which is convergent.

Thus, the given series is convergent for $|1-x|>1$ and for $x=2$; divergent for $|1-x|<1$ and for $x=0$, except at $|1-x|=0$ for which it is not defined.

EXERCISE 3.8

Find the interval of convergence for the following series.

1. $x-\frac{x^2}{\sqrt{2}}+\frac{x^3}{\sqrt{3}}-\frac{x^4}{\sqrt{4}}+\ldots$

2. $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots$

3. $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots$

4. $\frac{x}{1+\sqrt{1}}+\frac{x^2}{2+\sqrt{2}}+\frac{x^3}{3+\sqrt{3}}+\ldots$

5. $\sum\frac{(x+2)^n}{3^n.n}$

6. $\sum\frac{x^n}{(2n-1)^2 2^n}$

7. $\sum\frac{nx^n}{(n+1)(n+2)}\ (x>0)$

8. $\sum\frac{2^2.4^2.6^2\ldots(2n)^2}{3^2.5^2.7^2\ldots(2n+1)^2}x^n$

9. $\sum_{n=0}^{\infty}\frac{(3x+5)^n}{(n+1)!}$

10. $\sum\frac{(-1)^n(x-1)^{2n}}{n.3^n}$

ANSWERS

Exercise 3.1 (p. 156)

1. (a) $\frac{n}{n+1}$ (b) $\frac{n^2}{n!}$ (c) $\frac{1}{n(n+1)}$ (d) $\left(\frac{n+1}{n}\right)^n$
2. (a) Convergent (b) Convergent (c) Convergent (d) Divergent
3. (a) Divergent (b) Convergent (c) Oscillatory (d) Oscillatory
(e) Divergent (f) Divergent.

Exercise 3.2 (p. 161)

1. Convergent 2. Divergent 3. Convergent 4. Convergent
5. Divergent 6. Convergent 7. Divergent 8. Divergent
9. Convergent for $x<\frac{1}{e}$ and divergent for $x\geq\frac{1}{e}$ 10. Divergent
11. Convergent 12. Convergent 13. Convergent 14. Divergent
15. Convergent for $q>1+p$, and divergent for $q\leq 1+p$.

Exercise 3.3 (p. 165)

1. Convergent
2. Convergent
3. Convergent for $x \leq 1$, and divergent for $x > 1$
4. Convergent
5. Convergent for $x < 1$; and divergent for $x \geq 1$
6. Convergent if $\beta > \alpha > 0$; and divergent if $\alpha \geq \beta > 0$
7. Convergent if $x < 1$; and divergent if $x \geq 1$
8. Convergent for $x \leq 1$, and divergent for $x > 1$
9. Convergent for $x < 1$, and divergent for $x \geq 1$
10. Convergent for $x < 1$ or $x > 1$, and divergent for $x = 1$
11. Convergent for $x > a$, and divergent for $x \leq a$
12. Convergent
13. Convergent
14. Divergent
15. Convergent.

Exercise 3.4 (p. 171)

1. Divergent
2. Convergent for $x \leq 1$, and divergent for $x > 1$
3. Convergent for $x < \frac{1}{e}$, and divergent for $x \geq \frac{1}{e}$
4. Convergent for $x \leq 1$, and divergent for $x > 1$
5. Convergent for $x^2 < 4$, and divergent for $x^2 \geq 4$
6. Convergent for $x < \frac{1}{e}$, and divergent for $x \geq \frac{1}{e}$
7. Convergent for $x < 1$, and divergent for $x \geq 1$
8. Divergent
9. Convergent for $x < 1$ or at $x = 1$ for $\gamma > \alpha + \beta$, and divergent for $x > 1$ or at $x = 1$ for $\gamma \leq \alpha + \beta$
10. Convergent for $x < 1$, and divergent for $x \geq 1$
11. Convergent
12. Convergent for $x^2 < 1$, and divergent for $x^2 \geq 1$
13. Convergent for $x > 2$, and divergent for $x \leq 2$
14. Convergent for $x < 1$ or for $x = 1$ and $b > a + d$, and divergent for $x > 1$ or for $x = 1$ and $b \leq a + d$
15. Convergent for $x < 1$, and divergent for $x \geq 1$.

Exercise 3.5 (p. 174)

1. Convergent
2. Convergent
3. Convergent for $x < 1$, and divergent for $x \geq 1$
4. Convergent
5. Convergent for $x < 1$, and divergent for $x \geq 1$
6. Convergent
7. Convergent for $x < e$, and divergent for $x \geq e$
8. Divergent
9. Convergent for $a, b, < 1$, and divergent for $a, b \geq 1$
10. Convergent for $k > 1$, and divergent for $k \leq 1$.

Exercise 3.6 (p. 176)

1. Oscillatory
2. Convergent
3. Convergent
4. Oscillatory
5. Convergent
6. Convergent
7. Convergent
8. Convergent
9. Not convergent.

Exercise 3.7 (p. 178)

1. Absolutely convergent
2. Absolutely convergent
3. Conditionally convergent
4. Absolutely convergent
5. Absolutely convergent

Exercise 3.8 (p. 181)

1. $-1 < x \leq 1$
2. $-1 < x \leq 1$
3. $x < \frac{1}{e}$
4. $x < 1$
5. $x < 1$
6. $x < 2$
7. $x < 1$
8. $x < 1$
9. $-\infty < x < \infty$
10. $1 - \sqrt{3} < x < 1 + \sqrt{3}$.